The Wave Equation and Rotation

Gary D. Simpson

email: gsim100887@aol.com

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Summary

A vector solution to the spherical wave equation is presented. It is based upon three-dimensional versions of Euler's Equation. A requirement of the solution method is rotation of the wave media. The rotation itself is believed by the author to account for the strong force. The solution resembles electro-magnetism and it resembles gravity. Applicability of the solution to the weak force is presently unclear. The solution creates a mechanism to account for action at a distance. Because interactions are the result of rotation, action at a distance is not limited by the wave velocity in the medium. A modified form of the wave equation is presented to more clearly illustrate the rotational feature. A solution is presented that satisfies both the classical wave equation and the Schrödinger Wave Equation. Equation 16 is the most significant result presented.

Preface

The reader is encouraged to study the Appendices. The author is neither a mathematician nor a physicist. The mathematical tools used are quaternions, differential equations (specifically, separation of variables), and Euler's equation. These tools are perhaps intimidating to the reader. Hopefully the author has presented them in a manner that is readily understandable. It is hoped by the author that the quaternion based solution method will find application elsewhere.

Discussion

The wave equation is written as follows[1]:

Equation 1

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

The left-hand-side of equation 1 is known as the Laplacian[2]. The use of the Laplacian is convenient because it allows the user to move easily between coordinate systems. Equation 2 is equation 1 written in Laplacian form.

Equation 2

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

It is desired by the author to work with one dimensional spherical coordinates. Therefore, the appropriate substitution[3] is made for the Laplacian and equation 2 is written as equation 3.

Equation 3

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Wolff[4] presents an exact scalar solution for equation 3. This text combines that solution with quaternions to form a vector solution to the spherical wave equation.

Quaternions were developed by Sir William Rowan Hamilton as a method to rotate vectors. The most important concepts regarding quaternions are presented by Hamilton[5]. These concepts are restated in Appendix A with several useful identities. Appendix A continues by presenting rotations of the unit vector **i** about the **j** axis and the **k** axis.

Euler's equation is written as follows[6]:

Equation 4

$$e^{ix} = \cos x + i\sin x$$

In equation 4, i is understood to be a generic sqrt(-1) as opposed to the vector $\mathbf{i}$. This distinction will be lost in Appendix B.

Appendix B combines Euler's equation with the results from Appendix A. The result is four vector equations and their opposites. These equations are actually three-dimensional forms of Euler's equation and are presented below in equations 5-8:

Equation 5 (counter-clockwise)

$$F_{i,k} = \mathbf{i}e^{-\mathbf{k}\theta_k} = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

Equation 6 (clock-wise)

$$F_{i,k}^{*} = \mathbf{i}e^{+\boldsymbol{k}\theta_k} = \mathbf{i}\cos(\theta_k) - \mathbf{j}\sin(\theta_k)$$

Equation 7 (clock-wise)

$$F_{i,j} = \mathbf{i}e^{+\mathbf{j}\theta_j} = \mathbf{i}\cos(\theta_j) + \mathbf{k}\sin(\theta_j)$$

Equation 8 (counter-clockwise)

$$F_{i,j}^{*} = \mathbf{i}e^{-\mathbf{j}\theta_j} = \mathbf{i}\cos(\theta_j) - \mathbf{k}\sin(\theta_j)$$

In equations 5-8, **i**, **j**, and **k** are unit vectors. The symbols Θ_k and Θ_j represent rotations about the **k** and **j** axes respectively. In spherical coordinates these would roughly equate with θ and ϕ. The superscripted * is used to indicate conjugation. The subscripts on the F are used to designate which vector is being rotated and the axis about which the rotation is occurring. It would be equally valid to derive a set of vector functions based upon rotating **j** or **k** instead of **i**.

Appendix C uses equations 5-8 in the Laplacian to solve the spherical wave equation. The vector **i** is taken to be in the direction of r. The solution requires that $\Theta_k = \Theta_j = r$. This requirement is not possible because the measuring units of angles and distance are not equal (i.e., radians vs. meters). Therefore, the wave equation must be revised from the way it was originally written for the solution to be valid. Appendix C shows that if the wave equation is written as shown in equation 9 below, then the solutions are as shown in equations 10 and 11 below. In equations 9-11, the rotation is governed by $\Theta_k = \Theta_j = \beta r$. In principle, β could be a positive or a negative constant and θ_k and θ_j could have opposite β's. The separation of variables constant is α^2. The letter l is used to designate the sqrt(-1) as used in the time portion of the equations to prevent confusion with the three unit vectors (**i**, **j**, and **k**).

Equation 9

$$\frac{1}{\beta^2}\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Equation 10

$$\psi(r,t) = \pm\frac{2}{r}\left(e^{+\alpha c \mathrm{l} t} + e^{-\alpha c \mathrm{l} t}\right)\begin{bmatrix} +\sin(\alpha\beta r) & +\sin(\alpha\beta r) \\ +\sin(\alpha\beta r) & -\sin(\alpha\beta r)\end{bmatrix}\begin{bmatrix}\mathbf{j} \\ \mathbf{k}\end{bmatrix}$$

Note that these four solutions sum to zero. For the case where r >> 0, equation 9 may be alternately solved to yield the following:

Equation 11

$$\psi(r,t) = \frac{4\mathbf{i}}{r}\cos(\alpha\beta r)\left(e^{+\alpha clt} + e^{-\alpha clt}\right)$$

In the discussion thus far, the vector **i** has been rotated about the **j** axis and the **k** axis. The results of these rotations were incorporated into the Laplacian portion of the wave equation and shown to produce a solution. Now, rotations of the vectors **j** and **k** about the **i** axis are considered with the objective being to insert them into the time portion of the wave equation. Appendix D develops these rotations and the results are as follows:

Equation 12 (counter-clockwise)

$$F_{j,i} = \mathbf{j}e^{-\mathbf{i}\theta_i} = \mathbf{j}\cos(\theta_i) + \mathbf{k}\sin(\theta_i)$$

Equation 13 (clock-wise)

$$F^*_{j,i} = \mathbf{j}e^{+\mathbf{i}\theta_i} = \mathbf{j}\cos(\theta_i) - \mathbf{k}\sin(\theta_i)$$

Equation 14 (clock-wise)

$$F_{k,i} = \mathbf{k}e^{+\mathbf{i}\theta_i} = \mathbf{k}\cos(\theta_i) + \mathbf{j}\sin(\theta_i)$$

Equation 15 (counter-clockwise)

$$F^*_{k,i} = \mathbf{k}e^{-\mathbf{i}\theta_i} = \mathbf{k}\cos(\theta_i) - \mathbf{j}\sin(\theta_i)$$

Appendix E uses equations 12-15 as the basis for the time portion of the wave equation solution. The result is presented below in equation 16. The rotation angle θ_i is set equal to αct.

Equation 16

$$\psi(r,t) = R(r) \times T(t)$$

Where R(r) and T(t) are given by the following:

Equation 16.1

$$R(r) = \pm\frac{2}{r}\begin{bmatrix} +\sin(\alpha\beta r) & +\sin(\alpha\beta r) \\ +\sin(\alpha\beta r) & -\sin(\alpha\beta r) \end{bmatrix}\begin{bmatrix} \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

Equation 16.2

$$T(t) = \begin{bmatrix} (\cos(\alpha ct) + \sin(\alpha ct)) & (\cos(\alpha ct) - \sin(\alpha ct)) \\ (\cos(\alpha ct) - \sin(\alpha ct)) & (\cos(\alpha ct) + \sin(\alpha ct)) \end{bmatrix}\begin{bmatrix} \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

In the development of equation 16.2, the i that is used to form the complex exponential with time is the unit vector **i**. The direction of this T vector rotates with respect to time in either the clock-wise or counter-clockwise direction when viewed down the **i** axis. Figure 1 at the end of the text illustrates the vectors that result from R(r) and T(t). The vectors resulting from R have a constant direction but a variable length that is dependent upon r. They are linearly polarized. The maximum length of the R(r) vectors is sqrt(2). The vectors that result from T have a constant length but a variable direction that is dependent upon t. They are circularly polarized. These vectors have a constant length equal to sqrt(2). The interaction of these two sets of vectors may very well account for what we perceive as forces. The vectors associated with R(r) may be visualized as the threads on a machine bolt while the vectors associated with T(t) are the threads on the matching nut. Forces would then be viewed as the interaction of rotations associated with separate sources.

Equation 16 can be simplified if desired into four (actually eight) alternate forms. These are presented below in equations 16.A.1 – 16.A.4. The derivations are presented in Appendix F. The matrix multiplication performed is a set of vector cross product multiplications rather than a series of row-column multiplications.

