Open access • Journal Article • DOI:10.4171/JEMS/928

The weak harnack inequality for the boltzmann equation without cut-off

— Source link

Cyril Imbert, Luis Silvestre

Institutions: École Normale Supérieure

Published on: 04 Nov 2019 - Journal of the European Mathematical Society (Banff International Research Station for Mathematical Innovation and Discovery)

Topics: Harnack's inequality, Boltzmann equation, Bounded function and Upper and lower bounds

The weak Harnack inequality for the Boltzmann equation without cut-off

Cyril Imbert, Luis Silvestre

THE WEAK HARNACK INEQUALITY FOR THE BOLTZMANN EQUATION WITHOUT CUT-OFF

CYRIL IMBERT AND LUIS SILVESTRE

ABSTRACT. We obtain the weak Harnack inequality and Hölder estimates for a large class of kinetic integro-differential equations. We prove that the Boltzmann equation without cut-off can be written in this form and satisfies our assumptions provided that the mass density is bounded away from vacuum and mass, energy and entropy densities are bounded above. As a consequence, we derive a local Hölder estimate and a quantitative lower bound for solutions of the (inhomogeneous) Boltzmann equation without cut-off.

CONTENTS

1. INTRODUCTION

The main result in this article is a version of the weak Harnack inequality, in the style of De Giorgi, Nash and Moser, for kinetic integro-differential equations. As a consequence, we derive local Hölder estimates and a quantitative lower bound for the inhomogeneous Boltzmann equation without cut-off.

Our estimates are local in the sense that they only require the equation to hold in a bounded domain.

The Boltzmann equation has the form

$$f_t + v \cdot \nabla_x f = Q(f,f) \quad \text{for } t \in (-1,0],\ x \in B_1,\ v \in B_1.$$

Here, the function $f = f(t,x,v)$ must be defined for $t \in (-1,0]$, $x \in B_1$ and $v \in \mathbb{R}^d$ in order to make sense of the nonlocal right hand side $Q(f,f)$.

We recall that Boltzmann's collision operator $Q(f,f)$ is defined as follows

$$Q(f,f) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f(v'_*)f(v') - f(v_*)f(v)) B(|v-v_*|, \cos\theta) \, \mathrm{d}v_* \, \mathrm{d}\sigma$$

where v'_* and v' are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma \quad \text{and} \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma$$

Date: December 8, 2018.
LS is supported in part by NSF grants DMS-1254332 and DMS-1362525.

and $\cos\theta$ (and $\sin(\theta/2)$) is *defined* as

$$\cos\theta := \frac{v-v_*}{|v-v_*|}\cdot\sigma \qquad \left(\text{and} \quad \sin(\theta/2) := \frac{v'-v}{|v'-v|}\cdot\sigma\right).$$

We assume that the cross-section B satisfies

$$B(r,\cos\theta) = r^\gamma b(\cos\theta) \quad \text{with} \quad b(\cos\theta) \approx |\sin(\theta/2)|^{-(d-1)-2s} \tag{1.1}$$

with $\gamma \in (-d,1]$ and $s \in (0,1)$.

The equation describes the density of particles at a specific time t, point in space x and with velocity v. This model stands at a mesoscopic level, in between the microscopic description of interactions between individual particles, and the macroscopic models of fluid dynamics.

We define the hydrodynamic quantities

$$\begin{aligned}
&\text{(mass density)} && M(t,x) := \int f(t,x,v)\,\mathrm{d}v,\\
&\text{(energy density)} && E(t,x) := \int f(t,x,v)|v|^2\,\mathrm{d}v,\\
&\text{(entropy density)} && H(t,x) := \int f\ln f(t,x,v)\,\mathrm{d}v.
\end{aligned}$$

These are the only quantities associated with a solution f which are meaningful at a macroscopic scale. Under some asymptotic regime, the hydrodynamic quantities in the Boltzmann equation formally converge to solutions of the compressible Euler equation, which is known to develop singularities in finite time (see for example [11]). Because of this fact, one could speculate that the Boltzmann equation may develop singularities as well. From this point of view, the best regularity result that one would expect is that if the hydrodynamic quantities are under appropriate control, then the solution f will be smooth. In other words, that every singularity of f would be observable at the macroscopic scale.

It is proved in [59] that when $M(t,x)$, $E(t,x)$ and $H(t,x)$ are uniformly bounded above, and in addition $M(t,x)$ is bounded below by a positive constant, then the solution f satisfies the L^∞ a priori estimate depending on those bounds only. The result in this paper goes a step further by proving a Hölder modulus of continuity, in all variables, under the same assumptions.

Theorem 1.1 (Hölder continuity). *Assume $s \in (0,1)$, $\gamma \in (-d,1]$, $\gamma + 2s \leqslant 2$ and let f be a non-negative solution of the Boltzmann equation for all $t \in (-1,0]$, $x \in B_1$ and $v \in B_1$. Assume that f is essentially bounded in $(-1,0] \times B_1 \times \mathbb{R}^d$ and there are positive constants M_0, M_1, E_0 such that for all (t,x) we have $M_1 \leqslant M(t,x) \leqslant M_0$ and $E(t,x) \leqslant E_0$ for all $(t,x) \in (-1,0] \times B_1$, then f is Hölder continuous in $(-1/2,0] \times B_{1/2} \times B_{1/2}$ with*

$$\|f\|_{C^\alpha((-1/2,0]\times B_{1/2}\times B_{1/2})} \leqslant C$$

where $C > 0$ and $\alpha \in (0,1)$ are constants depending on dimension, the L^∞ bound of f, M_0, M_1 and E_0.

Remark 1.2. Theorem 1.1 also holds true in any cylinder $Q \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$. In this case, constants C and γ also depends on the center of the cylinder and its radius.

Note that the value of the entropy $H(t,x)$ is bounded above by some constant H_0 depending only on M_0, E_0 and $\|f\|_{L^\infty}$ so we do not need to include the hypothesis $H(t,x) \leqslant H_0$ in Theorem 1.1 . Recall also that $\|f\|_{L^\infty}$ is bounded above for $t > 0$ in terms of M_0, M_1, E_0 and H_0, according to the result in [59], provided that $\gamma + 2s > 0$. So, at least in this range of values of γ, the Hölder modulus of continuity depends on the values of M_0, M_1, E_0 and H_0 only.

The best regularity results previously available for the inhomogeneous Boltzmann equation without cut-off give us C^∞ regularity depending on the assumption that the solution has infinite moments and belongs to the space H^5 with respect to all variables (v, x and t) [7], [3], [28]. Of course this is a much more stringent assumption than what we need for our Theorem 1.1 to hold. We make further comments about these and other related results in Section 1.3.

We also obtain a quantitative lower bound for the solution f.

Theorem 1.3 (Lower bound)**.** *Let f be a non-negative supersolution of the Boltzmann equation in $[0,T] \times B_R \times B_R$. Under the same assumptions on γ, s and f as in Theorem 1.1, we have the lower bound*

$$\inf_{[T/2,T]\times B_{R/2}\times B_{R/2}} f \geqslant c(R).$$

The constant $c(R)$ depends on T, R, γ, s, d, M_0, M_1, E_0, and $\|f\|_{L^\infty}$.

It has been a longstanding issue to find appropriate lower bounds for the solutions of the Boltzmann equation. The best result available is perhaps from the work of Mouhot [53]. He obtains an explicit exponentially decaying lower bound for the Boltzmann equation without cut-off. He makes strong a priori regularity assumptions on the solution f, in addition to the assumptions that we make in this paper. We do not provide an explicit formula for $c(R)$. Its precise decay as $R \to \infty$ will be the subject of future work.

Remark 1.4. If $\gamma + 2s > 2$, similar results can be obtained by further assuming that the $(\gamma+2s)$-momentum of the function f is finite at every point (t,x).

1.1. **A linear kinetic integro-differential equation.** The main result of this paper concerns a general kinetic integro-differential equation. The results for the Boltzmann equation described above follow as corollaries. We study an equation of the form

$$f_t + v \cdot \nabla_x f = L_v f + h \tag{1.2}$$

for $t \in (-1,0]$, $x \in B_1$ and $v \in B_1$, where $L_v f$ is a linear integro-differential operator in the velocity variable of the following form

$$L_v f(t,x,v) = PV \int_{\mathbb{R}^d} (f(t,x,v') - f(t,x,v)) K(t,x,v,v') \, \mathrm{d}v'$$

for a locally bounded function h and a measurable kernel $K : [-1,0] \times B_1 \times B_{\bar R} \times \mathbb{R}^d \to [0,+\infty)$ satisfying appropriate assumptions that we describe below.

For every value of t and x, the kernel $K(t,x,v,w)$ is a non-negative function of v and w. We assume that the following conditions hold for every value of t and x (we omit t and x dependence to clean up the notation).

Let us fix a $\bar R \geqslant 1$. We will make assumptions on the kernel $K(v,v')$ for $v \in B_{\bar R}$. We need to pick $\bar R$ slightly larger than one for technical reasons that will be apparent in Section 5.

Our first assumption is a coercivity condition on L_v. We assume that there exists $\lambda > 0$ and $\Lambda > 0$ such that, for any function $f : \mathbb{R}^d \to \mathbb{R}$ supported in $B_{\bar R}$,

$$\lambda \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|f(v)-f(v')|^2}{|v-v'|^{d+2s}} \, \mathrm{d}v \, \mathrm{d}v' \leqslant -\int_{\mathbb{R}^d} L_v f(v)\, f(v) \, \mathrm{d}v + \Lambda \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{1.3}$$

This coercivity condition is well known to hold for the Boltzmann equation when the function f has bounded mass, energy and entropy density, and its mass is also away from vacuum (see [48, 62, 1] and the discussion below). The proofs in the literature are based on Fourier analysis. We provide a proof in the appendix which follows by a direct geometric computation in physical variables.

In the case $s < 1/2$, we also make the following nondegeneracy assumption.

$$\inf_{|e|=1} \int_{B_r(v)} ((v'-v)\cdot e)_+^2 K(v,v') \, \mathrm{d}v' \geqslant \lambda r^{2-2s} \qquad \text{for every value of } v \in B_{\bar R}. \tag{1.4}$$

Here, when we write $(w\cdot e)_+^2$, we mean $((w\cdot e)_+)^2 = \max(w\cdot e, 0)^2$.

The coercivity condition would be obviously true if K is symmetric (i.e. $K(v,v') = K(v',v)$) and $K(v,v') \geqslant \lambda |v-v'|^{-d-2s}$. These assumptions are not satisfied by the Boltzmann kernel a priori.

For some kernels (not necessarily coming from the Boltzmann equation) it might be difficult to check whether the coercivity condition (1.3) holds. The nondegeneracy assumption (1.4) is usually very easy to check in explicit examples of kernels K. We do not know of any example of a kernel which satisfies (1.4) but not (1.3). It is natural to conjecture this implication (modulo adjusting λ by a fixed factor).

The second assumption is a weak upper bound on the kernel K.

$$\begin{cases} (i) & \int_{\mathbb{R}^d \setminus B_r(v)} K(v,v') \, \mathrm{d}v' \leqslant \Lambda r^{-2s} \text{ for any } r > 0 \text{ and } v \in B_{\bar R} \\ (ii) & \int_{B_{\bar R} \setminus B_r(v')} K(v,v') \, \mathrm{d}v \leqslant \Lambda r^{-2s} \text{ for any } r > 0 \text{ and } v' \in B_{\bar R}. \end{cases} \tag{1.5}$$

Note that if $K(v,v') \lesssim |v-v'|^{-d-2s}$, then the assumption (1.5) holds. Our assumption only concerns average values of K on the complementary set of balls. Therefore, a kernel containing a singular part is allowed. We will see that the Boltzmann kernel satisfies (1.5) even though $K(v,v') \lesssim |v-v'|^{-d-2s}$ may not hold a priori.

Note that both inequalities in (1.5) would be the same if K were symmetric. But we do *not* assume symmetry of the kernel. That is $K(v,v') \neq K(v',v)$ in general. The symmetry assumption is very common for integro-differential equations because it represents the fact that the equation is in *divergence form*. It is equivalent to the operator L_v being self adjoint. We explain this concept in Subsection 1.3.3.

The following assumptions provide a mild control on the anti-symmetric part of the kernel.

We assume that

$$\forall v \in B_{7\bar{R}/8}, \qquad \left| PV \int_{B_{\bar{R}/8}(v)} \big(K(v,v') - K(v',v)\big) \, \mathrm{d}v' \right| \leqslant \Lambda. \tag{1.6}$$

Moreover, if $s \geqslant 1/2$, we need to assume the following extra cancellation.

$$\forall r \in (0, \bar{R}/8], \forall v \in B_{7\bar{R}/8}, \qquad \left| PV \int_{B_r(v)} (v-v') \big(K(v,v') - K(v',v)\big) \, \mathrm{d}v' \right| \leqslant \Lambda(1 + r^{1-2s}). \tag{1.7}$$

When K is symmetric, the left hand sides in (1.6) and (1.7) are identically zero and therefore the assumptions trivially hold.

When $s > 1/2$, if the assumption (1.7) holds for $r = \bar{R}$ and in addition (1.5) holds, then we observe that (1.7) automatically holds for all $r \in (0, \bar{R}]$. The requirement that the inequality (1.7) holds for all $r \in (0, \bar{R}]$, as opposed to only $r = \bar{R}$, only makes a difference for the case $s = 1/2$. We discuss the scaling properties of our assumptions in subsections 2.2 and 2.3.

When we apply our results to the Boltzmann equation, the kernel K depends on the solution f and is determined by the formula

$$\int_{\mathbb{R}^d} \int_{\partial B_1} f(v'_*)(g(v') - g(v)) B(|v - v_*|, \theta) \, \mathrm{d}\sigma \, \mathrm{d}v_* = \int_{\mathbb{R}^d} (g(v') - g(v)) K_f(v,v') \, \mathrm{d}v'.$$

In this way,

$$Q(f,g) = \int (g(v') - g(v)) K_f(v,v') \, \mathrm{d}v' + (\text{lower order terms}).$$

The constant λ in the assumption (1.3) depend only on the mass, energy and entropy densities of f. The constant Λ in (1.5), (1.6) and (1.7), depends only on the mass and energy density of f when $\gamma \in [0,1]$. It depends on further integrability properties of f when $\gamma < 0$ (they are bounded in terms of $\|f\|_{L^\infty}$ for example). All these assumptions will be verified in Section 3.

1.2. **Main results.**

The notion of weak solution will be made precise by the end of Section 5.

Theorem 1.5 (Hölder continuity)**.** *Assume the kernel is non-negative and there exist $\lambda > 0$ and $\Lambda > 0$ such that (1.3), (1.5) and (1.6) hold true with $\bar{R} = 2$. If $s \geqslant 1/2$, we also assume (1.7); if $s < 1/2$, we also assume (1.4). Let f be a solution of (1.2) for all $t \in (-1,0]$, $x \in B_1$ and $v \in B_1$. Assume that f is essentially bounded in $(-1,0] \times B_1 \times \mathbb{R}^d$. Then f is Hölder continuous in $(-1/2,0] \times B_{1/2} \times B_{1/2}$ with*

$$\|f\|_{C^\gamma((-1/2,0] \times B_{1/2} \times B_{1/2})} \leqslant C \left(\|f\|_{L^\infty((-1,0] \times B_1 \times \mathbb{R}^d)} + \|h\|_{L^\infty((-1,0] \times B_1 \times B_1)} \right)$$

where $C > 0$ and $\gamma \in (0,1)$ are constants only depending on dimension, λ and Λ.

This theorem is in fact derived from the following estimate.

Theorem 1.6 (Weak Harnack inequality)**.** *There are constants r_0, $R_1 > 1$, ε and C so that the following proposition holds. Assume the kernel is non-negative and there exist $\lambda > 0$ and $\Lambda > 0$ such that (1.3), (1.5) and (1.6) holds true with $\bar{R} = 2R_1$. If $s \geqslant 1/2$, we also assume (1.7); if $s < 1/2$, we also assume (1.4). Assume that f is a non-negative supersolution of (1.2) in $(-1,0] \times B_{R_1^{1+2s}} \times B_{R_1}$. Then*

$$\left(\int_{Q^-} f^\varepsilon(t,x,v) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \right)^{1/\varepsilon} \leqslant C \left(\inf_{Q^+} f + \|h\|_{L^\infty((-1,0] \times B_1 \times B_1)} \right)$$

where

$$Q^+ = (-r_0^{2s}, 0] \times B_{r_0^{1+2s}} \times B_{r_0} \quad \text{and} \quad Q^- = (-1, -1 + r_0^{2s}] \times B_{r_0^{1+2s}} \times B_{r_0}$$

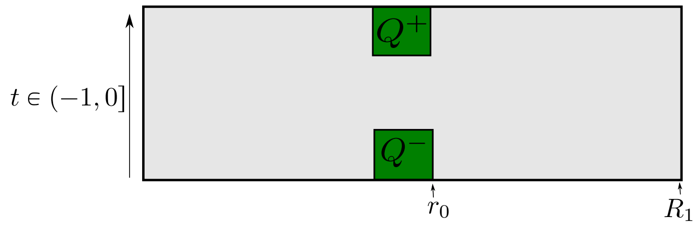

FIGURE 1. The geometric setting of the weak Harnack inequality

(see Figure 1) and the constants $C > 0$, $\varepsilon > 0$, only depend on dimension, s, λ and Λ. The constants r_0 and R_1 depend on dimension and s only (not on λ and Λ).

1.3. **Comments on the results and related works.**

1.3.1. *Difficulties related to this problem.* This subsection is our attempt to explain and compare the main challenges that we faced proving the main results in this paper, and the new ideas that were introduced.

We start by reviewing some recent developments about parabolic kinetic equations in divergence form, with rough coefficients. In some sense, our main theorems are an integro-differential counterpart of these previous results. The equations have the form

$$f_t + v \cdot \nabla_x f = \frac{\partial}{\partial v_i}\left(a_{ij}\frac{\partial}{\partial v_j} f\right).$$

The diffusion coefficient $a_{ij} = a_{ij}(t, x, v)$ is assumed to be uniformly elliptic. No regularity assumption should be made in a_{ij}, otherwise the equation may fit into the more classical hypoelliptic theory, and would not imply such interesting results for the Landau equation. Pascucci and Polidoro [55] obtained the local L^∞ estimate for this equation using Moser's method. Continuing in that direction, Wang and Zhang obtained Hölder estimates in [49], [64] and [65]. Their proof is quite involved. A highly nontrivial step is to obtain an appropriate formulation of a Poincaré inequality adapted to the Lie group action related to the equation.

A simplified proof, following the method of De Giorgi, was recently obtained by Golse, Imbert, Mouhot and Vasseur [36]. In this paper a version De Giorgi's isoperimetric inequality is obtained by a compactness argument. We use that idea for the case $s \in [1/2, 1)$. This general method is not applicable to the case $s < 1/2$, since it uses crucially that the characteristic function of a nontrivial set can never be in H^s. We do not use velocity averaging lemmas like in [36] anywhere in this paper. Instead, we take advantage of more elementary properties of the fractional Kolmogorov equation.

The first step in the proof of De Giorgi, Nash and Moser, which consists of a local L^∞ estimate, needs to be formulated appropriately to hold for integro-differential equations with degenerate kernels. Our proof in Section 6 follows a properly adapted version of De Giorgi's iteration. We do not use either averaging lemmas, or hypoelliptic estimates for the x variable (like in [36] or [55]). Instead, we iterate an improvement of integrability obtained directly from the fundamental solution to the fractional Kolmogorov equation.

In the second part of the proof of the theorem of De Giorgi, Nash and Moser we take different strategies depending on whether $s \in (0, 1/2)$ or $s \in [1/2, 1)$. In the first case, we construct a barrier function to propagate lower bounds as in the method by Krylov and Safonov for nondivergence equations. When $s \in [1/2, 1)$, the proof is based on a measure estimate of intermediate sets (as in De Giorgi's original work) obtained by compactness (as in [36]), but using a more direct approach based on the fractional Kolmogorov equation instead of hypoelliptic estimates and averaging lemmas. We could not find a single method that works for the full range $s \in (0, 1)$ for general integro-differential equations. However, in the case of the Boltzmann equation, the method used for the range $s \in (0, 1/2)$ actually works for the full range, as we explain below.

The kernel K_f, from the Boltzmann equation, satisfies the extra symmetry condition $K(v, v + w) = K(v, v - w)$ which we do not use in this paper. We chose not to take advantage of this condition in order to have the most natural result for general integro-differential equations. Using this assumption would allow us to simplify some of the proofs. Most importantly, the barriers of Section 7 would hold for the full range $s \in (0, 1)$ and therefore the results from Section 8 would be unnecessary. Moreover, the proof of Lemma 6.4 could be done more easily using a similar function g as in the proof of Lemma 6.2. The commutator estimates of Lemmas 4.10 and 4.11 would not be necessary anywhere.

One of the main ideas in the work of Caffarelli, Chan and Vasseur [21] about parabolic integro-differential equations (not kinetic) is how they formulate De Giorgi's isoperimetric lemma in the integro-differential setting. Their original method is purely nonlocal. It does not work for second order equations. It uses crucially that $\mathcal{E}(g_+, g_-) \gtrsim \|g_+\|_{L^1} \|g_-\|_{L^1}$, where $\mathcal{E}$ is a bilinear form like the one we define in Section 4. In our context, this is not true for two reasons. First, because we have the additional variable x that plays no explicit role in the integral diffusion and is not *seen* by the bilinear form $\mathcal{E}$. Secondly, because the assumptions that we make in the kernels are too mild for this condition to hold even in the space homogeneous case. In [21], they assume that the kernel K is symmetric and $K(v, v') \gtrsim \lambda |v - v'|^{-d-2s}$ for every value of v and v'.

Our equation (1.2) involves three different variables: t, x and v. It reduces to a more standard parabolic integro-differential equation when f is constant in x. The diffusion takes place with respect to the variable v only. The equation includes the kinetic transport term $v \cdot \nabla_x f$, which somehow transfers the regularization effect from the v variable to the x variable. The variable x has to be dealt with differently to the t and v variables. For example it has a different scaling and it is affected by translations of the function with respect to the v variable. One major difficulty that it brings is in the proof of the ink-spots theorem. The original ink-spots covering by Krylov and Safonov was for non-kinetic parabolic equations without the extra variable x. Including this extra variable changes the geometry. The natural parabolic cylinders, which are invariant by the Lie group acting on the equation, are oblique in the variable x. With this geometry, there is no chance to apply a Calderón-Zygmund decomposition like in [40] because we cannot tile the space with slanted cylinders with varying slopes. We need a custom made version of the ink-spots covering theorem, which is developed in Section 10. See that section for further explanation on the difficulties and ideas involved in this covering result.

When we apply our main results to the Boltzmann equation in Theorems 1.3 and 1.1, we only want to assume a priori some minimal physically relevant information on f. We assume a control, for all t and x, of the mass, energy and entropy densities. Under these assumptions, there is very little one can say about the Boltzmann collision kernel K_f. We are forced to work with very general, non-symmetric, and possibly singular kernels. This paper would be much simpler if we made a convenience assumption like $K(v, v') = K(v', v) \approx |v - v'|^{-d-2s}$, but it would not suffice to apply the result to the Boltzmann equation. It is not a priori obvious what assumptions the Boltzmann kernel will satisfy. In Section 3, we prove that K_f satisfies (1.3), (1.5), (1.6) and (1.7). Our assumptions (1.6) and (1.7) allow us to consider non-symmetric kernels whose anti-symmetric part is as singular as the symmetric one in absolute value, but contains some cancellation. Up to the authors' knowledge, this is the first time such a condition appears in the literature of integro-differential equations.

The estimate for the bilinear form given in Theorem 4.1 is interesting in itself and new. It tells us that the bilinear form $\langle L_v f, g \rangle$ is bounded in $H^s \times H^s$ assuming the very mild, and easy to check, conditions on the kernel K given in (1.5), (1.6) and (1.7). Such an estimate is reminiscent of some others proved specifically for the Boltzmann equation, see for instance [9], [4], [51], [28]. Here, the estimate is proved for a very general bilinear form associated with a non-symmetric integro-differential operator. Note that in previous works in integro-differential equations, the upper bound of Theorem 4.1 was included as an assumption together with (1.5) and symmetry (see [43]).

1.3.2. *Boltzmann without cut-off.* The main results of this paper apply to the Boltzmann equation without cut-off in the inhomogeneous setting.

In the case of moderately soft potentials, which corresponds to $\gamma + 2s > -2$, an a priori estimate in L^∞ is given in [59]. In that case, we obtain a Hölder modulus of continuity depending on the bounds on $M(t, x)$, $E(t, x)$ and $H(t, x)$ only. For very soft potentials, Theorem 1.1 gives us a Hölder modulus of continuity provided that we know a priori that f is bounded. Note that our estimates do not depend on any further regularity assumption on the initial data.

Since Carlo Cercignani in 1969, it is believed that the Boltzmann collision operator without cut-off has a regularizing effect. Some similarities with the fractional Laplacian operator in the velocity variable have been observed in the form of coercivity estimates. This is the first time that ideas originating in the work of De Giorgi and Nash for parabolic equations are applied in the context of the Boltzmann equation.

The first results for the Boltzmann equations without cut-off that indicate a regularization effect appear in the study of the entropy dissipation. A lower bound for the entropy dissipation with respect to a fractional Sobolev norm is first obtained [48] and improved in [62]. The optimal space H^s is finally obtained in [1]. We can also deduce a coercivity estimate from the proof in this paper. The coercivity estimate, which we mention in Proposition 3.3, essentially says that the Boltzmann collision operator satisfies the assumption (1.3). It plays an essential role in most of the works concerning the regularization effect of the Boltzmann equation without cut-off. The proof of the coercivity estimate in [1] is done using Fourier analysis after reducing the problem to the case of Maxwellian molecules ($\gamma = 0$). There is a simplified proof, also using Fourier analysis and in particular the Littlewood-Paley decomposition, in [5] and [6]. These proofs are considerably easier in the Maxwellian case ($\gamma = 0$), because they use Bobylev's formula. We give a new alternative proof in the Appendix A based on the geometric understanding of the Boltzmann kernel. All computations are done in physical variables. Our proof works in the same way for any value of γ. It transparently gives us an estimate with respect to the same anisotropic weighted Sobolev spaces as in [37].

The coercivity estimate implies some gain of regularity for the Boltzmann equation without cut-off. In the space homogeneous case, iterating this gain of regularity, it is known that solutions belong to the Schwartz class for all positive times. This result holds under rather general cross section assumptions, including essentially hard and moderately soft potentials in the non-cut-off case. See [32], [5], [6], [39], [51] and [27].

For the spacially inhomogeneous case without cut-off, one can also obtain some regularization effect combining the coercivity with hypoelliptic estimates. Iterating such estimates leads to the C^∞ regularity of solutions. However, it is necessary to impose significant conditional regularity in order to start the iteration. The best regularity results available require the assumptions that $\langle v\rangle^k f(t,x,v)$ belongs to $H^5([0,T],\mathbb{R}^3,\mathbb{R}^3)$ for all values of $k \in \mathbb{N}$, and in addition the mass density is assumed to be bounded below. Under these assumptions, they prove that f belongs to the Schwartz class for positive time in [7], [9], [28].

It may be interesting to compare the current state of the regularity results for the Boltzmann equation with the classical development of nonlinear elliptic equations. Hilbert's 19^{th} problem consisted in the regularity of minimizers of smooth convex functionals in H^1 (see [66]). These minimizers solve a nonlinear elliptic equation in divergence form. From the beginning of the century (starting by the work of Bernstein [18]), people proved that solutions were analytic provided that some conditional regularity assumption was satisfied. The assumptions were progressively improved through the years. By iterating the Schauder estimates, it was possible to prove that solutions were analytic starting from a $C^{1,\alpha}$ estimate. However, variational techniques only provided a weak solution in H^1. It was a long standing problem to bridge that gap, and it was finally achieved independently by De Giorgi [30] and Nash [54]. Our result in this paper plays the role, in the context of the inhomogeneous non-cut-off Botlzmann equation, of the results of De Giorgi and Nash for elliptic and parabolic equations. Unfortunately, there is still a gap between what we prove (C^α regularity) and what is necessary to iteratively obtain C^∞ regularity of the solution by current methods (H^5 regularity plus infinite moments). So, more work is necessary.

In [59], results from general integro-differential equations are applied to the Boltzmann equation. There is an L^∞ estimate, a Hölder estimate and a lower bound. However, the last two apply only to the space homogeneous case. The results in this paper are proved with different techniques compared to [59]. In this work, we develop a result in the flavor of De Giorgi, Nash and Moser theorem for equations in divergence form. The results in [59] use the methods from [56] which are in the flavor of Krylov-Safonov theory for equations in nondivergence form. The coercivity estimate plays no role in [59], and it certainly does here. Our result in Theorem 1.5 complements the L^∞ estimate from [59].

In [3], the authors prove that if the inital data is sufficiently nice, the Boltzmann equation admits a unique smooth solution locally in time. For small perturbations around a Maxwellian, the equation is known to have global smooth solutions [38], [37], [8], [2]. As far as existence of weak solutions is concerned, Alexandre and Villani prove in [10] the existence of a certain type of renormalized solution. Neither the uniqueness nor the regularity of these solutions is well understood. They prove that the family of solutions is compact using the entropy dissipation estimate.

The study of the regularity of solutions is relevant for most aspects of the qualitative analysis of the Boltzmann equation without cutoff. For example, Desvillettes and Villani prove in [31] that the solutions converge to equilibrium, at a specific rate, provided that the solution remains smooth.

We consider this paper to be an important step towards a longer term goal to prove the following conjecture. We believe that if f is a solution to the Boltzmann equation with $\gamma + 2s \in (0,2]$ and such that $0 < M_1 \leqslant M(t,x) \leqslant M_0$, $E(t,x) \leqslant E_0$ and $H(t,x) \leqslant H_0$, then f should be C^∞ for positive time.

It is not at all clear whether the assumption $\gamma + 2s > 0$ is necessary to obtain regularity. However, the L^∞ estimate for very soft potentials is out of reach by current methods without further assumptions. This is also the case for the space homogeneous Boltzmann equation.

It would be possible to study the precise behavior of the constants λ and Λ for which (1.3), (1.5), (1.6) and (1.7) hold and obtain a global weighted C^α estimate using a scaling argument as in Remark A.7. However, this estimate also depends on the L^∞ norm of f. It is to be expected that the solution f should decay exponentially for large velocities, in addition to the L^∞ bound given in [59]. See [35] for a result in that direction in the space homogeneous case. A better decay in f for large velocities would imply a better C^α estimate for large velocities. Because of that, we postpone the analysis of large velocities to future work when the decay of f is better understood. The local result provided in Theorem 1.5 provides the right tool to study the C^α estimate for large velocities in terms of the decay of f.

1.3.3. *Regularity theory for integro-differential equations.* The study of Hölder estimates and the Harnack inequality for integro-differential equations of the form

$$f_t(t,v) = \int_{\mathbb{R}^d} (f(t,v') - f(t,v))K(t,v,v')\,\mathrm{d}v'$$

is a very active area of current research. It developed originally motivated by problems in probability, with applications to mathematical finance [61] and physics [50]. The main technical novelty of this work is our study of a kinetic equation with this kind of diffusion. Our equation has the extra variable x, and the transport term $v \cdot \nabla_x f$, without any explicit diffusion in x. Previous Hölder estimates for integro-differential equations may be applied to the Boltzmann equation, at most, in the space homogeneous case only. Yet, even in the space homogeneous case, the results in this paper present novelties. The assumptions we make on the kernel (1.3), (1.5), (1.6) and (1.7) are more general than in previous works about integro-differential equations. Because of that, our main results in Theorems 1.5 and 1.6 are new even in the space homogeneous case. In this subsection, we review and compare the literature about integro-diferential diffusions. We stress that all previous results apply to the space homogeneous case only.

The interest in Hölder estimates and Harnack inequalities started from the study of regularization properties of classical parabolic equations of second order. For equations in divergence form (like $f_t = \partial_i a_{ij}(t,v)\partial_j f$), the estimates were originally obtained independently by De Giorgi [30] and Nash [54], and later reproved by Moser [52]. For equations in nondivergence form (like $f_t = a_{ij}(t,v)\partial_{ij} f$) the result was obtain much later by Krylov and Safonov [46]. The techniques used for equations in divergence or nondivergence form are very different. In the former case, the equation's structure is amenable to variational methods, and energy estimates in Sobolev spaces. In the latter case, tools like the Alexandroff estimate and explicit barrier functions are used for the proofs. Both types of results, with their corresponding approaches, have their counterparts for integro-differential equations. In this paper, we use the variational structure of the equation and work with localized energy estimates. These are ideas for equations in divergence form. However, we use some ideas that originated in the study of equations in nondivergence form, like the ink-spots theorem and barrier functions. Below, we review other results for integro-differential equations following each approach.

A second order operator in divergence form $f \mapsto \partial_i(a_{ij}(t,v)\partial_j f)$ is characterized by the fact that it is self-adjoint in L^2. For integro-differential operators, this is reflected in a symmetry condition for the kernel: $K(v,v') = K(v',v)$. A second order operator in nondivergence form $f \mapsto a_{ij}(t,v)\partial_{ij} f$ has the convenient property that it returns a bounded function when evaluated in a smooth function f. For integro-differential operators, this is reflected in a different symmetry condition $K(v,v+w) = K(v,v-w)$. The Boltzmann collision kernel has the symmetry condition that corresponds to equations in nondivergence form. This structure is exploited in [59] to obtain Hölder estimates in the space homogeneous case, and L^∞ estimates for the full equation. In this paper we apply techniques for equations in divergence form. We include assumptions (1.6) and (1.7) which measure how much the kernel K is allowed to depart from being symmetric (as in $K(v,v') = K(v',v)$).

The Harnack inequality and Hölder estimates for integro-differential equations in *divergence* form has a long history with several major contributions. Some results in this direction are [44], [14], [41], [29], [21], [34], [43] and [33]. There is a small survey on the subject in [43]. In these papers the kernel K satisfies the symmetry condition $K(v,v') = K(v',v)$ plus some ellipticity assumptions. It is perhaps clear that there is some room in the methods for a lower order asymetric part in K. Our assumptions (1.6) and (1.7) allow us to consider a non-symmetric kernel K whose asymetric part is as singular as the symmetric part. We require a control of the asymetric part in terms of cancellation conditions, which is new.

