THE WAVEGUIDE EIGENVALUE PROBLEM AND THE TENSOR INFINITE ARNOLDI METHOD

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We consider the following PDE–eigenvalue problem, which arises in the study of waves traveling in a periodic medium [1]: determine a non–trivial function $u(x,z)$ and a complex number γ such that

$$\begin{aligned} \Delta u(x,z) &+ 2\gamma u_z(x,z) + (\gamma^2 + \kappa(x,z)^2)u(x,z) = 0, \ \ (x,z) \in \mathbb{R}^2, && \text{(1a)} \\ u(x,z) &= u(x,z+1) \ \text{ for all } (x,z) \in \mathbb{R}^2, && \text{(1b)} \\ u(x,\cdot) &\to 0 \text{ when } |x| \to \infty. && \text{(1c)} \end{aligned}$$

The function $\kappa(x,z)$ is piecewise constant and is assumed to satisfy: $\kappa(x,z) = \kappa_-$ when $x \leq x_-$, $\kappa(x,z) = \kappa_+$ when $x \geq x_+$ and $\kappa(x,z) = \kappa(x,z+1)$. This problem can be rephrased as an equivalent problem on a finite domain by means of a Dirichlet–to–Neumann map. A particular type of finite-element discretization of the finite-domain problem leads to the following nonlinear eigenvalue problem, which consists of finding pairs $(\gamma, v) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{0\})$ such that

$$\begin{pmatrix} Q(\gamma) & C_1(\gamma) \\ C_2^T & R\Lambda(\gamma)R^{-1} \end{pmatrix} v = 0. \tag{2}$$

The matrices $Q(\gamma)$ and $C_1(\gamma)$ are polynomials of second degree in γ. The matrix $\Lambda(\gamma)$ is diagonal and involves square roots of polynomials in γ. The problem (2) is a large–scale nonlinear eigenvalue problem of the type extensively studied in recent literature [2]. The algorithm we propose is based on the infinite Arnoldi method [3], which can be interpreted as the standard Arnoldi method applied to a linear and infinite dimensional eigenvalue problem. In the new algorithm, we suggest to represent the basis of the Krylov subspace as a factorization involving a tensor. This factorization allows us to reduce the memory requirements and the computation time. By construction, this new algorithm, which we call the tensor infinite Arnoldi method, is mathematically equivalent to the infinite Arnoldi method. The infinite Arnoldi method requires efficient procedures to compute the derivatives of the functions that define the nonlinear eigenvalue problem. For this problem such derivatives can be computed with a closed and efficient formula. Moreover we exploit sparsity and low–rank structure of the nonlinear eigenvalue problem. The matrix-vector product corresponding to R and R^{-1} can be computed with the Fast Fourier Transform (FFT).

References

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