Equation 16.A.1 (clock-wise)

$$\pm\frac{4\sin(\alpha\beta r)}{r}(-\cos(\alpha ct)-\ \mathbf{i}\sin(\alpha ct))$$

Equation 16.A.2 (clock-wise)

$$\pm\frac{4\sin(\alpha\beta r)}{r}(-\sin(\alpha ct)+\ \mathbf{i}\cos(\alpha ct))=\ \pm\frac{4\sin(\alpha\beta r)}{r}\left(-\cos\left(\frac{\pi}{2}-\alpha ct\right)+\ \mathbf{i}\sin\left(\frac{\pi}{2}-\alpha ct\right)\right)$$

Equation 16.A.3 (counter-clockwise)

$$\pm\frac{4\sin(\alpha\beta r)}{r}(-\cos(\alpha ct)+\ \mathbf{i}\sin(\alpha ct))$$

Equation 16.A.4 (counter-clockwise)

$$\pm\frac{4\sin(\alpha\beta r)}{r}(+\sin(\alpha ct)+\ \mathbf{i}\cos(\alpha ct))=\ \pm\frac{4\sin(\alpha\beta r)}{r}\left(+\cos\left(\frac{\pi}{2}-\alpha ct\right)+\ \mathbf{i}\sin\left(\frac{\pi}{2}-\alpha ct\right)\right)$$

If the time portions of equations 16.A.1-16.A.4 are labeled as T_1-T_4 respectively, it is quite easy to show that $\mathbf{i}T_1$ = -T_2, $\mathbf{i}T_2$ = T_1, $\mathbf{i}T_3$ = -T_4, and $\mathbf{i}T_4$ = T_3. Therefore, equations 16.A.1-16.A.4 may also be written as equations 16.B.1 and 16.B.2. Note that T_3 is the conjugate of T_1.

Equation 16.B.1 (clock-wise)

$$\psi^*(r,t)=\ +\frac{4\sin(\alpha\beta r)}{r}(-\cos(\alpha ct)-\ \mathbf{i}\sin(\alpha ct))\mathbf{i}^n\ where\ n=0,1,2,3$$

Equation 16.B.2 (counter-clockwise)

$$\psi(r,t) = +\frac{4\sin(\alpha\beta r)}{r}(-\cos(\alpha ct) + \mathbf{i}\sin(\alpha ct))\mathbf{i}^n \; where \; n = 0, 1, 2, 3$$

Figures 2 and 3 and the end of the text illustrate the solution to equation 16.B.2 for n = 0.

Now Schrödinger will be considered. Equation 17 is a one dimensional form of the Schrödinger Wave Equation[7]. In equation 17, h is actually ħ but h is used instead of ħ because of font limitations.

Equation 17

$$-ih\frac{\partial\psi}{\partial t} = \frac{h^2}{2m}\frac{\partial^2\psi}{\partial x^2}$$

The main difference between the classical wave equation and Schrödinger is the time derivative. In Schrödinger, the 1'st derivative is used and there is a multiplication factor of i included. In the classical equation, the 2'nd time derivative is used. Since the time portion of the solution is complex, taking the 2'nd time derivative of Schrödinger will convert the –i into +1.

Equations 9 and 17 can both be solved for the Laplacian of ψ and then the 1'st and 2'nd time derivatives can be set equal to each other. This is done in Appendix G with the result being that equation 16 will satisfy both the classical wave equation and the Schrödinger Equation if either equation 18.1 or equation 18.2 is true.

Equation 18.1 (counter-clockwise rotation)

$$\alpha\beta^2 = \frac{2mc}{h}$$

Equation 18.2 (clock-wise rotation)

$$-\alpha\beta^2 = \frac{2mc}{h}$$

Equations 18.1 and 18.2 seem to imply that only one part of equation 16.2 can be valid when both the Schrödinger Wave Equation and the classical wave equation are true. Perhaps this is the difference between matter and anti-matter. If this is true, then matter and anti-matter should have reverse rotations with one favoring counter-clockwise and the other favoring clock-wise. Since forces may be viewed as the interaction of these rotations, the gravitational force between matter and anti-matter would be repulsive while the gravitational force between matter and matter or between anti-matter and anti-matter would be attractive. Perhaps this accounts for Einstein's cosmological constant. Distant galaxies might be anti-matter!

The author will further speculate that equation 16 contains at least 8 solutions. The two solutions associated with the T matrix represent matter and anti-matter. The two solutions associated with the R matrix represent up-spin and down-spin. The (+/-) sign associated with the R matrix represents charge.

Other slight variations would result from selecting β as positive or negative. Please note that in this view, the positron might not be considered anti-matter.

A quaternion is capable of changing the length of a vector in addition to performing a rotation. Upon first consideration, this seems rather trivial. Equation 19 was taken from Thomas[8] and may be used to produce an infinite series solution.

Equation 19

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$$

Using equation 19, equations 5-8 and equations 12-15 may be rewritten as equations 20-27:

Equation 20

$$F_{i,k} = \sum_{n=1}^{\infty} \frac{\mathbf{i}e^{-\mathbf{k}(\theta_k+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}(\mathbf{i}\cos(\theta_k + 2\pi n) + \mathbf{j}\sin(\theta_k + 2\pi n))$$

Equation 21

$$F_{i,k}^{*} = \sum_{n=1}^{\infty} \frac{\mathbf{i}e^{+\mathbf{k}(\theta_k+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}(\mathbf{i}\cos(\theta_k + 2\pi n) - \mathbf{j}\sin(\theta_k + 2\pi n))$$

Equation 22

$$F_{i,j} = \sum_{n=1}^{\infty} \frac{\mathbf{i}e^{+\mathbf{j}(\theta_j+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\mathbf{i}\cos\left(\theta_j + 2\pi n\right) + \mathbf{k}\sin\left(\theta_j + 2\pi n\right)\right)$$

Equation 23

$$F_{i,j}^{*} = \sum_{n=1}^{\infty} \frac{\mathbf{i}e^{-\mathbf{j}(\theta_j+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}\left(\mathbf{i}\cos\left(\theta_j + 2\pi n\right) - \mathbf{k}\sin\left(\theta_j + 2\pi n\right)\right)$$

Equation 24

$$F_{j,i} = \sum_{n=1}^{\infty} \frac{\mathbf{j}e^{-\mathbf{i}(\theta_i+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}(\mathbf{j}\cos(\theta_i + 2\pi n) + \mathbf{k}\sin(\theta_i + 2\pi n))$$

Equation 25

$$F_{j,i}^{*} = \sum_{n=1}^{\infty} \frac{\mathbf{j}e^{+\mathbf{i}(\theta_i+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}(\mathbf{j}\cos(\theta_i + 2\pi n) - \mathbf{k}\sin(\theta_i + 2\pi n))$$

Equation 26

$$F_{k,i} = \sum_{n=1}^{\infty} \frac{\mathbf{k}e^{+\mathbf{i}(\theta_i+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}(\mathbf{k}\cos(\theta_i + 2\pi n) + \mathbf{j}\sin(\theta_i + 2\pi n))$$

Equation 27

$$F_{k,i}^* = \sum_{n=1}^{\infty} \frac{\mathbf{k}e^{-\mathbf{i}(\theta_i+2\pi n)}}{2^n} = \sum_{n=1}^{\infty} \frac{1}{2^n}(\mathbf{k}\cos(\theta_k + 2\pi n) - \mathbf{j}\sin(\theta_i + 2\pi n))$$

Equations 20-27 are not used in this text. They are presented merely for completeness.

In the development of equation 9, it is stated that ϕ = θ = βr. This traces out a spiraling string. It is reasonable to wonder regarding the length of this string and if the mass of a particle can be correlated to this length. This is done in Appendix H and equations 28 and 29 are presented. Essentially, a particle is viewed as a spinning ball of string.

Equation 28

$$L = \beta\sqrt{2}\left(\frac{R}{2}\sqrt{\frac{1}{2\beta^2} + R^2} + \frac{1}{4\beta^2}\sinh^{-1}\left(\beta R\sqrt{2}\right)\right)$$

For the condition that beta is very large, equation 28 is approximated by equation 29.

Equation 29

$$L = \frac{1}{2}\beta R^2\sqrt{2}$$

As a reminder, h is actually ħ but ħ is not used due to font limitations.