A natural ellipticity condition on the kernel is to assume that it is comparable with the fractional Laplacian. The classical assumption would be $K(v,v') \approx |v-v'|^{-d-2s}$ for every value of v and v'. This assumption is made in [44], [14], [41], [29] and [21]. The results were extended to a much more general class of kernels in [34], [43] and [33]. The assumptions there are essentially equivalent to our assumptions (1.3) (the lower bound on the bilinear form in the symmetric case) and (1.5) (the upper bound on the kernel), plus the result of our Lemma 4.2 (the upper bound for the bilinear form). It is a new contribution of this paper that Lemma 4.2 follows from (1.5). We also prove in Theorem 4.1 that the integro-differential operator L_v is bounded in H^s to H^{-s} for a non-symmetric kernel satisfying (1.5), (1.6) and (1.7). The proof is significantly more complicated in the non-symmetric case.

The study of integro-differential equations in *nondivergence* form followed a parallel path using different tools. These are the Hölder estimates and the Harnack inequality for kernels satisfying the other symmetry condition: $K(v,v+w) = K(v,v-w)$. There are also many important results in this direction including [17], [60], [16], [15], [57], [22], [58], [13], [12], [19], [23], [25], [47], [26], [42] and [56]. The majority of these results make the pointwise assumption on the kernel $K(v,v') \approx |v-v'|^{-d-2s}$, and therefore are not directly applicable to the Boltzmann equation. It is only in [19], [42] and [56] that more singular kernels are considered. The assumptions in [56] are sufficiently general to be applicable to the space homogeneous Boltzmann equation. Our result is for equations in *divergence* form, and thus none of these papers either implies or follows from ours. Interestingly, we use some of the ideas for nondivergence equations. Most importantly, the ink-spots theorem that we develop in Section 10 is a generalization of a similar covering argument in [56].

We stress that our main regularity result in Theorem 1.5 requires the equation to hold in a bounded domain only. The parameters λ and Λ in the assumptions (1.3), (1.5), (1.6) and (1.7) will deteriorate as $|v| \to \infty$ in the case of the Boltzmann equation.

1.4. **Organization of the article.** We set our notation and further analyze our assumptions in Section 2. The relationship between our main results and the Boltzmann equation is discussed in Section 3, where we prove in particular that the Boltzmann kernel satisfies the assumptions listed above. The analysis of the operator L_v and its associated bilinear form $\mathcal{E}$ is done in Section 4. This section should be interesting in itself. This is where the generality of our assumptions on the kernels is reflected. All the results in Section 4 would be straight forward if we assumed that the kernels satisfy $K(v,v') = K(v',v)$ and $K(v,v') \approx |v-v'|^{-d-2s}$. The core of the proof of the Weak Harnack inequality and Hölder estimates for integro-differential equations is done in sections 6, 7, 8, 9, 10 and 11. Section 5 contains fairly unsurprising statements that are technically necessary for the completeness of the rest of our proofs. Experts will probably skim through this section quickly. The appendix A contains a new proof of the coercivity bound for the Boltzmann equation (Subsection A.1) and some technical lemmas (Subsection A.2).

2. Preliminaries

2.1. **Notation.** For a real number a, $a^+ = \max(a,0)$.

A constant is called *universal* if it only depends on dimension and the constants s, λ and Λ in the assumptions (1.3), (1.5), (1.6) and (1.7).

When we write $a \lesssim b$, we mean that there exists a universal constant C, so that $a \leqslant Cb$. We write $a \approx b$ when both $a \lesssim b$ and $b \lesssim a$ hold.

When we write $\dot H^s(\Omega)$ for some $\Omega \subset \mathbb{R}^d$, we mean the space whose norm is given by

$$\|f\|^2_{\dot H^s(\Omega)} := \iint_{\Omega\times\Omega} \frac{|f(v')-f(v)|^2}{|v-v'|^{d+2s}} \,\mathrm{d}v' \,\mathrm{d}v.$$

The space $H^s(\Omega)$ is the one corresponding to the norm

$$\|f\|^2_{H^s(\Omega)} := \|f\|^2_{\dot H^s(\Omega)} + \|f\|^2_{L^2(\Omega)}.$$

The space $H^s_0(\Omega)$ is obtained by completing the space of C^∞ functions in $\mathbb{R}^d$ supported in Ω with respect to the norm $\|\cdot\|_{H^s(\Omega)}$. When $\Omega = \mathbb{R}^d$, $H^s_0(\Omega) = H^s(\Omega)$. We also define $H^{-s}(\Omega)$ as the dual of $H^s_0(\Omega)$.

It is well known that $\|f\|^2_{\dot H(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\xi|^{2s}|\hat f(\xi)|^2 \,\mathrm{d}\xi$. Moreover, $f \in H^{-s}(\mathbb{R}^d)$ if and only if $f = g_1 + (-\Delta)^{s/2} g_2$ with $g_1, g_2 \in L^2(\mathbb{R}^d)$. Similarly, f is in the dual of $\dot H^s(\mathbb{R}^d)$ if $f = (-\Delta)^{s/2} g$ for some function $g \in L^2(\mathbb{R}^d)$.

Note also that if $f : \mathbb{R}^d \to \mathbb{R}$ is supported in B_1, then $\|f\|_{H^s(\mathbb{R}^d)}$, $\|f\|_{H^s(B_2)}$ and $\|f\|_{\dot H^s(B_2)}$ are all equivalent.

2.2. **First consequences of assumptions.** After an obvious readjustment of constants (depending on d and s), the assumption (1.5) is equivalent to the following

$$\begin{cases} (i) & \int_{B_{2r}(v)\setminus B_r(v)} K(v,v')\,\mathrm{d}v' \leqslant \Lambda r^{-2s} \text{ for any } r>0 \text{ and } v \in B_{\bar R} \\ (ii) & \int_{B_{\bar R}\cap B_{2r}(v')\setminus B_r(v')} K(v,v')\,\mathrm{d}v \leqslant \Lambda r^{-2s} \text{ for any } r>0 \text{ and } v' \in B_{\bar R}. \end{cases}$$

It is also equivalent to

$$\begin{cases} (i) & \int_{B_r(v)} |v-v'|^2 K(v,v')\,\mathrm{d}v' \leqslant \Lambda r^{2-2s} \text{ for any } r>0 \text{ and } v \in B_{\bar R} \\ (ii) & \int_{B_{\bar R}\cap B_r(v')} |v-v'|^2 K(v,v')\,\mathrm{d}v \leqslant \Lambda r^{2-2s} \text{ for any } r>0 \text{ and } v' \in B_{\bar R}. \end{cases}$$

We use the three forms of the assumption (1.5) indistinctively in different parts of the paper.

As we mentioned before, when $s > 1/2$, if the assumption (1.7) holds for some value of $r = r_0$ and also (1.5) holds, then (1.7) also holds for any other value $r \in (0, \bar R/8]$. The reason is the following computation. We write it for the case $r < r_0$. The case $r > r_0$ follows similarly.

$$\begin{aligned} \left| PV \int_{B_r} (v-v')\left(K(v,v') - K(v',v)\right)\,\mathrm{d}v' \right| &\leqslant \left| PV \int_{B_{r_0}} (v-v')\left(K(v,v') - K(v',v)\right)\,\mathrm{d}v' \right| \\ &\quad + \left| PV \int_{B_{r_0}\setminus B_r} (v-v')\left(K(v,v') - K(v',v)\right)\,\mathrm{d}v' \right|, \\ &\leqslant \Lambda(1 + r_0^{1-2s}) + \int_{\mathbb{R}^d\setminus B_r} |v-v'|(K(v,v') + K(v',v))\,\mathrm{d}v', \\ &\leqslant \Lambda(1 + r_0^{1-2s} + r^{1-2s}). \end{aligned}$$

The last inequality is a consequence of (1.5). Note that, for the case $s = 1/2$, this last integral may be divergent and thus the assumption (1.7) is made so that the inequality holds for all values of r in the range $(0, \bar R/8]$.

2.3. **Invariant transformations.** If f satisfies the equation (1.2) for some kernel K satisfying (1.3), (1.5), (1.6) and (1.7), then the scaled function $f_r(t,x,v) = f(r^{2s}t, r^{2s+1}x, rv)$ satisfies a modified equation

$$\partial_t f_r + v\nabla_x f_r + \tilde{L_v} f_r = h_r,$$

where

$$\begin{aligned} h_r(t,x,v) &= r^{2s} h(r^{2s}t, r^{2s+1}x, rv), \\ K_r(t,x,v,v') &= r^{d+2s} K(r^{2s}t, r^{2s+1}x, rv). \end{aligned}$$

For any $r \in [0,1]$, the kernel K_r satisfies the assumptions (1.3), (1.5), (1.6) and (1.7) with a larger radius $\bar R/r$ instead of $\bar R$. Moreover, $\|h_r\|_{L^\infty(Q_1)} \leqslant r^{2s}\|h\|_{L^\infty(Q_1)} \leqslant \|h\|_{L^\infty(Q_1)}$.

The equation is also invariant under the family of transformations $\mathcal{T}_{z_0}$. Here $z_0 = (t_0, x_0, v_0) \in \mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$.

$$\begin{aligned} \mathcal{T}_{z_0}(t,x,v) &= (t_0 + t, x_0 + x + tv_0, v_0 + v) = z_0 \circ z, \\ \mathcal{T}_{z_0}^{-1}(t,x,v) &= (t - t_0, x - x_0 - (t-t_0)v_0, v - v_0) = z_0^{-1} \circ z \end{aligned}$$

(see Figure 2). Indeed, the product $\circ$ induces a Lie group structure on $\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d$. We remark that

$$(T,0,0) \circ (t,x,v) = (t+T, x, v),$$

that is to say, translation in time coincides with a left Lie product.

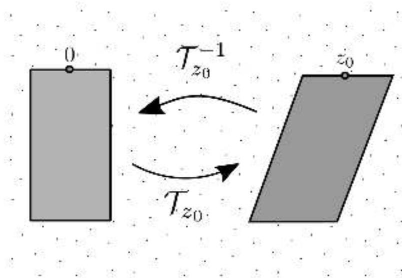

FIGURE 2. The transformation leaving the equation invariant. On the left, a straight cylinder centered at the origin. On the right a slanted cylinder centered at z_0.

Because of the scaling and the group action that keep our class of equations invariant, we are forced to work with *slanted* cylinders: for a given center $z_0 = (t_0, x_0, v_0)$ and some radius $r > 0$ by the following formula

$$Q_r(z_0) = \{(t,x,v) : -r^{2s} \leqslant t - t_0 \leqslant 0, |v - v_0| < r, |x - x_0 - (t-t_0)v_0| < r^{1+2s}\}. \tag{2.1}$$

Remark that for $z_0 = 0$,

$$Q_r = Q_r(0) = (-r^{2s}, 0] \times B_{r^{2s+1}} \times B_r.$$

2.4. **The fractional Kolmogorov equation.** In this subsection we review the fractional Kolmogorov's equation:

$$f_t + v \cdot \nabla_x f + (-\Delta_v)^s f = h. \tag{2.2}$$

The previous Lie group structure also preserves this equation. There is a fundamental solution $J(t,x,v)$ which has the following form

$$J(t,x,v) = c_d \, \frac{1}{t^{d+d/s}} \, \mathcal{J}\left(\frac{x}{t^{1+1/2s}}, \frac{v}{t^{1/2s}}\right).$$

The function $\mathcal{J}$ can be computed explicitly in Fourier variables by the formula

$$\hat{\mathcal{J}}(\varphi, \xi) = \exp\left(-\int_0^1 |\xi - \tau\varphi|^{2s} \, \mathrm{d}\tau\right).$$

In the physical variables x and v, the formula for $\mathcal{J}$ is not explicit. However, some simple properties can be deduced from classical considerations. We collect them in the following proposition.

Proposition 2.1 (Fundamental solution of the fractional Kolmogorov equation)**.** *The functions J and $\mathcal{J}$ have the following properties.*

(1) *The function $\mathcal{J}$ is C^∞ and decays polynomially at infinity. Moreover, $\mathcal{J}$ and all its derivatives are integrable in $\mathbb{R}^{2d}$.*
(2) *For every $t > 0$, $\int_{R^{2d}} J(t,x,v) \, \mathrm{d}v \, \mathrm{d}x = 1$.*
(3) *Both functions are nonnegative: $J \geqslant 0$ and $\mathcal{J} \geqslant 0$.*
(4) *For any $p \geqslant 1$, we have*

$$\|J(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^{2d})} = t^{-d(1+1/s)(1-1/p)} \, \|\mathcal{J}\|_{L^p(\mathbb{R}^{2d})},$$
$$\|(-\Delta)^{s/2} J(t,\cdot,\cdot)\|_{L^p(\mathbb{R}^{2d})} = t^{-d(1+1/s)(1-1/p)-1/2} \, \|(-\Delta)^{s/2}\mathcal{J}\|_{L^p(\mathbb{R}^{2d})}.$$

In particular, for $p_\star = (2d(1+s)+2s)/(2d(1+s)+s) \in (1,2)$, we have $\|J(t,\cdot,\cdot)\|_{L^{p_\star}(R^{2d})} \leqslant Ct^{1/2-1/p_\star}$ and

$$\|(-\Delta)_v^{s/2} J(t,\cdot,\cdot)\|_{L^{p_\star}(\mathbb{R}^{2d})} \leqslant Ct^{-1/p_\star}.$$

The initial value problem (2.2) is solved by the formula

$$\begin{aligned} f(t,x,v) = &\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} f_0(y,w) J(t, x-y-tw, v-w) \, \mathrm{d}w \, \mathrm{d}y \\ &+ \int_0^t \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} h(\tau,y,w) J(t-\tau, x-y-(t-\tau)w, v-w) \, \mathrm{d}w \, \mathrm{d}y \, \mathrm{d}\tau \end{aligned} \tag{2.3}$$

We define the *modified* convolution $*_t$ by the formula

$$h *_t j(x,v) = \iint h(y,w) j(x-y-tw, v-w)\, \mathrm{d}w\, \mathrm{d}y.$$

If we make the change of variables $\tilde{j}(x,v) = j(x+tv,v)$, then $h *_t j(x,v) = h * \tilde{j}(x-tv,v)$. Thus, the *modified* convolution is the same as the usual convolution conjugated by that change of variables (of Jacobian one). We observe that this convolution satisfies the usual Young's inequality:

$$\left\| \iint h(y,w) j(x-y-tw, v-w)\, \mathrm{d}w\, \mathrm{d}y \right\|_{L^r_{x,v}} \leqslant \|h\|_{L^p_{x,v}} \|j\|_{L^q_{x,v}} \qquad \text{independenly of } t. \tag{2.4}$$

Here $1 + 1/r = 1/p + 1/q$.

The following proposition is simply a consequence of Young's inequality.

Proposition 2.2 (Gain of integrability)**.** *Let f be the solution of (2.2) in $[0,T] \times \mathbb{R}^{2d}$, with $f(0,x,v) = f_0(x,v) \in L^2(\mathbb{R}^{2d})$. Assume $h \in L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$. Then*

$$\|f\|_{L^q([0,T]\times\mathbb{R}^{2d})} \leqslant C(T) \left(\|f_0\|_{L^2(\mathbb{R}^{2d})} + \|h\|_{L^2([0,T]\times\mathbb{R}^d, H^{-s}(\mathbb{R}^d))} \right)$$

for any q such that $1/q > 1/p_\star - 1/2$ and $p_\star$ is the one from Proposition 2.1.

Proof. Since $h \in L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$, then there exists h_1 and h_2 in $L^2([0,T] \times \mathbb{R}^{2d})$. so that $h = h_1 + (-\Delta)^{s/2}_v h_2$ and

$$\|h_1\|_{L^2} + \|h_2\|_{L^2} \approx \|h\|_{L^2([0,T]\times\mathbb{R}^d, H^{-s}(\mathbb{R}^d))}.$$

We use the formula (2.3) to solve (2.2). Let us write $f(t,x,v) = f_1(t,x,v) + f_2(t,x,v) + f_3(t,x,v)$, where

$$\begin{aligned} f_1(t,\cdot,\cdot) &:= f_0 *_t J(t,\cdot,\cdot), \\ f_2(t,\cdot,\cdot) &:= \int_0^t h_1(\tau) *_{(t-\tau)} J(t-\tau,\cdot,\cdot)\, \mathrm{d}\tau, \\ f_3(t,\cdot,\cdot) &:= \int_0^t h_2(\tau) *_{(t-\tau)} (-\Delta)^{s/2}_v J(t-\tau,\cdot,\cdot)\, \mathrm{d}\tau. \end{aligned}$$

Let $p \in [1, p_\star)$ be the number such that $1/q = 1/p - 1/2$. Applying Young's inequality for each value of t, we have

$$\begin{aligned} \|f_1(t,\cdot,\cdot)\|_{L^q} &\leqslant \|f_0\|_{L^2} \|\mathcal{J}\|_{L^p} t^{1/2-\alpha}, \\ \|f_2(t,\cdot,\cdot)\|_{L^q} &\leqslant \int_0^t \|h_1(\tau)\|_{L^2} \|\mathcal{J}\|_{L^p} (t-\tau)^{1/2-\alpha}\, \mathrm{d}\tau, \\ \|f_3(t,\cdot,\cdot)\|_{L^q} &\leqslant \int_0^t \|h_2(\tau)\|_{L^2} \|(-\Delta)^{s/2}\mathcal{J}\|_{L^p} (t-\tau)^{-\alpha}\, \mathrm{d}\tau. \end{aligned}$$

Here $\alpha = d(1+1/s)(1-1/p) + 1/2 < 1/p_\star < 1/p$ since $p < p_\star$. Moreover, $q(1/2-\alpha) > -1$ so $f_1 \in L^q([0,T] \times R^{2d})$ with

$$\|f_1\|_{L^q([0,T]\times R^{2d})} \leqslant C T^{(1/2-\alpha)+1/q} \|f_0\|_{L^2}.$$

We estimate the other two terms applying Young's inequality once again

$$\begin{aligned} \|f_2\|_{L^q([0,T]\times\mathbb{R}^{2d})} &\leqslant C \|h_1\|_{L^2([0,T],R^{2d})} T^{1/2+1/p-\alpha}, \\ \|f_3\|_{L^q([0,T]\times\mathbb{R}^{2d})} &\leqslant C \|h_2\|_{L^2([0,T],R^{2d})} T^{1/p-\alpha}. \end{aligned}$$

This finishes the proof. □

Remark 2.3. The power p in Lemma 2.2 can also be taken equal to $p_\star$ by using the weak-type Young's inequality in place of the usual Young's inequality for convolutions and a finer analysis of the $L^{p_\star,\infty}$ norm of J. Since we do not need a sharp result in this paper, we prefer to keep this lemma as elementary as possible.

3. The Boltzmann kernel

In this subsection, we explain why the Boltzmann collision operator associated with inverse power-law potentials (see (1.1)) satisfy the assumptions we made on the kernel as soon the quantities $M(t,x)$, $E(t,x)$ and $H(t,x)$ defined in the introduction are under control.

3.1. **The collision operator as an integro-differential operator plus a lower order term.** It is classical to observe that B can be replaced with any $\tilde{B}$ satisfying for all $k, \sigma \in \mathbb{S}^{d-1}$,

$$B(r, k \cdot \sigma) + B(r, -k \cdot \sigma) = \tilde{B}(r, k \cdot \sigma) + \tilde{B}(r, -k \cdot \sigma).$$

For this reason, we can (and do) follow [59] and assume

$$\text{If } k \cdot \sigma < 0, \text{ then } \quad B(r, k \cdot \sigma) \approx r^{\gamma} |\cos(\theta/2)|^{\gamma+2s+1} \tag{3.1}$$

where $\cos(\theta/2) := \frac{v-v_*}{|v-v_*|} \cdot \frac{v-v_*'}{|v-v_*-|}$.

We split Q in Q_1 and Q_2 as follows: $Q(f,g) = Q_1(f,g) + Q_2(f,g)$ with

$$\begin{cases} Q_1(f,g) &= \iint f_*'(g'-g) B \, \mathrm{d}v_* \, \mathrm{d}\sigma, \\ Q_2(f,g) &= \left(\iint (f_*' - f_*) B \, \mathrm{d}v_* \, \mathrm{d}\sigma \right) g. \end{cases}$$

Such a decomposition appears for instance in [63, 59].

The term Q_1 can be rewritten using Carleman coordinates [24].

Lemma 3.1 (The integro-differential operator [59]). *The term $g \mapsto Q_1(f,g)$ corresponds to some linear operator $L_v g$ with $K = K_f$ given by*

$$K_f(v, v') = \frac{2^{d-1}}{|v'-v|} \int_{w \perp v'-v} f(v+w) B(r, \cos\theta) r^{-d+2} \, \mathrm{d}w \tag{3.2}$$

where

$$r^2 = |v'-v|^2 + |w|^2 \quad \textit{and} \quad \cos\theta = \frac{v-v'-w}{|v-v'-w|} \cdot \frac{v'-v-w}{|v'-v-w|}.$$

The proof of the previous lemma is simply a change of variables to Carleman coordinates, see [59]. It is recalled in Appendix for the reader's convenience, see Lemma A.9. The term $Q_2(f,g)$ is of lower order because of the cancellation lemma [62],[1].

Lemma 3.2 (Cancellation [62], [1]). *The following formula holds true for any $v \in \mathbb{R}^d$,*

$$\iint (f_*' - f_*) B \, \mathrm{d}v_* \, \mathrm{d}\sigma = C_b |\cdot|^{\gamma} \star f(v)$$

with

$$C_b = \int_{\mathbb{S}^{d-1}} \left\{ \frac{2^{(d+\gamma)/2}}{(1+\sigma \cdot e)^{(d+\gamma)/2}} - 1 \right\} b(\sigma \cdot e) \, \mathrm{d}\sigma$$

for any $e \in \mathbb{S}^{d-1}$.

3.2. **Coercivity bound.** We prove this lower bound in the Appendix A.1. This is a well known result in the Boltzmann literature.

Proposition 3.3 (Lower bound [1, 37]). *Let $g : \mathbb{R}^d \to \mathbb{R}$ be a function supported in $B_{\bar{R}}$. Then*

$$c\|g\|^2_{\dot{H}^s(\mathbb{R}^d)} \leqslant -\int_{\mathbb{R}^d} Q(f,g)(v) g(v) \, \mathrm{d}v + C\|g\|^2_{L^2(\mathbb{R}^d)}.$$

The constants c and C depend on the mass, energy and entropy of f, the dimension d and the radius $\bar{R}$. In other words, K_f satisfies (1.3) as soon as mass, energy and entropy of f are bounded. In the case of the mass, we also need it to be bounded below.

The assumption (1.4), which we need in the case $s \in (0, 1/2)$ is clearly satisfied by the Boltzmann kernel. This follows as consequence of Lemma 4.8 in [59].

3.3. **Upper bounds.** In this paragraph, we justify that the Boltzmann kernel satisfies (1.5). We recall that (1.5)-(i) was already proved in [59]. Recall the equivalent formulations of (1.5) explained in Section 2.

Lemma 3.4 (Upper bound (1.5)-(i) [59, Corollary 4.4])**.** *Assume $\gamma+2s \leqslant 2$. Then for all $r>0$ and $v \in B_{\bar R}$,*

$$\int_{B_{2r}(v)\setminus B_r(v)} K_f(v,v')\,\mathrm{d}v' \lesssim r^{-2s}\left(\int_{\mathbb{R}^d} f(z)|z-v|^{\gamma+2s}\,\mathrm{d}z\right).$$

In particular, K_f satisfies (1.5)-(i) with Λ that depends only on $\|f \star |\cdot|^{\gamma+2s}\|_{L^\infty(B_2)}$. More precisely, if $\gamma+2s \in [0,2]$, then Λ in (1.5)-(i) depends only on mass and energy; if $\gamma+2s \leqslant 0$, then it depends on mass, dimension, γ, s and $\|f\|_{L^\infty}$.

We can now derive (1.5)-(ii).

Lemma 3.5 (Upper bound (1.5)-(ii))**.** *Assume $\gamma+2s \leqslant 2$. Then for all $v' \in B_{\bar R}$ and $r>0$,*

$$\int_{\mathbb{R}^d\setminus B_r(v')} K_f(v,v')\,\mathrm{d}v \lesssim r^{-2s}\left(\int_{\mathbb{R}^d} f(z)|z-v'|^{\gamma+2s}\,\mathrm{d}z\right).$$

In particular, K_f satisfies (1.5)-(ii) with Λ that only depends on $\|f \star |\cdot|^{\gamma+2s}\|_{L^\infty(B_{\bar R})}$. More precisely, if $\gamma+2s \in [0,2]$, it depends only on mass and energy; if $\gamma+2s \leqslant 0$, Λ then it depends on mass, dimension, γ, s and $\|f\|_{L^\infty}$.

Proof. According to the formula for $K(v,v')$ in terms of f (Corollary 4.2 in [59]),

$$K_f(v,v') \approx |v-v'|^{-d-2s}\left(\int_{w\perp(v-v')} f(v+w)|w|^{\gamma+2s+1}\,\mathrm{d}w\right).$$

Without loss of generality, let us take $v'=0$ in order to simplify the notation. Therefore

$$\int_{\mathbb{R}^d\setminus B_r} K_f(v,0)\,\mathrm{d}v \lesssim \int_r^\infty \rho^{-d-2s}\int_{\partial B_\rho}\int_{w\perp v} f(v+w)|w|^{\gamma+2s+1}\,\mathrm{d}w\,\mathrm{d}S(v)\,\mathrm{d}\rho.$$

Applying (A.6) from Lemma A.10,

$$\begin{aligned}\int_{\mathbb{R}^d\setminus B_r} K_f(v,0)\,\mathrm{d}v &\lesssim \int_r^\infty \rho^{-2s-1}\int_{\mathbb{R}^d\setminus B_\rho} f(z)\frac{(|z|^2-\rho^2)^{\frac{d-2+\gamma+2s}{2}}}{|z|^{d-2}}\,\mathrm{d}z\,\mathrm{d}\rho,\\ &= \int_{\mathbb{R}^d\setminus B_r}\frac{f(z)}{|z|^{d-2}}\left(\int_r^{|z|}\rho^{-2s-1}(|z|^2-\rho^2)^{\frac{d-2+\gamma+2s}{2}}\,\mathrm{d}\rho\right)\mathrm{d}z,\\ &\leqslant \int_{\mathbb{R}^d\setminus B_r}\frac{f(z)}{|z|^{d-2}}\left(r^{-2s}|z|^{d-2+\gamma+2s}\right)\mathrm{d}z,\\ &= r^{-2s}\int_{\mathbb{R}^d\setminus B_r} f(z)|z|^{\gamma+2s}\,\mathrm{d}z.\end{aligned}$$

□

3.4. **The cancellation assumptions.** In this paragraph, we justify that the kernel associated with the Boltzmann equation satisfies the cancellation assumptions (1.6) and (1.7).

The first cancellation condition, assumption (1.6), is essentially the cancellation lemma, which is well known in the kinetic community.

Lemma 3.6 (Classical cancellation lemma)**.** *The kernel K_f satisfies for all $v \in \mathbb{R}^d$,*

$$\left|PV\int_{\mathbb{R}^d}(K_f(v,v')-K_f(v',v))\,\mathrm{d}v'\right| \leqslant C\left(\int_{\mathbb{R}^d} f(z)|z-v|^\gamma\,\mathrm{d}z\right).$$

In particular, K_f satisfies (1.6) with Λ that only depends on $\|f\star|\cdot|^\gamma\|_{L^\infty(B_{\bar R})}$. More precisely, if $\gamma \in [0,2]$, Λ in (1.6) depends only on upper bounds on mass and energy; if $\gamma \leqslant 0$, it depends on mass, dimension, γ and $\|f\|_{L^\infty}$.

Proof. Let $P(v)$ denote $PV \int (K_f(v',v) - K_f(v,v'))\, \mathrm{d}v'$. In view of the definition of K_f, we have

$$P(v) = 2^{d-1} \int_{\mathbb{R}^d} \mathrm{d}v' \left(\int_{w \perp v'-v} f(v+w) \frac{B(r,\cos\theta)}{|v'-v| r^{d-2}}\, \mathrm{d}w - \int_{w \perp v'-v} f(v'+w) \frac{B(\tilde r, \cos\tilde\theta)}{|v'-v|\tilde r^{d-2}}\, \mathrm{d}w \right)$$
$$= 2^{d-1} \int_{\mathbb{R}^d} \int_{w \perp v'-v} (f(v+w) - f(v'+w)) \frac{B(r,\cos\theta)}{|v'-v| r^{d-2}}\, \mathrm{d}w\, \mathrm{d}v'$$

since $r = \tilde r$ and $\cos\theta = \cos\tilde\theta$. Using now (A.3) from Lemma A.9, we get

$$P(v) = \int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} (f(v'_*) - f(v_*)) B(r,\cos\theta)\, \mathrm{d}\sigma\, \mathrm{d}v_*.$$

The cancellation Lemma 3.2 tells us that

$$P(v) = c \int_{\mathbb{R}^d} |v-w|^\gamma f(w)\, \mathrm{d}w.$$

The proof is now complete. □

Lemma 3.7 (More subtle cancellation lemma)**.** *The two following properties hold true for any $R > 0$,*

$$PV \int_{B_R(v)} (v'-v) K_f(v,v')\, \mathrm{d}v' = 0, \tag{3.3}$$

$$\left| PV \int_{B_R(v)} (v'-v) K_f(v,v')\, \mathrm{d}v \right| \leqslant C \left(\int_{\mathbb{R}^d} f(z) |z-v'|^{1+\gamma}\, \mathrm{d}z \right). \tag{3.4}$$

In particular, the kernel satisfies (1.7) and Λ only depends on $\|f \star |\cdot|^{1+\gamma}\|_{L^\infty(B_{\bar R})}$. More precisely, if $\gamma \in [-1,1]$, Λ in (1.7) depends only on upper bounds on mass and energy; if $\gamma \leqslant -1$, it depends on mass, dimension, γ and $\|f\|_{L^\infty}$.

Proof. The first identity (3.3) is obvious from the symmetry property: $K_f(v, v+w) = K_f(v, v-w)$. The difficulty is thus to justify the second identity (3.4).

Without loss of generality, let us assume $v' = 0$. In view of Lemma 3.1 (coming from [59]), the kernel K_f can be written for $v' = 0$ as follows,

$$K_f(v,0) = \frac{2^{d-1}}{|v|} \int_{\{w : w \perp v\}} f(v+w) B(r,\cos\theta) \frac{1}{r^{d-2}}\, \mathrm{d}w$$

where $r^2 = |v|^2 + |w|^2 = |z|^2$ and $z = v + w$ and

$$\cos\theta = \widehat{v+w} \cdot \widehat{w-v} = \frac{|w|^2 - |v|^2}{|v+w|^2} = \frac{|z|^2 - 2|v|^2}{|z|^2}.$$

The way $b(\cos\theta)$ is modified for $\cos\theta < 0$, implies that

$$\frac{1}{|v| r^{d-2}} B(r,\cos\theta) \approx |v|^{-d-2s} |w|^{\gamma+2s+1}.$$

Since r and $\cos\theta$ only depend on $|z|$ and $|v|$, this implies that there exists $A(|z|,|v|)$ such that

$$\frac{1}{|v| r^{d-2}} B(r,\cos\theta) = A(|z|,|v|) |v|^{-d-2s} |w|^{\gamma+2s+1}$$

and a constant $C_A > 1$ such that for all z, v,

$$C_A^{-1} \leqslant A(|z|,|v|) \leqslant C_A.$$

In the following computation, the definition of r changes. We write $r = |v|$. We integrate in v first on spheres ∂B_r and then with respect to the radius r.

$$\left| PV \int_{B_R} v K_f(v,0)\, \mathrm{d}v \right| = \left| \int_0^R r^{-d-2s} \int_{\partial B_r} \int_{w \perp v} v A(|v+w|,|v|) f(v+w) |w|^{\gamma+2s+1}\, \mathrm{d}w\, \mathrm{d}v\, \mathrm{d}r \right|$$

We use the change of variables (A.7) of Lemma A.10. Note that $|w|^2 + r^2 = |z|^2$.

$$\begin{aligned}\left|PV\int_{B_R} vK_f(v,0)\,\mathrm{d}v\right| &= \omega_{d-2}\left|\int_0^R r^{1-2s}\int_{\mathbb{R}^d\setminus B_r} A(|z|,r)zf(z)\frac{(|z|^2-r^2)^{\frac{d-2+\gamma+2s}{2}}}{|z|^d}\,\mathrm{d}z\,\mathrm{d}r\right|\\ &= \omega_{d-2}\left|\int_{\mathbb{R}^d} zf(z)|z|^{-d}\left(\int_0^{\min(|z|,R)} r^{1-2s}A(|z|,r)(|z|^2-r^2)^{\frac{d-2+\gamma+2s}{2}}\,\mathrm{d}r\right)\,\mathrm{d}z\right|\\ &\leqslant \omega_{d-2}C_A\int_{\mathbb{R}^d} f(z)|z|^{-1+\gamma+2s}\left(\int_0^{\min(|z|,R)} r^{1-2s}\,\mathrm{d}r\right)\,\mathrm{d}z\\ &\leqslant C\int_{\mathbb{R}^d} f(z)|z|^{1+\gamma}\,\mathrm{d}z.\end{aligned}$$

The proof is now complete. □

Remark 3.8. There is a subtle cancellation that allows this proof to work. The whole point of this lemma is that the principal value of the integral is bounded around the origin. The reader will notice that here we end up with an integral of the form $\int_0^{|z|} r^{1-2s}\,\mathrm{d}r$. In the proof of Lemma 3.5, we end up with an intergrand r^{-1-2s} which is not integrable around the origin. The difference originates in Lemma A.10 given in Appendix. The third identity in that lemma incorporates an extra cancellation due to the fact that the average values of $v \in \partial B_r$ so that $v + w = z$, for some $w \perp v$, is $r^2 z/|z|^2$.

Remark 3.9. Note that the cancellation condition given in Lemma 3.7 is slightly stronger than (1.7) since the right hand side is bounded independently of R even when $s > 1/2$. Moreover, a rate of convergence to zero as $R \to 0$ can be deduced from the proof.

3.5. **Proofs of Theorems 1.1 and 1.3.** In this subsection we explain how Theorems 1.1 and 1.3 follow from Theorems 1.5 and 1.6. Theorem 1.1 is indeed a straight forward application of Theorem 1.5.

Proof of Theorem 1.1. The Boltzmann equation can be written in the form

$$f_t + v\cdot\nabla_x f = \left(\int_{\mathbb{R}^d}(f(v')-f(v))K_f(v,v')\,\mathrm{d}v'\right) + c\left(\int_{\mathbb{R}^d} f(v-w)|w|^\gamma\,\mathrm{d}w\right)f.$$

Thus, if we define

$$h := c\left(\int_{\mathbb{R}^d} f(v-w)|w|^\gamma\,\mathrm{d}w\right)f,$$

then $h \in L^\infty$ with its norm bounded in terms of $\|f\|_{L^\infty}$ and M_0.