Solving equation 18.1 for β yields the following:

Equation 30

$$\beta = \pm\sqrt{\frac{2mc}{\alpha h}}$$

Equation 30 may be substituted into either equation 28 or 29 to produce a relation between string length and mass. Combining the positive root with equation 29 produces the following:

Equation 31

$$L = R^2\sqrt{\frac{mc}{\alpha h}}$$

Equation 31 may be used to compare the mass of particles. Doing so yields the following:

Equation 32

$$\frac{m_2}{m_1} = \frac{L_2^2 R_1^4 \alpha_2}{L_1^2 R_2^4 \alpha_1}$$

So what does all of this suggest? The author believes that spin is the only truly intrinsic property. Protons, neutrons, and electrons all have spin – and all have the same spin - because the space within which they are contained is rotating. Further, the author believes that every point in the universe is rotating in this way. We perceive the interactions of these rotations as particles or forces or both. It is hoped that the parameters α, β, and R combined with equations 16 and 29 will allow for the description of all possible particles and forces. Ideally, there should be some basis or justification for the parameter values selected.

Consider the strong force. If two rotating objects come into contact with each other, they can either be pulled together or pushed apart depending upon the relative directions of the rotation at the point of contact. Within this model, the strong force is thought to be the result of a clock-wise rotation around one wave center interacting with a counter-clockwise rotation around another wave center or vice-versa.

Next, consider electro-magnetism. By using dot-products, it is easy to show that the vector **i** is perpendicular to all of the solutions presented in equation 10. It is also easy to show that within each pair of solutions, the individual solutions are perpendicular to each other. These features are consistent with electro-magnetism.

Lastly, consider gravity. The solution presented as equation 11 acts in the **i** direction. Also, since the wave interactions are the result of rotation rather than longitudal waves, action at a distance is not limited by the wave velocity through the medium.

For the ideas presented above to have any credibility, it must be possible to use rotation to form some type of stable structure. This can be done in two-dimensions (x, y) as follows:

1. Begin at point (0, 0) with r being in the +x direction. Apply r = ϕ = θ and move from r = 0 to r = +2π. This will trace a string in three-dimensional space that will end at the point (+2π, 0). At the end point, the trace will be moving in the +y direction.
2. Now continue with r being in the +y direction. Apply r = ϕ = θ and move from r = 0 to r = +2π. This will trace a string in three-dimensional space that will end at the point (+2π, +2π). At the end point, the trace will be moving in the -x direction.
3. Now continue with r being in the -x direction. Apply r = ϕ = θ and move from r = 0 to r = +2π. This will trace a string in three-dimensional space that will end at the point (0, +2π). At the end point, the trace will be moving in the -y direction.
4. Lastly, continue with r being in the -y direction. Apply r = ϕ = θ and move from r = 0 to r = +2π. This will trace a string in three-dimensional space that will end at the point (0, 0). At the end

point, the trace will be moving in the +x direction. This completes the structure and illustrates how the four R(r) solutions may be combined. Whether this structure represents an electron, the vacuum, or something else is not clear to the author. Other structures are also possible. Please note that in this construction, the string itself is continuous, but the various derivatives might not be continuous.

It should also be possible to use the concept of rotation to gain some insight into the structure of the proton and neutron. The following is a fairly loose argument but is very interesting and thought provoking none the less. The volume of a sphere is $(4/3)\pi r^3$. Rotation implies $\beta r = \phi = \theta$ and converts the volume equation into something that is proportional to π^4. The ratio of the mass of the proton to the mass of the electron as reported on Wikipedia is 1836.15267245(75)[9]. Taking this value and dividing by π^4 yields ~18.849911. If this quotient is divided again by π, the result is 6.00011. The author submits that there is a geometric argument that the mass ratio of the proton to the electron is $6\pi^5$ (~1836.12) and that this argument somehow invokes rotation. The author does not currently claim to know the nature of this geometric argument. As an additional piece of evidence, it is claimed by Reinhold[10] *et al* that this mass ratio has increased by .002% over the last 12 billion years. If this adjustment is made to the value of $6\pi^5$ then the result becomes 1836.15. This is accurate to within roughly one part per million.

Conclusion

The work of Wolff[4] is extended into a vector solution of the spherical wave equation. This solution is shown to be qualitatively consistent with the Dirac Wave Equation, the Schrödinger Wave Equation, the strong force, electro-magnetism, and gravity. A rotating wave media is an integral part of the solution method. Lastly, a closed structure is presented that incorporates the four R(r) solutions presented for the wave equation. All of this, taken together, leads the author to conclude that there is a tangible wave medium, that this wave medium is subjected to point-wise rotation at unimaginable speed, and that the result of the interaction of these rotational waves is our observed universe. The author believes that the wave medium itself is the source of everything that we perceive as matter and energy.

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The author acknowledges gratefully the works of Don Hotson and Milo Wolff. Don Hotson's work was not explicitly referenced herein, but it was very influential in the thinking that resulted in this text. Milo Wolff's work was explicitly cited, but that one small reference does not begin to do justice to his efforts. The author also thanks Charlie Papazian for the book "The Complete Joy of Home Brewing". Relax. Don't worry. Have a homebrew.

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Appendix A

Quaternions:

A quaternion Q is defined by Hamilton[5] as follows:

Equation A.1

$$Q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$$

Hamilton[5] further made the following definitions:

Equation A.2

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$$

It is essential to understand that within Hamilton's system, the sequence of multiplication is important. Therefore, **ij** ≠ **ji** but rather **ij** = -**ji**. The identities within equation A.2 allow the following statements:

Equation A.3

$$\mathbf{i}(\mathbf{ijk}) = -\mathbf{i}\ therefore\ \mathbf{jk} = \mathbf{i}$$

Equation A.4

$$(\mathbf{ijk})\mathbf{k} = -\mathbf{k}\ therefore\ \mathbf{ij} = \mathbf{k}$$

Equation A.5

$$\mathbf{i}(\mathbf{ijk})\mathbf{k} = -\mathbf{ik}\ therefore\ \mathbf{j} = -\mathbf{ik}$$

Now consider a vector r**i** in the **i**-**j** plane. It is desired to rotate this vector counter-clockwise about the **k** axis to form a new vector of length r. The problem is constructed as follows:

$$r\mathbf{i}Q = r\mathbf{i}\cos\theta_k + r\mathbf{j}\sin\theta_k$$

$$r\mathbf{i}(q_0 + q_3\mathbf{k}) = r\mathbf{i}\cos(\theta_k) + r\mathbf{j}\sin(\theta_k)$$

In the quaternion above, q_1 and q_2 are zero because the rotation is only about the **k** axis. The r term can be eliminated by dividing both sides of the equation by r. This gives the following:

$$q_0\mathbf{i} + q_3\mathbf{ik} = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

Since **ik** = -**j**, it follows that:

$$q_0\mathbf{i} - q_3\mathbf{j} = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

By inspection, $q_0 = \cos(\Theta_k)$ and $q_3 = -\sin(\Theta_k)$. Substituting these values back into the rotation yields:

Equation A.6

$$\mathbf{i}(\cos(\theta_k) - \mathbf{k}\sin(\theta_k)) = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

If these steps are repeated for a clock-wise rotation about **k**, the result is as follows:

Equation A.7

$$\mathbf{i}(\cos(\theta_k) + \mathbf{k}\sin(\theta_k)) = \mathbf{i}\cos(\theta_k) - \mathbf{j}\sin(\theta_k)$$

Next consider clockwise rotation of the vector r**i** in the **i**-**k** plane about the **j** axis. The problem is stated as follows:

$$r\mathbf{i}Q = r\mathbf{i}\cos(\theta_j) + r\mathbf{k}\sin(\theta_j)$$

$$r\mathbf{i}(q_0 + q_2\mathbf{j}) = r\mathbf{i}\cos(\theta_j) + r\mathbf{k}\sin(\theta_j)$$

Repeating the process described above produces:

$$q_0\mathbf{i} + q_2\mathbf{ij} = \mathbf{i}\cos(\theta_j) + \mathbf{k}\sin(\theta_j)$$

By inspection, $q_0 = \cos(\Theta_j)$ and since **ij** = **k**, $q_2 = \sin(\Theta_j)$. Substitution into the rotation yields:

Equation A.8

$$\mathbf{i}\left(\cos(\theta_j) + \mathbf{j}\sin(\theta_j)\right) = \mathbf{i}\cos(\theta_j) + \mathbf{k}\sin(\theta_j)$$

Repeating this exercise for a counter-clockwise rotation produces:

Equation A.9

$$\mathbf{i}\left(\cos(\theta_j) - \mathbf{j}\sin(\theta_j)\right) = \mathbf{i}\cos(\theta_j) - \mathbf{k}\sin(\theta_j)$$

Appendix B

Euler's equation is typically written as follows[6]:

Equation B.1

$$e^{ix} = \cos(x) + i\sin(x)$$

In this equation it is typically understood that i is simply a generic sqrt(-1). Also, sin(-x) = -sin(x). Therefore, equation B.1 can also be written as follows:

Equation B.2

$$e^{-ix} = \cos(x) - i\sin(x)$$

Compare equations B.1 and B.2 with equations A.6-A.9. Euler's equation is essentially the quaternion in each of these four rotations! Making use of this fact and taking care to preserve the unit vectors produces the following four equations and their opposites:

Equation B.3 (counter-clockwise)

$$F_{i,k} = \mathbf{i}e^{-\mathbf{k}\theta_k} = \mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k)$$

Equation B.4 (clock-wise)

$$F_{i,k}^{*} = \mathbf{i}e^{+\mathbf{k}\theta_k} = \mathbf{i}\cos(\theta_k) - \mathbf{j}\sin(\theta_k)$$

Equation B.5 (clock-wise)

$$F_{i,j} = \mathbf{i}e^{+\mathbf{j}\theta_j} = \mathbf{i}\cos(\theta_j) + \mathbf{k}\sin(\theta_j)$$

Equation B.6 (counter-clockwise)

$$F_{i,j}^{*} = \mathbf{i}e^{-\mathbf{j}\theta_j} = \mathbf{i}\cos(\theta_j) - \mathbf{k}\sin(\theta_j)$$

The ultimate objective is to use equations B.3-B.6 to solve the wave equation. Therefore the 1'st and 2'nd derivatives of each are needed.

Equation B.3.1

$$\frac{dF_{i,k}}{d\theta_k} = -\mathbf{j}e^{-\mathbf{k}\theta_k} = -\mathbf{i}\sin(\theta_k) + \mathbf{j}\cos(\theta_k)$$

Equation B.3.2

$$\frac{d^2F_{i,k}}{d\theta_k^2} = -\mathbf{i}e^{-\mathbf{k}\theta_k} = -\mathbf{i}\cos(\theta_k) - \mathbf{j}\sin(\theta_k) = -F_{i,k}$$

Equation B.4.1

$$\frac{dF_{i,k}^*}{d\theta_k} = \mathbf{j}e^{+\mathbf{k}\theta_k} = -\mathbf{i}\sin(\theta_k) - \mathbf{j}\cos(\theta_k)$$

Equation B.4.2

$$\frac{d^2F_{i,k}^*}{d\theta_k^2} = -\mathbf{i}e^{+\mathbf{k}\theta_k} = -\mathbf{i}\cos(\theta_k) + \mathbf{j}\sin(\theta_k) = -F_{i,k}^*$$

Equation B.5.1

$$\frac{dF_{i,j}}{d\theta_j} = -\mathbf{k}e^{+\mathbf{j}\theta_j} = -\mathbf{i}\sin(\theta_j) + \mathbf{k}\cos(\theta_j)$$

Equation B.5.2

$$\frac{d^2F_{i,j}}{d\theta_j^2} = -\mathbf{i}e^{+\mathbf{j}\theta_j} = -\mathbf{i}\cos(\theta_j) - \mathbf{k}\sin(\theta_j) = -F_{i,j}$$

Equation B.6.1

$$\frac{dF_{i,j}^*}{d\theta_j} = \mathbf{k}e^{-\mathbf{j}\theta_j} = -\mathbf{i}\sin(\theta_j) - \mathbf{k}\cos(\theta_j)$$

Equation B.6.2

$$\frac{d^2F_{i,j}^*}{d\theta_j^2} = -\mathbf{i}e^{-\mathbf{j}\theta_j} = -\mathbf{i}\cos(\theta_j) + \mathbf{k}\sin(\theta_j) = -F_{i,j}^*$$

Appendix C

Begin with the wave equation in spherical coordinates:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

The separation of variables method assumes that the solution to a problem such as this has the following form:

Equation C.1

$$\psi(r,t) = R(r)T(t)$$

R(r) is a function of only r and T(t) is a function of only t. Therefore, R(r) is a constant for the t derivatives and T(t) is a constant for the r derivatives. This assumption concerning ψ also allows the partial derivatives to be replaced by total derivatives.

$$\frac{\partial\psi}{\partial t} = R\frac{dT}{dt}$$

$$\frac{\partial^2\psi}{\partial t^2} = R\frac{d^2T}{dt^2}$$

$$\frac{\partial\psi}{\partial r} = T\frac{dR}{dr}$$

Making these substitutions into the wave equation and rearranging gives the following:

Equation C.2

$$\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{1}{T}\frac{1}{c^2}\frac{d^2T}{dt^2} = -\alpha^2$$

The α term is included because the only way for the two sets of differentials to be equal is if they are both equal to the same constant. To understand this, you simply need to remember that the solution is assumed to be of the form shown in equation C.1.

Differential equations are seldom solved by integration. Instead they are solved by guessing a solution and performing the necessary differentiations and then substituting the results into the differential equation. For the wave equation, the author will assume that equations B.3-B.6 are solutions to the R(r) portion of the problem. The time portion will simply be the sum of two exponentials.

Scalar Solution to Wave Equation:

Prior to attempting the vector solution, it is helpful to revisit the scalar solution. This was presented by Wolff[4].

$$R(r) = \frac{1}{r}\left(Ae^{+\alpha ir} + Be^{-\alpha ir}\right)$$

The function R must have a finite value at r = 0. Remembering Euler, it is necessary that B = -A. This will cause the cosine terms to cancel each other. For the sine terms, sin(r) divided by r has a finite limiting value at r = 0. This can be shown using L'Hôpital's Rule. Simplifying R gives the following:

$$R(r) = \frac{A}{r}\left(e^{+\alpha ir} - e^{-\alpha ir}\right)$$

The next task is to determine the 1'st and 2'nd derivatives with respect to r.

$$\frac{dR}{dr} = \frac{A}{r}\left(\alpha ie^{+\alpha ir} + \alpha ie^{-\alpha ir}\right) - \frac{A}{r^2}\left(e^{+\alpha ir} - e^{-\alpha ir}\right)$$

$$\frac{dR}{dr} = \frac{1}{r}\left(A\left(\alpha ie^{+\alpha ir} + \alpha ie^{-\alpha ir}\right) - R\right)$$

$$\frac{d^2R}{dr^2} = \frac{1}{r}\left(A\left(-\alpha^2 e^{+\alpha ir} + \alpha^2 e^{-\alpha ir}\right) - \frac{dR}{dr}\right) - \frac{1}{r^2}\left(A\left(\alpha ie^{+\alpha ir} + \alpha ie^{-\alpha ir}\right) - R\right)$$

$$\frac{d^2R}{dr^2} = -\alpha^2 R - \frac{1}{r}\frac{dR}{dr} - \frac{1}{r}\frac{dR}{dr}$$

Equation C.3

$$\frac{d^2R}{dr^2} = -\alpha^2 R - \frac{2}{r}\frac{dR}{dr}$$

The next task is to take the R portion of equation C.2 and expand it.

Equation C.4

$$\frac{1}{R}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \frac{1}{R}\frac{1}{r^2}\left(r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right) = \frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{2}{r}\frac{dR}{dr}\right)$$

Next substitute equation C.3 into equation C.4 and simplify the result. This will eliminate the 1'st derivative term and demonstrate that R satisfies equation C.2.

The T portion of equation C.2 is much simpler. Assume T as follows:

$$T(t) = De^{+\alpha cit} + Ee^{-\alpha cit}$$

Since R is of the form isin(r), it follows that T must also be of this form in order for RT to be real (i.e., both must include i so that their product will be real). Therefore, E = -D is chosen to cause the cosine terms to cancel.

$$T(t) = D\left(e^{+\alpha cit} - e^{-\alpha cit}\right)$$

The 1'st and 2'nd derivatives are easily found:

$$\frac{dT}{dt} = D\left(\alpha cie^{+\alpha cit} + \alpha cie^{-\alpha cit}\right)$$

Equation C.5

$$\frac{d^2T}{dt^2} = D\left(-\alpha^2c^2e^{+\alpha cit} + \alpha^2c^2e^{-\alpha cit}\right) = -\alpha^2c^2T$$

Substituting C.5 into C.2 demonstrates that T satisfies equation C.2. Therefore, a solution to the wave equation is RT as follows:

Equation C.6

$$\psi(r,t) = \frac{AD}{r}\left(e^{+\alpha ir} - e^{-\alpha ir}\right)\left(e^{+\alpha cit} - e^{-\alpha cit}\right)$$

Vector Solution to Wave Equation:

Euler's equation is a solution to the wave equation as seen above. Equations B.3-B.6 are based upon Euler's equation. Therefore, it seems reasonable to expect that equations B.3-B.6 would also be solutions. But where is r in these equations? Instead of r, these equations have Θ_k and Θ_j. For these equations to be applied to the wave equation, it is necessary that these angles of rotation be functions of r. The simplest way to do this is simply to make the statement that $\Theta_k = \Theta_j = r$. But this has a small problem. Specifically, the unit of measure used for an angle is not equal to the unit of measure used for a distance. Therefore, there must be a conversion factor present. The statement used instead will be $\Theta_k = \Theta_j = \beta r$. In principle, β can be a positive or negative constant and θ_k and θ_j could have opposite β's. Now the wave equation has a problem because a β^2 term will appear after differentiating twice. This is easily fixed by stating the wave equation as follows:

Equation C.7

$$\frac{1}{\beta^2}\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Solving C.7 using equations B.3-B.6 is identical to the scalar solution presented above, except the function R is as follows:

Equation C.8

$$R(r) = \frac{1}{r}\left(AF_{i,k} - AF^{*}_{i,k} + BF_{i,j} - BF^{*}_{i,j}\right)$$

By selecting the coefficients to be A, -A, B, and –B, the cosine terms will all cancel leaving only the sine terms. This will force the solution to have a finite value at r = 0. Essentially, equation C.8 is evaluating the difference between a counter-clockwise rotation and a clockwise rotation about the **k** axis and adding it to the difference between a clock-wise and counter-clockwise rotation about the **j** axis.

There is no reason why nature would favor **j** over **k** or vice-versa. Therefore, it is pretty certain that A and B are equal in magnitude although one could be positive and the other negative. So, it seems the possible combinations for A & B are: both A and B are positive, A is positive and B is negative, both A and B are negative, and A is negative and B is positive.

Now we must examine the far right-hand side of equations B.3-B.6. For the case where A and B are both +1, the sum of the four functions will be +2**j**sin(αβr) + 2**k**sin(αβr). For A = +1 and B = -1, the sum becomes +2**j**sin(αβr) – 2**k**sin(αβr). For A and B both equal to -1, the sum will be -2**j**sin(αβr) - 2**k**sin(αβr). For the case with A = -1 and B = +1, the sum becomes -2**j**sin(αβr) + 2**k**sin(αβr). For compactness, this can be written in matrix form as follows:

$$\pm 2A\begin{bmatrix} +\sin(\alpha\beta r) & +\sin(\alpha\beta r) \\ +\sin(\alpha\beta r) & -\sin(\alpha\beta r) \end{bmatrix}\begin{bmatrix} \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

Now T must be determined. For the scalar solution, D and E were chosen to produce a function that included i so that it would combine with the i in the R function to produce a real solution. For the vector solution, this is not required. Instead, E = D is chosen to produce a real T since R is real.

$$T(t) = De^{+\alpha cit} + De^{-\alpha cit}$$

Combining R with T produces the following:

Equation C.9

$$\psi(r,t) = \pm\frac{2AD}{r}\left(e^{+\alpha clt} + e^{-\alpha clt}\right)\begin{bmatrix} +\sin(\alpha\beta r) & +\sin(\alpha\beta r) \\ +\sin(\alpha\beta r) & -\sin(\alpha\beta r) \end{bmatrix}\begin{bmatrix} \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

For the situation where r >> 0, a simpler solution can be obtained by selecting the A's and B's such that the cosine terms are kept and the sine terms all cancel. This solution is:

Equation C.10

$$\psi(r,t) = \frac{4AD\mathbf{i}}{r}\cos(\alpha\beta r)\left(e^{+\alpha clt} + e^{-\alpha clt}\right)$$

Appendix D

Consider the vector **j** to be rotated about the **i** axis in the counter-clockwise direction. The quaternion problem is stated as follows:

$$\mathbf{j}(q_0 + q_1\mathbf{i}) = \mathbf{j}\cos(\theta_i) + \mathbf{k}\sin(\theta_i)$$

$$q_0\mathbf{j} + q_1\mathbf{ji} = \mathbf{j}\cos(\theta_i) + \mathbf{k}\sin(\theta_i)$$

$$q_0\mathbf{j} - q_1\mathbf{k} = \mathbf{j}\cos(\theta_i) + \mathbf{k}\sin(\theta_i)$$

$$q_0 = \cos(\theta_i) \ and \ q_1 = -\sin(\theta_i)$$

Equation D.1 (counter-clockwise)

$$F_{j,i} = \mathbf{j}e^{-\mathbf{i}\theta_i} = \mathbf{j}\cos(\theta_i) + \mathbf{k}\sin(\theta_i)$$

Next, rotate **j** in the clock-wise direction.

$$\mathbf{j}(q_0 + q_1\mathbf{i}) = \mathbf{j}\cos(\theta_i) - \mathbf{k}\sin(\theta_i)$$

$$q_0\mathbf{j} + q_1\mathbf{ji} = \mathbf{j}\cos(\theta_i) - \mathbf{k}\sin(\theta_i)$$

$$q_0\mathbf{j} - q_1\mathbf{k} = \mathbf{j}\cos(\theta_i) - \mathbf{k}\sin(\theta_i)$$

$$q_0 = \cos(\theta_i) \ and \ q_1 = \sin(\theta_i)$$

Equation D.2 (clock-wise)

$$F^*_{j,i} = \mathbf{j}e^{+\mathbf{i}\theta_i} = \mathbf{j}\cos(\theta_i) - \mathbf{k}\sin(\theta_i)$$

Now, rotate **k** clock-wise about the **i** axis.

$$\mathbf{k}(q_0 + q_1\mathbf{i}) = \mathbf{k}\cos(\theta_i) + \mathbf{j}\sin(\theta_i)$$

$$q_0\mathbf{k} + q_1\mathbf{ki} = \mathbf{k}\cos(\theta_i) + \mathbf{j}\sin(\theta_i)$$

$$q_0\mathbf{k} + q_1\mathbf{j} = \mathbf{k}\cos(\theta_i) + \mathbf{j}\sin(\theta_i)$$

$$q_0 = \cos(\theta_i) \ and \ q_1 = \sin(\theta_i)$$

Equation D.3 (clock-wise)

$$F_{k,i} = \mathbf{k}e^{+\mathbf{i}\theta_i} = \mathbf{k}\cos(\theta_i) + \mathbf{j}\sin(\theta_i)$$

And lastly, rotate **k** counter-clockwise about the **i** axis.

$$\mathbf{k}(q_0 + q_1\mathbf{i}) = \mathbf{k}\cos(\theta_i) - \mathbf{j}\sin(\theta_i)$$

$$q_0\mathbf{k} + q_1\mathbf{ki} = \mathbf{k}\cos(\theta_i) - \mathbf{j}\sin(\theta_i)$$

$$q_0\mathbf{k} + q_1\mathbf{j} = \mathbf{k}\cos(\theta_i) - \mathbf{j}\sin(\theta_i)$$

$$q_0 = \cos(\theta_i) \ and\ q_1 = -\sin(\theta_i)$$

Equation D.4 (counter-clockwise)

$$F^*_{k,i} = \mathbf{k}e^{-\mathbf{i}\theta_i} = \mathbf{k}\cos(\theta_i) - \mathbf{j}\sin(\theta_i)$$

It is desired to incorporate these into the wave equation. Therefore the derivatives will be needed.

Equation D.1.1

$$\frac{dF_{j,i}}{d\theta_i} = -\mathbf{k}e^{-\mathbf{i}\theta_i} = -\mathbf{j}\sin(\theta_i) + \mathbf{k}\cos(\theta_i)$$

Equation D.1.2

$$\frac{d^2F_{j,i}}{d\theta_i^2} = -\mathbf{j}e^{-\mathbf{i}\theta_i} = -\mathbf{j}\cos(\theta_i) - \mathbf{k}\sin(\theta_i) = -F_{j,i}$$

Equation D.2.1

$$\frac{dF^*_{j,i}}{d\theta_i} = \mathbf{k}e^{+\mathbf{i}\theta_i} = -\mathbf{j}\sin(\theta_i) - \mathbf{k}\cos(\theta_i)$$

Equation D.2.2

$$\frac{d^2F^*_{j,i}}{d\theta_i^2} = -\mathbf{j}e^{+\mathbf{i}\theta_i} = -\mathbf{j}\cos(\theta_i) + \mathbf{k}\sin(\theta_i) = -F^*_{j,i}$$

Equation D.3.1

$$\frac{dF_{k,i}}{d\theta_i} = -\mathbf{j}e^{+\mathbf{i}\theta_i} = -\mathbf{k}\sin(\theta_i) + \mathbf{j}\cos(\theta_i)$$

Equation D.3.2

$$\frac{d^2 F_{k,i}}{d\theta_i^2} = -\mathbf{k}e^{+\mathbf{i}\theta_i} = -\mathbf{k}\cos(\theta_i) - \mathbf{j}\sin(\theta_i) = -F_{k,i}$$

Equation D.4.1

$$\frac{dF_{k,i}^*}{d\theta_i} = \mathbf{j}e^{-\mathbf{i}\theta_i} = -\mathbf{k}\sin(\theta_i) - \mathbf{j}\cos(\theta_i)$$

Equation D.4.2

$$\frac{d^2 F_{k,i}^*}{d\theta_i^2} = -\mathbf{k}e^{-\mathbf{i}\theta_i} = -\mathbf{k}\cos(\theta_i) + \mathbf{j}\sin(\theta_i) = -F_{k,i}^*$$

Appendix E

Begin by restating equations D.1 – D.4 with E.3 and E.4 being in slightly different form.