Moreover, from Proposition 3.3 and Lemmas 3.4, 3.5, 3.6 and 3.7, the kernel K_f satisfies the assumptions (1.3), (1.5), (1.6) and (1.7). Thus, the proof is finished as a corollary of Theorem 1.5. □

Theorem 1.3 follows mostly from Theorem 1.6. We use some other results which are presented later in this article which allow us to extend the lower bound to an arbitrary radius $R > 0$.

Proof of Theorem 1.3. Without loss of generality, we assume $T = 4$. The general case follows by scaling.

Like in the proof above of Theorem 1.1, we have that f is a supersolution of (1.2) for some $h \geqslant 0$. In particular,

$$f_t + v\cdot\nabla_x f \geqslant L_v f.$$

According to Lemma A.2, there is an $R_0 > 0$, $m > 0$ and $\ell > 0$ so that for all (t,x),

$$|\{v \in B_{R_0} : f(t,x,v) \geqslant \ell\}| \geqslant m.$$

Let r_0 be the one from Theorem 1.6. We have that

$$\int_{[0,r_0^{2s}]\times B_{r_0^{1+2s}}\times B_{R_0}} f^\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \geqslant \ell^\varepsilon m r_0^{2s+(1+2s)d}.$$

It is possible to cover the set $[0, r_0^{2s}] \times B_{r_0^{1+2s}} \times B_{R_0}$ with N slanted cylinders $Q_{r_0}(z)$ with $N \leqslant (R_0/r_0)^{2d}/c$ for some universal constant $c > 0$. This implies that there must be some point $z = (r_0^{2s}, x, v) \in \{r_0^{2s}\} \times B_{r_0^{2s}(r_0+R_0)} \times B_{R_0}$ so that

$$\int_{Q_{r_0}(z)} f^\varepsilon \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \geqslant c\ell^\varepsilon m r_0^{2s+(3+2s)d}/R_0^{2d}.$$

Applying Theorem 1.6 (properly translated), we get

$$\inf_{Q_{r_0}(\tilde{z})} f \geqslant c,$$

for some constant $c > 0$ and $\tilde{z} \in \{1\} \times B_{r_0^{1+2s}+R_0} \times B_{R_0}$.

This bound below in $Q_{r_0}(\tilde{z})$ is propagated to $[2,4] \times B_R \times B_R$, for any arbitrary $R > 0$ using the barrier function from Lemma 7.1 if $s < 1/2$ or the combination of Lemmas 8.3 and 6.6 if $s \geqslant 1/2$.

Note that the geometric setting of Lemmas 7.1, 8.3 and 6.6 are independent of the constants λ and Λ. This is important since these ellipticity constants depend on R. □

4. Study of a bilinear form

This section is devoted to the study of a general bilinear form $\mathcal{E}$ associated with a kernel K through the following formula,

$$\mathcal{E}(\varphi, g) = -\int (L_v\varphi)(v) g(v) \, \mathrm{d}v = \lim_{\varepsilon \to 0} \left(\iint_{|v-v'|>\varepsilon} (\varphi(v) - \varphi(v')) g(v) K(v, v') \, \mathrm{d}v' \, \mathrm{d}v \right).$$

In the remainder of this section, we abuse notation by ignoring the limit as $\varepsilon \to 0$. This means that some integrals corresponding to the odd part of K may need to be understood in the principal value sense. Indeed, we recall that the operator $L_v\varphi$ is given by the formula

$$L_v\varphi(v) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d \setminus B_\varepsilon(v)} (\varphi(v') - \varphi(v)) K(v, v') \, \mathrm{d}v'. \tag{4.1}$$

The limit does not necessarily converge for every value of v, even if φ is smooth. The correct understanding of $L_v\varphi$ as a distribution is obtained through the analysis of the bilinear form $\mathcal{E}$ done in this section.

When we study the equation (1.2), the bilinear form $\mathcal{E}$ will be computed for functions φ and g depending on values of t and x. The kernel, and consequently also the bilinear form, depend on t and x. In this section we study properties of bilinear forms like this that will be applied for every fixed value of t and x.

In this section, we also assume that the kernel K is defined for all values of $v \in \mathbb{R}^d$ and our assumptions hold uniformly. This is convenient for the exposition and some of the proofs. In Section 5, we will show that any kernel satisfying (1.3), (1.5), (1.6) and (1.7) can be extended to all values of $v \in \mathbb{R}^d$ to satisfy a global version of these assumptions. So, a posteriori, our approach is not limiting.

Since it is not necessary for K to be non-negative for the results in this section to hold, and they may be used elsewhere, we restate here our main assumption allowing sign changing kernels. We make the following assumptions for some parameter $s \in (0,1)$ and a constant Λ.

$$\forall v \in \mathbb{R}^d, \forall r > 0, \qquad \begin{cases} (i) & \int_{\mathbb{R}^d \setminus B_r(v)} |K(v, v')| \, \mathrm{d}v' \leqslant \Lambda r^{-2s} \\ (ii) & \int_{\mathbb{R}^d \setminus B_r(v)} |K(v', v)| \, \mathrm{d}v' \leqslant \Lambda r^{-2s}. \end{cases} \tag{4.2}$$

We also state a global version of the cancellation assumptions (1.6) and (1.7).

$$\forall v \in \mathbb{R}^d, \qquad \left| PV \int_{\mathbb{R}^d} (K(v, v') - K(v', v)) \, \mathrm{d}v' \right| \leqslant \Lambda. \tag{4.3}$$

In the case $s \geqslant 1/2$, we also assume that for all $R > 0$,

$$\forall v \in \mathbb{R}^d, \qquad \left| PV \int_{B_R(v)} (K(v, v') - K(v', v))(v - v') \, \mathrm{d}v' \right| \leqslant \Lambda(1 + R^{1-2s}). \tag{4.4}$$

The main result of this section will be that the bilinear form $\mathcal{E}$ is bounded in $H^s \times H^s$ provided that (4.2), (4.3) and (4.4) hold. We also show some other estimates that we will use.

4.1. **Estimates in H^s.** The main result of this section is the fact that the bilinear form $\mathcal{E}$ is bounded in $H^s \times H^s$ as soon as our assumptions (4.2), (4.3) and (4.4) hold. We state it in the following theorem.

Theorem 4.1 (Estimate in H^s)**.** *Let K satisfy (4.2) and (4.3). If $s \geqslant 1/2$, we also assume that it satisfies (4.4). There then exists a constant C depending only on s, d and Λ, so that*

$$\mathcal{E}(f,g) \leqslant C\|f\|_{H^s}\|g\|_{H^s}.$$

It is convenient for some of our proofs to spit $\mathcal{E}$ into the symmetric and anti-symmetric part of K. Let

$$\mathcal{E}(\varphi,g) = \mathcal{E}^{\text{sym}}(\varphi,g) + \mathcal{E}^{\text{skew}}(\varphi,g)$$

with

$$\mathcal{E}^{\text{sym}}(\varphi,g) = \frac{1}{2}\iint (\varphi(v)-\varphi(v'))(g(v)-g(v'))K(v,v')\,\mathrm{d}v'\,\mathrm{d}v,$$
$$\mathcal{E}^{\text{skew}}(\varphi,g) = \frac{1}{2}PV\iint (\varphi(v)-\varphi(v'))(g(v)+g(v'))K(v,v')\,\mathrm{d}v'\,\mathrm{d}v.$$

Note that $\mathcal{E} = \mathcal{E}^{\text{sym}}$ and $\mathcal{E}^{\text{skew}} = 0$ when the symmetry condition $K(v,v') = K(v',v)$ holds. Likewise, when K is anti-symmetric (i.e. $K(v,v') = -K(v',v)$) then $\mathcal{E}^{\text{sym}} = 0$ and $\mathcal{E}^{\text{skew}} = \mathcal{E}$. Consequently, writing K as the sum of its symmetric plus anti-symmetric part corresponds to writing $\mathcal{E}$ as the sum of $\mathcal{E}^{\text{sym}}$ and $\mathcal{E}^{\text{skew}}$.

We will prove Theorem 4.1 estimating $\mathcal{E}^{\text{sym}}$ and $\mathcal{E}^{\text{skew}}$ separately. Note that, because of the density of smooth functions in H^s, it suffices to prove Theorem 4.1 when g and φ are smooth.

4.1.1. *Estimate of the symmetric part.*

Lemma 4.2 (Estimate of the symmetric part)**.** *Let K be a kernel satisfying (4.2). Then, there exists a constant depending only on Λ, s and dimension, so that for any function $g \in H^s(\mathbb{R}^d)$,*

$$\mathcal{E}^{sym}(g,g) \leqslant C\|g\|^2_{\dot{H}^s}.$$

Proof. Without loss of generality, we can assume $K \geqslant 0$ here. Otherwise, the value of $\mathcal{E}^{\text{sym}}(g,g)$ would only increase if we replace $K(v,v')$ by $|K(v,v')|$. We write

$$\mathcal{E}^{\text{sym}}(g,g) = \sum_{k=-\infty}^{\infty} P(2^k) \tag{4.5}$$

where, for any $r > 0$,

$$P(r) := \iint_{\{(v,v')\in\mathbb{R}^d\times\mathbb{R}^d : r\leqslant|v-v'|<2r\}} |g(v')-g(v)|^2 K(v,v')\,\mathrm{d}v'\,\mathrm{d}v.$$

The key of this proof is to estimate $P(r)$ with a similar expression involving the kernel $|v-v'|^{-d-2s}$ instead.

For any values of v and v', let $m = (v+v')/2$, we introduce an auxiliary point $w \in B_{r/4}(m)$. From the triangle inequality $|g(v')-g(v)|^2 \leqslant 2|g(v')-g(w)|^2 + 2|g(w)-g(v)|^2$. Then

$$P(r) \lesssim \frac{1}{r^d}\iint_{\{(v,v')\in\mathbb{R}^d\times\mathbb{R}^d : r\leqslant|v-v'|<2r\}}\int_{B_{r/4}(m)} \left(|g(v')-g(w)|^2 + |g(w)-g(v)|^2\right) K(v,v')\,\mathrm{d}w\,\mathrm{d}v'\,\mathrm{d}v,$$

we change the order of integration for each term in the integrand,

$$\leqslant \frac{1}{r^d}\iint_{r/4\leqslant|v'-w|<5r/4} |g(v')-g(w)|^2 \left(\int_{\Omega_{v',w}} K(v,v')\,\mathrm{d}v\right)\mathrm{d}w\,\mathrm{d}v'$$
$$+ \frac{1}{r^d}\iint_{r/4\leqslant|v-w|<5r/4} |g(w)-g(v)|^2 \left(\int_{\Omega_{v,w}} K(v,v')\,\mathrm{d}v'\right)\mathrm{d}w\,\mathrm{d}v.$$

Here the set $\Omega_{v,w}$ contains all values of v' that correspond to any given pair (v,w). We will only use that $\Omega_{v,w} \subset B_{2r}(v)\backslash B_r(v)$. Both terms are bounded by the same expression using each line in (4.2). Thus,

$$P(r) \lesssim \frac{\Lambda}{r^{d+2s}}\iint_{r/4\leqslant|v-w|<5r/4} |g(v')-g(w)|^2\,\mathrm{d}w\,\mathrm{d}v \lesssim \Lambda\iint_{r/4\leqslant|v-w|<5r/4} \frac{|g(v')-g(w)|^2}{|v-w|^{d+2s}}\,\mathrm{d}w\,\mathrm{d}v.$$

Applying this estimate for each term in (4.5), we get the desired estimate. □

4.1.2. *Estimates of the anti-symmetric part.* Finding an appropriate upper bound for $\mathcal{E}$ when K is not symmetric is more complicated than for $\mathcal{E}^{\text{sym}}$. The cancellation assumptions (4.3) and (4.4) are necessary. We will prove the estimates differently for the case $s \in (0,1/2)$ and $s \in [1/2,1)$. Note that the hypothesis (4.4) is only used in the later case.

Lemma 4.3 (Estimate of $L_v f$ for $s < 1/2$)**.** *Assume $s \in (0,1/2)$. Let K be a kernel satisfying (4.2). The following estimate holds*

$$\|L_v f\|_{L^2} \leqslant C \|f\|_{L^2}^{\frac{1-2s}{1+2s}} \|f\|_{\dot{H}^{s+1/2}}^{\frac{4s}{1+2s}}.$$

Proof. For some $R > 0$, to be determined below, let us write $L_v f = \ell_0 + \ell_1 + \ell_2$, where

$$\begin{aligned}
\ell_0(v) &= \int_{B_R(v)} (f(v') - f(v)) K(v,v') \,\mathrm{d}v', \\
\ell_1(v) &= \int_{\mathbb{R}^d \setminus B_R(v)} f(v') K(v,v') \,\mathrm{d}v', \\
\ell_2(v) &= -\left(\int_{\mathbb{R}^d \setminus B_R(v)} K(v,v') \,\mathrm{d}v' \right) f(v).
\end{aligned}$$

We prove the estimate for each one of the three terms.

Let us start with ℓ_2, which is the easiest. In this case, obviously,

$$\|\ell_2\|_{L^2} \leqslant \left(\sup_v \int_{\mathbb{R}^d \setminus B_R(v)} K(v,v') \,\mathrm{d}v' \right) \|f\|_{L^2} \leqslant \Lambda R^{-2s} \|f\|_{L^2}.$$

The estimate for ℓ_1 involves the Cauchy-Schwarz inequality and an application of Fubini's theorem. In this case we use the second line of (4.2).

$$\begin{aligned}
\|\ell_1\|_{L^2}^2 &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d \setminus B_R(v)} f(v') K(v,v') \,\mathrm{d}v' \right)^2 \mathrm{d}v, \\
&\leqslant \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d \setminus B_R(v)} K(v,v') \,\mathrm{d}v' \right) \left(\int_{\mathbb{R}^d \setminus B_R(v)} f(v')^2 K(v,v') \,\mathrm{d}v' \right) \mathrm{d}v, \\
&\leqslant \Lambda R^{-2s} \int_{\mathbb{R}^d} f(v')^2 \left(\int_{\{v : |v'-v| > R\}} K(v,v') \,\mathrm{d}v \right) \mathrm{d}v' \leqslant \Lambda^2 R^{-4s} \|f\|_{L^2}^2.
\end{aligned}$$

We estimate ℓ_0 using the Cauchy-Schwarz inequality together with (4.2) and comments from Subsection 2.2.

$$\begin{aligned}
\|\ell_0\|_{L^2}^2 &= \int_{\mathbb{R}^d} \left(\int_{B_R(v)} (f(v') - f(v)) K(v,v') \,\mathrm{d}v' \right)^2 \mathrm{d}v, \\
&\leqslant \int_{\mathbb{R}^d} \left(\int_{B_R(v)} (f(v') - f(v))^2 |v-v'|^{-1} K(v,v') \,\mathrm{d}v' \right) \left(\int_{B_R(v)} |v-v'| K(v,v') \,\mathrm{d}v' \right) \mathrm{d}v, \\
&\leqslant \Lambda R^{1-2s} \iint_{|v-v'|<r} (f(v') - f(v))^2 |v-v'|^{-1} K(v,v') \,\mathrm{d}v' \,\mathrm{d}v.
\end{aligned}$$

The kernel $|v-v'|^{-1} K(v,v')$ satisfies (1.5) with $s + 1/2$ instead of s. Then, we apply Lemma 4.2 to get

$$\|\ell_0\|_{L^2}^2 \lesssim R^{1-2s} \|f\|_{\dot{H}^{s+1/2}}^2.$$

The proof is finished choosing $R = (\|f\|_{L^2} / \|f\|_{H^{s+1/2}})^{2/(1+2s)}$. □

The estimate for $\|L_v f\|_{L^2}$ when $s \geqslant 1/2$ is harder to obtain. We will use the following auxiliary kernel.

$$A(v,w) = \int_{\{v' \in B_R(v) : (v'-v)\cdot(w-v) \geqslant |w-v|^2\}} \frac{|v'-v|^{d-2} K(v,v')}{|w-v|^{d-2} |w-v'|^{d-2}} \,\mathrm{d}v'.$$

Lemma 4.4 (Estimates on the auxiliary kernel)**.** *Let K be a kernel satisfying* (4.2) *and $s \geqslant 1/2$. We have*

$$\int_{B_R(v)} |A(v,w)| \,\mathrm{d}w \lesssim R^{2-2s} \quad \textit{and} \quad \int_{B_R(w)} |A(v,w)| \,\mathrm{d}v \lesssim R^{2-2s}. \tag{4.6}$$

Proof. The first of the two inequalities in (4.6) is a relatively straight forward computation using (4.2). Let us choose $v = 0$ without loss of generality. We have

$$\begin{aligned}
\int_{B_R} |A(0,w)| \,\mathrm{d}w &\leqslant \int_{B_R} \int_{\{v' \in B_R : v' \cdot w \geqslant |w|^2\}} \frac{|v'|^{d-2} |K(0,v')|}{|w|^{d-2}|w-v'|^{d-2}} \,\mathrm{d}v' \,\mathrm{d}w, \\
&= \int_{B_R} |v'|^{d-2} |K(0,v')| \left(\int_{B_{|v'|/2}(v'/2)} \frac{1}{|w|^{d-2}|w-v'|^{d-2}} \,\mathrm{d}w \right) \mathrm{d}v', \\
&= C \int_{B_R} |K(0,v')| |v'|^2 \,\mathrm{d}v' \lesssim C R^{2-2s}.
\end{aligned}$$

Let us move to the second inequality in (4.6). Assume without loss of generality that $w = 0$. We have

$$\int_{B_R} |A(v,0)| \,\mathrm{d}v \leqslant \int_{B_R} \int_{\{v' \in B_R(v) : v' \cdot v \leqslant 0\}} \frac{|v'-v|^{d-2} |K(v,v')|}{|v|^{d-2}|v'|^{d-2}} \,\mathrm{d}v' \,\mathrm{d}v \leqslant I_1 + I_2.$$

From the triangle inequality $|v'-v|^{d-2} \lesssim |v|^{d-2} + |v'|^{d-2}$, we can estimate the above integral by $I_1 + I_2$, where I_1 and I_2 are defined below. We analyze both terms using (4.2) and Fubini's theorem.

$$\begin{aligned}
I_1 :&= \int_{B_R} \int_{\{v' : |v'-v| < R \text{ and } v' \cdot v \leqslant 0\}} \frac{|v|^{d-2} |K(v,v')|}{|v|^{d-2}|v'|^{d-2}} \,\mathrm{d}v' \,\mathrm{d}v, \\
&= \int_{B_{2R}} \int_{\{v : |v-v'| < R \text{ and } v' \cdot v \leqslant 0\} \cap B_R} \frac{|K(v,v')|}{|v'|^{d-2}} \,\mathrm{d}v \,\mathrm{d}v', \\
&\lesssim R^{-2s} \int_{B_{2R}} |v'|^{2-d} \,\mathrm{d}v' \lesssim R^{2-2s}.
\end{aligned}$$

We now consider

$$I_2 := \int_{B_R} \int_{\{v' : |v'-v| < R \text{ and } v' \cdot v \leqslant 0\}} \frac{|v'|^{d-2} |K(v,v')|}{|v|^{d-2}|v'|^{d-2}} \,\mathrm{d}v' \,\mathrm{d}v.$$

The computation that proves that $I_2 \lesssim R^{2-2s}$ is almost identical integrating in v' first and in v second.

This concludes the estimate for every term involved in (4.6). □

We will use the following lemma from multivariate calculus when proving estimates on the operator $L_v f$ associated with the kernel K.

Lemma 4.5 (The multi-path lemma)**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be any twice differentiable function. The following inequality holds for any pair of points $v, v' \in \mathbb{R}^d$.*

$$\left| f(v') - f(v) - \frac{\nabla f(v) + \nabla f(v')}{2} \cdot (v'-v) \right| \leqslant \frac{1}{d\omega_d} |v-v'|^{d-2} \int_{B_R(m)} \frac{|D^2 f(w)|}{|w-v|^{d-2}|w-v'|^{d-2}} \,\mathrm{d}w. \tag{4.7}$$

Here $R = |v-v'|/2$ and $m = (v+v')/2$. Thus, $B_R(m)$ is the ball with diameter from v to v'.

Proof. For any $w \in B_R(m)$, we write

$$|f(w) - f(v) - \nabla f(v) \cdot (w-v)| \leqslant |w-v| \int_0^{|w-v|} |D^2 f|(v + z\widehat{w-v}) \,\mathrm{d}z.$$

where $\widehat{w-v} = (w-v)/|w-v|$. In particular, computing with spherical coordinates the integral in the first line below with origin at $w = v$,

$$\begin{aligned}\left|\left(\fint_{B_R(m)} f(w)\,\mathrm{d}w\right) - f(v) - \nabla f(v)\cdot\frac{(v'-v)}{2}\right| &= \left|\fint_{B_R(m)} f(w) - f(v) - \nabla f(v)\cdot(w-v)\,\mathrm{d}w\right|,\\ &\leqslant \fint_{B_R(m)} |w-v|\left\{\int_0^{|w-v|} |D^2 f|(v+\zeta\widehat{w-v})\,\mathrm{d}\zeta\right\}\mathrm{d}w\\ &\lesssim \int_{B_R(m)} \frac{|D^2 f|(w)}{|w-v|^{d-2}}\,\mathrm{d}w.\end{aligned}$$

This implies that

$$\left|\left(\fint_{B_R(m)} f(w)\,\mathrm{d}w\right) - f(v) - \nabla f(v)\cdot\frac{(v'-v)}{2}\right| \lesssim R^{d-2}\int_{B_R(m)} \frac{|D^2 f|(w)}{|w-v|^{d-2}|w-v'|^{d-2}}\,\mathrm{d}w.$$

Exchanging the role of v and v' and subtracting the resulting inequalities yields (4.7). □

Lemma 4.6 (Estimate of $L_v f$ for $s \geqslant 1/2$)**.** *Assume $s \in [1/2, 1)$. Let K be an anti-symmetric kernel (i.e. $K(v,v') = -K(v',v)$) satisfying (4.2) and (4.4). The following estimate holds*

$$\|L_v f\|_{L^2} \leqslant C\|f\|_{L^2}^{1-s}\|D^2 f\|_{L^2}^{s} + \Lambda\|\nabla f\|_{L^2}.$$

Proof. We write $L_v f = \ell_0 + \ell_1 + \ell_2$ like in Lemma 4.3. The estimates $\|\ell_1\|_{L^2} \leqslant \Lambda R^{-2s}\|f\|_{L^2}$ and $\|\ell_2\|_{L^2} \leqslant \Lambda R^{-2s}\|f\|_{L^2}$ follow an identical proof. The estimate for $\|\ell_0\|_{L^2}$ is different.

Recall that

$$\ell_0(v) = \int_{B_R(v)} (f(v') - f(v))K(v,v')\,\mathrm{d}v'$$

We write $\ell_0 = \ell_0^0 + \ell_0^1 + \ell_0^2$ with

$$\begin{aligned}\ell_0^0 &= \frac{1}{2}\int_{B_R(v)} \big(\nabla f(v') - \nabla f(v)\big)(v'-v)K(v,v')\,\mathrm{d}v',\\ \ell_0^1 &= \nabla f(v)\int_{B_R(v)} (v'-v)K(v,v')\,\mathrm{d}v',\\ \ell_0^2 &= \int_{B_R(v)} \left(f(v') - f(v) - \frac{\nabla f(v) + \nabla f(v')}{2}\cdot(v'-v)\right)K(v,v')\,\mathrm{d}v'.\end{aligned}$$

The same argument that gives us the upper bound for ℓ_0 in Lemma 4.3 gives us in this case

$$\|\ell_0^0\|_{L^2} \leqslant CR^{1-s}\|\nabla f\|_{\dot H^s} \lesssim R^{1-s}\|f\|_{L^2}^{(1-s)/2}\|D^2 f\|_{L^2}^{(1+s)/2}.$$

Indeed, ℓ_0^0 equals the same as ℓ_0 in the proof of Lemma 4.3 with ∇f instead of f and $(v-v')K(v,v')$ instead of $K(v,v')$. The fact that these are vector valued functions does not affect the proof. Note that $(v-v')K(v,v')$ satisfies (4.2) with $s-1/2$ instead of s. The second inequality is an elementary interpolation.

The cancellation assumption (4.4) says that

$$\begin{aligned}\|\ell_0^1\|_{L^2} &\leqslant \Lambda(R^{1-2s}+1)\|\nabla f\|_{L^2}\\ &\leqslant \Lambda R^{1-2s}\|f\|_{L^2}^{\frac12}\|D^2 f\|_{L^2}^{\frac12} + \Lambda\|\nabla f\|_{L^2}.\end{aligned}$$

In order to estimate ℓ_0^2, we use Lemma 4.5. We write

$$\begin{aligned}\|\ell_0^2\|_{L^2}^2 &= \int_{\mathbb{R}^d}\left(\int_{B_R(v)}\left(f(v') - f(v) - \frac{\nabla f(v) + \nabla f(v')}{2}\cdot(v'-v)\right)K(v,v')\,\mathrm{d}v'\right)^2\mathrm{d}v,\\ &\lesssim \int_{\mathbb{R}^d}\left(\int_{B_R(v)}\int_{B_r(m)} |D^2 f(w))|\,\frac{|v'-v|^{d-2}K(v,v')}{|w-v|^{d-2}|w-v'|^{d-2}}\,\mathrm{d}w\,\mathrm{d}v'\right)^2\mathrm{d}v,\end{aligned}$$

using Fubini's theorem

$$= \int_{\mathbb{R}^d} \left(\int_{B_R(v)} |D^2 f(w)| \left(\int_{\{v' : (v'-v)\cdot(w-v) \geqslant |w-v|^2\}} \frac{|v'-v|^{d-2} K(v,v')}{|w-v|^{d-2}|w-v'|^{d-2}} \, \mathrm{d}v' \right) \mathrm{d}w \right)^2 \mathrm{d}v.$$

In view of the definition of $A(v,w)$, we can use (4.6) and get

$$\begin{aligned}
\|\ell_0^2\|_{L^2}^2 &\leqslant \int_{\mathbb{R}^d} \left(\int_{B_R} |D^2 f(w)| A(v,w) \, \mathrm{d}w \right)^2 \mathrm{d}v, \\
&\leqslant \int_{\mathbb{R}^d} \left(\int_{B_R} A(v,w) \, \mathrm{d}w \right) \left(\int_{B_R} |D^2 f(w)|^2 A(v,w) \, \mathrm{d}w \right) \mathrm{d}v, \\
&\leqslant C R^{2-2s} \int_{\mathbb{R}^d} |D^2 f(w)|^2 \left(\int_{|v-w|<\mathbb{R}} A(v,w) \, \mathrm{d}v \right) \mathrm{d}w \leqslant C R^{2(2-2s)} \|D^2 f\|_{L^2}^2.
\end{aligned}$$

Choosing $R = \|f\|_{L^2}^{\frac{1}{2}} / \|D^2 f\|_{L^2}^{\frac{1}{2}}$ completes the proof. □

We can now prove the main result of this section.

Proof of Theorem 4.1. We prove the upper bound applying Lemmas 4.3 and 4.6 to both operators L_v and its adjoint L_v^t, and doing some sort of interpolation. Note that L_v^t has the same form as L_v plus a correction which is bounded from L^2 to L^2 (thanks to the cancellation assumption (4.3)), so Lemmas 4.3 and 4.6 apply to L_v^t as well. Indeed,

$$L_v^t f(v) = \int_{\mathbb{R}^d} (f(v') - f(v)) K(v', v) \, \mathrm{d}v' + \left(\int_{\mathbb{R}^d} K(v',v) - K(v,v') \, \mathrm{d}v' \right) f(v).$$

The following interpolation is probably classical. We prove it using Littlewood-Paley theory. Since we have already obtained the estimate for $\mathcal{E}^{\mathrm{sym}}$ in Lemma 4.2, we are only left to prove the estimate for $\mathcal{E}^{\mathrm{skew}}$. In the case $s \in (0, 1/2)$, the proof below gives the estimate for $\mathcal{E}$ right away. For $s \in [1/2, 1)$ the proof below applies to $\mathcal{E}^{\mathrm{skew}}$ only.

Let Δ_i be the Littlewood-Paley projectors. We use the convention that all low modes are enclosed in Δ_0. That is $f = \sum_{i=0}^{\infty} \Delta_i f$, with the index i being non-negative. We use the fact that for any $s \geqslant 0$,

$$\|\Delta_i f\|_{H^s} \approx 2^{is} \|\Delta_i f\|_{L^2}$$

Moreover, from Lemma 4.3, if $s \in (0, 1/2)$,

$$\|L_v \Delta_i f\|_{L^2} \lesssim \|\Delta_i f\|_{L^2}^{\frac{1-2s}{1+2s}} \|\Delta_i f\|_{\dot{H}^{s+1/2}}^{\frac{4s}{1+2s}} \lesssim 2^{si} \|\Delta_i f\|_{H^s}$$

From Lemma 4.6, if $s \in [1/2, 1)$,

$$\|L_v \Delta_i f\|_{L^2} \lesssim \|\Delta_i f\|_{L^2}^{1-s} \|\Delta_i f\|_{H^2}^{s} + \|\Delta_i f\|_{H^1} \lesssim 2^{si} \|\Delta_i f\|_{H^s}$$

The same estimates hold for L_v^t in the place of L_v.

Therefore,

$$\begin{aligned}
\mathcal{E}(f,g) &= \sum_{ij} \mathcal{E}(\Delta_i f, \Delta_j g), \\
&= \sum_{i \leqslant j} \langle L_v \Delta_i f, \Delta_j g \rangle + \sum_{i > j} \langle \Delta_i f, L^t \Delta_j g \rangle, \\
&\lesssim \sum_{i,j} 2^{-s|i-j|} \|\Delta_i f\|_{H^s} \|\Delta_j g\|_{H^s}, \\
&= \sum_{k=0}^{\infty} 2^{-sk} \sum_{i=0}^{\infty} \|\Delta_i f\|_{H^s} \|\Delta_{i+k} g\|_{H^s} + \|\Delta_{i+k} f\|_{H^s} \|\Delta_i g\|_{H^s}, \\
&\leqslant \sum_{k=0}^{\infty} 2^{-sk+1} \left(\sum_{i=0}^{\infty} \|\Delta_i f\|_{H^s}^2 \right)^{1/2} \left(\sum_{i=0}^{\infty} \|\Delta_j g\|_{H^s}^2 \right)^{1/2}, \\
&\lesssim \|f\|_{H^s} \|g\|_{H^s}.
\end{aligned}$$

The proof is now complete. □

4.2. **A generalized cancellation lemma.** As a preparation for the next subsection, we prove the following generalized cancellation lemma.

Lemma 4.7 (Generalized cancellation)**.** *Let K be a kernel satisfying (4.2); if $s \geqslant 1/2$, we also assume that K satisfies (4.4). Let φ be a bounded C^2 function. Then*

$$PV \int_{\mathbb{R}^d} (\varphi(v') - \varphi(v))[K(v,v') - K(v',v)] \, \mathrm{d}v' \leqslant C \|\varphi\|_{C_2},$$

for some constant C depending on Λ and dimension.

Proof. The proof is a direct computation. We estimate the tail of the integral using (4.2) together with the boundedness of φ. Then, we estimate the integral in B_1 using (4.4) and the smoothness of φ. We write the proof for the case $2s \geqslant 1$ first, and later indicate its simplification when $2s < 1$.

$$\begin{aligned} PV \int & (\varphi(v') - \varphi(v))[K(v,v') - K(v',v)] \, \mathrm{d}v' \\ &\leqslant PV \int_{B_1} (\varphi(v') - \varphi(v))[K(v,v') - K(v',v)] \, \mathrm{d}v' + C\Lambda \|\varphi\|_{L^\infty}, \\ &\leqslant PV \int_{B_1} (v'-v)\nabla\varphi(v)[K(v,v') - K(v',v)] \\ &\quad + \|D^2\varphi\|_\infty |v-v'|^2 |K(v,v') - K(v',v)| \, \mathrm{d}v' + C\Lambda\|\varphi\|_{L^\infty}, \\ &\leqslant C\Lambda \|\varphi\|_{C^2}. \end{aligned}$$

For the last inequality we used that thanks to (4.2),

$$\int_{B_1} |v'-v|^2 [K(v,v') - K(v',v)] \, \mathrm{d}v' \lesssim \Lambda,$$

and thanks to (4.4),

$$PV \int_{B_1} (v'-v)[K(v,v') - K(v',v)] \, \mathrm{d}v' \leqslant \Lambda.$$

When $s < 1/2$, we do not need to use (4.4). We simply use (4.2) to get

$$\begin{aligned} \int_{B_1} (\varphi(v') - \varphi(v))[K(v,v') - K(v',v)] \, \mathrm{d}v' &\leqslant \int_{B_1} |v-v'| [\varphi]_{C^1} [K(v,v') - K(v',v)] \, \mathrm{d}v', \\ &\leqslant C\Lambda [\varphi]_{C^1}. \end{aligned}$$ □

Remark 4.8. Lemma 4.7 tells us in particular that when K is anti-symmetric, the operator $L_v f$ is well defined pointwise. The same cannot be said for a symmetric kernel of $s \geqslant 1/2$. When K is a symmetric kernel assuming only (4.2), the value of $L_v f(v)$ is not necessarily defined pointwise, even if f is smooth. It is only through $\mathcal{E}^{\text{sym}}$ that we can define L_v as an operator from H^s to H^{-s}.

4.3. **Estimate focusing on the smoothness of only one function.** In this section we obtain an estimate for $\mathcal{E}(\varphi, g)$ taking maximum advantage of the smoothness of φ, and not so much on the smoothness of g.

Lemma 4.9 (Second upper bound for $\mathcal{E}$)**.** *Let K satisfy (4.2) and (4.3). If $s \geqslant 1/2$, we also assume (4.4). For any two functions $g \in H^s(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ and $\varphi \in C^2$ with $g \geqslant 0$ and any $\varepsilon > 0$, we have*

$$\mathcal{E}(\varphi, g) \leqslant \varepsilon \|g\|^2_{\dot{H}^s} + C\varepsilon^{-1} \|\varphi\|^2_{C^1} |\{v \in \mathbb{R}^d : g(v) > 0\}| + C\|\varphi\|_{C^2} \|g\|_{L^1}. \tag{4.8}$$

Proof. Recall that $\mathcal{E} = \mathcal{E}^{\text{sym}} + \mathcal{E}^{\text{skew}}$. We estimate each term separately.