Equation E.1 (counter-clockwise)

$$F_{j,i} = \mathbf{j}e^{-\mathbf{i}\theta_i} = +\mathbf{j}\cos(\theta_i) + \mathbf{k}\sin(\theta_i)$$

Equation E.2 (clock-wise)

$$F_{j,i}^{*} = \mathbf{j}e^{+\mathbf{i}\theta_i} = +\mathbf{j}\cos(\theta_i) - \mathbf{k}\sin(\theta_i)$$

Equation E.3 (clock-wise)

$$F_{k,i} = \mathbf{k}e^{+\mathbf{i}\theta_i} = +\mathbf{j}\sin(\theta_i) + \mathbf{k}\cos(\theta_i)$$

Equation E.4 (counter-clockwise)

$$F_{k,i}^{*} = \mathbf{k}e^{-\mathbf{i}\theta_i} = -\mathbf{j}\sin(\theta_i) + \mathbf{k}\cos(\theta_i)$$

Equations E.1 and E.4 are counter-clockwise rotations and equations E.2 and E.3 are clockwise rotations. Adding the two clock-wise rotations together and the two counter-clockwise rotations together gives the following expressions:

Equation E.5 (clock-wise)

$$F_{j,i}^{*} + F_{k,i} = (\mathbf{j} + \mathbf{k})e^{+\mathbf{i}\theta_i} = \mathbf{j}(\cos(\theta_i) + \sin(\theta_i)) + \mathbf{k}(\cos(\theta_i) - \sin(\theta_i))$$

Equation E.6 (counter-clockwise)

$$F_{j,i} + F_{k,i}^{*} = (\mathbf{j} + \mathbf{k})e^{-\mathbf{i}\theta_i} = \mathbf{j}(\cos(\theta_i) - \sin(\theta_i)) + \mathbf{k}(\cos(\theta_i) + \sin(\theta_i))$$

Please note that equations E.5 and E.6 are actually rotations of the entire **j**-**k** plane. In Appendix D it was shown that each of the four individual equations is a solution to the time portion of the wave equation since $(d^2\psi/d\theta_i^2) = -\psi$. Therefore, equations E.5 and E.6 are also solutions.

The next step is to express θ_i in terms of t. This is easily done as follows:

Equation E.7

$$\theta_i = \alpha c t$$

Substituting into equations E.5 and E.6 gives the following:

Equation E.8 (clock-wise)

$$F_{j,i}^{*} + F_{k,i} = (\mathbf{j}+\mathbf{k})e^{+\mathbf{i}\alpha ct} = \mathbf{j}(\cos(\alpha ct) + \sin(\alpha ct)) + \mathbf{k}(\cos(\alpha ct) - \sin(\alpha ct))$$

The 1'st and 2'nd time derivatives are:

Equation E.8.1

$$\frac{d}{dt}\left(F_{j,i}^{*} + F_{k,i}\right) = +\alpha c\mathbf{i}(\mathbf{j}+\mathbf{k})e^{+\mathbf{i}\alpha ct} = +\alpha c(\mathbf{k}-\mathbf{j})e^{+\mathbf{i}\alpha ct}$$
$$= +\alpha c[\mathbf{j}(-\sin(\alpha ct) + \cos(\alpha ct)) + \mathbf{k}(-\sin(\alpha ct) - \cos(\alpha ct))]$$

Equation E.8.2

$$\frac{d^2}{dt^2}\left(F_{j,i}^{*} + F_{k,i}\right) = -\alpha^2 c^2(\mathbf{j}+\mathbf{k})e^{+\mathbf{i}\alpha ct} = -\alpha^2 c^2[\mathbf{j}(\cos(\alpha ct) + \sin(\alpha ct)) + \mathbf{k}(\cos(\alpha ct) - \sin(\alpha ct))]$$
$$= -\alpha^2 c^2\left(F_{j,i}^{*} + F_{k,i}\right)$$

Equation E.9 (counter-clockwise)

$$F_{j,i} + F_{k,i}^{*} = (\mathbf{j}+\mathbf{k})e^{-\mathbf{i}\alpha ct} = \mathbf{j}(\cos(\alpha ct) - \sin(\alpha ct)) + \mathbf{k}(\cos(\alpha ct) + \sin(\alpha ct))$$

Equation E.9.1

$$\frac{d}{dt}\left(F_{j,i} + F_{k,i}^{*}\right) = -\alpha c\mathbf{i}(\mathbf{j}+\mathbf{k})e^{-\mathbf{i}\alpha ct} = -\alpha c(\mathbf{k}-\mathbf{j})e^{-\mathbf{i}\alpha ct}$$
$$= +\alpha c[\mathbf{j}(-\sin(\alpha ct) - \cos(\alpha ct)) + \mathbf{k}(-\sin(\alpha ct) + \cos(\alpha ct))]$$

Equation E.9.2

$$\frac{d^2}{dt^2}\left(F_{j,i} + F_{k,i}^{*}\right) = -\alpha^2 c^2(\mathbf{j}+\mathbf{k})e^{-\mathbf{i}\alpha ct} = -\alpha^2 c^2[\mathbf{j}(\cos(\alpha ct) - \sin(\alpha ct)) + \mathbf{k}(\cos(\alpha ct) + \sin(\alpha ct))]$$
$$= -\alpha^2 c^2\left(F_{j,i} + F_{k,i}^{*}\right)$$

Expressing the far right hand side of E.8 and E.9 in matrix form gives:

Equation E.10

$$T(t) = \begin{bmatrix} (\cos(\alpha ct) + \sin(\alpha ct)) & (\cos(\alpha ct) - \sin(\alpha ct)) \\ (\cos(\alpha ct) - \sin(\alpha ct)) & (\cos(\alpha ct) + \sin(\alpha ct)) \end{bmatrix} \begin{bmatrix} \mathbf{j} \\ \mathbf{k} \end{bmatrix}$$

Appendix F

Begin by presenting the matrix solution.

Equation F.1

$$\pm\frac{2AD}{r}\begin{bmatrix}+\sin(\alpha\beta r) & +\sin(\alpha\beta r)\\ +\sin(\alpha\beta r) & -\sin(\alpha\beta r)\end{bmatrix}\begin{bmatrix}\mathbf{j}\\ \mathbf{k}\end{bmatrix}\times\begin{bmatrix}(\cos(\alpha ct)+\ \sin(\alpha ct)) & (\cos(\alpha ct)-\ \sin(\alpha ct))\\ (\cos(\alpha ct)-\ \sin(\alpha ct)) & (\cos(\alpha ct)+\sin(\alpha ct))\end{bmatrix}\begin{bmatrix}\mathbf{j}\\ \mathbf{k}\end{bmatrix}$$

To simplify the appearance of F.1, make the following substitutions:

$$a=\ \cos(\alpha ct)$$

$$b=\ \sin(\alpha ct)$$

$$d=\ \sin(\alpha\beta r)$$

Equation F.2

$$\pm\frac{2AD}{r}\begin{bmatrix}+d & +d\\ +d & -d\end{bmatrix}\begin{bmatrix}\mathbf{j}\\ \mathbf{k}\end{bmatrix}\cdot\begin{bmatrix}(a+b) & (a-b)\\ (a-b) & (a+b)\end{bmatrix}\begin{bmatrix}\mathbf{j}\\ \mathbf{k}\end{bmatrix}$$

Perform the multiplication (dot products) in the following order:

1. Top row of R multiplied by top row of T
2. Bottom row of R multiplied by top row of T
3. Top row of R multiplied by bottom row of T
4. Bottom row of R multiplied by bottom row of T

$$\pm\frac{2AD}{r}(+d\mathbf{j}+d\mathbf{k})[(a+b)\mathbf{j}+\ (a-b)\mathbf{k}]$$

$$\pm\frac{2AD}{r}[d(a+b)\mathbf{j}^2+\ d(a+b)\mathbf{kj}+\ d(a-b)\mathbf{jk}+\ d(a-b)\mathbf{k}^2]$$

$$\pm\frac{2AD}{r}[-(da+db)+\ -(da+db)\mathbf{i}+\ (da-db)\mathbf{i}+\ -(da-db)]$$

$$\pm\frac{2AD}{r}(-2da-2db\mathbf{i})$$

Equation F.3 (clock-wise)

$$\pm\frac{4ADd}{r}(-a\ -\ b\mathbf{i})$$

Repeating this process for the other three multiplications yields:

Equation F.4 (clock-wise)

$$\pm\frac{4ADd}{r}(-b\ +\ a\mathbf{i})$$

Equation F.5 (counter-clockwise)

$$\pm\frac{4ADd}{r}(-a+\ b\mathbf{i})$$

Equation F.6 (counter-clockwise)

$$\pm\frac{4ADd}{r}(+b+a\mathbf{i})$$

Substituting back into F.3-F.6 gives:

Equation F.7 (clock-wise)

$$\pm\frac{4AD\sin(\alpha\beta r)}{r}(-\cos(\alpha ct)-\ \mathbf{i}\sin(\alpha ct))$$

Equation F.8 (clock-wise)

$$\pm\frac{4AD\sin(\alpha\beta r)}{r}(-\sin(\alpha ct)+\ \mathbf{i}\cos(\alpha ct))$$

Equation F.9 (counter-clockwise)

$$\pm\frac{4AD\sin(\alpha\beta r)}{r}(-\cos(\alpha ct)+\ \mathbf{i}\sin(\alpha ct))$$

Equation F.10 (counter-clockwise)

$$\pm\frac{4AD\sin(\alpha\beta r)}{r}(+\sin(\alpha ct)+\ \mathbf{i}\cos(\alpha ct))$$

Appendix G

Solve equation 9 for the Laplacian.

Equation G.1

$$\nabla^2\psi = \frac{\beta^2}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

Solve equation 17 for the Laplacian.

Equation G.2

$$\nabla^2\psi = -i\frac{2m}{h}\frac{\partial\psi}{\partial t}$$

Set equation G.1 equal to equation G.2.

Equation G.3

$$\frac{\beta^2}{c^2}\frac{\partial^2\psi}{\partial t^2} = -i\frac{2m}{h}\frac{\partial\psi}{\partial t}$$

F = RT can be substituted into equation G.3 and the R's can be eliminated to leave only T(t).

Equation G.4

$$\frac{\beta^2}{c^2}\frac{\partial^2 T}{\partial t^2} = -\mathbf{i}\frac{2m}{h}\frac{\partial T}{\partial t}$$

Now the appropriate time derivatives from Appendix E may be substituted into equation G.4. This will determine the conditions needed for a solution to the classic wave equation to also be a solution to the Schrödinger Equation.

For the clock-wise rotation:

$$\frac{\beta^2}{c^2}\left(-\alpha^2c^2(\mathbf{j}+\mathbf{k})e^{+\mathbf{i}\alpha ct}\right) = -\mathbf{i}\frac{2m}{h}\left(\alpha c\mathbf{i}(\mathbf{j}+\mathbf{k})e^{+\mathbf{i}\alpha ct}\right)$$

$$-\frac{\beta^2\alpha}{c} = +\frac{2m}{h}$$

Rearranging slightly gives the following:

Equation G.5 (clock-wise)

$$-\alpha\beta^2 = \frac{2mc}{h}$$

For counter-clockwise rotation:

$$\frac{\beta^2}{c^2}\left(-\alpha^2 c^2 (\mathbf{j}+\mathbf{k}) e^{-\mathbf{i}\alpha ct}\right) = -\mathbf{i}\frac{2m}{h}\left(-\alpha c\mathbf{i}(\mathbf{j}+\mathbf{k}) e^{-\mathbf{i}\alpha ct}\right)$$

$$-\frac{\beta^2\alpha}{c} = -\frac{2m}{h}$$

Rearranging slightly gives the following:

Equation G.6 (counter-clockwise)

$$+\alpha\beta^2 = +\frac{2mc}{h}$$

Appendix H

$$\phi = \theta = \beta r$$

$$d\phi = d\theta = \beta dr$$

$$dL = \sqrt{(dr)^2 + (rd\phi)^2 + (rd\theta)^2}$$

$$dL = \sqrt{(dr)^2 + (r\beta dr)^2 + (r\beta dr)^2}$$

$$dL = \sqrt{1 + (\beta r)^2 + (\beta r)^2}\, dr$$

$$dL = \sqrt{1 + 2\beta^2 r^2}\, dr$$

$$dL = \pm\beta\sqrt{2}\sqrt{\frac{1}{2\beta^2} + r^2}\, dr$$

$$\int_0^L dL = \pm\beta\sqrt{2}\int_0^R \sqrt{\frac{1}{2\beta^2} + r^2}\, dr$$

The following indefinite integral is presented by Thomas[11].

$$\int \sqrt{a^2 + x^2}\, dx = \frac{x}{2}\sqrt{a^2 + x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + Constant$$

By comparison $a^2 = (1/2\beta^2)$ and $x^2 = r^2$.

The lower limit of the definite integral evaluates to zero due to the (x/2) term and the $\sinh^{-1}(x/a)$ term.

Therefore the solution is as follows:

Equation H.1

$$L = \pm\beta\sqrt{2}\left[\frac{R}{2}\sqrt{\frac{1}{2\beta^2} + R^2} + \frac{1}{4\beta^2}\sinh^{-1}\left(\pm\beta R\sqrt{2}\right)\right]$$

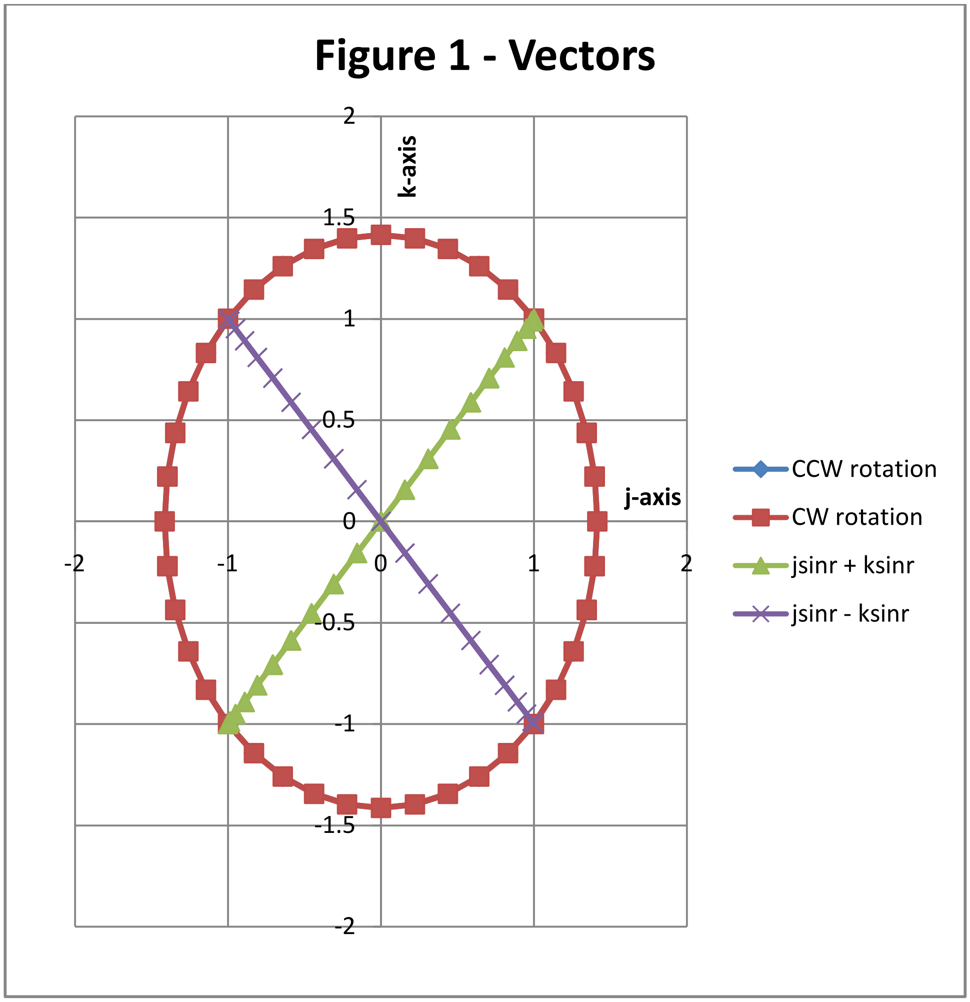
Figure 1 - Vectors
k-axis
j-axis
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2
-1
0
1
2
CCW rotation
CW rotation
jsinr + ksinr
jsinr - ksinr