In order to estimate $\mathcal{E}^{\text{sym}}$, we apply the following elementary identity

$$|g(v) - g(v')| \leqslant (\chi_{g>0}(v) + \chi_{g>0}(v'))|g(v) - g(v')|;$$

we then get

$$\begin{aligned}\mathcal{E}^{\text{sym}}(\varphi,g) &\leqslant \iint |\varphi(v)-\varphi(v')|\chi_{g>0}(v)|g(v)-g(v')|K(v,v')\,\mathrm{d}v'\,\mathrm{d}v,\\ &\leqslant \varepsilon\iint (g(v)-g(v'))^2K(v,v')\,\mathrm{d}v'\,\mathrm{d}v + (4\varepsilon)^{-1}\iint(\varphi(v)-\varphi(v'))^2\chi_{g>0}(v)K(v,v')\,\mathrm{d}v'\,\mathrm{d}v,\\ &= \varepsilon\mathcal{E}^{\text{sym}}(g,g)+(4\varepsilon)^{-1}\int\chi_{g>0}(v)\left(\int(\varphi(v)-\varphi(v'))^2K(v,v')\,\mathrm{d}v'\right)\mathrm{d}v,\end{aligned}$$

using Lemma 4.2 and the assumption (4.2),

$$\leqslant \varepsilon C\|g\|^2_{\dot H^s}+C\varepsilon^{-1}\|\varphi\|^2_{C^1}\int\chi_{g>0}\,\mathrm{d}v.$$

As far as $\mathcal{E}^{\text{skew}}$ is concerned, we first rewrite it as follows

$$\begin{aligned}\mathcal{E}^{\text{skew}}(\varphi,g) =& \frac14\iint(\varphi(v)-\varphi(v'))(g(v)+g(v'))(K(v,v')-K(v',v))\,\mathrm{d}v'\,\mathrm{d}v\\ =&\frac12\iint(\varphi(v)-\varphi(v'))g(v)\left(K(v,v')-K(v',v)\right)\,\mathrm{d}v'\,\mathrm{d}v\\ =&\frac12\int g(v)\left\{PV\int_{\mathbb{R}^d}(\varphi(v)-\varphi(v'))\left(K(v,v')-K(v',v)\right)\,\mathrm{d}v'\right\}\mathrm{d}v,\end{aligned}$$

using Lemma 4.7,

$$\leqslant C\|\varphi\|_{C^2}\int g(v)\,\mathrm{d}v.$$

Combining the upper bounds for $\mathcal{E}^{\text{sym}}$ and $\mathcal{E}^{\text{skew}}$, we conclude the proof. □

4.4. **Commutator estimates.**

Lemma 4.10 (Commutator estimate for $s \in (0,1/2)$)**.** *Let us assume $s\in(0,1/2)$ and that K satisfies (4.2). Let D be a closed set and Ω open so that $D \Subset \Omega \subset \mathbb{R}^d$. Let φ be a smooth function supported in D and $f \in H^s(\Omega)\cap L^\infty(\mathbb{R}^d)$. We have the following commutator estimate*

$$L_v[\varphi f]-\varphi L_v f = h_1+h_2,$$

with

$$\begin{aligned}\|h_1\|_{L^\infty(\mathbb{R}^d)} &\lesssim \|\varphi\|_{L^\infty}\|f\|_{L^\infty(\mathbb{R}^d)}d(D,\mathbb{R}^d\backslash\Omega)^{-2s},\\ \|h_1\|_{L^2(\mathbb{R}^d\backslash\Omega)} &\lesssim \|\varphi\|_{L^\infty}\|f\|_{L^2(D)}d(D,\mathbb{R}^d\backslash\Omega)^{-2s},\\ \|h_2\|_{L^2(\mathbb{R}^d)} &\lesssim \|\varphi\|_{C^1}\|f\|_{L^2(\Omega)}.\end{aligned}$$

Moreover, $h_2=0$ outside Ω. Whenever $\Omega=\mathbb{R}^d$, we can consider $d(D,\mathbb{R}^d\backslash\Omega)=+\infty$ and $h_1=0$.

Proof. From the formula (4.1), we get

$$C[\varphi,f](v) := L_v[\varphi f](v)-\varphi(v)L_vf(v) = \int_{\mathbb{R}^d} f(v')(\varphi(v')-\varphi(v))K(v,v')\,\mathrm{d}v'.$$

Let $r = d(D,\mathbb{R}^d\backslash\Omega)/2$, and let $E = D + B_r$. Thus, we have $D \Subset E \Subset \Omega$, with $d(D,\mathbb{R}^d\backslash E) = r$ and $d(D,\mathbb{R}^d\backslash\Omega)=r$.

We define

$$h_1(v) = \int_{\mathbb{R}^d\backslash B_r(v)} f(v')(\varphi(v')-\varphi(v))K(v,v')\,\mathrm{d}v', \qquad h_2(v) = \int_{B_r(v)} f(v')(\varphi(v')-\varphi(v))K(v,v')\,\mathrm{d}v'.$$

From (4.2), for any value of $v\in\mathbb{R}^d$, we have

$$|h_1(v)| \leqslant 2\|f\|_{L^\infty}\|\varphi\|_{L^\infty}\Lambda r^{-2s}$$

which is the first inequality.

When $v \notin D$, we have $\varphi(v) = 0$. Therefore, the integrand in $C[\varphi, f](v)$ is nonzero only for $v' \in D$. We thus have for $v \notin \Omega \supset D$,

$$h_1(v) = \int_D f(v')\varphi(v')K(v,v')\,\mathrm{d}v'$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\Omega} h_1(v)^2\,\mathrm{d}v &= \int_{\mathbb{R}^d\setminus\Omega}\left(\int_D f(v')\varphi(v')K(v,v')\,\mathrm{d}v'\right)^2\mathrm{d}v,\\ &\leqslant \|\varphi\|_{L^\infty}^2\int_{\mathbb{R}^d\setminus\Omega}\left(\int_D f(v')^2|K(v,v')|\,\mathrm{d}v'\right)\left(\int_D |K(v,v')|\,\mathrm{d}v'\right)\mathrm{d}v,\end{aligned}$$

using (4.2),

$$\begin{aligned}&\leqslant \|\varphi\|_{L^\infty}^2\Lambda r^{-2s}\int_{\mathbb{R}^d\setminus\Omega}\int_D f(v')^2|K(v,v')|\,\mathrm{d}v'\,\mathrm{d}v,\\ &\leqslant \|\varphi\|_{L^\infty}^2\Lambda r^{-2s}\int_D f(v')^2\left(\int_{|v-v'|>r}|K(v,v')|\,\mathrm{d}v\right)\mathrm{d}v = \Lambda^2 r^{-4s}\|\varphi\|_{L^\infty}^2\|f\|_{L^2(D)}^2.\end{aligned}$$

This gives us the second inequality.

In order to estimate $\|h_2\|_{L^2}$, we use Cauchy Schwarz.

$$\begin{aligned}\|h_2\|_{L^2}^2 &= \int_E\left(\int_{B_r(v)} f(v')(\varphi(v')-\varphi(v))K(v,v')\,\mathrm{d}v'\right)^2\mathrm{d}v,\\ &\leqslant \int_E\left(\int_{B_r(v)} f(v')^2|\varphi(v')-\varphi(v)|\,|K(v,v')|\,\mathrm{d}v'\right)\left(\int_{B_r(v)}|\varphi(v')-\varphi(v)|\,|K(v,v')|\,\mathrm{d}v'\right)\mathrm{d}v,\end{aligned}$$

Since φ is bounded and C^1, then (4.2) implies (note that $s < 1/2$),

$$\begin{aligned}&\int_{\mathbb{R}^d}|\varphi(v')-\varphi(v)|\,|K(v,v')|\,\mathrm{d}v' \lesssim \|\varphi\|_{C^1} \qquad \text{for every value of } v\in\mathbb{R}^d,\\ &\int_{\mathbb{R}^d}|\varphi(v')-\varphi(v)|\,|K(v,v')|\,\mathrm{d}v \lesssim \|\varphi\|_{C^1} \qquad \text{for every value of } v'\in\mathbb{R}^d.\end{aligned}$$

Therefore,

$$\begin{aligned}\|h_2\|_{L^2}^2 &\lesssim \|\varphi\|_{C^1}\int_E\left(\int_{B_r(v)} f(v')^2|\varphi(v')-\varphi(v)|\,|K(v,v')|\,\mathrm{d}v'\right)\mathrm{d}v,\\ &\lesssim \|\varphi\|_{C^1}\int_\Omega f(v')^2\left(\int_{E\cap B_r(v')}|\varphi(v')-\varphi(v)|\,|K(v,v')|\,\mathrm{d}v\right)\mathrm{d}v',\\ &\lesssim \|\varphi\|_{C^1}^2\|f\|_{L^2(\Omega)}^2.\end{aligned}$$

□

Lemma 4.11 (Commutator estimate for $s \in [1/2, 1)$)**.** *Let us assume $s \in [1/2, 1)$ and that K satisfies (4.2) and (4.4). Let D be a closed set, and Ω open so that $D \Subset \Omega \subset \mathbb{R}^d$. Let φ be a smooth function supported in D and $f \in H^s(\Omega) \cap L^\infty(\mathbb{R}^d)$. We have the following commutator estimate*

$$L_v[\varphi f] - \varphi L_v f = h_1 + h_2 + (-\Delta)^{s/2}h_3,$$

with

$$\begin{aligned}\|h_1\|_{L^\infty(\mathbb{R}^d)} &\lesssim \|\varphi\|_{L^\infty}\|f\|_{L^\infty(\mathbb{R}^d)}(d(D,\mathbb{R}^d\setminus\Omega)+d(v,D))^{-2s},\\ \|h_1\|_{L^2(\mathbb{R}^d\setminus\Omega)} &\lesssim \|\varphi\|_{L^\infty}\|f\|_{L^2(D)}d(D,\mathbb{R}^d\setminus\Omega)^{-2s},\\ \|h_2\|_{L^2(\mathbb{R}^d)} &\lesssim \|\varphi\|_{C^2}\|f\|_{H^s(\Omega)},\\ \|h_3\|_{L^2(\mathbb{R}^d)} &\lesssim \|\varphi\|_{C^2}\|f\|_{L^2(\Omega)}.\end{aligned}$$

Moreover, $h_2 = 0$ outside Ω. Whenever $\Omega = \mathbb{R}^d$, we can consider $d(D, \mathbb{R}^d \backslash \Omega) = +\infty$ and $h_1 = 0$.

Proof. We define h_1 and $\tilde{h}_2$ by the expressions of h_1 and h_2 in the proof of Lemma 4.10. The estimates for h_1 follow identically. We will split $\tilde{h}_2 = h_2 + (-\Delta)^{s/2} h_3$, and need to prove the estimate for each term.

Note that, by construction, $\tilde{h}_2(v) = 0$ for any $v \notin E$.

Let us write K as the sum of its symmetric plus antisymmetric parts: $K = K_s + K_a$. We start by estimating the antisymmetric contribution.

Because of Lemma 4.7, we have that $\|L_v^a \varphi\|_{L^\infty} \lesssim \|\varphi\|_{C^2}$. Then

$$\begin{aligned}\|\tilde{h}_2^a\|_{L^2(\Omega)} &:= \left| \int_{B_r(v)} f(v')(\varphi(v') - \varphi(v)) K_a(v,v') \, \mathrm{d}v' \right|_{L^2(\Omega)}, \\ &\leqslant \left| \int_{B_r(v)} (f(v') - f(v))(\varphi(v') - \varphi(v)) K_a(v,v') \, \mathrm{d}v' \right|_{L^2(\Omega)} + C\|\varphi\|_{C^2} \|f\|_{L^2(\Omega)}.\end{aligned}$$

With respect to the first term, we apply Cauchy-Schwarz and Lemma 4.2 to obtain

$$\begin{aligned}&\left| \int_{B_r(v)} (f(v') - f(v))(\varphi(v') - \varphi(v)) K_a(v,v') \, \mathrm{d}v' \right|^2_{L^2(\Omega)} \\ &\qquad \leqslant \int_E \left(\int_{B_r(v)} (f(v') - f(v))^2 |K_a(v,v')| \, \mathrm{d}v' \right) \left(\int_{B_r(v)} (\varphi(v') - \varphi(v))^2 |K_a(v,v')| \, \mathrm{d}v' \right) \mathrm{d}v, \\ &\qquad \lesssim \|\varphi\|_{C^2} \iint_{\Omega \times \Omega} (f(v') - f(v))^2 |K_a(v,v')| \, \mathrm{d}v' \, \mathrm{d}v \lesssim \|\varphi\|_{C^2} \|f\|^2_{\dot{H}^s(\Omega)}.\end{aligned}$$

Therefore, we conclude the estimate for the antisymmetric contribution $\|\tilde{h}_2^a\|_{L^2(\mathbb{R}^d)} \leqslant C\|\varphi\|_{C^2} \|f\|_{H^s(\Omega)}$.

Now we need to analyse the contribution of K_s to $\tilde{h}_2$, which we call $\tilde{h}_2^s$. We estimate it by duality. Let $g \in H^s(\mathbb{R}^d)$, recall that $\tilde{h}_s^2$ is supported in E and consider

$$\begin{aligned}\int_E \tilde{h}_2^s(v) g(v) \, \mathrm{d}v &= \int_E \int_{B_r(v)} g(v) f(v')(\varphi(v') - \varphi(v)) K_s(v,v') \, \mathrm{d}v' \, \mathrm{d}v, \\ &= \frac{1}{2} \int_E f(v) \left(\int_{B_r(v)} (g(v) - g(v'))(\varphi(v') - \varphi(v)) K_s(v,v') \, \mathrm{d}v' \, \mathrm{d}v \right) \\ &\quad + g(v) \left(\int_{B_r(v)} (f(v') - f(v))(\varphi(v') - \varphi(v)) K_s(v,v') \, \mathrm{d}v' \, \mathrm{d}v \right) \mathrm{d}v.\end{aligned}$$

Applying the Cauchy-Schwarz inequality and Lemma 4.2 as above, we get

$$\int_\Omega \tilde{h}_2^s(v) g(v) \, \mathrm{d}v \lesssim \|\varphi\|_{C^2} \left(\|f\|_{\dot{H}^s} \|g\|_{L^2} + \|f\|_{L^2} \|g\|_{\dot{H}^s} \right).$$

Therefore, $\tilde{h}_2^s$ can be written as a sum $\hat{h}_2^s + (-\Delta) h_3$ with $\|\hat{h}_2^s\|_{L^2(\mathbb{R}^d)} \lesssim \|\varphi\|_{C^2} \|f\|_{H^s(\Omega)}$ and $\|h_3\|_{L^2(\mathbb{R}^d)} \lesssim \|\varphi\|_{C^2} \|f\|_{L^2(\Omega)}$.

We finish the proof by letting $h_2 = \tilde{h}_2^a + \hat{h}_2^s$. □

5. Reduction to global kernels and weak solutions

The assumptions (1.3), (1.5), (1.6) and (1.7) are given in terms of values of $v \in B_{\bar{R}}$ only. It is natural that if we consider the equation (1.2) to hold for $v \in B_1$ and we intend to prove local regularity estimates, it should be useless to make assumptions for $K(v,v')$ when $v \notin B_2$. It is confortable for the proofs of a few lemmas (in particular the results in Section 4 above and Lemma 6.1 below) to have a kernel that is globally defined and satisfies all these assumptions for all values of v and v'. In this section we explain how to extend a kernel to the full space in order to have that.

5.1. **Reduction to global kernels.**

Proposition 5.1 (A kernel defined globally). *Assume that $K : B_{\bar{R}} \times \mathbb{R}^d \to \mathbb{R}$ satisfies (1.3), (1.5), (1.6) and (1.7). There exists a kernel $\tilde{K} : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following global version of assumptions (1.3), (1.5), (1.6) and (1.7).*

- *$\tilde{K}(v,v') = K(v,v')$ whenever v and v' belong to $B_{2\bar{R}/3}$. Moreover $\tilde{K}(v,v') \geqslant 0$ for all $v,v' \in \mathbb{R}^d$ and for all $v \in B_{\bar{R}/2}$,*

$$\int_{\mathbb{R}^d} |K(v,v') - \tilde{K}(v,v')| \, \mathrm{d}v' \leqslant C\Lambda. \tag{5.1}$$

- *For any function $f \in H^s(\mathbb{R}^d)$,*

$$\lambda \|f\|^2_{\dot{H}^s} \leqslant - \int_{\mathbb{R}^d} \tilde{L_v} f(v)\, f(v) \, \mathrm{d}v + \Lambda \|f\|^2_{L^2(\mathbb{R}^d)}. \tag{5.2}$$

  Here $\tilde{L_v}$ is the integro-differential operator corresponding to the kernel $\tilde{K}$.
- *The assumptions (4.2), (4.3) and (4.4) hold for $\tilde{K}$ with a constant $C\Lambda$ instead of Λ, where C depends on s, $\bar{R}$, and dimension only.*

Proof. Let $\eta : \mathbb{R}^d \to [0,1]$ be a smooth radial function so that $\eta = 1$ in $B_{3\bar{R}/4}$ and $\eta = 0$ outside $B_{7\bar{R}/8}$. We define

$$\tilde{K}(v,v') = \eta(v)\eta(v')K(v,v') + \Lambda(1 - \eta(v)\eta(v'))|v-v'|^{-d-2s}.$$

Note that even though $K(v,v')$ is not defined when $v \notin B_{\bar{R}}$, since we have the factor $\eta(v) = 0$ there, there is no ambiguity in the definition of $\tilde{K}(v,v')$.

The first item in the Proposition is obvious by construction. We start by checking (4.2). For any $v \in \mathbb{R}^d$ and $r > 0$, we have

$$\begin{aligned}\int_{\mathbb{R}^d \setminus B_r(v)} \tilde{K}(v,v') \, \mathrm{d}v' &= \int_{\mathbb{R}^d \setminus B_r(v)} \eta(v)\eta(v')K(v,v') + \Lambda(1-\eta(v)\eta(v'))|v-v'|^{-d-2s} \, \mathrm{d}v', \\ &\leqslant \eta(v) \int_{\mathbb{R}^d \setminus B_r(v)} K(v,v') \, \mathrm{d}v' + (1-\eta(v))\Lambda \int_{\mathbb{R}^d \setminus B_r(v)} |v-v'|^{-d-2s} \, \mathrm{d}v' \lesssim \Lambda.\end{aligned}$$

For any $v' \in \mathbb{R}^d$ and $r > 0$, we do almost the same computation

$$\begin{aligned}\int_{\mathbb{R}^d \setminus B_r(v')} \tilde{K}(v,v') \, \mathrm{d}v &= \int_{\mathbb{R}^d \setminus B_r(v)} \eta(v)\eta(v')K(v,v') + \Lambda(1-\eta(v)\eta(v'))|v-v'|^{-d-2s} \, \mathrm{d}v', \\ &\leqslant \eta(v') \int_{B_{\bar{R}} \setminus B_r(v')} K(v,v') \, \mathrm{d}v + (1-\eta(v'))\Lambda \int_{\mathbb{R}^d \setminus B_r(v')} |v-v'|^{-d-2s} \, \mathrm{d}v \lesssim \Lambda.\end{aligned}$$

This justifies (4.2). We now verify (4.4). This only applies when $2s \geqslant 1$. Given any $r > 0$, we compute

$$\begin{aligned}&\left| PV \int_{B_r(v)} (v-v')\big(\tilde{K}(v,v') - \tilde{K}(v',v)\big) \, \mathrm{d}v' \right| \\ &\quad = \eta(v) \left| PV \int_{B_r(v) \cap B_{\bar{R}}} (v-v')\eta(v') \big(K(v,v') - K(v',v)\big) \, \mathrm{d}v' \right|, \\ &\quad \leqslant \eta(v) \Bigg(\eta(v)\Lambda(1 + \min(r, (\bar{R} - |v|))^{1-2s}) \\ &\qquad + \left| \int_{B_r(v) \cap B_{\bar{R}}} (v-v')(\eta(v') - \eta(v)) \big(K(v,v') - K(v',v)\big) \, \mathrm{d}v' \right| \Bigg),\end{aligned}$$

Note that $\eta(v) = 0$ if $|v| > 7\bar{R}/8$, therefore $\eta(v)(\bar{R} - |v|)^{1-2s} \leqslant C$ for some constant depending on $\bar{R}$.

$$\begin{aligned}&\leqslant C\eta(v) \left(\Lambda(1 + r^{1-2s}) + \int_{B_r(v) \cap B_{\bar{R}}} |v-v'|^2 |K(v,v') - K(v',v)| \, \mathrm{d}v' \right), \\ &\leqslant C\Lambda(1 + r^{1-2s}).\end{aligned}$$

This proves (4.4).

We now move on to (4.3). When $s \in [0,1/2)$ the proof is similar to the computation above for (4.4). Indeed,

$$\begin{aligned}\left|PV\int_{\mathbb{R}^d}\left(\tilde K(v,v')-\tilde K(v',v)\right)\mathrm{d}v'\right| &= \eta(v)\left|PV\int_{B_{\bar R}}\eta(v')\left(K(v,v')-K(v',v)\right)\,\mathrm{d}v'\right|,\\ &\leqslant \eta(v)\left(\Lambda+\left|PV\int_{B_{\bar R}}(\eta(v')-\eta(v))\left(K(v,v')-K(v',v)\right)\,\mathrm{d}v'\right|\right),\\ &\leqslant \eta(v)\left(\Lambda+C\int_{B_{\bar R}}|v'-v||K(v,v')-K(v',v)|)\,\mathrm{d}v'\right)\leqslant C\Lambda\eta(v).\end{aligned}$$

The last inequality follows from (1.5) because $s \in [0,1/2)$.

In the case $s \in [1/2,1)$, we modify the estimate of the last line. We have

$$\begin{aligned}\left|PV\int_{\mathbb{R}^d}\left(\tilde K(v,v')-\tilde K(v',v)\right)\mathrm{d}v'\right| &\leqslant \eta(v)\left(\Lambda+\left|PV\int_{B_{\bar R}}(\eta(v')-\eta(v))\left(K(v,v')-K(v',v)\right)\,\mathrm{d}v'\right|\right),\\ &\leqslant \eta(v)\Bigg(\Lambda+\left|\nabla\eta(v)\cdot PV\int_{B_{\bar R}}(v'-v)\left(K(v,v')-K(v',v)\right)\,\mathrm{d}v'\right|\\ &\qquad +\int_{B_{\bar R}}C|v'-v|^2|K(v,v')-K(v',v)|\,\mathrm{d}v'\Bigg)\leqslant C\Lambda\eta(v).\end{aligned}$$

For the last inequality, we apply (1.5) and (1.7).

We now justify (5.2). We see that

$$\begin{aligned}-\int \tilde L_v f(v)f(v)\,\mathrm{d}v &= \mathcal{E}^{\tilde{\text{sym}}}(f,f)+\mathcal{E}^{\tilde{\text{skew}}}(f,f),\\ &= \iint |f(v)-f(v')|^2\tilde K(v,v')\,\mathrm{d}v\,\mathrm{d}v' + \int f(v)^2\left(PV\int(\tilde K(v,v')-\tilde K(v',v))\,\mathrm{d}v'\right)\mathrm{d}v,\\ &\geqslant \iint_{\mathbb{R}^{2d}}|f(v)-f(v')|^2\tilde K(v,v')\,\mathrm{d}v\,\mathrm{d}v' - \Lambda\|f\|^2_{L^2}.\end{aligned}$$

Let $3\bar R/4 < r_1 < r_2 < \bar R$ so that $\eta(v) < 2/3$ if $|v| > r_1$ and $\eta(v) > 1/3$ if $|v| < r_2$. The first term in the definition of $\tilde K(v,v')$ of bounded below by $K(v,v')/9$ when both v and v' belong to B_{r_2}. When v and v' do not belong to B_{r_1}, we can estimate $\tilde K(v,v')$ from below by $\Lambda|v-v'|^{-d-2s}/3$. If v and v' belong to $B_{r_2}\backslash B_{r_1}$, the value of $\tilde K(v,v')$ is bounded below by the sum of the two previous terms. We have,

$$\begin{aligned}\iint_{\mathbb{R}^{2d}}|f(v)-f(v')|^2\tilde K(v,v')\,\mathrm{d}v\,\mathrm{d}v' &\geqslant \frac19\iint_{B_{r_2}\times B_{r_2}}|f(v)-f(v')|^2K(v,v')\,\mathrm{d}v\,\mathrm{d}v'\\ &\quad+\frac{\Lambda}{3}\iint_{\mathbb{R}^{2d}\backslash(B_{r_1}\times B_{r_1})}|f(v)-f(v')|^2|v-v'|^{-d-2s}\,\mathrm{d}v\,\mathrm{d}v'.\end{aligned}$$

We need to estimate the first term using (1.3). Let φ be a smooth radial function so that $\varphi = 1$ in B_{r_1} and $\varphi = 0$ outside B_{r_2}. Using Lemma 4.7 after some arithmetic manipulations, we see that

$$\begin{aligned}\mathcal{E}(\varphi f,\varphi f) &= \iint_{B_{\bar R}\times B_{\bar R}}\varphi(v)\varphi(v')(f(v)-f(v'))^2K(v,v')\,\mathrm{d}v'\,\mathrm{d}v\\ &\quad+2\int_{B_{\bar R}}f(v)^2\varphi(v)\left(PV\int_{B_{\bar R}}(\varphi(v)-\varphi(v'))(K(v,v')-K(v',v))\,\mathrm{d}v'\right)\mathrm{d}v,\\ &\quad+2\int_{B_{\bar R}}\varphi(v)^2f(v)^2\left(\int_{\mathbb{R}^d\backslash B_{\bar R}}K(v,v')\,\mathrm{d}v'\right)\mathrm{d}v,\\ &\leqslant \iint_{B_{r_2}\times B_{r_2}}|f(v)-f(v')|^2K(v,v')\,\mathrm{d}v\,\mathrm{d}v' + C\|f\|^2_{L^2}.\end{aligned}$$

Combining the last three displayed inequalities with (1.3), we obtain

$$\begin{aligned}
\mathcal{E}(f,f) &= -\int \tilde{L_v} f(v) f(v)\,\mathrm{d}v,\\
&\geqslant \iint_{\mathbb{R}^{2d}} |f(v)-f(v')|^2 \tilde{K}(v,v')\,\mathrm{d}v\,\mathrm{d}v' - \Lambda\|f\|_{L^2}^2,\\
&\geqslant \frac{1}{9}\iint_{B_{r_2}\times B_{r_2}} |f(v)-f(v')|^2 K(v,v')\,\mathrm{d}v\,\mathrm{d}v'\\
&\qquad + \frac{\Lambda}{3}\iint_{\mathbb{R}^{2d}\setminus(B_{r_1}\times B_{r_1})} |f(v)-f(v')|^2|v-v'|^{-d-2s}\,\mathrm{d}v\,\mathrm{d}v' - \Lambda\|f\|_{L^2}^2,\\
&\geqslant \frac{1}{9}\mathcal{E}(\varphi f,\varphi f) - C\|f\|_{L^2}^2 + \frac{\Lambda}{3}\iint_{\mathbb{R}^{2d}\setminus(B_{r_1}\times B_{r_1})} |f(v)-f(v')|^2|v-v'|^{-d-2s}\,\mathrm{d}v\,\mathrm{d}v',\\
&\geqslant \min(\lambda/9,\Lambda/3)\left(\iint_{\mathbb{R}^d\times\mathbb{R}^d} |f(v)-f(v')|^2|v-v'|^{-d-2s}\,\mathrm{d}v\,\mathrm{d}v'\right) - C\|f\|_{L^2}^2.
\end{aligned}$$

□

The extended kernel $\tilde{K}$ can be used to reduce many results to the case of globally defined kernels. The following results, which we will need later, are examples.

Corollary 5.2 (The operator L_v maps H^s into H^{-s})**.** *Assume $K : B_{\bar{R}} \times \mathbb{R}^d \to \mathbb{R}$ is a non-negative kernel that satisfies (1.5) and (1.6); if $s \geqslant 1/2$, we also assume that K satisfies (1.7). For any $f \in H^s(\mathbb{R}^d)$ and $g \in H^s(\mathbb{R}^d)$ supported in $B_{\bar{R}/2}$,*

$$\mathcal{E}(f,g) = -\int_{B_{\bar{R}/2}} L_v f(v) g(v)\,\mathrm{d}v \leqslant \Lambda \|f\|_{H^s(\mathbb{R}^d)}\|g\|_{H^s(\mathbb{R}^d)} \tag{5.3}$$

for some positive constant Λ depending on dimension.

Corollary 5.3 (Second upper bound for $\mathcal{E}$)**.** *Let K satisfy (1.5), (1.6). If $s \geqslant 1/2$, we also assume (1.7). For any two functions $g \in H^s(B_{\bar{R}/2}) \cap L^1(\mathbb{R}^d)$ and $\varphi \in C^2$, both compactly supported in $B_{\bar{R}/2}$, with $g \geqslant 0$ and any $\varepsilon > 0$, we have*

$$\mathcal{E}(\varphi,g) \leqslant \varepsilon\|g\|_{H^s}^2 + C\varepsilon^{-1}\|\nabla\varphi\|_{L^\infty}^2|\{v \in \mathbb{R}^d : g(v) > 0\}| + C\|\varphi\|_{C^2}\|g\|_{L^1} + C\varepsilon\|g\|_{L^2}^2. \tag{5.4}$$

Corollary 5.4. *Let K satisfy (1.5), D, Ω, φ and f be as in Lemma 4.10. Assume that $B_{\bar{R}/2} \supset \Omega$. We extend the operator $\tilde{L}$ as in Proposition 5.1. Then,*

$$\tilde{L}[\varphi f] - \varphi L f = h_1 + h_2,$$

where h_1 and h_2 satisfy the same estimates as in Lemma 4.10.

Corollary 5.5. *Let K satisfy (1.5) and (1.7), D, Ω, φ and f be as in Lemma 4.11. Assume that $B_{\bar{R}/2} \supset \Omega$. We extend the operator $\tilde{L}$ as in Proposition 5.1. Then,*

$$\tilde{L}[\varphi f] - \varphi L f = h_1 + h_2 + (-\Delta)^{s/2} h_3,$$

where h_1, h_2 and h_3 satisfy the same estimates as in Lemma 4.11.

The justifications of the two lemmas above are almost identical. We explain the latter one.

Proof of Corollary 5.5. Let $\bar{K}$ be the extended kernel according to Proposition 5.1.

Applying Lemma 4.11, we obtain that

$$\tilde{L}[\varphi f] - \varphi \tilde{L} f = \tilde{h}_1 + h_2 + (-\Delta)^{s/2} h_3.$$

For this corollary, we want to replace $\varphi \tilde{L} f$ by $\varphi L f$. Since φ is supported in D, these two expressions only differ when $v \in D$. In this case, we have

$$|\varphi(v)\tilde{L}f(v) - \varphi(v)Lf(v)| = \left|\varphi(v)\int_{\mathbb{R}^d} [f(v)-f(v')]\left(K(v,v') - \tilde{K}(v,v')\right)\,\mathrm{d}v\right| \leqslant C\varphi(v)\|f\|_{L^\infty}\delta^{-2s}.$$

This difference is absorbed by the term h_1 by setting $h_1 = \tilde{h}_1 + \varphi(v)\tilde{L}f(v) - \varphi(v)Lf(v)$. □

5.2. **Definition of weak solutions.** We now discuss the concept of weak solutions. In order to justify the definition we are going to give below, we start with the following preparatory lemma.

Lemma 5.6 (The bilinear form $\mathcal{E}$ in the local case)**.** *Let* $\operatorname{supp}\varphi \Subset B_{\bar R/2}$ *and* $\varphi \in H^s(\mathbb{R}^d)$. *Assume* K *satisfies* (1.5), (1.6) *and* (1.7). *Then for all* $f \in L^\infty(\mathbb{R}^d \setminus B_{\bar R/2}) + H^s(\mathbb{R}^d)$

$$\mathcal{E}(f,\varphi) \leqslant C\|f\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar R/2})+H^s(\mathbb{R}^d)}\|\varphi\|_{H^s(\mathbb{R}^d)},$$

where the constant C *depends on* Λ, d, s *and the support of* φ. *Here,*

$$\|f\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar R/2})+H^s(\mathbb{R}^d)} = \inf\left\{\|f_1\|_{L^\infty(\mathbb{R}^d\setminus B_{\bar R/2})} + \|f_2\|_{H^s(\mathbb{R}^d)} : f = f_1 + f_2 \text{ and } f_1 = 0 \text{ in } B_{\bar R/2}\right\}.$$

More precisely, the inequality holds for smooth functions, and therefore it allows the bilinear form to be extended to the appropriate spaces of functions.

Note that the restriction $f \in L^\infty(\mathbb{R}^d\setminus B_{\bar R/2}) + H^s(\mathbb{R}^d)$ imposes some fractional Sobolev regularity in $B_{\bar R/2}$ but not so much outside. In particular, any function $f \in H^s(B_{\bar R/2+\varepsilon}) \cap L^\infty(\mathbb{R}^d\setminus B_{\bar R/2+\varepsilon})$ is in this space.

Proof. As mentioned above, we assume for the proof that both f and φ are smooth. Afterwards, the inequality is obtained by density when $f \in L^\infty(\mathbb{R}^d\setminus B_{\bar R/2}) + H^s(\mathbb{R}^d)$ and $\varphi \in H^s(\mathbb{R}^d)$ is compactly supported in $B_{\bar R/2}$

Let $f = f_1 + f_2$ as in the definition of the norm in $L^\infty(\mathbb{R}^d\setminus B_{\bar R/2}) + H^s(\mathbb{R}^d)$. Applying Corollary 5.2, $|\mathcal{E}(f_2,\varphi)| \lesssim \|f_2\|_{H^s}\|\varphi\|_{H^s}$. We are left to compute $\mathcal{E}(f_1,\varphi)$. We have

$$\begin{aligned}\mathcal{E}(f_1,\varphi) &= \lim_{\varepsilon\to 0}\iint_{|v'-v|>\varepsilon} (f_1(v') - f_1(v))\varphi(v)K(v,v')\,\mathrm{d}v'\,\mathrm{d}v,\\ &= \lim_{\varepsilon\to 0}\int_{\operatorname{supp}\varphi}\left(\int_{\mathbb{R}^d\setminus B_\varepsilon(v)} f_1(v')K(v,v')\,\mathrm{d}v'\right)\varphi(v)\,\mathrm{d}v,\\ &= \int_{\operatorname{supp}\varphi}\left(\int_{\mathbb{R}^d\setminus B_\delta(v)} f_1(v')K(v,v')\,\mathrm{d}v'\right)\varphi(v)\,\mathrm{d}v.\end{aligned}$$

Here δ is the distance between the support of φ and $\mathbb{R}^d\setminus B_{\bar R/2}$.

$$\leqslant \Lambda\delta^{-2s}\|f_1\|_{L^\infty}\left(\int_{\operatorname{supp}\varphi}\varphi(v)\,\mathrm{d}v\right) \leqslant C\Lambda\delta^{-2s}\|f_1\|_{L^\infty}\|\varphi\|_{H^s}.$$

□

Another way to describe Lemma 5.6 is that L_v is a bounded operator from $L^\infty(\mathbb{R}^d\setminus B_{\bar R/2}) + H^s(\mathbb{R}^d)$ to $H^{-s}(B_{\bar R})$. We will use this to define the concept of weak solution.

Definition 5.7 (Weak solutions)**.** *Assume* K *satisfies* (1.5), (1.6) *and* (1.7). *Given the cylinder* $Q = (0,T)\times B_{(\bar R/2)^{1+2s}} \times B_{\bar R/2}$, *We say that a function* $f : [0,T]\times B_{(\bar R/2)^{1+2s}}\times\mathbb{R}^d \to \mathbb{R}$ *is a* subsolution *of* (1.2) *in the cylinder* Q *if*

$$f \in C^0((0,T), L^2(B_{(\bar R/2)^{1+2s}}\times B_{\bar R/2})) \cap L^2((0,T)\times B_{(\bar R/2)^{1+2s}}, L^\infty(\mathbb{R}^d\setminus B_{\bar R/2}) + H^s(\mathbb{R}^d)),$$
$$f_t + v\cdot\nabla_x f \in L^2((0,T)\times B_{(\bar R/2)^{1+2s}}, H^{-s}(B_{\bar R/2})),$$

and for all non-negative test function $\varphi \in L^2((0,T)\times B_{(\bar R/2)^{1+2s}}, H^s(\mathbb{R}^d))$ *so that for every* t *and* x, $\varphi(t,x\cdot)$ *is compactly supported in* $B_{\bar R/2}$,

$$\iiint (f_t + v\cdot\nabla_x f)\varphi + \iint \mathcal{E}(f,\varphi) - \iiint h\varphi \leqslant 0. \tag{5.5}$$

A function f *is a* supersolution *of* (1.2) *in* Q *if* $-f$ *is a subsolution of* (1.2) *in* Q. *A function* f *is a* solution *of* (1.2) *in* Q *if it is both a sub- and a supersolution.*

Remark 5.8. Assuming that $f \in C^0((0,T), L^2(B_{(\bar R/2)^{1+2s}}\times B_{\bar R/2}))$ and $f \in L^2((0,T)\times B_{(\bar R/2)^{1+2s}}, H^s(B_{\bar R/2}))$ is rather natural in view of the energy estimates one can easily get from the coercivity assumption.

Note that the bilinear form $\iint \mathcal{E}(f,\varphi)$ in (5.5) is well defined because of Lemma 5.6.

6. The first lemma of De Giorgi

This section is devoted to the first intermediate result in the proof of the weak Harnack inequality. It is referred to as the first lemma of De Giorgi. It consists in controlling a local pointwise bound in the interior of a cylinder by an integral quantity in the cylinder. Its proof (see Subsection 6.2) relies on a global energy estimate (See Subsection 6.1).

For degenerate integral equations, the situation is different than for equations of second order. It is not true that the maximum of a nonnegative subsolution can be bounded by above by a multiple of its L^2 norm. One needs to impose an extra global restriction (in this case we assume $0 \leqslant f \leqslant 1$ globally). This is because of nonlocal effects, since the positive values of the function outside of the domain of the equation may *pull* the maximum upwards. The strong Harnack inequality fails in general. This fact is well documented and there are counterexamples (see [20]).

6.1. **Energy estimates.** The proof of the first lemma of De Giorgi relies on an iteration of energy estimates applied to certain truncated functions. For kinetic equations, the energy estimate naturally gives us some regularization with respect to the v variable. We use the fractional Kolmogorov equation to translate this regularization in v to a higher degree of integrability of the function.

Lemma 6.1 (Global energy inequality and gain of integrability)**.** *Assume $\tilde{K}$, and its corresponding operator $\tilde{L_v}$, satisfy (5.2), (4.2), (4.3) and (4.4). Let $G \geqslant 0$ be a weak sub-solution of*

$$\begin{cases} (\partial_t + v\cdot\nabla_x)G - \tilde{L_v}G \leqslant H_1 + (-\Delta)_v^{s/2} H_2 & \text{in } [0,T]\times\mathbb{R}^{2d}, \\ G(0,x,v) = G_0(x,v) & \text{in } \mathbb{R}^{2d} \end{cases} \tag{6.1}$$

with a source terms $H_1, H_2 \in L^2([0,T]\times\mathbb{R}^{2d})$. Then,

$$\sup_{\tau\in[0,T]} \|G(\tau)\|^2_{L^2(\mathbb{R}^{2d})} + \|G\|^2_{L^2([0,T]\times\mathbb{R}^d, \dot{H}^s(\mathbb{R}^d))} \leqslant C\left(\|G_0\|^2_{L^2(\mathbb{R}^{2d})} + \|H_1\|^2_{L^2([0,T]\times\mathbb{R}^{2d})} + \|H_2\|^2_{L^2([0,T]\times\mathbb{R}^{2d})}\right). \tag{6.2}$$

Moreover, there exists $p > 2$ (only depending on dimension and s) such that

$$\|G\|^2_{L^p([0,T]\times\mathbb{R}^{2d})} \leqslant C\left(\|G_0\|^2_{L^2(\mathbb{R}^{2d})} + \|H_1\|^2_{L^2([0,T]\times\mathbb{R}^{2d})} + \|H_2\|^2_{L^2([0,T]\times\mathbb{R}^{2d})}\right), \tag{6.3}$$

for some constant C depending on λ, Λ, d, p, s, and T.

Proof. Multiplying the equation by G and integrating on the time interval $[0,\tau]$ for $\tau \in [0,T]$, we get

$$\frac{1}{2}\|G(\tau)\|^2_{L^2(\mathbb{R}^{2d})} + \int_0^\tau \int_{\mathbb{R}^d} \mathcal{E}(G,G)\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\|G_0\|^2_{L^2(\mathbb{R}^{2d})} + \int_{[0,T]\times\mathbb{R}^{2d}} (H_1 + (-\Delta)_v^{s/2} H_2)G.$$

Using (5.2) from Proposition 5.1, we have

$$\begin{aligned} \frac{1}{2}\|G(\tau)\|^2_{L^2(\mathbb{R}^{2d})} + \int_0^\tau \int_{\mathbb{R}^d} \lambda\|G\|^2_{\dot{H}^s} - \Lambda\|G\|^2_{L^2}\,\mathrm{d}x\,\mathrm{d}t \leqslant & \frac{1}{2}\|G_0\|^2_{L^2(\mathbb{R}^{2d})} \\ & + \int_0^T \|H_1(t)\|_{L^2}\|G(t)\|_{L^2} + \|H_2(t)\|_{L^2}\|G(t)\|_{\dot{H}^s}\,\mathrm{d}t. \end{aligned} \tag{6.4}$$

Therefore,

$$\frac{1}{2}\|G(\tau)\|^2_{L^2(\mathbb{R}^{2d})} + \int_0^\tau \int_{\mathbb{R}^d} -\frac{\Lambda}{2}\|G\|^2_{L^2}\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{2}\|G_0\|^2_{L^2(\mathbb{R}^{2d})} + C\int_0^T \|H_1(t)\|^2_{L^2} + \|H_2(t)\|^2_{L^2}\,\mathrm{d}t.$$

Integrating against $\exp(-\Lambda\tau/2)$ with respect to τ yields

$$\|G\|^2_{L^2([0,T]\times\mathbb{R}^{2d})} \leqslant C\left(\|G_0\|^2_{L^2(\mathbb{R}^{2d})} + \|H_1\|^2_{L^2([0,T]\times\mathbb{R}^{2d})} + \|H_2\|^2_{L^2([0,T]\times\mathbb{R}^{2d})}\right).$$

Using this information back into (6.4), we finally get

$$\sup_{\tau\in[0,T]} \|G(\tau)\|^2_{L^2(\mathbb{R}^{2d})} + \|G\|^2_{L^2_{t,x}\dot{H}^s_v([c,b]\times\mathbb{R}^{2d})} \leqslant C\left(\|G_0\|^2_{L^2(\mathbb{R}^{2d})} + \|H_1\|^2_{L^2([0,T]\times\mathbb{R}^{2d})} + \|H_2\|^2_{L^2([0,T]\times\mathbb{R}^{2d})}\right). \tag{6.5}$$

The function G is also a subsolution of the fractional Kolmogorov equation with an appropriate right hand side

$$G_t + v \cdot \nabla_x G + (-\Delta)^s G \leqslant (-\Delta)^s G + \tilde{L_v} G + H_1 + (-\Delta)^{s/2} H_2.$$

Thus, G is smaller or equal to the exact solution of this equation. Theorem 4.1 ensures that $\tilde{L_v} G \in L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$. We then can apply Proposition 2.2 to G with $h = H_1 + (-\Delta)^{s/2} H_2 + \tilde{L_v} G + (-\Delta)^s G$ so that

$$\|h\|_{L^2([0,T]\times\mathbb{R}^d, H^{-s}(\mathbb{R}^d))} \leqslant \|H_1\|_{L^2} + \|H_2\|_{L^2} + C\|G\|_{L^2_{t,x}\dot{H}^s_v([0,T]\times\mathbb{R}^{2d})}.$$

and get (6.3). □

Let us analyze a localized version of the energy dissipation.

Lemma 6.2 (Local energy dissipation)**.** *Let f be a subsolution of (1.2) in $[0,T] \times B_{R^{1+2s}} \times B_R$ with $h = 0$. Assume $0 \leqslant f \leqslant 1$ almost everywhere in $[0,T] \times B_{R^{1+2s}} \times \mathbb{R}^d$. Assume K satisfies (1.3), (1.5), (1.6) and (1.7) with $\bar{R} = 2R$. Then, for any $\delta \in (0,1)$, we have*

$$\sup_{t\in[0,T]} \iint_{B_{(R-\delta)^{1+2s}} \times B_{R-\delta}} f(t,x,v)^2 \, \mathrm{d}v \, \mathrm{d}x + \int_0^T \int_{B_{(R-\delta)^{1+2s}}} \|f\|^2_{H^s(B_{R-\delta})} \, \mathrm{d}x \, \mathrm{d}t$$
$$\leqslant \iint_{B_{R^{1+2s}} \times B_R} f(0,x,v)^2 \, \mathrm{d}v \, \mathrm{d}x + C\delta^{-2} |\{f > 0\} \cap [0,T] \times B_{R^{1+2s}} \times B_R|. \tag{6.6}$$

Remark 6.3. The factor in δ^{-2} can be improved in terms of s (probably to δ^{-2s}). The optimal power is irrelevant for the rest of our proof.

Proof. Let $\varphi : \mathbb{R}^{2d} \to [0,1]$ be C^∞, supported in $B_{R^{1+2s}} \times B_R$, so that $\varphi = 1$ in $B_{(R-\delta)^{1+2s}} \times B_{R-\delta}$. It is not hard to check that we can construct such φ with $\|\varphi\|_{C^2} \leqslant \delta^{-2}$.

Let $g = (\varphi + f - 1)_+$ and $\tilde{g} = (1 - \varphi - f)_-$ We use g as a test function for (1.2) and obtain for a.e. $t \in [0,T]$,

$$\begin{aligned} 0 &\geqslant \iint (f_t + v\nabla_x f) g \, \mathrm{d}v \, \mathrm{d}x + \int \mathcal{E}(f,g) \, \mathrm{d}x, \\ &= -\frac{1}{2}\frac{d}{dt} \iint g^2 \, \mathrm{d}v \, \mathrm{d}x + \int \mathcal{E}(g,g) \, \mathrm{d}x - \int \mathcal{E}(\tilde{g},g) \, \mathrm{d}x - \int \mathcal{E}(\varphi,g) \, \mathrm{d}x + \int g(\partial_t \varphi + v \cdot \nabla_x \varphi) \, \mathrm{d}v \, \mathrm{d}x. \end{aligned}$$

We used the fact that $\nabla_x(g^2) = 2g(-\nabla_x \varphi + \nabla_x f)$. Remarking that $\mathcal{E}(\tilde{g}, g) \leqslant 0$ and using (1.3) and (5.4) from Corollary 5.3 yields for any $t_0 \in [0,T]$,

$$\begin{aligned} &\frac{1}{2} \iint g^2(t_0,x,v) \, \mathrm{d}v \, \mathrm{d}x + \lambda \int_0^{t_0} \int \|g\|^2_{H^s} \, \mathrm{d}x \, \mathrm{d}t \\ &\leqslant \iint g^2(0,x,v) \, \mathrm{d}v \, \mathrm{d}x + \varepsilon \int_0^{t_0} \int \|g\|^2_{H^s} \, \mathrm{d}x \, \mathrm{d}t + C\varepsilon^{-1} [\varphi]^2_{C^1} |\{g > 0\} \cap \{0 \leqslant t \leqslant t_0\}| + C\|\varphi\|_{C^2} \|g\|_{L^1} + C\|g\|^2_{L^2} \\ &\qquad + \int_0^{t_0} \iint g(\partial_t \varphi + v \cdot \nabla_x \varphi) \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

Recall that $\|\varphi\|_{C^1} \lesssim \delta^{-1}$ and $\|\varphi\|_{C^2} \lesssim \delta^{-2}$. Also $g(t,x,v) \in [0,1]$ for all (t,x,v), therefore $\|g\|_{L^1}$ and $\|g\|^2_{L^2}$ are both bounded by $|\{g > 0\}| \leqslant |\{f > 0\} \cap [0,T] \times B_{R^{1+2s}} \times B_R|$. Therefore, taking supremum in t_0,

$$\begin{aligned} &\sup_{t\in[0,T]} \frac{1}{2} \iint g^2(t,x,v) \, \mathrm{d}v \, \mathrm{d}x + \int_0^T \int \|g\|^2_{H^s} \, \mathrm{d}x \, \mathrm{d}t \\ &\leqslant \iint g^2(0,x,v) \, \mathrm{d}v \, \mathrm{d}x + \varepsilon \int_0^T \int \|g\|^2_{H^s} \, \mathrm{d}x \, \mathrm{d}t + \left(\varepsilon^{-1}\delta^{-2} + \delta^{-2} + 2\right) |\{f > 0\} \cap [0,T] \times B_{R^{1+2s}} \times B_R|. \end{aligned}$$

Note that $g = f$ in $B_{(R-\delta)^{1+2s}} \times B_{R-\delta}$, $g \leqslant f$ everywhere, and $g = 0$ outside of $B_{R^{1+2s}} \times B_R$. We thus conclude the proof picking $\varepsilon > 0$ small. □

Lemma 6.4 (Local gain of integrability)**.** *Let f be a subsolution of (1.2) in $[0,T]\times B_{R^{1+2s}}\times B_R$ with $h=0$. Assume $0\leqslant f\leqslant 1$ almost everywhere in $[0,T]\times B_{R^{1+2s}}\times\mathbb{R}^d$. Assume K satisfies (1.3), (1.5), (1.6) and (1.7) with $\bar R=2R$. Then for any $\delta\in(0,1)$ and $\delta<R$,*

$$\left(\int_0^T\iint_{B_{(R-\delta)^{1+2s}}\times B_{R-\delta}} f^p\,\mathrm{d}t\,\mathrm{d}v\,\mathrm{d}x\right)^{2/p} \tag{6.7}$$
$$\leqslant \delta^{-2}\int_{B_{R^{1+2s}}\times B_R} f(0,x,v)^2\,\mathrm{d}v\,\mathrm{d}x + C\delta^{-4}\left|\{f>0\}\cap([0,T]\times B_{R^{1+2s}}\times B_R)\right|$$

where $p>2$ is some universal constant (explicit).

Remark 6.5. The exponents in the factors δ^{-2} and δ^{-4} are most certainly not optimal. This is not important for the rest of our proof.

Proof. Let us start by the following simple observation. Wherever $f(t,x,v)=0$, we have $f_t+v\cdot\nabla_x f=0$ (a.e.) and $L_v f\geqslant 0$. In particular, the following equation also holds and contains slightly more information than (1.2).

$$f_t+v\cdot\nabla_x f-L_v f\leqslant -(L_v f)\chi_{\{f=0\}} = -\left(\int_{\mathbb{R}^d} f(v')K(v,v')\,\mathrm{d}v'\right)\chi_{\{f=0\}}. \tag{6.8}$$

Let us call

$$N := \delta^{-2}\int_{B_{R^{1+2s}}\times B_R} f(0,x,v)^2\,\mathrm{d}v\,\mathrm{d}x + C\delta^{-4}\left|\{f>0\}\cap([0,T]\times B_{R^{1+2s}}\times B_R)\right|.$$

From Lemma 6.2, we know that

$$\int_0^T\int_{B_{(R-\delta/2)^{1+2s}}}\|f\|^2_{H^s(B_{R-\delta/2})}\,\mathrm{d}x\,\mathrm{d}t\leqslant\delta^2 N.$$

Let $\varphi:\mathbb{R}^{2d}\to[0,1]$ be C^∞, supported in $B_{(R-\delta/2)^{1+2s}}\times B_{R-\delta/2}$, so that $\varphi=1$ in $B_{(R-\delta)^{1+2s}}\times B_{R-\delta}$. It is not hard to check that we can construct such φ with $\|D\varphi\|_{L^\infty}\lesssim\delta^{-1}$ and $\|D^2\varphi\|_{L^\infty}\lesssim\delta^{-2}$.

Let us analyse what equation the funtion $g=\varphi f$ satisfies. Combining Corollaries 5.4 and 5.5 with (6.8), we have

$$[\partial_t+v\cdot\nabla_x-\tilde L_v]g\leqslant f(v\cdot\nabla_x\varphi)-\varphi(L_v f)\chi_{\{f=0\}}-h_1-h_2-(-\Delta)^{s/2}h_3 \qquad \text{in } [0,T]\times B_{(R-\delta/2)^{1+2s}}\times B_{R-\delta/2}.$$

We want to verify that the right hand side belongs to $L^2([0,T]\times\mathbb{R}^d,H^{-s}(\mathbb{R}^d))$ with norm bounded above by N.

Following the proofs of Lemmas 4.10 and 4.11 and Corollaries 5.4 and 5.5, we have

$$h_1=\int_{\mathbb{R}^d\setminus B_{\delta/2}(v)}\varphi(v)f(v)[K(v,v')-\tilde K(v,v')]+f(v')(\varphi(v')K(v,v')-\varphi(v)\tilde K(v,v'))\,\mathrm{d}v'.$$

Therefore, at the points in $\{f=0\}$ we have

$$-\varphi(L_v f)\chi_{\{f=0\}}-h_1\leqslant\int_{\mathbb{R}^d\setminus B_{\delta/2}(v)} f(v')\left((\varphi(v)-1)K(v,v')-\varphi(v')\tilde K(v,v')\right)\,\mathrm{d}v'\leqslant 0.$$

This allows us to simplify the equation to

$$[\partial_t+v\cdot\nabla_x-\tilde L x_v]g\leqslant f(v\cdot\nabla_x\varphi)-h_1\chi_{\{f>0\}}-h_2-(-\Delta)^{s/2}h_3 \qquad \text{in } [0,T]\times B_{(R-\delta/2)^{1+2s}}\times B_{R-\delta/2}. \tag{6.9}$$

Corollaries 5.4 and 5.5 tell us that

$$\|h_2\|_{L^2},\|h_3\|_{L^2([0,T]\times\mathbb{R}^d,L^2(\mathbb{R}^d))}\lesssim\delta^{-2}\|f\|_{L^2([0,T]\times B_{R^{1+2s}},H^s(B_R))}\leqslant N.$$

Since $(v\cdot\nabla_x\varphi)$ is bounded and supported in $B_{(R-\delta/2)^{1+2s}}\times B_{R-\delta/2}$, and $0\leqslant f\leqslant 1$, we clearly have $\|f(v\cdot\nabla_x\varphi)\|_{L^2}\leqslant N$. Likewise $\|h_1\chi_{\{f>0\}}\|_{L^2}\leqslant N$.

We conclude the proof applying Lemma 6.1 to (6.9). □

6.2. **De Giorgi's iteration.** This subsection is devoted to the proof of the following lemma.

Lemma 6.6 (First lemma of De Giorgi). *Let $\tilde{Q} = [-\tau, 0] \times B_{R_1^{1+2s}} \times B_{R_1}$ and $\hat{Q} = [-\hat{\tau}, 0] \times B_{R_2^{1+2s}} \times B_{R_2}$ with $0 < \tilde{\tau} < \hat{\tau}$ and $R_1 \leqslant R_2$. There exists $\varepsilon_0 > 0$ (depending on τ, $\hat{\tau}$, R_1, R_2, dimension, s, λ and Λ) such that for all supersolution f of $f_t + v \cdot \nabla_x f - L_v f \geqslant 0$ in $\hat{Q}$ such that $f \geqslant 0$ almost everywhere in $[-\hat{\tau}, 0] \times \mathbb{R}^{2d}$ and*

$$\int_{\hat{Q}} (2 - f)_+^2 \, \mathrm{d}t \, \mathrm{d}v \, \mathrm{d}x \leqslant \varepsilon_0, \tag{6.10}$$

we have

$$f \geqslant 1 \quad a.e. \ in \ \tilde{Q}.$$

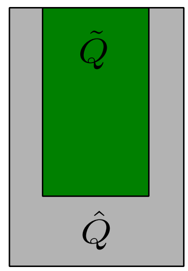

Figure 3. The cylinders $\hat{Q}$ and $\tilde{Q}$

After Lemma 6.4, the proof of Lemma 6.6 follows by the relatively standard De Giorgi's iteration.

Proof of Lemma 6.6. Let us consider the sequences

$$\ell_k = 1 + 2^{-k}, \qquad r_k = R_1 + (R_2 - R_1) 2^{-k}, \qquad t_k = \tau - 2^{-k}(\hat{\tau} - \tau).$$

We define

$$A_k := \int_{t_k}^0 \iint_{B_{r_k^{1+2s}} \times B_{r_k}} (\ell_k - f)_+^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t.$$

The assumption (6.10) tells us that $A_0 \leqslant \varepsilon_0$. The strategy of De Giorgi's iteration is to prove that $A_k \to 0$ as $k \to \infty$ provided that ε_0 is sufficiently small. The conclusion clearly follows from that.

In order to prove that A_k converges towards 0, we are going to prove that

$$A_{k+1} \leqslant C 2^{Ck} A_k^{1+\varepsilon} \tag{6.11}$$

for some $\varepsilon > 0$.

We first pick $t_{k+\frac{1}{2}} \in [t_k, t_{k+1}]$ such that

$$\iint_{B_{r_k^{1+2s}} \times B_{r_k}} (\ell_k - f(t_{k+\frac{1}{2}}, x, v))_+^2 \, \mathrm{d}v \, \mathrm{d}x \leqslant \frac{1}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} \iint_{B_{r_k^{1+2s}} \times B_{r_k}} (\ell_k - f)_+^2 \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \leqslant C 2^k A_k.$$

Note that $(\ell_{k+1} - f)_+$ is a subsolution with values in $[0, 2]$ (in particular half of it takes values in $[0, 1]$). We then apply Lemma 6.4, and obtain the following inequality (note that $\ell_{k+1} \leqslant \ell_k$)

$$\left(\int_{t_{k+\frac{1}{2}}}^0 \int_{B_{r_{k+1}^{1+2s}} \times B_{r_{k+1}}} (\ell_{k+1} - f)_+^p \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t \right)^{2/p} \leqslant C 4^k A_k + C 16^k |\{f < \ell_{k+1}\} \cap ([t_{k+1/2}, 0] \times B_{r_k^{1+2s}} \times B_{r_k})|. \tag{6.12}$$

We now estimate $|\{f < \ell_{k+1}\} \cap ([t_{k+1/2}, 0] \times B_{r_k} \times B_{r_k})|$ in terms of A_k. We use Chebyshev inequality and get

$$\begin{aligned} |\{f < \ell_{k+1}\} \cap ([t_{k+1/2}, 0] \times B_{r_k^{1+2s}} \times B_{r_k})| &= |\{(\ell_k - f)_+ > 2^{-k-1}\} \cap ([t_{k+1/2}, 0] \times B_{r_k^{1+2s}} \times B_{r_k})|, \\ &\leqslant 16^{k+1} A_k. \end{aligned} \tag{6.13}$$

Combining (6.12) and (6.13), we get

$$\left(\int_{t_{k+1}}^0 \iint_{B_{r_{k+1}^{1+2s}} \times B_{r_{k+1}}} (\ell_{k+1} - f)_+^p \, \mathrm{d}t \, \mathrm{d}v \, \mathrm{d}x\right)^{\frac{2}{p}} \leqslant C 2^{8k} A_k$$

(we used that $t_{k+\frac{1}{2}} \leqslant t_{k+1} \leqslant 0$). We can now combine this estimate with (6.13) and get

$$\begin{aligned} A_{k+1} &\leqslant \left(\int_{t_{k+1}}^0 \iint_{B_{r_{k+1}^{1+2s}} \times B_{r_{k+1}}} (\ell_{k+1} - f)_+^p \, \mathrm{d}t \, \mathrm{d}v \, \mathrm{d}x\right)^{\frac{2}{p}} |\{f < \ell_{k+1}\} \cap ([t_{k+1}, 0] \times B_{r_{k+1}^{1+2s}} \times B_{r_{k+1}})|^{1-\frac{2}{p}} \\ &\leqslant C 2^{8k} A_k^{1+\frac{2-p}{p}}. \end{aligned}$$

This yields (6.11) with $\varepsilon = \frac{2-p}{p} > 0$. The proof is now complete. □

7. Barrier functions for $s < 1/2$

A remarkable difference between the range $s < 1/2$ and $s \geqslant 1/2$ is that, in the former, the integral expression in the definition of $L_v f(v)$ is computable pointwise for all smooth functions f provided that K satisfies the first line in (1.5). The reason for this is simply that from the Lipschitz continuity of f we get

$$\int_{B_{2r}(v) \setminus B_r(v)} |f(v) - f(v')| K(v, v') \, \mathrm{d}v' \leqslant r \|f\|_{\mathrm{Lip}} \int_{B_{2r(v)} \setminus B_r(v)} K(v, v') \, \mathrm{d}v' \leqslant \Lambda \|f\|_{\mathrm{Lip}} r^{1-2s}. \tag{7.1}$$

This is summable for $r = 2^{-k}$ as k ranges accross the natural numbers when $s < 1/2$.

If we assumed further than K is symmetric in the *non-divergence* sense $K(v, v+h) = K(v, v-h)$, then the same analysis as above would hold for $s \in (0,1)$ and $f \in C^{1,1}$ (instead of $f \in$ Lip) and the results in this section could be extended to the full range $s \in (0,1)$. Note that the Boltzmann kernel satisfies this symmetry, but we do not make that assumption in Theorems 1.6 and 1.5.

We build barrier functions using crucially the assumption (1.4).

Lemma 7.1 (Existence of barriers)**.** *For any $r > 0$, $R > 0$, $\tau > 0$ and $T > 0$, there exist constants $\theta > 0$ and $R_1 > 0$, and a function $\varphi : [0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d \to [0, 1]$ such that*

- *we have $\varphi \in C^{1,1}([0, \infty) \times \mathbb{R}^{2d})$; moreover, φ is smooth in the open set $\{\varphi > 0\}$;*
- *for any kernel $K(t, x, v)$ that satisfies (1.5) and (1.4) with $\bar{R} = R_1$, and all $(t, x, v) \in \Omega \subset [0, \infty) \times \mathbb{R}^{2d}$, we have*
$$\varphi_t + v \cdot \nabla_x \varphi - L_v \varphi \leqslant 0 \qquad \text{in } \Omega;$$
- *at the initial time, the support of $\varphi(0, \cdot, \cdot)$ is contained in $B_{r^{1+2s}} \times B_r$;*
- *we have the following lower bound: $\varphi(t, x, v) \geqslant \theta$ if $t \in [\tau, T]$, $x \in B_{R^{1+2s}}$ and $v \in B_R$;*
- *the function $\varphi(t, x, v)$ vanishes if $t \in [0, T]$ and $(x, v) \notin B_{R_1^{1+2s}} \times B_{R_1}$.*

The function φ depends on r, R, τ, T, dimension d, λ, Λ and s (which should be in $(0,1)$). The radius R_1 depends on r, R, τ, T, dimension d, and s (but not λ and Λ).

Lemma 7.1 will be proved by the end of this section. We remark that we only use the first line in (1.5).

It is convenient to define the extremal (Pucci type) operators which correspond to the supremum and infimum of all values of $L_v f(v)$ for any kernel K satisfying (1.5) and (1.4).

Let us say that a nonnegative kernel $K : \mathbb{R}^d \to [0, +\infty]$ belongs to the class $\mathcal{K}_0$ if

$$K \in \mathcal{K}_0 \Leftrightarrow \begin{cases} \int_{\mathbb{R}^d \setminus B_r} K(w) \, \mathrm{d}w \leqslant \Lambda r^{-2s}, \\ \inf_{|e|=1} \int_{B_r} (w \cdot e)_+^2 K(w) \, \mathrm{d}w \geqslant \lambda r^{2-2s}. \end{cases}$$

Correspondingly, we define the extremal operators $\mathcal{M}^+$ and $\mathcal{M}^-$.

$$\mathcal{M}^+ f(v) = \sup \left\{ \int_{\mathbb{R}^d} (f(v') - f(v)) K(v' - v) \, \mathrm{d}v' : K \in \mathcal{K}_0 \right\},$$

$$\mathcal{M}^- f(v) = \inf \left\{ \int_{\mathbb{R}^d} (f(v') - f(v)) K(v' - v) \, \mathrm{d}v' : K \in \mathcal{K}_0 \right\}.$$

Note that the infimum and supremum are taken only with respect to a family of translation invariant linear operators, whose kernels depend only on $v'-v$. However, the kernel which achieves the extremal value will be different at every value of v. Therefore, effectively, the operators $\mathcal{M}^+ f$ and $\mathcal{M}^- f$ correspond to the supremum and infimum value of $L_v f$ for all kernels $K(v,v')$ satisfying the first line in (1.5) and (1.4).

We start by pointing out a simple continuity property of $\mathcal{M}^+$ and $\mathcal{M}^-$.

Lemma 7.2. *Let f and g be two bounded functions that are Lipschitz in $B_r(v)$, then*

$$|\mathcal{M}^- f(v) - \mathcal{M}^- g(v)| \leqslant C_r \left(\|f-g\|_{L^\infty(\mathbb{R}^d)} + \|f-g\|_{\mathrm{Lip}(B_r(v))}\right).$$

The same holds for $\mathcal{M}^+$.

Remark 7.3. Note that the norm $\|f-g\|_{L^\infty(\mathbb{R}^d)}$ can be weighted. Indeed, the same estimate holds with $\|(1+|v|)^{-\sigma}(f(v)-g(v))\|_{L^\infty(\mathbb{R}^d)}$ instead provided that $\sigma < 2s$.

Proof. It is enough to notice that each linear operator in the infimum of the definition of $\mathcal{M}^-$ satisfies the continuity estimate. □

Corollary 7.4. *If f_n is a sequence of functions so that $f_n \to f$ uniformly in $\mathbb{R}^d$ and $f_n \to f$ in $\mathrm{Lip}(\Omega)$, then $\mathcal{M}^+ f_n$ and $\mathcal{M}^- f_n$ converge to $\mathcal{M}^+ f$ and $\mathcal{M}^- f$ uniformly in compact sets of Ω.*

The following is perhaps not strictly a corollary of Lemma 7.2, since it requires a slightly sharper analysis (but standard and elementary).

Corollary 7.5. *Let f be a bounded continuous function in $\mathbb{R}^d$ and Lipchitz in some open set Ω. The functions $\mathcal{M}^- f$ and $\mathcal{M}^+ f$ are continuous in Ω.*

Since the operators $\mathcal{M}^+$ and $\mathcal{M}^-$ are a supremum and infimum of linear ones, then they are also sub- and super-additive respectively. That means that for any f and g,

$$\mathcal{M}^-(f+g)(v) \geqslant \mathcal{M}^- f(v) + \mathcal{M}^- g(v), \qquad \mathcal{M}^+(f+g)(v) \leqslant \mathcal{M}^+ f(v) + \mathcal{M}^+ g(v).$$

Lemma 7.6 (The function φ_1)**.** *Let $\varphi_1 : \mathbb{R}^d \to [0,1]$ be a nonnegative, radially symmetric function, so that*

- *$\{\varphi_1 > 0\} = B_1$, $\varphi_1 \in C^2(B_1)$, $\varphi_1 = 1$ in $B_{1/2}$, and $v \cdot \nabla\varphi_1(v) \leqslant 0$;*
- *$\varphi_1 \in C^2(B_1)$ and $\varphi \in C^{1,1}(\mathbb{R}^d)$; more precisely, there is a discontinuity of $D^2\varphi_1$ on ∂B_1 so that $\lim_{r\to 1^-} D^2\varphi_1(re) = e \otimes e$ for any $|e| = 1$.*

Then, there exist two constants $\delta > 0$ and $\theta > 0$ so that

$$\mathcal{M}^-\varphi_1(v) \geqslant \theta \text{ for any } v \in B_1 \text{ so that } \varphi_1(v) < \delta.$$

Remark 7.7. We can choose any function $\varphi_1(x) = \Psi(|x|)$ with Ψ non-increasing in $\mathbb{R}$, positive and C^2 in $[0,1]$, supported in $[0,1]$, $\Psi \equiv 1$ in $[0,1/2]$, and $\Psi'(1) = 0$ and $\Psi''(1) = 1$.

Proof. Since $\mathcal{M}^-\varphi_1$ is continuous in B_1, it is enough to prove that $\mathcal{M}^-\varphi_1$ is strictly positive on ∂B_1. From radial symmetry, we are left to show that $\mathcal{M}^-\varphi_1(e) > 0$ for $e = (1,0,\dots,0)$.