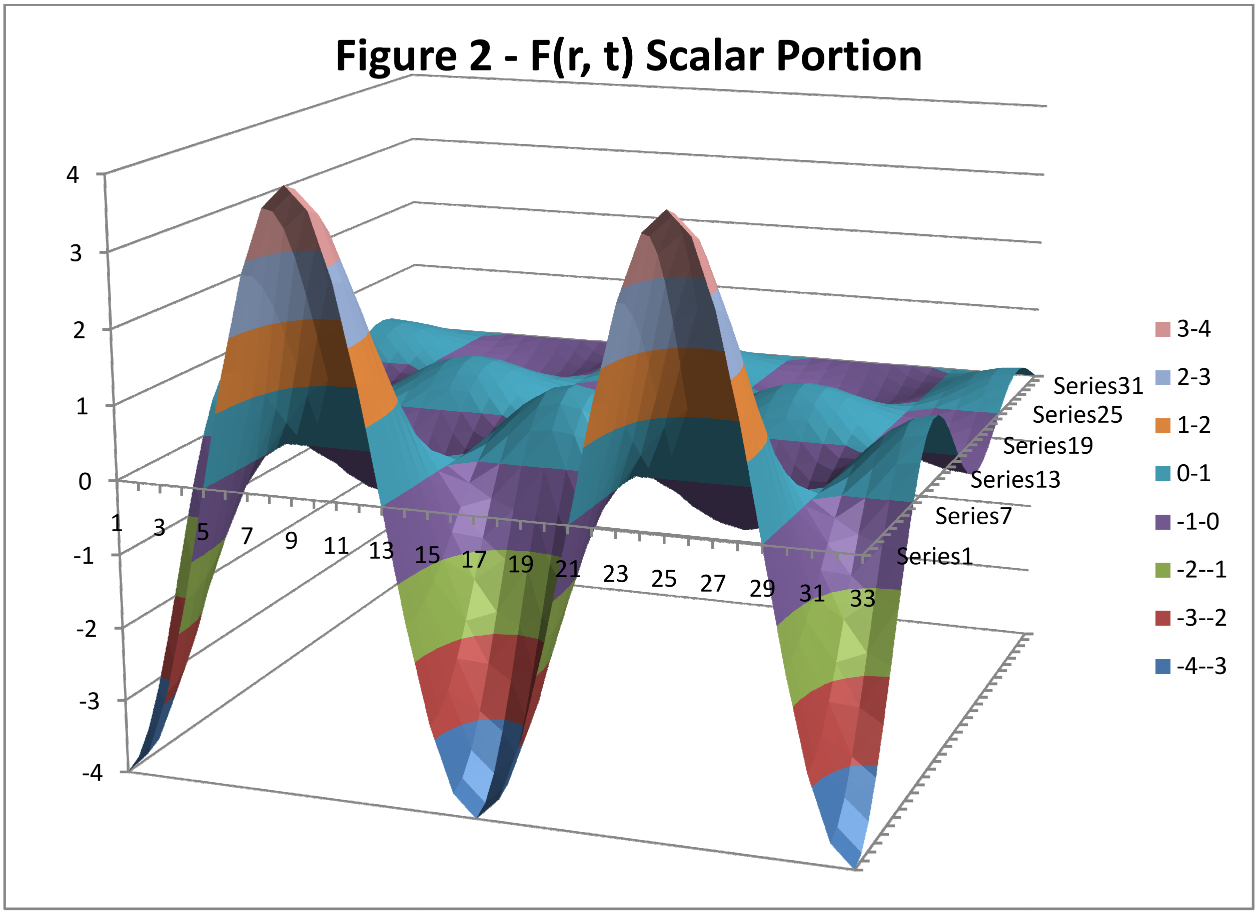
Figure 2 - F(r, t) Scalar Portion
4
3
2
1
0
-1
-2
-3
-4
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
31
33
Series31
Series25
Series19
Series13
Series7
Series1
3-4
2-3
1-2
0-1
-1-0
-2--1
-3--2
-4--3

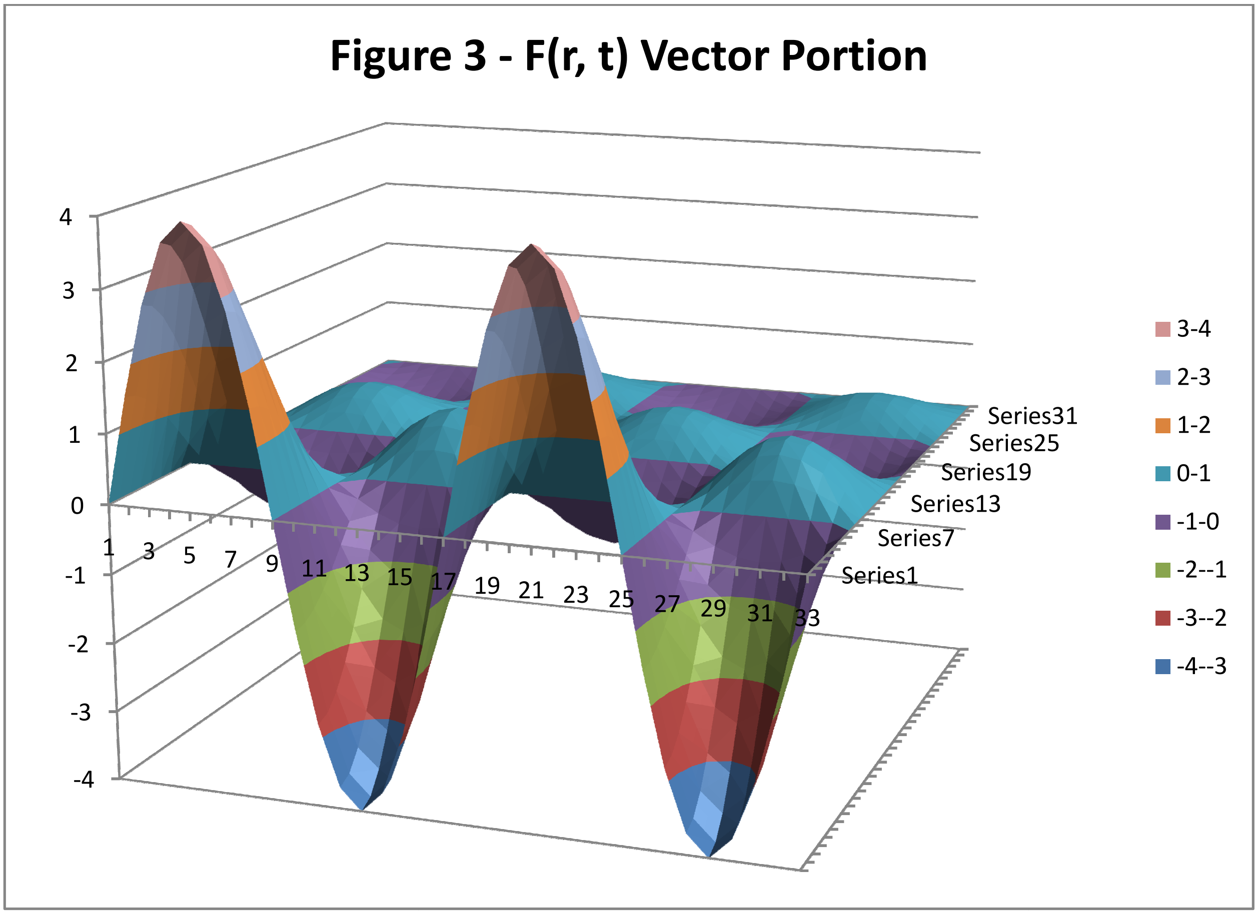

Figure 3 - F(r, t) Vector Portion
4
3
2
1
0
-1
-2
-3
-4
1
3
5
7
9
11
13
15
17
19
21
23
25
27
29
31
33
Series31
Series25
Series19
Series13
Series7
Series1
3-4
2-3
1-2
0-1
-1-0
-2--1
-3--2
-4--3