Let $\varepsilon > 0$. From the super-additivity of $\mathcal{M}^-$, we have

$$\mathcal{M}^-\varphi_1(e) \geqslant \mathcal{M}^-(\varphi_1\chi_{B_\varepsilon(e)})(e) + \mathcal{M}^-(\varphi_1\chi_{\mathbb{R}^d\setminus B_\varepsilon(e)})(e).$$

For any $K \in \mathcal{K}_0$, since $K \geqslant 0$ and $\varphi_1 \geqslant 0$, we have

$$\int_{\mathbb{R}^d} ((\varphi_1\chi_{\mathbb{R}^d\setminus B_\varepsilon(e)})(v') - (\varphi_1\chi_{\mathbb{R}^d\setminus B_\varepsilon(e)})(e)))K(v'-e)\,\mathrm{d}v' = \int_{\mathbb{R}^d\setminus B_\varepsilon(e)} \varphi_1(v')K(v'-e)\,\mathrm{d}v' \geqslant 0.$$

Therefore $\mathcal{M}^-(\varphi_1\chi_{\mathbb{R}^d\setminus B_\varepsilon(e)})(e) \geqslant 0$.

We now show that $\mathcal{M}^-(\varphi_1\chi_{B_\varepsilon(e)})(e)$ is bounded below for $\varepsilon > 0$ small. Essentially this follows because $\varphi_1(v')$ is approximately $((v'-e)\cdot(-e))_+^2$ in $B_\varepsilon(e)$.

Indeed, let the scaled function φ_ε be

$$\varphi_\varepsilon(w) = \begin{cases} \varepsilon^{-2}\varphi_1(e+\varepsilon w) & \text{if } |w| < 1, \\ 0 & \text{if } |w| \geqslant 1. \end{cases}$$

Thus

$$\mathcal{M}^-(\varphi_1\chi_{B_\varepsilon(e)})(e) = \varepsilon^{2-2s}\mathcal{M}^-\varphi_\varepsilon(0).$$

From the definition of φ_1 ,we know that

$$\varphi_\varepsilon(w) \to q(w) := \begin{cases} (-w\cdot e)_+^2 & \text{for } |w|<1, \\ 0 & \text{for } |w|\geqslant 1, \end{cases}$$

uniformly in $\mathbb{R}^d$ and also in $\mathrm{Lip}(B_{1/2})$. Therefore, using Corollary 7.4,

$$\mathcal{M}^-\varphi_\varepsilon(0) \to \mathcal{M}^- q(0) \geqslant \lambda.$$

The last inequality comes from the non-degeneracy condition (1.4).

Therefore, choosing ε sufficiently small,

$$\mathcal{M}^-\varphi_1(e) \geqslant \mathcal{M}^-(\varphi_1\chi_{B_\varepsilon(e)})(e) \geqslant \frac{\lambda}{2}\varepsilon^{2-2s} > 0.$$

This concludes the proof. □

Lemma 7.8 (The function φ_2)**.** *Let $t_0 > (0,1)$ be arbitrary and φ_1 be a function as in Lemma 7.6. Let $A = \left(5+\frac{1}{2s}\right)$. Let us define the function $\varphi_2 : \mathbb{R}^{2d} \to [0,1]$ to be*

$$\varphi_2(x,v) := \varphi_1(x)\varphi_1(v-Ax).$$

There exists a constant $\delta > 0$ so that if at some point (x,v), $\varphi_2(x,v) < \delta$, then

$$\left(-1-\frac{1}{2s}\right) x\cdot\nabla_x\varphi_2 - \frac{1}{2s} v\cdot\nabla_v\varphi_2 + t_0\left(v\cdot\nabla_x\varphi_2 - \mathcal{M}_v^-\varphi_2\right) \leqslant 0. \tag{7.2}$$

Proof. Since $\min\varphi_2 = 0$, then $\mathcal{M}_v^-\varphi_2 \geqslant 0$ wherever $\varphi_2 = 0$. Thus, the inequality is trivial wherever $\varphi_2 = 0$. We are left to verify it at points where $\varphi_2 > 0$. Note that this is a bounded set since there $|x| \leqslant 1$ and $|v| \leqslant A|x| + 1 \leqslant A + 1$.

We expand the left hand of (7.2), in terms of φ_1, x and v, as the sum of two terms $T_1 + T_2$, where

$$T_1 = \varphi_1(x)\left\{\nabla\varphi_1(v-Ax)\cdot\left(A\left(1+\frac{1}{2s}\right)x - \left(t_0A+\frac{1}{2s}\right)v\right) - t_0\mathcal{M}^-\varphi_1(v-Ax)\right\},$$
$$T_2 = \varphi_1(v-Ax)\nabla\varphi_1(x)\cdot\left\{-\left(1+\frac{1}{2s}\right)x + t_0 v\right\}.$$

We first claim that

$$\text{there exist } \delta_1 > 0 \text{ such that } T_1 \leqslant 0 \text{ if } \varphi_1(v-Ax) < \delta_1. \tag{7.3}$$

Using Lemma 7.6, we pick δ_1 sufficiently small so that $\mathcal{M}^-\varphi_1(v-Ax) \geqslant \theta$ whenever $\varphi_1(v-Ax) < \delta_1$. Thanks to the continuity of $\nabla\varphi_1$, we pick δ_1 smaller if necessary so that whenever $\varphi_1(v-Ax) < \delta_1$,

$$\nabla\varphi_1(v-Ax)\cdot\left(A\left(1+\frac{1}{2s}\right)x - \left(t_0A+\frac{1}{2s}\right)v\right) - t_0\mathcal{M}^-\varphi_1(v-Ax) < -\frac{t_0\theta}{2}.$$

Therefore, we have

$$T_1 \leqslant -\frac{t_0\theta\varphi_1(x)}{2} \text{ whenever } \varphi_1(v-Ax) < \delta_1.$$

In particular, (7.3) holds true.

We next claim that

$$\text{there exist } \delta_2 > 0 \text{ such that } T_2 \text{ if } \varphi_1(x) < \delta_2. \tag{7.4}$$

Because of the second derivative of φ_1 of ∂B_1, we have the following expansion

$$\nabla\varphi_1(x) = -(1-|x|)\frac{x}{|x|} + O((1-|x|)^2).$$

Whenever $\varphi_1(v-Ax) > 0$, also $v \in B_1(Ax)$, and therefore

$$\nabla\varphi_1(x)\cdot\left\{-\left(1+\frac{1}{2s}\right)x + v\right\} \leqslant (1-|x|)(-4|x|+1) + C(1-|x|)^2 < -(1-|x|) + C(1-|x|)^2.$$

Thus,

$$T_2 \leqslant -\varphi_1(v-Ax)(1-|x|)/2 \leqslant 0 \text{ whenever } \varphi_1(x) < \delta_2$$

and δ_2 is sufficiently small. In particular, (7.4) holds true.

In view of (7.3) and (7.4), $T_1 + T_2 \leqslant 0$ if $\varphi_1(v - Ax) < \delta_1$ and $\varphi_1(x) < \delta_2$.

Let us analyse the case $\varphi_1(x) \geqslant \delta_2$; in this case consider $\varphi_1(v - Ax) < \delta_{11} < \delta_1$ so that

$$T_1 + T_2 \leqslant -t_0\delta_2\theta/2 + C\delta_{11}.$$

Picking δ_{11} sufficiently small (depending on the previous choice of δ_2), we assure $T_1 + T_2 < 0$ in this case.

We are left with the case $\varphi_1(v - Ax) \geqslant \delta_1$. In this case we have for $\varphi_1(x) < \delta_{21}$,

$$\begin{aligned} T_1 + T_2 &\leqslant C\varphi_1(x) - \varphi_1(v - Ax)(1 - |x|)/2, \\ &\leqslant C\delta_{21} - \delta_1(1 - |x|)/2 < 0, \end{aligned}$$

provided $|x|$ is sufficiently close to 1, which follows if $\varphi_1(x) < \delta_{21} < \delta_2$ with δ_{21} sufficiently small.

Finally, we finish the proof picking $\delta = \delta_{11}\delta_{21}$ to ensure that at least one of the three cases above holds. □

Lemma 7.9 (The function φ_3). *Let φ_2 be the function from Lemma 7.8 and $t_0 > 0$. The function $\varphi_3(t, x, v)$ given by*

$$\varphi_3(t, x, v) = \frac{t_0^p}{(t + t_0)^p}\varphi_2\left(\left(\frac{t_0}{t + t_0}\right)^{1+\frac{1}{2s}} x, \left(\frac{t_0}{t + t_0}\right)^{\frac{1}{2s}} v\right),$$

is a subsolution of the equation

$$\partial_t\varphi_3 + v \cdot \nabla_x\varphi_3 - \mathcal{M}^-\varphi_3 \leqslant 0,$$

provided that p is sufficiently large (depending on φ_1, λ, Λ, s and d, but not t_0).

Proof. We write the equation in terms of φ_2. We have

$$\begin{aligned} \partial_t\varphi_3 + v \cdot \nabla_x\varphi_3 - \mathcal{M}^-\varphi_3 = \frac{t_0^p}{(t + t_0)^{p+1}}\Big\{ &- p\varphi_2(X, V) \\ &+ \left(-1 - \frac{1}{2s}\right) X \cdot \nabla_x\varphi_2(X, V) - \frac{1}{2s}V \cdot \nabla_v\varphi_2(X, V) \\ &+ t_0 V \cdot \nabla_x\varphi_2(X, V) - t_0\mathcal{M}_v^-\varphi_2(X, V)\Big\}, \end{aligned}$$

where $X = (t_0/(t + t_0))^{1+\frac{1}{2s}}x$ and $V = (t_0/(t + t_0))^{\frac{1}{2s}}v$.

Let $\delta > 0$ be as in Lemma 7.8, so that the right hand side is non-positive when $\varphi_2 < \delta$. We choose p large so that the term $p\varphi_2 \geqslant p\delta$ is larger than all the others terms when $\varphi_2 \geqslant \delta$. Thus, the right hand side is never positive. □

Proof of Lemma 7.1. Note that $\varphi_3(0, x, v) = \varphi_2(x, v)$, where φ_2 and φ_3 are the functions in Lemmas 7.8 and 7.9 respectively. Note that these fuctions depends on the choice of t_0 which will be made below. Also, the value of p depends on t_0. The function φ_2 is supported in $B_1 \times B_{A+1}$. We must rescale φ_3 in order to obtain a function so that $\varphi(0, x, v)$ is supported in $B_r \times B_r$. We pick $\rho > 0$ small and let

$$\varphi(t, x, v) = \varphi_3(\rho^{-2s}t, \rho^{-2s-1}x, \rho^{-1}v),$$

so that $\rho(A + 1) \leqslant r$. This ensures the first three items in Lemma 7.1. Indeed, the function φ satisfies

$$\varphi_t + v \cdot \nabla_x\varphi - \mathcal{M}^-\varphi \leqslant 0.$$

In particular, also

$$\varphi_t + v \cdot \nabla_x\varphi - L_v\varphi \leqslant 0,$$

since $L_v\varphi \geqslant \mathcal{M}^-\varphi$ in Ω.

In order to obtain the lower bound in $[\tau, T] \times B_R$, we are going to choose the parameter t_0 accordingly. Note that the value of t_0 does not affect $\varphi(0, x, v)$.

From the construction of φ_1 and φ_2, we have $\varphi_2(x, v) = 1$ whenever $|x| < \frac{1}{4A}$ and $|v| < 1/4$. Picking t_0 sufficiently small, for $(t, x, v) \in [\tau, T] \times B_{R^{1+2s}} \times B_R$, we have

$$\begin{aligned} \rho^{-2s-1}\left(\frac{t_0}{t + t_0}\right)^{1+\frac{1}{2s}} |x| &\leqslant \rho^{-2s-1}\left(\frac{t_0}{\tau}\right)^{1+\frac{1}{2s}} R^{1+2s} < \frac{1}{4A}, \\ \rho^{-1}\left(\frac{t_0}{t + t_0}\right)^{\frac{1}{2s}} |v| &\leqslant \rho^{-1}\left(\frac{t_0}{\tau}\right)^{\frac{1}{2s}} R < \frac{1}{4}. \end{aligned}$$

Therefore, when $(t,x,v) \in [\tau, T] \times B_R \times B_R$, we have

$$\varphi(t,x,v) = \frac{t_0^p}{(\rho^{-2s}t + t_0)^p} \geqslant \frac{t_0^p}{(\rho^{-2s}T + t_0)^p} =: \theta > 0.$$

This justifies the fourth item in Lemma 7.1.

Finally, for the last item, we just pick R_1 sufficiently large. The function φ_2 is supported in $B_1 \times B_{1+A}$. Depending on our choices of t_0 and ρ above, the function $\varphi(t,\cdot,\cdot)$ is supported inside $B_{R_1} \times B_{R_1}$ for all $t \in [0,T]$. This achieves the construction of the barrier.

Note that the only parameters in this construction that depend on λ and Λ are p and θ. □

8. The intermediate-value lemma for $s \geqslant \frac{1}{2}$

This section is devoted to the statement and proof of a version of De Giorgi's isoperimetric lemma in the case $s \geqslant \frac{1}{2}$. It is inspired by the compactness method in [36]. However, unlike [36], we do not use averaging lemmas. Instead, the analysis of the fractional Kolmogorov equation plays a critical role.

The first lemma of this section concerns a supersolution of the equation (1.2). In this case we add a nonnegative measure to the right hand side in order to have an exact solution. The purpose of this lemma is to provide a basic control of the total measure that we add.

Lemma 8.1 (A priori estimate on a nonnegative measure)**.** *Let $Q = [0,T] \times B_{R^{1+2s}} \times B_R$, $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$ be supported in Q. Assume also that*

$$f_t + v \cdot \nabla_x f + (-\Delta)^s f \geqslant h \qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathbb{R}^d,$$

for some $h \in L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$. Then

$$f_t + v \cdot \nabla_x f + (-\Delta)^s f = \tilde{h} + \mu \qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathbb{R}^d,$$

where μ is a nonnegative measure supported in $[0,T] \times B_{(2R)^{1+2s}} \times B_{2R}$ such that

$$\mu(Q) \leqslant C(1 + \|h\|_{L^2_{t,x}H^{-s}_v}),$$

and $\tilde{h} = h$ in $[0,T] \times B_{(2R)^{1+2s}} \times B_{2R}$ and

$$\|\tilde{h}\|_{L^2_{t,x}H^{-s}_v} \leqslant C(1 + \|h\|_{L^2_{t,x}H^{-s}_v}).$$

Proof. Note that for $(x,v) \notin B_{R^{1+2s}} \times B_R$, $f_t + v \cdot \nabla_x f = 0$. Moreover,

$$|(-\Delta)^s f(t,x,v)| = c\left|\int_{B_R} f(t,x,w)|w-v|^{-d-2s}\,\mathrm{d}w\right| \leqslant (|v| - R)^{-d-2s}|B_R|\chi_{|x| \leqslant R}.$$

Therefore, $f_t + v \cdot \nabla_x f + (-\Delta)^s f = (-\Delta)^s f$ is an L^2 function outside of $B_{R^{1+2s}} \times B_{3R/2}$.

Let $\varphi : \mathbb{R}^{2d} \to [0,1]$ be a smooth bump function so that $\varphi = 1$ in $B_{R^{1+2s}} \times B_{3R/2}$ and $\varphi = 0$ outside of $B_{(2R)^{1+2s}} \times B_{2R}$.

We first need to justify that there is $\tilde{h} \in L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$ so that

$$\tilde{h} = h\varphi + (1-\varphi)(-\Delta)^s f.$$

We clearly have that $(1-\varphi)(-\Delta)^s f$ is in $L^2_{t,x}H^{-s}_v$. We are left to justify that $h\varphi \in L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$. This follows by duality once we observe that for every $g \in L^2_{t,x}H^s_v$, we also have $\varphi g \in H^s(\mathbb{R}^d)$.

With this definition of $\tilde{h}$, we still have

$$f_t + v \cdot \nabla_x f + (-\Delta)^s f \geqslant \tilde{h} \qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathbb{R}^d,$$

with equality for $(x,v) \notin B_{(2R)^{1+2s}} \times B_{2R}$.

Let μ be the nonnegative measure, supported in $[0,T] \times B_{(2R)^{1+2s}} \times B_{2R}$, defined by

$$\mu := f_t + v \cdot \nabla_x f + (-\Delta)^s f - \tilde{h}.$$

In order to estimate the total measure of μ, we test it against a test function which is identically one on its support. Let $\tilde\varphi = 1$ in $B_{(2R)^{1+2s}} \times B_{2R}$ and be supported in $B_{(3R)^{1+2s}} \times B_{3R}$. We have

$$\begin{aligned}\mu([0,T] \times \mathbb{R}^d \times \mathbb{R}^d) &= \int_{[0,T]\times\mathbb{R}^{2d}} \tilde\varphi \,\mathrm{d}\mu,\\ &= \int_{[0,T]\times\mathbb{R}^{2d}} \tilde\varphi \left(f_t + v\cdot\nabla_x f + (-\Delta)^s f - \tilde h\right) \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t,\\ &= \int_{\mathbb{R}^{2d}} (f(T,x,v) - f(0,x,v))\tilde\varphi(x,v)\,\mathrm{d}v\,\mathrm{d}x\\ &\quad + \int_{[0,T]\times\mathbb{R}^{2d}} \left\{[-v\cdot\nabla_x\tilde\varphi + (-\Delta)^s\tilde\varphi]\,f - \tilde\varphi\tilde h\right\} \mathrm{d}v\,\mathrm{d}x\,\mathrm{d}t \leqslant C. \qquad \square\end{aligned}$$

Lemma 8.2 (Intermediate sets for the Kolmogorov equation)**.** *Let* $s \in [1/2, 1)$*. Let* $f : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$*. Assume* f *is a supersolution of the fractional Kolmogorov equation*

$$f_t + v\cdot\nabla_x f + (-\Delta)^s f \geqslant h \qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathbb{R}^d,$$

where $h \in L^2([0,T]\times\mathbb{R}^d, H^{-s}(\mathbb{R}^d))$*. Let* $r_1 > 0$*,* $r_2 > 0$*,* $0 < r_3 < r_4$ *and* $0 < t_1 < t_2 < T$ *such that* $r_3/2 > (r_1^{1+2s} + r_2^{1+2s})/(t_2 - t_1)$*. We define*

$$\begin{aligned} Q^1 &= [0,t_1] \times B_{r_1^{1+2s}} \times B_{r_1},\\ Q^2 &= [t_2,T] \times B_{r_2^{1+2s}} \times B_{r_2},\\ Q^3 &= [0,T] \times B_{r_3^{1+2s}} \times B_{r_3},\\ Q^4 &= [0,T] \times B_{r_4^{1+2s}} \times B_{r_4}.\end{aligned}$$

Let us assume that f *is supported in* Q^4 *and* $f \in L^2([0,T]\times\mathbb{R}^d, H^s(\mathbb{R}^d)) \cap C([0,T], L^2(\mathbb{R}^d\times\mathbb{R}^d))$*. For every pair of positive numbers* δ_1, δ_2*, there exist* $\theta > 0$ *and* $\mu > 0$ *so that whenever*

$$|\{f = 1\} \cap Q^1| \geqslant \delta_1 \qquad \text{and} \qquad |\{f \leqslant \theta\} \cap Q^2| \geqslant \delta_2,$$

then

$$|\{\theta < f < 1\} \cap Q^3| \geqslant \mu.$$

Here, the constants θ *and* μ *depend on* δ_1*,* δ_2*,* $\|h\|_{L^2_{t,x}H^{-s}_v}$*,* $\|f\|_{L^2_{t,x}H^s_v}$*,* t_1*,* t_2*,* T*,* r_1*,* r_2*,* r_3*,* r_4*,* s *and* d*.*

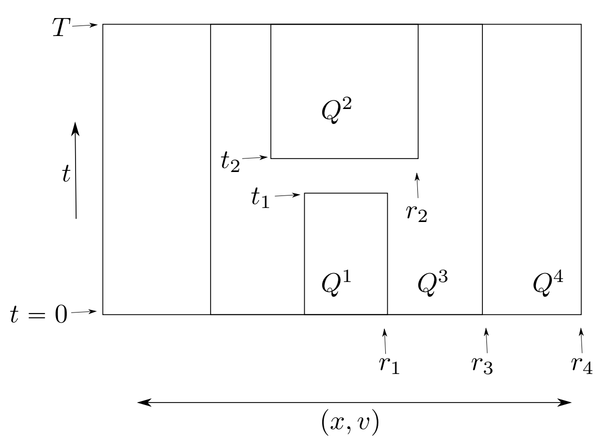

FIGURE 4. The geometric setting of Lemma 8.2.

Proof. Assume the contrary. Then, there is a sequence of functions f_i, uniformly bounded in $L^2([0,T]\times\mathbb{R}^d, H^s(\mathbb{R}^d))$, h_i uniformly bounded in $L^2([0,T]\times\mathbb{R}^d, H^{-s}(\mathbb{R}^d))$, and sequences of positive numbers $\theta_i \to 0$

and $\mu_i \to 0$ so that all hypotheses in the lemma hold, however

$$
\begin{aligned}
|\{f_i = 1\} \cap Q^1| &\geqslant \delta_1, \\
|\{f_i \leqslant \theta_i\} \cap Q^2| &\geqslant \delta_2, \\
|\{\theta_i < f_i < 1\} \cap Q^3| &< \mu_i.
\end{aligned} \tag{8.1}
$$

We will find a contradiction *by compactness*. That is, we will find a subsequence that converges and find a limit function f_∞ which only takes the values 1 and 0 in Q^3. We will derive a contradiction there provided $s \in [1/2, 1)$.

According to Lemma 8.1, there are measures μ_i, supported in $[0,T] \times B_{(2r_4)^{1+2s}} \times B_{2r_4}$, and modified right hand sides $\tilde{h}_i$ so that

$$[\partial_t + v \cdot \nabla_x + (-\Delta)^s] f_i = \tilde{h}_i + \mu_i.$$

Moreover, $\mu_i([0,T] \times \mathbb{R}^d \times \mathbb{R}^d) \leqslant C$, $\|\tilde{h}_i\|_{L^2_{t,x} H^{-s}_v} \leqslant C$.

Let us write $\tilde{h}_i = h^i_1 + (-\Delta)^{s/2} h^i_2$ for h^i_1 and h^i_1 in $L^2([0,T] \times \mathbb{R}^{2d})$, with $\|h^i_1\|_{L^2} \leqslant C$ and $\|h^i_2\|_{L^2} \leqslant C$.

Up to extracting a subsequence, we can assume that $f_i(0,\cdot,\cdot)$ converges weakly in $L^2(\mathbb{R}^{2d})$, f_i, h^i_1 and h^i_2 converge weakly in $L^2([0,T] \times \mathbb{R}^{2d})$ to f_∞, h^∞_1 and h^∞_2, and μ_i converges weakly-$*$ in the space of Radon measures $\mathbb{M}([0,T] \times \mathbb{R}^{2d})$ to some measure μ_∞.

Using the formula (2.3), we can write $f_i = T_0 f_i(0,\cdot,\cdot) + T_1 \mu_i + T_2 h^i_1 + T_3 h^i_2$. Here, the operators $T_0 : L^2(B_{r_4}) \to L^1(Q^4)$, $T_1 : \mathbb{M}([0,T] \times \mathbb{R}^{2d}) \to L^1(Q^4)$, $T_2, T_3 : L^2([0,T] \times \mathbb{R}^{2d}) \to L^2(Q^4)$ and are given by

$$
\begin{aligned}
T_0 f_0 &:= f_0 *_t J(t,\cdot,\cdot). \\
T_1 \mu &:= \int_0^t \mu(t) *_t J(t-\tau,\cdot,\cdot) \,\mathrm{d}\tau, \\
T_2 h_1 &:= \int_0^t h_1(t) *_t J(t-\tau,\cdot,\cdot) \,\mathrm{d}\tau, \\
T_3 h_2 &:= \int_0^t h_2(t) *_t (-\Delta)^{s/2} J(t-\tau,\cdot,\cdot) \,\mathrm{d}\tau.
\end{aligned}
$$

Note that T_1, T_2 and T_3 are exactly convolutions in all variables (t,x,v) with respect to the natural Lie group structure. Also T_0 is the same as T_1 applied to a singular measure concentrated on $t = 0$ with marginal density f_0.

The operators T_1, T_2 and T_3 are compact simply because they are convolutions with the L^1 functions J and $(-\Delta)^{s/2} J$. Therefore $f_i = T_0 f_i(0,\cdot,\cdot) + T_1 \mu_i + T_2 h^i_1 + T_3 h^i_2$ converges strongly in $L^1(Q^4)$ to some function f_∞. Since we have $0 \leqslant f_i \leqslant 1$, then in fact f_i converges strongly to f_∞ in $L^p(Q^4)$ for any $p \in [1,+\infty)$.

The function f_∞ solves, in the sense of distributions,

$$[\partial_t + v \cdot \nabla_x + (-\Delta)^s] f_\infty \geqslant h^\infty_1 + (-\Delta)^{s/2} h^\infty_2.$$

Moreover, since $f_i \to f_\infty$ in L^1, from (8.1) we deduce that

$$
\begin{aligned}
|\{f_\infty = 1\} \cap Q^1| &\geqslant \delta_1, \\
|\{f_\infty = 0\} \cap Q^2| &\geqslant \delta_2, \\
|\{0 < f_\infty < 1\} \cap Q^3| &= 0.
\end{aligned} \tag{8.2}
$$

Then, f_∞ only takes the values 0 and 1, almost everywhere in Q^3. Moreover, we have $\|f_\infty\|_{L^2([0,T] \times \mathbb{R}^d, H^s(\mathbb{R}^d))} \leqslant C$. Thus $f_\infty(t,x,\cdot) \in H^s(B_{r_3})$ almost everywhere in $[0,T] \times B_{r_3^{1+2s}}$. Since $s \geqslant 1/2$, this implies that $f(t,x,\cdot)$ is either constant 1 or constant 0 in B_{r_3} for $(t,x) \in [0,T] \times B_{r_3^{1+2s}}$. From this point on, we write $f_\infty(t,x) := f_\infty(t,x,v)$ provided that (t,x,v) is restricted to Q^3. Note that $(-\Delta)^s f_\infty$ is not constant in Q^3 due to the nonlocality of $(-\Delta)^s$.

Let $\varphi : \mathbb{R}^d \to [0,+\infty)$ be a smooth bump function supported in $B_{r_3/2}$, such that

$$\int_{\mathbb{R}^d} \varphi(v) \,\mathrm{d}v = 1, \qquad \int_{\mathbb{R}^d} \varphi(v)\, v \,\mathrm{d}v = 0.$$

For any $v_0 \in B_{r_3/2}$ and $(t,x) \in [0,T] \times B_{r_3}$, we have

$$f_\infty(t,x) = \int_{\mathbb{R}^d} f_\infty(t,x)\varphi(v-v_0)\,\mathrm{d}v.$$

Therefore, using the equation

$$\begin{aligned}\partial_t f_\infty(t,x) &\geqslant \int_{\mathbb{R}^d} \big[-v\cdot\nabla_x f_\infty(t,x) - (-\Delta)^s f_\infty(t,x,v) + h_1^\infty(t,x,v) + (-\Delta)^{s/2} h_2^\infty(t,x,v)\big]\varphi(v-v_0)\,\mathrm{d}v,\\ &= -v_0\cdot\nabla_x f_\infty(t,x) + \int_{\mathbb{R}^d} \{(-f_\infty(t,x,v)+h_2^\infty(t,x,v))(-\Delta)^s\varphi(v-v_0) + h_1^\infty\varphi(v-v_0)\}\,\mathrm{d}v.\end{aligned}$$

Thus, for any $v_0 \in B_{r_3/2}$, $f_\infty(t,x)$ satisfies the transport equation

$$\partial_t f_\infty + v_0\cdot\nabla_x f_\infty \geqslant H_{v_0}(t,x).$$

where H_{v_0} is the function in $L^2([0,T]\times B_{r_3^{1+2s}})$ given by

$$H_{v_0}(t,x) = \int_{\mathbb{R}^d} (-f_\infty(t,x,v)+h_2^\infty(t,x,v))(-\Delta)^s\varphi(v-v_0) + h_1^\infty\varphi(v-v_0)\,\mathrm{d}v.$$

From (8.2), we know that there exist some $\tau_1 \in [0,t_1]$ and $\tau_2 \in [t_2,T]$ so that

$$\begin{aligned}|\{x : f_\infty(\tau_1,x) = 1\}\cap B_{r_1}| &\geqslant \frac{\delta_1}{t_1},\\ |\{x : f_\infty(\tau_2,x) = 0\}\cap B_{r_2}| &\geqslant \frac{\delta_2}{T-t_2}.\end{aligned}$$

Let $S_1 = \{x : f_\infty(\tau_1,x)=1\}\cap B_{r_1^{1+2s}}$ and $S_2 = \{x : f_\infty(\tau_2,x)=0\}\cap B_{r_2^{1+2s}}$. Since

$$\|\chi_{S_1} * \chi_{-S_2}\|_{L^1} = |S_1||S_2| \geqslant \frac{\delta_1\delta_2}{t_1(T-t_2)},$$

then, there exists one vector $w_0 \in B_{r_1^{1+2s}+r_2^{1+2s}}$ such that

$$|S_1\cap(S_2-w_0)| \geqslant \frac{\delta_1\delta_2}{t_1(T-t_2)|B_{r_1^{1+2s}+r_2^{1+2s}}|} =: c_0.$$

Let $v_0 = w_0/(\tau_2-\tau_1)$. We have $|v_0| \leqslant |w_0|/(t_2-t_1) \leqslant r_3/2$.

Since the right hand side $H_{v_0} \in L^2([0,T]\times B_{r_3^{1+2s}})$, in particular, for almost all $x \in S_1\cap(S_2-w_0)$, the function $t \mapsto H_{v_0}(t, x+(t-\tau_1)v_0)$ is in $L^2(\tau_1,\tau_2)$.

Because of the transport equation that f_∞ satisfies in Q^3, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} f_\infty(t,x+(t-\tau_1)v_0) \geqslant H_{v_0}(t,x+(t-\tau_1)v_0).$$

In particular, for almost every $x \in B_{r_1}$, there is a constant $C(x) > 0$ so that

$$f_\infty(\tilde t_2, x+(\tilde t_2-\tau_1)v_0) - f_\infty(\tilde t_1, x+(\tilde t_1-\tau_1)v_0) \geqslant -C(x)(\tilde t_2-\tilde t_1)^{1/2}, \qquad \text{for any } t_1 < \tilde t_1 < \tilde t_2 < t_2.$$

However, since $f_\infty(t,x+(t-\tau_1)v_0)$ only takes the values 0 and 1, and $f_\infty(\tau_1,x) = 1$ for every $x \in S_1\cap(S_2-w_0)$, then $f_\infty(t,x+(t-\tau_1)v_0) = 1$ for every $x \in S_1\cap(S_2-w_0)$ and $t \in [\tau_1,T]$.

We arrive to a contradiction since $f_\infty(\tau_2,x+(\tau_2-\tau_1)v_0) = f_\infty(\tau_2,x+w_0) = 0$ for every $x \in S_1\cap(S_2-w_0)$. This achieves the proof. □

Lemma 8.3 (Intermediate sets for local super-solutions)**.** *Let $s \in [1/2,1)$. Let r_1, r_2, r_3, r_4, t_1, t_2, T, Q^1, Q^2, Q^3 and Q^4 be like in Lemma 8.2. Let $f : [0,T]\times B_{r_4}\times\mathbb{R}^d \to [0,+\infty)$. Assume f is a supersolution of*

$$f_t + v\cdot\nabla_x f - L_v f \geqslant 0 \qquad \text{in } Q^4.$$

For every pair of positive numbers δ_1,δ_2, there exists $\theta > 0$ and $\mu > 0$ so that whenever

$$|\{f \geqslant 1\}\cap Q^1| \geqslant \delta_1 \qquad \text{and} \qquad |\{f \leqslant \theta\}\cap Q^2| \geqslant \delta_2,$$

then

$$|\{\theta < f < 1\}\cap Q^3| \geqslant \mu.$$

Here, the constants θ and μ depend on δ_1, δ_2, t_1, t_2, T, r_1, r_2, r_3, r_4, λ, Λ, s and d.

Proof. By replacing f with $\min(f,1)$ (see Lemma A.11 in Appendix), we can assume that $0 \leqslant f \leqslant 1$ everywhere.

Let $\rho > 0$ so that $2\rho > r_4 - r_3$.

Applying Lemma 6.2 to $1-f$, we obtain that $f \in L^2([0,T] \times B_{(r_4-\rho)^{1+2s}}, H^s(B_{r_4-\rho}))$, with

$$\int_0^T \int_{B_{r_4-\rho}} \|f\|^2_{H^s(B_{r_4-\rho})} \, dx \, dt \leqslant C,$$

for some constant C depending only on r_4, ρ, d, λ, Λ and s.

Let $\varphi : \mathbb{R}^d \times \mathbb{R}^d \to [0,1]$ be a smooth bump function supported in $B_{(r_4-2\rho)^{1+2s}} \times B_{r_4-2\rho}$ and such that $\varphi = 1$ in $B_{r_3^{1+2s}} \times B_{r_3}$. We now have

$$\int_0^T \int_{\mathbb{R}^d} \|\varphi f\|^2_{H^s(\mathbb{R}^d)} \, dx \, dt \leqslant C. \tag{8.3}$$

From a direct computation, we get

$$[\partial_t + v \cdot \nabla_x - L_v](\varphi f) \geqslant (v \cdot \nabla_x \varphi) f - (L_v(\varphi f) - \varphi L_v f) \qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

The term $(v \cdot \nabla_x \varphi) f$ is bounded by one, and supported in $B_{(r_4-2\rho)^{1+2s}} \times B_{r_4-2\rho}$. The second term is a commutator, which is also bounded in $L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$ because of Lemmas 4.10 and 4.11. Let

$$h_0 := (v \cdot \nabla_x \varphi) f - (L_v[\varphi f] - \varphi L_v f) \in L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d)).$$

Now, we rewrite the equation for φf as a fractional Kolmogorov equation

$$[\partial_t + v \cdot \nabla_x + (-\Delta)^s_v](\varphi f) \geqslant h_0 + (-\Delta)^s_v(\varphi f) + L_v(\varphi f).$$

Because of (8.3), there is a function $h_1 \in L^2([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ such that $(-\Delta)^s_v(\varphi f) = (-\Delta)^{s/2}_v h_1$.

Also because of (8.3) and applying Corollary 5.2, $L_v(\varphi f)$ belongs to $H^{-s}(\mathbb{R}^d)$.

Summarizing, (φf) is a supersolution to a fractional Kolmogorov equation with a right hand side in $L^2([0,T] \times \mathbb{R}^d, H^{-s}(\mathbb{R}^d))$,

$$[\partial_t + v \cdot \nabla_x + (-\Delta)^s_v](\varphi f) \geqslant h_0 + (-\Delta)^{s/2}_v h_1 + L_v(\varphi f) \qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathbb{R}^d.$$

We finish the proof applying Lemma 8.2 with $r_4 - 2\rho$ instead of r_4. □

Lemma 8.4 (Propagation in measure)**.** *Under the same assumptions as in Lemma 8.3, For every pair of positive numbers δ_1, δ_2, there exists $\theta > 0$ so that whenever*

$$|\{f \geqslant 1\} \cap Q^1| \geqslant \delta_1 \qquad \textit{then} \qquad |\{f \leqslant \theta\} \cap Q^2| < \delta_2,$$

Here, the constant θ depends on δ_1, δ_2, t_1, t_2, T, r_1, r_2, r_3, r_4, λ, Λ, s and d.

Proof. Let $\tilde\theta$ and $\mu > 0$ be the values from Lemma 8.3. In this lemma we choose $\theta := \tilde\theta^k$ where k is the smallest integer larger than $|Q^3|/\mu$.

Assume the conclusion of the lemma was not true. Then for all values of $j = 0, 1, \dots, k-1$, the function $\tilde\theta^{-j} f$ would satisfy the hypothesis of Lemma 8.3. Therefore, for every $j = 0, 1, \dots, k-1$,

$$|\{\tilde\theta^{j+1} < f < \tilde\theta^j\} \cap Q^3| \geqslant \mu.$$

This is clearly impossible since all these are disjoint sets contained in Q^3, so their measures cannot add up to more than $|Q^3|$. □

9. The propagation lemma

We call *propagation lemma* a result that says that as soon as a (super)solution is above a large constant in most of a cylinder, then it is bounded from below by 1, say, for later times.

The difference between this propagation lemma and the first De Giorgi lemma proved in Section 6 lies in the sets of points where the estimates hold. Essentially, the propagation lemma is the result of De Giorgi's first lemma, combined with a propagation of the lower bound to later times and larger sets. This propagation is obtained using the barrier function of section 7 when $s \in (0,1/2)$ or the intermediate-value lemma from Section 8 when $s \in [1/2, 1)$.

Lemma 9.1 (Propagation lemma). *There exist $R_1 > 0$ (large, only depending on dimension and s), $\delta > 0$ (small, universal) and $M \geqslant 1$ (large, universal) such that for $\tilde T = 2^{2s}$, if f is a supersolution*

$$f_t + v \cdot \nabla_x f - L_v f \geqslant 0 \qquad \text{in } [-1, \tilde T] \times B_{R_1^{1+2s}} \times B_{R_1},$$

which is non-negative in $[-1, \tilde T] \times \mathbb{R}^{2d}$ and such that

$$|\{f > M\} \cap Q_1| \geqslant (1-\delta)|Q_1| \tag{9.1}$$

then $f \geqslant 1$ in $\tilde Q$ where $\tilde Q = [0, \tilde T] \times B_{2^{2s+1}} \times B_2$ (see Figure 5).

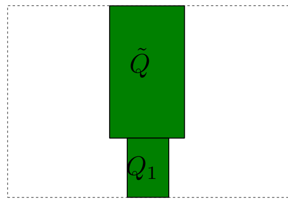

FIGURE 5. Geometric setting of the propagation Lemma 9.1.

Proof. We will prove the equivalent result that if $|\{f < 1\} \cap Q_1| < \delta$ then $f \geqslant 1/M$ in $\tilde Q$. The proof will be different depending on whether $s \in (0, 1/2)$ or $s \in [1/2, 1)$.

Let us start with the case $s \in (0, 1/2)$. We combine De Giorgi's first lemma with a barrier function.

We first apply Lemma 6.6 to $2f$, shifted in time, with $\hat Q = [-1, -1/2] \times B_1 \times B_1$ and $\tilde Q = [-3/4, -1/2] \times B_{1/2} \times B_{1/2}$. For δ suffiently small, we obtain that $f \geqslant 1/2$ in $\tilde Q$. In particular $f(-1/2, x, v) \geqslant 1/2$ for all $(x, v) \in B_{1/2} \times B_{1/2}$.

Let φ be the barrier of Lemma 7.1 with $T = 3/2$, $\tau = 1/2$ and $r = 1/2$. Lemme 7.1 also gives us the value of R_1. We apply the comparison principle to get that $f \geqslant \frac{1}{2}\varphi(t + 1/2, \cdot, \cdot)$ in $[-1/2, T] \times B_{R_1} \times B_{R_1}$ and conclude the proof. In this case $M = 2/\theta$, where $\theta > 0$ is the constant from Lemma 7.1.

For the case $s \in [1/2, 1)$, we combine the intermediate set lemma with De Giorgi's first lemma.

We apply Lemma 8.4 to $f(t-1, x, v)$, with $r_1 = 1$, $r_2 = 3$, $r_3 = 4$, $r_4 = R_1 = 5$, $t_1 = 1/2$, $t_2 = 3/4$, $T = \tilde T + 1$, arbitrary $\delta_1 = \delta > 0$ and $\delta_2 > 0$ sufficiently small. We obtain that there is a $\theta_1 > 0$ so that

$$\left| \{f > \theta_1\} \cap ([-1/4, \tilde T] \times B_{3^{1+2s}} \times B_3) \right| < \delta_2.$$

Then we apply Lemma 6.6 to $2f/\theta_1$ (again shifted in time) with $\hat Q = [-1/4, \tilde T] \times B_{3^{1+2s}} \times B_3$ and $\tilde Q = [0, \tilde T] \times B_{2^{1+2s}} \times B_2$. This concludes the proof. □

The propagation lemma implies the following corollaries.

Corollary 9.2 (Stacked propagation). *Let R_1 and δ be the constants from Lemma 9.1. Let $k \geqslant 1$, $T_k = \sum_{i=1}^k 2^{2si}$ and $R_k = 2^k R_1$. If f is a supersolution of (1.2) with $h = 0$ in $[-1, T_k] \times B_{R_k^{1+2s}} \times B_{R_k}$ and*

$$|\{f > M^k\} \cap Q_1| > (1-\delta)|Q_1|,$$

then $f \geqslant 1$ in $Q[k] := [T_{k-1}, T_k] \times B_{2^{(1+2s)k}} \times B_{2^k}$.

Proof. This is simply an iteration of Lemma 9.1. Indeed, getting $f \geqslant 1$ in $\tilde Q$ implies that $|\{\tilde f > M\} \cap \tilde Q| > (1-\delta)|\tilde Q|$ where $\tilde f = Mf$. Choosing $\tilde Q$ as the new cylinder Q in the basic propagation lemma yields that f is bounded from below by M^{-1} in a new cylinder. Iterating this estimate, we get the corollary. □

Corollary 9.3 (Propagation of minima). *Let R_1 and δ as in Lemma 9.1. Let f be a supersolution of (1.2) with $h = 0$ in $Q = [-1, 0] \times B_{R_1^{1+2s}} \times B_{R_1}$. Let $Q_r(t_0, x_0, v_0) \subset Q_1$ such that*

$$|\{f > A\} \cap Q_r(t_0, x_0, v_0)| > (1-\delta)|Q_r|.$$

Then, there exists some $p > 0$ and $c > 0$ so that

$$f(t, x, v) \gtrsim A \left(1 + \frac{t - t_0}{r^{2s}}\right)^{-p},$$

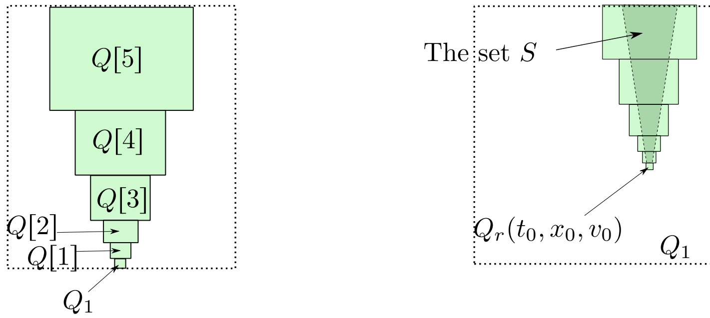

FIGURE 6. Geometric setting of Corollaries 9.2 and 9.3.

whenever (t, x, v) *belongs to the set*

$$S = S(t_0, x_0, v_0) := \Big\{(t, x, v) : t > t_0, |x - x_0 - (t - t_0)v_0| < \big((1 - 2^{-2s})(t - t_0) + r^{2s}\big)^{1+\frac{1}{2s}}, \text{ and } |v - v_0| < \big((1 - 2^{-2s})(t - t_0) + r^{2s}\big)^{\frac{1}{2s}}\Big\}.$$

Proof. Let $t_k = t_0 + \sum_{i=1}^{k} (2^i r)^{2s}$, $r_k = (2^k r)$ and

$$\tilde{Q}[k] := Q_{r_k}(t_k, x_0 + t_k v_0, v_0).$$

The change of variables $(t, x, v) \mapsto (r^{2s}(t - t_0), r^{1+2s}(x - x_0 - (t - t_0)v_0), r(v - v_0))$, which preserves the equation, transforms the cylinder Q_1 into $Q_r(t_0, x_0, v_0)$ and the cylinders $Q[k]$ of Corollary 9.2 into $\tilde{Q}[k]$.

We can easily check that $S \subset \bigcup \tilde{Q}[k]$. Corollary 9.2 tells us (after the change of variables above) that $f \geqslant A/M^k$ in $\tilde{Q}[k]$. Observe that $(t - t_0 + r^{2s}) \approx (2^k r)^{2s}$ in $\tilde{Q}[k]$, therefore

$$f(t, x, v) \geqslant AM^{-k} \gtrsim A \left(\frac{t - t_0 + r^{2s}}{r^{2s}}\right)^{-p},$$

where $p = \frac{\log(M)}{\log(2^{2s})}$. □

Remark 9.4. It is possible that in the proof of Corollary 9.3 some cylinder $\tilde{Q}[k]$ extends pass the time $t = 0$ and thus it is not strictly contained in Q. This is not a problem since we are dealing with a parabolic equation and future values of f do not affect earlier values. Indeed, we can readily verify that Lemma 9.1 also holds for any value of $\tilde{T} \in (0, 2^{2s})$. The only thing that matters is that R_1 is sufficiently large.

10. The ink-spots theorem for slanted cylinders

This section is dedicated to the statement and proof of a theorem involving a covering argument in the flavor of Krylov-Safonov growing ink spots theorem, or the Calderón-Zygmund decomposition. Such a theorem is used in the proof of the weak Harnack inequality. The statement of the theorem involves stacked (and slanted) cylinders:

$$\bar{Q}^m(z_0, r) = \{(t, x, v) : 0 < t - t_0 \leqslant mr^{2s}, |v - v_0| < r, |x - x_0 - (t - t_0)v_0| < (m + 2)r^{1+2s}\} \tag{10.1}$$

(see Figure 7).

The cylinder $\bar{Q}^m$ is a delayed version of Q. It starts immediately at the end of Q. Its duration in time is m times as long as Q. Its radius in x is $(m+2)$ times the radius in Q. It shares the same values of velocities v as Q.

Theorem 10.1 (The ink-spots theorem). *Let $E \subset F$ be two bounded measurable sets. We make the following assumption for some constant $\mu \in (0, 1)$.*

- *$E \subset Q_1$ and $|E| < (1 - \mu)|Q_1|$.*
- *If any cylinder $Q \subset Q_1$ such that $\bar{Q}^m \subset Q_1$ satisfies $|Q \cap E| \geqslant (1 - \mu)|Q|$, then $\bar{Q}^m \subset F$.*

Then $|E| \leqslant \frac{m+1}{m}(1 - c\mu)|F|$ for some constant $c \in (0, 1)$ depending on s and dimension only.

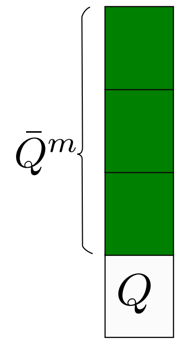

FIGURE 7. Stacked cylinders

There is no chance to adapt the Calderón-Zygmund decomposition to this context. It would require that we split a larger piece into smaller pieces of the same type. Even if we replace balls with cubes, the different slopes, depending on the center velocities, make this tiling condition impossible.

What we do is a variation of the growing ink-spots theorem. The original construction by Krylov and Safonov can be found (in English) in the Appendix 1 of [45]. Here, we have one extra dimension, x, which plays a different role and presents additional difficulies. The most significant difficulty is to go from a lower bound on the measure of the union of disjoint cylinders Q (Lemma 10.7) to a lower bound on the measure of the union of their delayed versions $\bar Q^m$ (Theorem 10.1). The problem is that if the center velocities of two cubes flow towards each other, they may create extra overlaps in their delayed versions. This is addressed essentially in Lemma 10.9, using that we expand the radius in x only by a fixed factor.

The values of x that belong to a slanted cylinder $Q_r(t_0, x_0, v_0)$ change for different values of t. They are contained in a ball with radius r^{1+2s} which flows in the direction of v_0 and is shifted a total distance $|v_0|r^{2s}$ from the initial to the end time. For small values of r, the lenght of this shift is an order of magnitude larger than the radius of the ball r^{1+2s}. Dealing with this shift is non-trivial, and that is the main difference between the covering argument described in this section and the usual ink-spots theorem.

The following corollary will be used when we need to confine both E and F to stay within a fixed cylinder.

Corollary 10.2 (Ink-spots theorem with leakage)**.** *Let $E \subset F$ be two bounded measurable sets. We make the following assumption for some constant $\mu \in (0, 1)$.*

- $E \subset Q_1$.
- *If any cylinder $Q \subset Q_1$ satisfies $|Q \cap E| \geqslant (1 - \mu)|Q|$, then $\bar Q^m \subset F$ and also $Q = Q_r(t, x, v)$ for some $r < r_0$.*

Then $|E| \leqslant \frac{m+1}{m}(1-c\mu)(|F \cap Q_1| + Cmr_0^{2s})$ for some constants c and C depending on s and dimension only.

10.1. **Stacked cylinders and scaling.** For any factor k, we define the scaled cylinder kQ_r by

$$kQ_r = Q_{kr}\left(\frac{k^{2s}-1}{2}r^{2s}, 0, 0\right).$$

Here, we scaled the radius r by a factor k and kept the same center of the cylinder. Note that the point (t_0, x_0, v_0) in $Q_r(t_0, x_0, v_0)$ refers to the top of the cylinder, not its center. In order to keep the center fixed, we updated the top.

Consistently with the Lie group action, we define

$$\begin{aligned} kQ_r(t_0, x_0, v_0) &:= \mathcal{T}_{(t_0,x_0,v_0)} kQ_r, \\ &= \{(t, x, v) : -\frac{k^{2s}+1}{2}r^{2s} < t - t_0, \leqslant \frac{k^{2s}-1}{2}r^{2s}, \\ &\qquad |v - v_0| < kr, \\ &\qquad |x - x_0 - (t - t_0)v_0| < (kr)^{1+2s}\}. \end{aligned}$$

Note that $|kQ_r(t_0, x_0, v_0)| = k^{2(sd+s+d)}|Q_r|$.

The first version of the growing ink-spots lemma uses essentially a variation of the Vitali covering lemma together with a generalized Lebesgue differentiation theorem.

10.2. **A generalized Lebesgue differentiation theorem.** In [40], a generalized Lebesgue differentiation theorem was derived for parabolic cylinders. Here, even though we have one additional variable (x), the proof is the essentially the same. It relies on an adaptation of Vitali's covering lemma (Lemma 10.5 below) and a maximal inequality (Lemma 10.6 below).

Theorem 10.3. *Let $f \in L^1(\Omega,\ \mathrm{d}x \otimes\ \mathrm{d}v \otimes\ \mathrm{d}t)$ where Ω is an open set of $\mathbb{R}^{2d+1}$. Then for a.e. $(t,x,v) \in \Omega$,*

$$\lim_{r \to 0^+} \fint_{Q_r(t,x,v)} |f - f(t,x,v)|\ \mathrm{d}x\ \mathrm{d}v\ \mathrm{d}t = 0.$$

Theorem 10.3 is obtained from Lemma 10.6 exactly as in [40]. For the reader's convenience we will provide below a proof of the maximal inequality.

In our setting, the cylinders $Q_r(t_0,x_0,v_0)$ are not the balls of any metric. The important properties of cylinders are explicitly given by the following lemma.

Lemma 10.4. *Let $Q_{r_0}(t_0,x_0,v_0)$ and $Q_{r_1}(t_1,x_1,v_1)$ be two cylinders with nonempty intersection. Assume that $2r_0 \geqslant r_1$. Then*

$$Q_{r_1}(t_1,x_1,v_1) \subset k Q_{r_0}(t_0,x_0,v_0),$$

for some universal constant k (it depends on s only).

Proof. Since all our definitions are invariant by the action of the Lie group, we can assume without loss of generality that $(t_0,x_0,v_0) = 0$ (the general case is reduced to this applying $\mathcal{T}^{-1}_{(t_0,x_0,v_0)}$).

We need to choose the constant k so that

$$\begin{aligned} k &\geqslant 5, \\ k^{2s} &\geqslant 1 + 2 \cdot 2^{2s}, \\ k^{1+2s} &\geqslant 1 + 2 \cdot 2^{1+2s}. \end{aligned}$$

The first inequality implies the other two when $s \geqslant 1/2$. The second inequality implies the other two when $s \leqslant 1/2$. In particular the third inequality is always redundant. In any case, we pick the smallest k satisfying these inequalities, which depends only on s.

Let $(t_2,x_2,v_2) \in Q_{r_0} \cap Q_{r_1}(t_1,x_1,v_1)$. Let $(t,x,v) \in Q_{r_1}(t_1,x_1,v_1)$. Then all the following hold

$$\begin{aligned} & t \leqslant t_1 < t_2 + r_1^{2s} \leqslant (2r_0)^{2s} \leqslant \frac{k^{2s}-1}{2} r_0^{2s}, \\ & t \geqslant t_1 - r_1^{2s} \geqslant t_2 - r_1^{2s} \geqslant -r_0^{2s} - (2r_0)^{2s} \geqslant -\frac{k^{2s}+1}{2} r_0^{2s}, \\ & |v| \leqslant |v - v_1| + |v_1 - v_2| + |v_2| \leqslant 2r_1 + r_0 \leqslant 5r_0 \leqslant k r_0, \\ & |x| \leqslant |x - x_1 - (t-t_1)v_1| + |x_2 - x_1 - (t_2 - t_1)v_1| + |x_2| \leqslant 2r_1^{2s+1} + r_0^{2s+1} \leqslant k^{2s+1} r_0^{2s+1}. \end{aligned}$$

Thus, we get that $(t,x,v) \in kQ_{r_0}$ and we conclude the proof. □

Lemma 10.5 (Vitali)**.** *Let $\{Q_j\}_{j \in J}$ be an arbitrary collection of slanted cylinders with bounded radius. Then, there exists a disjoint countable subcollection $\{Q_{j_i}\}$ so that*

$$\bigcup_{j \in J} Q_j = \bigcup_{i=1}^{\infty} k Q_{j_i}.$$

The proof of Lemma 10.5 is the same as the classical proof of the Vitali coverling lemma using Lemma 10.4 instead of the fact that in any metric space $B_{r_1}(x_1) \subset 5B_{r_0}$ if $B_{r_1}(x_1) \cap B_{r_0} \neq \varnothing$ and $r_1 \leqslant 2r_0$.

We next define the maximal function Mf as follows: for $(x,v,t) \in \Omega$,

$$Mf(t,x,v) = \sup_{Q \ni (x,v,t)} \fint_{Q \cap \Omega} |f|$$

where the supremum is taken over cylinders of the form $(y,w,s) + RQ_1$.

Lemma 10.6 (The maximal inequality)**.** *For all* $\lambda > 0$*,*

$$|\{Mf > \lambda\} \cap \Omega| \leqslant \frac{C}{\lambda}\|f\|_{L^1(\Omega)}.$$

Proof. For $(x,v,t) \in \{Mf > \lambda\} \cap \Omega$, there exists a cylinder $Q \ni (x,v,t)$ such that

$$\int_{Q\cap\Omega} |f| \geqslant \frac{\lambda}{2}|Q \cap \Omega|.$$

Then $\{Mf > \lambda\} \cap \Omega$ is covered with cylinders $\{Q_j\}$ such that the previous inequality holds. From Lemma 10.5, there exists a disjoint countable subcollection $\{Q_{j_i}\}$ so that

$$\{Mf > \lambda\} \cap \Omega\} \subset \bigcup_{i=1}^{\infty} kQ_{j_i}$$

for some integer k only depending on s.

We now write

$$\begin{aligned}\int_\Omega |f| \geqslant \int_{\Omega\cap\cup_i Q_{j_i}} |f| &= \sum_i \int_{\Omega\cap\ Q_{j_i}} |f| \\ &\geqslant \sum_i \frac{\lambda}{2}|Q_{j_i} \cap \Omega| = \frac{\lambda}{2}| \cup_i Q_{j_i} \cap \Omega| = \frac{\lambda}{2k^{2(d+ds+s)}}| \cup_i kQ_{j_i} \cap \Omega| \\ &\geqslant \frac{\lambda}{2k^{2(d+ds+s)}}|\{Mf > \lambda\} \cap \Omega|.\end{aligned}$$

We obtain the desired inequality with $C = 2k^{2(d+ds+s)}$. □

10.3. **Preliminary version without time delay.**

Lemma 10.7. *Let* $E \subset F \subset Q_1$ *be two measurable sets. Assume that there is a constant* $\mu > 0$ *such that*

- $|E| < (1-\mu)|Q_1|$.
- *if any cylinder* $Q \subset Q_1$ *satisfies* $|Q \cap E| \geqslant (1-\mu)|Q|$*, then* $Q \subset F$*.*

Then $|E| \leqslant (1-c\mu)|F|$ *for some constant* c *depending on* s *and dimension only.*

Proof. Using the generalized Lebesgue differentiation theorem (see [40] for an adaptation of the classical Lebesgue differentiation theorem 10.3, for almost all points $x \in E$, there is some cylinder Q^x containing x such that $|Q^x \cap E| \geqslant (1-\mu)|Q^x|$. For all Lebesgue points $x \in E$, let us choose a maximal slanted cylinder $Q^x \subset Q_1$ that contains x and such that $|Q^x \cap E| \geqslant (1-\mu)|Q^x|$. Here $Q^x = Q_{\bar r}(\bar t, \bar x, \bar v)$ for some $\bar r$, $\bar t$, $\bar x$ and $\bar v$. From one of the assumptions, we know that $Q^x \neq Q_1$ for any x. The other assumption tells us that $Q^x \subset F$.

We claim that $|Q^x \cap E| = (1-\mu)|Q^x|$. Otherwise, for $\delta > 0$ small enough, there would be a $\tilde Q$ such that $Q^x \subset \tilde Q \subset (1+\delta)Q^x$, $\tilde Q \subset Q_1$ and $|\tilde Q \cap E| > (1-\mu)|\tilde Q|$, contradicting the maximality of the choice of Q^x.

The family of cylinders Q^x covers the set E. By Lemma 10.5, we can select a finite subcollection of non overlapping cylinders $Q_j := Q^{x_j}$ such that $E \subset \bigcup_{j=1}^n kQ_j$.

Since $Q_j \subset F$ and $|Q_j \cap E| = (1-\mu)|Q_j|$, we have that $|Q_j \cap F \backslash E| = \mu|Q_j|$. Therefore

$$\begin{aligned}|F\backslash E| &\geqslant \sum_{j=1}^n |Q_j \cap F\backslash E| \\ &\geqslant \sum_{j=1}^n \mu|Q_j| \\ &= k^{-2(d+ds+s)}\mu \sum_{j=1}^n |kQ_j| \geqslant k^{-2(d+ds+s)}\mu|E|.\end{aligned}$$

We thus get

$$|F| \geqslant (1+\tilde c\mu)|E|$$

with $\tilde c = k^{-2(ds+d+s)}$. Since $\tilde c\mu \in (0,1)$, this implies

$$|E| \leqslant (1-c\mu)|F|$$

with $c = \tilde c/2$. □

10.4. **Stacked cylinders and leakage.** The following lemma can be deduced from Lemma 4.29 in [40] (There is a typo in the statement in that note, we embarrassingly apologize).

Lemma 10.8. *Consider a (possibly infinite) sequence of intervals $(a_j - h_k, a_j]$. Then*

$$\left|\bigcup_k (a_k, a_k + mh_k)\right| \geqslant \frac{m}{m+1}\left|\bigcup_k (a_k - h_k, a_k]\right|.$$

Proof. We first assume that $k = 1, \dots, N$ for some finite number N.

Let

$$\bigcup_{k=1}^N (a_k, a_k + mh_k) = \bigcup_\ell I_\ell,$$

for a disjoint family of intervals I_ℓ. Here, each I_ℓ is a union of intervals of the form $(a_i, a_i + mh_i]$. Let $a_0 - h_0$ be the minimum of $a_i - h_i$ and $a_1 + mh_1$ be the maximum of $a_i + mh_i$ respectively, for all i so that $(a_i, a_i + mh_i] \subset I_\ell$. Naturally, we have

$$|I_\ell| \geqslant (a_1 + mh_1) - a_0 \geqslant \frac{m}{m+1}((a_1 + mh_1) - (a_0 - h_0)) \geqslant \frac{m}{m+1}\left|\bigcup_{\{i:(a_i,a_i+mh_i]\subset I_\ell\}} (a_i - h_i, a_i]\right|,$$

Therefore

$$\begin{aligned}\left|\bigcup_{k=1}^N (a_k, a_k + mh_k]\right| &= |\bigcup_\ell I_\ell| = \sum_\ell |I_\ell|,\\ &\geqslant \frac{m}{m+1}\sum_\ell \left|\bigcup_{\{i:(a_i,a_i+mh_i]\subset I_\ell\}} (a_i - h_i, a_i]\right|,\\ &\geqslant \frac{m}{m+1}\left|\bigcup_{i=1,\dots,N} (a_i - h_i, a_i]\right|.\end{aligned}$$

It is now enough to let $N \to \infty$ to conclude. □

Lemma 10.9. *Let $\{Q_j\}$ be a collection of slanted cylinders, and $\bar{Q}_j^m$ be the corresponding delayed versions as in (10.1). Then*

$$\left|\bigcup_j \bar{Q}_j^m\right| \geqslant \frac{m}{m+1}\left|\bigcup_j Q_j\right|.$$

Proof. Because of Fubini's theorem, we know that for any set $A \subset \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d$,

$$|A| = \int |\{(t,x) : (t,x,v) \in A\}| \,\mathrm{d}v.$$

Therefore, in order to prove the lemma, it is enough to show that for every $v \in \mathbb{R}^d$,

$$\left|\left\{(t,x) : (t,x,v) \in \bigcup_j \bar{Q}_j^m\right\}\right| \geqslant \frac{m}{m+1}\left|\left\{(t,x) : (t,x,v) \in \bigcup_j Q_j\right\}\right|. \tag{10.2}$$

From now on, let v be any fixed $v \in \mathbb{R}^d$.

Any cylinder Q_j corresponds to $Q_{r_j}(t_j, x_j, v_j)$ for some choice of $r_j > 0$, $t_j \in \mathbb{R}$ and $x_j, v_j \in \mathbb{R}^d$. If $|v - v_j| < r_j$, we have

$$\left\{(t,x) : (t,x,v) \in \bar{Q}_j^m\right\} = \{(t,x) : 0 < t - t_j \leqslant mr_j^{2s}, |x - x_j - (t - t_j)v_j| < (m+2)r_j^{1+2s}\}.$$

The set in the left hand side would be empty when $|v - v_j| \geqslant r_j$.

When $|v - v_j| < r_j$, we have $|(t - t_j)(v_j - v)| < mr_j^{1+2s}$. Therefore, we can switch v_j with v in the last term, changing the right hand side, and we obtain a smaller set.

$$\left\{(t,x) : (t,x,v) \in \bar{Q}_j^m\right\} \supset \{(t,x) : 0 < t - t_j \leqslant mr_j^{2s}, |x - x_j - (t - t_j)v| < 2r_j^{1+2s}\}.$$

Let $z = x - tv$. The change of variables $(t,x) \mapsto (t,z)$ has Jacobian one. We will estimate the measure of the points (t,z) so that $(t, z+tv)$ belongs to the set above. Thus

$$\left|\left\{(t,x) : (t,x,v) \in \bigcup_{j:|v-v_j|<r_j} \bar{Q}_j^m\right\}\right| \geqslant \left|\bigcup_{j:|v-v_j|<r_j} \{(t,z) : 0 < t-t_j \leqslant m r_j^{2s}, |z-x_j+t_jv| < 2r_j^{1+2s}\}\right|. \tag{10.3}$$

Let $\tilde{Q}_j$ be the cylinders in $\mathbb{R} \times \mathbb{R}^d$ used in the right hand side of the inequality above,

$$\tilde{Q}_j = \{(t,z) : 0 < t - t_j \leqslant m r_j^{2s}, |z - x_j + t_j v| < 2r_j^{1+2s}\}.$$

Applying Fubini's theorem again,

$$\left|\bigcup_{|v-v_j|<r_j} \tilde{Q}_j\right| = \int_{\mathbb{R}^d} \left|\bigcup_{\substack{\{j:|v-v_j|<r_j,\\ |z-x_j+t_jv|<2r_j^{1+2s}\}}} (t_j, t_j + m r_j^{2s}]\right| \, \mathrm{d}z$$

Using lemma 10.8,

$$\begin{aligned}
\left|\bigcup_{|v-v_j|<r_j} \tilde{Q}_j\right| &\geqslant \frac{m}{m+1} \int_{\mathbb{R}^d} \left|\bigcup_{\substack{\{j:|v-v_j|<r_j,\\ |z-x_j+t_jv|<2r_j^{1+2s}\}}} (t_j - r_j^{2s}, t_j]\right| \, \mathrm{d}z, \\
&= \frac{m}{m+1} \left|\bigcup_{j:|v-v_j|<r_j} \{(t,x) : -r_j^{2s} < t - t_j \leqslant 0, |x - x_j - (t-t_j)v| \leqslant 2r_j^{1+2s}\}\right|, \qquad (10.4)\\
&\geqslant \frac{m}{m+1} \left|\bigcup_{j:|v-v_j|<r_j} \{(t,x) : -r_j^{2s} < t - t_j \leqslant 0, |x - x_j - (t-t_j)v_j| \leqslant r_j^{1+2s}\}\right|, \\
&= \frac{m}{m+1} \left|\left\{(t,x) : (t,x,v) \in \bigcup_j Q_j\right\}\right|.
\end{aligned}$$

For the last inequality we used that if $-r_j^{2s} < t - t_j \leqslant 0$, then $(t_j - t)|v - v_j| < r_j^{1+2s}$.

Combining (10.3) with (10.4), we obtain (10.2) and finish the proof. □

We can now turn to the proof of the main theorem.

Proof of Theorem 10.1. Let $\mathcal{Q}$ be the collection of all cylinders $Q \subset Q_1$ such that $|Q \cap E| \geqslant (1-\mu)|Q|$. Let $G := \bigcup_{Q \in \mathcal{Q}} Q$. By construction, the sets E and G satisfy the hypothesis of the Lemma 10.7. Therefore $(1 - c\mu)|G| \geqslant |E|$.

From our hypothesis $\bigcup_{Q\in\mathcal{Q}} \bar{Q}^m \subset F$. We conclude the proof applying Lemma 10.9,

$$|F| \geqslant \left|\bigcup_{Q\in\mathcal{Q}} \bar{Q}^m\right| \geqslant \frac{m}{m+1}\left|\bigcup_{Q\in\mathcal{Q}} Q\right| = \frac{m}{m+1}|G|.$$ □

Proof of Corollary 10.2. Note that the condition $|E| \leqslant (1-\delta)|Q_1|$ is implied by the second assumption when $r_0 < 1$. Moreover, the result is trivial for $r_0 \geqslant 1$ choosing C sufficiently large.

Let $\mathcal{Q}$ be the collection of all cylinders $Q \subset Q_1$ such that $|Q \cap E| \geqslant (1-\mu)|Q|$. Let $G := \bigcup_{Q\in\mathcal{Q}} \bar{Q}^m$. From Theorem 10.1, we have that $|E| \leqslant \frac{m}{m+1}(1 - c\mu)|G|$. Moreover, our hypothesis tell us that $G \subset F$.

In order to conclude the corollary, we will estimate the measure $G \backslash Q_1$ using the fact that each of the cubes $Q = Q_r(t,x,v) \subset Q_1$ has radius bounded by r_0. Recall that

$$\begin{aligned}
\bar{Q}^m = \{(\bar{t}, \bar{x}, \bar{v}) : 0 < \bar{t} - t \leqslant m r^{2s}&, \\
|\bar{v} - v| < r&, \\
|\bar{x} - x - (\bar{t} - t)v| < (m+2)r^{1+2s}&\}.
\end{aligned}$$

Since $Q \subset Q_1$, then $t < 0$. So $\bar{t} \leqslant mr_0^{2s}$. Moreover, $|\bar{v}| < 1$, since the velocities in $\bar{Q}^m$ are the same as in Q. Also, $|x| < 1$, so $|\bar{x}| \leqslant 1 + mr_0^{2s}$. Therefore, $\bar{Q}^m \subset (-1, mr_0^{2s}] \times B_{1+mr_0^{2s}} \times B_1$. The same thing applies to G.

$$G \subset (-1, mr_0^{2s}] \times B_{1+mr_0^{2s}} \times B_1.$$

Therefore $|F \cap Q_1| \geqslant |G \cap Q_1| \geqslant |G| - |G \backslash Q_1| \geqslant |G| - Cmr_0^{2s}$ and we conclude the proof. □

11. Proofs of the main results

In this section we complete the proofs of our main results. At this point, the main tools have already been established in previous sections. The weak Harnack inequality is proved combining the propagation lemma (Lemma 9.1) with our special version of the ink-spots theorem (Theorem 10.1). The structure of this proof is inspired by the work of Krylov and Safonov [46] for equations in nondivergence form.

11.1. **The weak Harnack inequality.**

Proof of Theorem 1.6. We choose R_1 to be the radius given in Lemma 9.1. We choose r_0 sufficiently small so that the set $S(t_0, x_0, v_0)$ from Corollary 9.3 contains Q_+ for any $(t_0, x_0, v_0) \in Q_-$ and $r \in (0, r_0)$.

Replacing f and h with cf and ch where the constant c is choosen as follows

$$c = (2 \inf_{Q^+} f + 2\|h\|_{L^\infty(Q_1)})^{-1},$$

we reduce to the case where $\inf_{Q^+} f \leqslant 1/2$ and $\|h\|_{L^\infty(Q_1)}) \leqslant 1/2$.

We can further reduce to the case $\inf_{Q^+} f \leqslant 1$ and $h = 0$. Indeed, if the function f is a supersolution of

$$f_t + v \cdot \nabla_x f - L_v f \geqslant -1/2,$$

then the function $\tilde{f}(t, x, v) = f(t, x, v) + (t + 1)/2$ is a nonnegative function in $[-1, 0] \times \mathbb{R}^{2d}$ which is a supersolution to (1.2) with $h = 0$. Moreover, $\inf_{Q^+} \tilde{f} \leqslant \inf_{Q^+} f + 1/2 \leqslant 1$ and $f^\varepsilon \leqslant \tilde{f}^\varepsilon$.

The proof relies on the application of the propagation lemmas 9.1 and Corollary 9.3. The constants M, δ in the remainder of the proof are chosen so that these propagation lemmas can be applied.

We are going to prove that in this case

$$\int_{Q^-} f^\varepsilon(t, x, v) \, dv \, dx \, dt \leqslant \tilde{C}_{\text{w.h.i.}}.$$

In order to do so, it is enough to prove that

$$\forall k \geqslant 1, \qquad |\{f > \bar{M}^k\} \cap Q^-| \leqslant C_{\text{w.h.i.}}(1 - \delta')^k \tag{11.1}$$

for some universal constants $\bar{M} \geqslant 1$, $C_{\text{w.h.i.}} \geqslant 1$ and $\delta' \in (0, 1)$.

Estimate (11.1) is proved by induction. For $k = 1$, we simply choose $C_{\text{w.h.i.}}$ and δ' so that

$$|Q^-| \leqslant C_{\text{w.h.i.}}(1 - \delta) \quad \text{and} \quad \delta' \leqslant \delta.$$

Note that by choosing a larger constant $C_{\text{w.h.i.}}$ we can make sure the inequality holds for arbitrarily many values of k.

Assume that the inequality holds true up to rank $k \geqslant 1$ and let us prove it for $k + 1$. We want to apply Corollary 10.2 of the growing ink spots theorem 10.1 with $\mu = \delta$, some integer $m \geqslant 1$ (to be fixed later, only depending on δ), and

$$\bar{M} = M^m$$

where δ and M given by the propagation lemma 9.1, and

$$E = \{f > \bar{M}^{k+1}\} \cap Q^- \quad \text{and} \quad F = \{f > \bar{M}^k\} \cap Q_1.$$

The sets E and F are bounded and measurable and $E \subset F \subset Q_1$. We consider a cylinder $Q = Q_r(z_0) \subset Q^-$ (in particular $r \in (0, r_0)$) such that $|Q \cap E| > (1 - \delta)|Q|$, that is to say

$$|\{f > \bar{M}^{k+1}\} \cap Q^-| > (1 - \delta)|Q|. \tag{11.2}$$

We now prove that r is small. Since we have $\inf_{Q^+} f \leqslant 1$ and $S(t_0, x_0, v_0)$ contains Q_+, Corollary 9.3 yields

$$\bar{M}^{k+1} \left(1 + \frac{1 - 2r_0^{2s}}{r^{2s}}\right)^{-p} \lesssim 1.$$

Therefore $r \lesssim \bar{M}^{-k/(2sp)}$. In particular $\bar{Q}^m \subset Q_1$ (at least for k large).

Now, we want to prove that, $\bar{Q}^m \subset F$, that is to say,

$$\bar{Q}^m \subset \{f > \bar{M}^k\}. \tag{11.3}$$

This follows simply from Corollary 9.2, with $k = m$ to the function $\tilde{f} = \bar{M}^{-k} f \circ \mathcal{T}_{z_0}$.

Applying Corollary 10.2 to E and F with $\mu = \delta$ and $r_0 = \bar{M}^{-k/(2sp)}$, we get

$$|\{f > \bar{M}^{k+1}\} \cap Q^-| \leqslant \frac{m+1}{m}(1 - c\delta)\left\{|\{f > \bar{M}^k\} \cap Q^-| + Cm\bar{M}^{-k/p}\right\}$$

where $C = C(s,d)$. We now use the induction hypothesis and get

$$|\{f > \bar{M}^{k+1}\} \cap Q^-| \leqslant \frac{m+1}{m}(1 - c\delta)\left\{C_{\text{w.h.i.}}(1-\delta')^k + Cm\bar{M}^{-k/p}\right\}.$$

Choosing δ' smaller than $\bar{M}^{-1/p}$, we have

$$|\{f > \bar{M}^{k+1}\} \cap Q^-| \leqslant C_{\text{w.h.i.}} \frac{m+1}{m}(1 - c\delta)\left\{1 + C_{\text{w.h.i.}}^{-1} Cm\right\}(1-\delta')^k.$$

We next pick m large enough (depending on δ) and then $C_{\text{w.h.i.}}$ large enough (depending on δ and m) so that

$$\frac{m+1}{m}(1 - c\delta)\left\{1 + C_{\text{w.h.i.}}^{-1} Cm\right\} \leqslant 1 - (c/2)\delta.$$

Now imposing $\delta' < (c/2)\delta$, we get the desired inequality:

$$|\{f > \bar{M}^{k+1}\} \cap Q^-| \leqslant C_{\text{w.h.i.}}(1-\delta')^{k+1}.$$

This achieves the proof of Estimate (11.1) and of the theorem. □

11.2. **The Hölder estimate.** In order to prove Theorem 1.5, we first prove two preparatory results, Lemma 11.1 has the flavor of a weak Harnack inequality, but for supersolutions that can take (controled) negative values. The lemma then implies Corollary 11.2 which is concerned with the improvement of oscillation of solutions with small forcing terms.

Lemma 11.1. *Let r_0 and R_1 as in Theorem 1.6 and $\rho = r_0/R_1$. Let $\tilde{Q}_-$ be*

$$\tilde{Q}_- := Q_\rho(-R_1^{-2s} + \rho^{2s}, 0, 0).$$

Let $f : (-1,0] \times B_1 \times \mathbb{R}^d \to \mathbb{R}$ be a function satisfying the following assumptions.

- $f_t + v \cdot \nabla_x f \geqslant L_v f + h$ *in Q_1, with $h \geqslant -\varepsilon_0$;*
- *For $t \in (-1,0]$, $x \in B_1$, $v \in B_2$, $f(t,x,v) \in [0,1]$;*
- *For $t \in (-1,0]$, $x \in B_1$ and $v \in \mathbb{R}^d \backslash B_2$, $f(t,x,v) \geqslant -\left(\frac{|v|}{2}\right)^{\alpha_0} + 1$;*
- $|\{f \geqslant 1/2\} \cap \tilde{Q}_-| \geqslant \frac{1}{2}|\tilde{Q}_-|$.

If $\alpha_0 > 0$, $\varepsilon_0 > 0$ and $\delta > 0$ are sufficiently small, then

$$f \geqslant \delta \text{ in } Q_\rho.$$

Proof. We can assume that $h \leqslant 0$ without loss of generality. Let us first scale the function by defining $\tilde{f}(t,x,v) = f(R_1^{-2s}t, R_1^{-1-2s}x, R_1^{-1}v)$. This function satisfies the equation

$$\partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} - \tilde{L}_v f \geqslant -\varepsilon_0 R_1^{-2s} \geqslant -\varepsilon_0,$$

in Q_{R_1}. The rescaled kernel in $\tilde{L}_v$ satisfies the same assumptions (1.3), (1.5), (1.6), (1.4) if $s < 1/2$, and (1.7) if $s \geqslant 1/2$. Note that Q_{R_1} contains $(-1,0] \times B_{R_1^{1+2s}} \times B_{R_1}$.

Let $\tilde{f}_+ = \max(\tilde{f}, 0)$. This function satisfies the following equation in Q_{R_1},

$$\begin{aligned}\partial_t \tilde{f}_+ + v \cdot \nabla_x \tilde{f}_+ &\geqslant \int_{\mathbb{R}^d} (\tilde{f}_+(w) - \tilde{f}_+(v))K(v,w)\,\mathrm{d}w + h - \int_{|w| \geqslant 2R_1} \tilde{f}_-(w)K(v,w)\,\mathrm{d}w,\\ &\geqslant \int_{\mathbb{R}^d} (\tilde{f}_+(w) - \tilde{f}_+(v))K(v,w)\,\mathrm{d}w - 2\varepsilon_0\end{aligned}$$

provided α_0 is small.

Applying Theorem 1.6 (the weak Harnack inequality), we get

$$\begin{aligned}\tilde f_+ &\geqslant \left(\int_{\tilde Q_-} \tilde f^\varepsilon\right)^{1/\varepsilon} - 2\varepsilon_0 && \text{in } Q_+\\ &\geqslant \frac{1}{2}|\tilde Q_-|^{1/\varepsilon} - 2\varepsilon_0 \\ &\geqslant \delta && \text{for } \varepsilon_0 \text{ and } \delta \text{ sufficiently small.}\end{aligned}$$

Rescaling back to f, we finish the proof. □

Corollary 11.2. *Let f be a solution of (1.2) in Q_1 with $|h| \leqslant \varepsilon_0$. Assume that*

$$\operatorname*{osc}_{(-1,0]\times B_1\times B_R} f \leqslant \left(\frac{R}{2}\right)^{\alpha_0} \qquad \text{for all } R \geqslant 2.$$

Then

$$\operatorname*{osc}_{Q_\rho} f \leqslant 1-\delta.$$

Here, $\varepsilon_0 > 0$, $\delta > 0$, $\alpha_0 > 0$ and $\rho > 0$ are the same constants as in Lemma 11.1.

Proof. Let $a = \operatorname{essinf}_{(-1,0]\times B_1\times B_2} f$ and $b = \operatorname{esssup}_{(-1,0]\times B_1\times B_2} f$. The values of $f(t,x,v)$ are either above or below the middle value $(a+b)/2$ in at last half of the points in $\tilde Q_-$. Thus, one of the following inequalities holds.

$$\left|\left\{f \geqslant \frac{a+b}{2}\right\} \cap \tilde Q_-\right| \geqslant \frac{1}{2}|\tilde Q_-| \qquad \text{or} \qquad \left|\left\{f \leqslant \frac{a+b}{2}\right\} \cap \tilde Q_-\right| \geqslant \frac{1}{2}|\tilde Q_-|.$$

Assume the former. The opposite case would follow from the same proof upside down.

Consider the function

$$\bar f(t,x,v) = 1 - b + f(t,x,v).$$

This choice is made so that $\operatorname{esssup}_{(-1,0]\times B_1\times B_2} \bar f = 1$.

Since $\operatorname{osc}_{(-1,0]\times B_1\times B_2} \bar f \leqslant 1$, then $\bar f \in [0,1]$ for $(t,x,v) \in (-1,0] \times B_1 \times B_2$.

Since $\operatorname{osc}_{(-1,0]\times B_1\times B_R} \bar f \leqslant \left(\frac{R}{2}\right)^{\alpha_0}$ for $R \geqslant 2$ and $\operatorname{esssup}_{(-1,0]\times B_1\times B_2} \bar f = 1$, then $\bar f(t,x,v) \geqslant 1 - \left(\frac{|v|}{2}\right)^{\alpha_0}$ for $t \in (-1,0]$, $x \in B_1$ and $v \in \mathbb{R}^d \setminus B_2$.

Thus, $\bar f$ satisfies the hypothesis of Lemma 11.1, $\bar f \in [\delta, 1]$ in B_ρ, and the corollary follows. □

Proof of Theorem 1.5. Without loss of generality we assume $\|f\|_{L^\infty((-1,0]\times B_1\times\mathbb{R}^d)} \leqslant 1$ and $\|h\|_{L^\infty(Q_1)} \leqslant \varepsilon_0$, where ε_0 is the constant from Lemma 11.1. Otherwise, we replace f by

$$\tilde f(t,x,v) = \frac{1}{\|f\|_{L^\infty((-1,0]\times B_1\times\mathbb{R}^d)} + \|h\|_{L^\infty(Q_1)}/\varepsilon_0} f(t,x,v).$$

We want to prove that there exists some universal constant C so that for all $r > 0$,

$$\operatorname*{osc}_{Q_r} f \leqslant C r^\alpha.$$

We choose $\alpha < \min(\alpha_0, \ln(1-\delta)/\ln(\rho/2))$, where ρ, δ and α_0 are the constants from Lemma 11.1.

Let $A(r) := \operatorname{osc}_{Q_r} f = \operatorname{esssup}_{Q_r} f - \operatorname{essinf}_{Q_r} f$. It is a monotone increasing function. We cannot assume a priori that A is a continuous function, but it is always left continuous. Since $|f| \leqslant 1$, we also have $A(r) \leqslant 2$ for all $r > 0$. Hence, we can choose C large enough so that $A(r) \leqslant Cr^\alpha$ for all $r \geqslant \rho$.

Assume the theorem is not true, then let

$$r_0 := \sup\{r : A(r) > Cr^\alpha\} \in (0,\rho).$$

Since $A(r)$ is left continuous, $A(r_0) \geqslant Cr_0^\alpha$.

Let f_0 be the rescaled function

$$f_0(t,x,v) = \frac{1}{C}\left(\frac{\rho}{2r_0}\right)^\alpha f\left(\left(\frac{r_0}{\rho}\right)^{2s} t, \left(\frac{r_0}{\rho}\right)^{2s+1} x, \frac{r_0}{\rho} v\right).$$

Since $A(r) \leqslant Cr^\alpha$ for $r > r_0$,

$$\operatorname*{osc}_{Q_R} f_0 \leqslant (R/2)^\alpha \qquad \text{for } R > \rho.$$

In particular, since $\alpha \leqslant \alpha_0$, we can apply Corollary 11.2 and obtain that

$$\operatorname*{osc}_{Q_\rho} f_0 \leqslant 1 - \delta.$$

Therefore, in terms of the original function f,

$$A(r_0) \leqslant C \left(\frac{2r_0}{\rho} \right)^\alpha (1-\delta).$$

This contradicts that $A(r_0) \geqslant C r_0^\alpha$ since $\alpha < \ln(1-\delta)/\ln(\rho/2)$, and we finish the proof. □

Appendix A. New proofs of known estimates and technical lemmas

A.1. **The coercivity estimate for the Boltzmann kernel.** In this appendix we give a geometric proof of the following coercivity estimate for the Boltzmann equation. It says that the Boltzmann kernel satisfies the assumption (1.3).

Proposition A.1. *Assume that the function f satisfies the inequalities*

$$M_1 \leqslant \int_{\mathbb{R}^d} f(v) \,\mathrm{d}v \leqslant M_0,$$

$$\int_{\mathbb{R}^d} |v|^2 f(v) \,\mathrm{d}v \leqslant E_0,$$

$$\int_{\mathbb{R}^d} f(v) \ln f(v) \,\mathrm{d}v \leqslant H_0.$$

Assume also that $f * |\cdot|^\gamma$ is bounded by some constant K_0. This bound is controlled by M_0 and E_0 if $\gamma \in [0,2]$, and it is an extra assumption when $\gamma < 0$.

Let $g : \mathbb{R}^d \to \mathbb{R}$ be a function supported in B_R. Then

$$-\int_{\mathbb{R}^d} L_v g(v) g(v) \,\mathrm{d}v \geqslant \lambda \|g\|^2_{\dot{H}^s} - \Lambda \|g\|^2_{L^2}.$$

The constants λ and Λ depend only on M_0, M_1, E_0, H_0, K_0 the dimension d and the radius R.

In particular, for an appropriately larger constant Λ,

$$\int_{\mathbb{R}^d} Q(f,g)(v) g(v) \,\mathrm{d}v \leqslant -\lambda \|g\|^2_{\dot{H}^s} + \Lambda \|g\|^2_{L^2}.$$

Note that the extra assumptions about the boundedness of $f * |\cdot|^\gamma$ comes from the usual condition for the classical cancellation Lemma to give us a bounded function. It is the same assumption as in Lemma 3.6.

The constant c above may go to zero as $R \to \infty$ depending on the value of γ. The precise optimal rate for this can be easily deduced from the proof. We explain this in remark A.7

The proofs of Proposition A.1 that can be found in the Boltzmann literature are done using Fourier analysis. Here, we present a relatively elementary proof based on a direct computation and a geometric argument in physical variables.

We define K_f, $Q_1(f,g) = L_v g$ and $Q_2(f,g) = c(f * |\cdot|^\gamma)\, g$ as described in Section 3.

In [59], there is an estimate for K_f in terms of a simplified integral expression. It says that

$$K_f(v,v') \approx \left(\int_{\{w\cdot(v'-v)=0\}} f(v+w)\ |w|^{\gamma+2s+1} \,\mathrm{d}w \right) |v'-v|^{-d-2s}. \tag{A.1}$$

A.1.1. *Lower bounds for K_f in a cone of directions.* We obtain a lower bound for $K_f(v,v')$ in a symmetric cone of directions with vertex v. This was done in [59]. It follows essentially from the following lemma.

Lemma A.2. *Let $f : \mathbb{R}^N \to R$ be as in Proposition A.1. There exists an $r > 0$, $\ell > 0$ and $m > 0$ depending on M_1, E_0 and H_0 such that*

$$|\{v : f(v) > \ell\} \cap B_r| \geqslant m$$

Combining Lemma A.2 with the expression (A.1), we deduce the following statement. It is essentially the same as Lemma 4.8 in [59], but with a more detailed description of the cone of directions where the lower bound holds.

Lemma A.3. *Let $f : \mathbb{R}^N \to R$ be non-negative and*

$$M_1 \leqslant \int_{\mathbb{R}^N} f(v) \, dv \leqslant M_0,$$

$$\int_{\mathbb{R}^N} |v|^2 f(v) \, dv \leqslant E_0,$$

$$\int_{\mathbb{R}^N} f(v) \ln f(v) \, dv \leqslant H_0.$$

For any $v \in \mathbb{R}^d$, there exists a set of directions $A = A(v) \in \partial B_1$, so that $K_f(v, v') \geqslant \lambda(1 + |v|)^{1+2s+\gamma}|v - v'|^{-d-2s}$ for all v' so that $(v' - v)/|v' - v| \in A$.

Moreover, this set of directions $A \subset S^{d-1}$ satisfies the following properties.

- *A is symmetric: $A = -A$.*
- *Any big circle in S^{d-1} intersects A on a set of (one dimensional) measure at least $c(1 + |v|)^{-1}$. In particular, the $(d - 1)$ dimensional measure of A is at least $\mu(v) := c(1 + |v|)^{-1}$.*
- *A is contained on a strip of width $\leqslant C(1 + |v|)^{-1}$ around the equator perpendicular to v.*

By a big circle, we mean a closed geodesic in S^{d-1}. They are the intersection of S^{d-2} with any 2-dimensional subspace.

The proof of Lemma A.3 is similar to the one of Lemma 4.8 in [59]. Here we have a more precise description than in that paper because we add a lower bound of the measure of the intersection of A with any big circle instead of only its total measure. The proof is relatively easy to explain with a picture on the blackboard, but perhaps somewhat cumbersome to write down.

Proof. Let $F = \{v : f(v) > \ell\} \cap B_r$ be the set described in Lemma A.2, which has measure at least m.

From the formula for K_f given in (A.1), one immediately sees that $\sigma \in A(v)$ when the hyperplane perpendicular to σ intersects F in a set of measure at least cm/r, with $\lambda = c\ell m/r$.

The three properties described in the lemma are simple geometric consequences of this construction using only that the measure of F is bounded below and $F \subset B_r$ for some given constant r. Indeed, as σ takes all values on a big circle in ∂B_1, its perpendicular hyperplanes swipe the space $\mathbb{R}^d$ (see Figure 8).

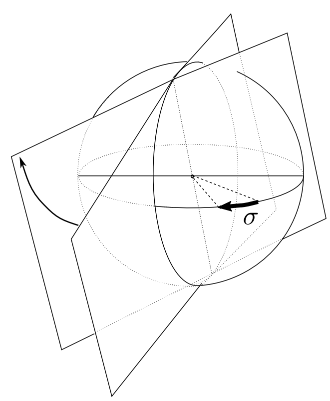

FIGURE 8. As σ moves along a big circle in ∂B_1, its perpendicular planes swipe the space

Because of Fubini's theorem, the points σ on that big circle for which its perpendicular hyperplane intersects F in a set of measure at least $cm/(1 + |v|)$ has to be at least of measure $cm/(1 + |v|)$.

Note that depending on the direction of the big circle, the lower bound on its intersection with $A(v)$ could be improved. For example, if the big circle is perpendicular to v, the measure of its intersection with $A(v)$ is bounded below independently of v. This fact will not be relevant to any of the computations below. □

Figure 9 is taken from [59] and shows all the elements in Lemma A.3.

We may call $\Xi(v)$ the symmetric cone of values of v' so that $(v' - v)/|v' - v| \in A$. In particular, the lower bound $K_f(v, v') \geqslant \lambda(1 + |v|)^{1+2s+\gamma}|v - v'|^{-d-2s}$ holds when $v' \in \Xi(v)$.

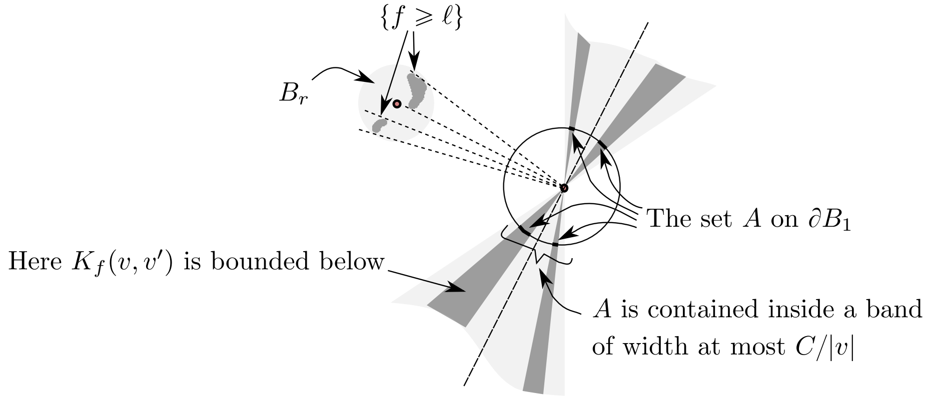

FIGURE 9. The geometric setting of Lemma A.3. The cone $\Xi(v)$ is generated by the set $\{f \geqslant l\}$.

The second item in Lemma A.3 says that there is a universal lower bound on the density of $\Xi(v)$ inside the cone of v' given by (A.2). This is all we will use in order to prove the coercivity estimate below.

The third item in the properties of A says that for any $v' \in \Xi(v)$,

$$|v \cdot (v' - v)| \leqslant C|v' - v|. \tag{A.2}$$

This third point plays no role in the local version of the coercivity estimate. It is useful to understand the global coercivity estimate as explained in Remark A.7.

A.1.2. *Proof of the lower bound.* It turns out that the conditions on the kernel K_f given by Lemma A.3 plus the cancellation lemma is all we need to obtain the bound from below of Proposition A.1. For some arbitrary $R > 0$, let us call $\mu := c(1+R)^{-1}$ to the lower bound on the one-dimensional measure of the intersection of $A(v)$ with big circles as in the last item of Lemma A.3.

We start with a few preparatory lemmas.

Lemma A.4. *(See Figure 10) Let $\Xi(v)$ be the cones corresponding to the sets of directions $A(v)$ as in Lemma A.3. Let $\mathcal{L}$ be a line in $\mathbb{R}^d$ at distance $\rho > 0$ from a point $v \in \mathbb{R}^d$. Then, for some constants c and C depending only on $\mu(v)$,*

$$|\Xi(v) \cap \mathcal{L} \cap B_{C\rho}| \geqslant c\rho.$$

Proof. The projection of the line $\mathcal{L} - v$ on the sphere $\mathbb{S}^{d-1}$ is half of a big circle. According to Lemma A.3, the intersection of this projection with the set of directions $A = A(v)$ has (one-dimensional) measure at least $\mu(v)/2$ (recall that $A(v)$ is symmetric). At least half of these directions form an angle with $\mathcal{L}$ of at least $\mu(v)/8$. For each of these points $z \in A(v) \subset \mathbb{S}^{d-1}$, there corresponds an actual intersection point in $v + \alpha z \in \Xi(v) \cap \mathcal{L}$, with $\alpha \in [\rho, 8\mu(v)^{-1}\rho]$. Thus, the one dimensional measure of the points $v + \alpha z \in \Xi(v) \cap \mathcal{L} \cap B_{C\rho}$ is bounded below by $c\rho$, where $C = 8\mu(v)^{-1}$ and $c = \mu(v)/4$.

□

Lemma A.5. *Let $\Xi(v)$ be the cones corresponding to the directions $A(v)$ as in Lemma A.3. Let v_1 and v_2 be two points in $\mathbb{R}^d$. Assume $|v_1| \geqslant |v_2|$. Let $\mu(v_1) \geqslant \mu_0$ and $\mu(v_2) \geqslant \mu_0$ for some $\mu_0 > 0$. We have*

$$|\Xi(v_1) \cap \Xi(v_2) \cap B_r(v_2)| \geqslant c|v_1 - v_2|^d,$$

where $r = C|v_1 - v_2|$, and c and C depend on μ_0 only.

Proof. The lines $\mathcal{L}$ contained in $\Xi(v_1)$ are indexed by their directions $e \in A(v_1)$. At least half of these lines form an angle $\theta(\mu) > 0$ with the vector $v_2 - v_1$. In particular, all such lines are at distance at least $c|v_1 - v_2|$ from v_2, where c depends on μ and d only. According to Lemma A.4, for all such $\mathcal{L}$,

$$|\Xi(v_2) \cap \mathcal{L} \cap B_r(v_2)| \geqslant c|v_1 - v_2|.$$

Integrating over all directions $e \in A(v_1)$ we conclude the lemma. □

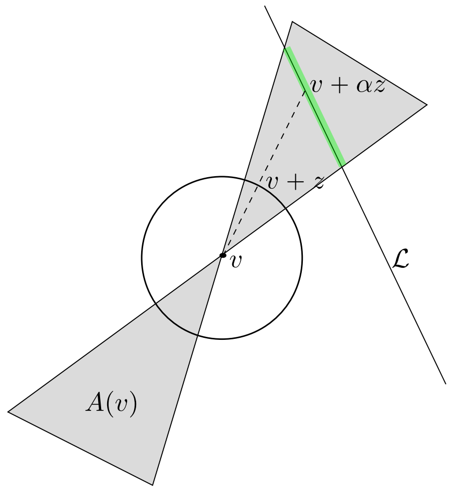

FIGURE 10. Intersection of a line $\mathcal{L}$ with a cone $\Xi(v)$.

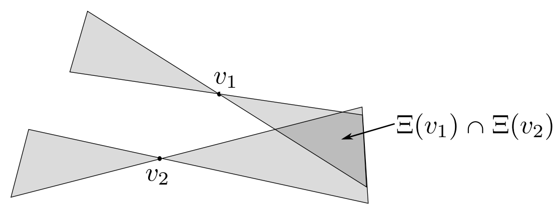

FIGURE 11. The intersection of two cones inside a ball.

The following lemma is the main estimate in the context of integro-differential equations, which implies Proposition A.1 when combined with the cancellation Lemma.

Lemma A.6. *Let $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a non-negative kernel like in Lemma A.3. Let $\mu = \inf\{\mu(v) : v \in B_R\}$.*

Then, there is a constant $c > 0$, depending only on μ , λ , d and s, so that for any function g supported in B_R,

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} (g(v) - g(v'))^2 K(v, v') \,\mathrm{d}v \,\mathrm{d}v' \geqslant c\|g\|^2_{\dot{H}^s}.$$

Proof. Symmetrizing the integral, we can replace $K(v, v')$ by $\frac{1}{2}(K(v, v') + K(v', v))$. Thus, we assume that K is symmetric.

From Lemma A.5, we have that for all $v_1, v_2 \in B_{2R}$, there is a constant C_0 (sufficiently large depending on R and the various constants involving f) so that

$$|\Xi(v_1) \cap \Xi(v_2) \cap B_{C_0|v_1-v_2|}(v_2)| \geqslant c|v_1 - v_2|^d.$$

Note that $B_{C_0|v_1-v_2|}(v_2) \subset B_{(C_0+1)|v_1-v_2|}(v_1)$. Moreover, we can choose c_0 small enough so that

$$|\Xi(v_1) \cap \Xi(v_2) \cap B_{C_0|v_1-v_2|}(v_2) \backslash B_{c_0|v_1-v_2|}(v_1) \backslash B_{c_0|v_1-v_2|}(v_2)| \geqslant c|v_1 - v_2|^d.$$

For the same choice of constants c_0 and C_0, let

$$N_r(v) := \int_{B_{C_0 r}(v) \backslash B_{c_0 r}(v)} |g(v) - g(w)|^2 K(v, w) \,\mathrm{d}w.$$

Therefore, for any $v_1, v_2 \in B_{2R}$, using that $|g(v_1) - g(z)|^2 + |g(v_2) - g(z)|^2 \geqslant |g(v_1) - g(v_2)|^2/2$, if we let $r = |v_1 - v_2|$,

$$\begin{aligned} N_r(v_1) + N_r(v_2) \geqslant c \Bigg(&\int_{\Xi(v_1)\cap B_{C_0 r}(v_1)\setminus B_{c_0 r}(v_1)} |g(v_1) - g(z)|^2 |v_1 - z|^{-d-2s} \,\mathrm{d}z \\ &+ \int_{\Xi(v_2)\cap B_{C_0 r}(v_2)\setminus B_{c_0 r}(v_2)} |g(v_1) - g(z)|^2 |v_2 - z|^{-d-2s} \,\mathrm{d}z \Bigg), \\ \geqslant c &\left(\int_{\Xi(v_1)\cap\Xi(v_2)\cap B_{C_0 r}(v_2)\setminus B_{c_0 r}(v_1)\setminus B_{c_0 r}(v_2)} |g(v_1) - g(v_2)|^2 r^{-d-2s} \,\mathrm{d}z \right), \\ \geqslant c &|g(v_1) - g(v_2)|^2 |v_1 - v_2|^{-2s}. \end{aligned}$$

Therefore

$$\begin{aligned} \|g\|_{H^s}^2 &\leqslant C \iint_{B_2\times B_2} |g(v_1) - g(v_2)|^2 |v_1 - v_2|^{-d-2s} \,\mathrm{d}v_1 \,\mathrm{d}v_2, \\ &\leqslant C \iint_{B_2\times B_2} (N_r(v_1) + N_r(v_2)) |v_1 - v_2|^{-d} \,\mathrm{d}v_1 \,\mathrm{d}v_2 \qquad \text{here } r = |v_1 - v_2|, \\ &= 2C \iint_{B_2\times B_2} N_r(v_1) |v_1 - v_2|^{-d} \,\mathrm{d}v_1 \,\mathrm{d}v_2, \\ &\leqslant C \int_{B_2} \int_{r=0}^{\infty} \int_{\mathbb{S}^{d-1}} N_r(v_1) r^{-1} \,\mathrm{d}\sigma \,\mathrm{d}r \,\mathrm{d}v_1, \qquad \text{using polar coordinates for } v_2, \\ &= C \int_{B_2} \int_{\mathbb{R}^d} |g(v_1) - g(z)|^2 K(v_1, z) \left(\int_{C_0^{-1}|v_1 - z|}^{c_0^{-1}|v_1 - z|} r^{-1} \,\mathrm{d}r \right) \mathrm{d}z \,\mathrm{d}v_1, \\ &= C \int_{B_2} \int_{\mathbb{R}^d} |g(v_1) - g(z)|^2 K(v_1, z) \,\mathrm{d}z \,\mathrm{d}v_1. \end{aligned}$$

This finishes the proof. □

Once we have Lemma A.6, we can derive Proposition A.1 as a corollary.

Proof of Proposition A.1. It follows from a simple computation using Lemma A.6 and Lemma 3.6.

$$\begin{aligned} -\int L_v g(v) g(v) \,\mathrm{d}v &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} (g(v) - g(v')) g(v) K_f(v, v') \,\mathrm{d}v' \,\mathrm{d}v, \\ &= \frac{1}{2} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (g(v) - g(v'))^2 K_f(v, v') \,\mathrm{d}v' \,\mathrm{d}v \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} g(v)^2 \left(\int_{\mathbb{R}^d} (K(v', v) - K(v, v')) \,\mathrm{d}v' \right) \mathrm{d}v, \\ &\geqslant \lambda \|g\|_{H^s}^2 - \Lambda \|g\|_{L^2}^2. \end{aligned}$$

The first term was bounded using Lemma A.6 and the second term using Lemma 3.6. □

Remark A.7. We sketch the precise asymptotics of the coercity estimate for large velocities. This computation plays no role in this paper, but it is interesting to see how the metric introduced in [37] arises naturally from the geometry described above. We only analyze the symmetric part of the bilinear form as in Lemma A.6. Analyzing the full bilinear form requires another similar computation for the cancellation estimate.

For large values of v, the cone $A(v)$ is approximately of width $1/|v|$ and perpendicular to v. The lower bound in Lemma A.6 depends only on a lower bound for $\mu(v)$ in B_R and the lower bound for $K(v, v')$ for $v' \in \Xi(v)$. It is easy to see how the estimate behaves for large velocities from a scaling argument. Indeed, let $v_0 \in \mathbb{R}^d$. For every $v \in B_1(v_0)$, the cone $\Xi(v)$ has measure $\mu(v) \gtrsim (1 + |v_0|)^{-1}$ and it is approximately perpendicular to v_0 in the sense described above. Let T be the linear change of variables

$$Tv = (1 + |v_0|)^{-1} P v + P^{\perp} v, \qquad \text{where } Pv = \frac{\langle v, v_0 \rangle}{|v_0|^2} v_0, \qquad P^{\perp} v = v - Pv.$$

Let $\tilde{g}(v) = g(Tv)$. So that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} (g(v)-g(v'))^2 K(v,v')\,\mathrm{d}v\,\mathrm{d}v' = \iint_{\mathbb{R}^d\times\mathbb{R}^d} (\tilde{g}(v)-\tilde{g}(v'))^2 \tilde{K}(v,v')\,\mathrm{d}v\,\mathrm{d}v',$$

where

$$\tilde{K}(v,v') = |\det(DT)|^2 K(Tv,Tv') = (1+|v_0|)^{-2} K(Tv,Tv').$$

The point of this change of variables is to make the non-degeneracy cone $\tilde{K}$ bounded below in measure for all $v \in B_1(T^{-1}(v_0))$, uniformly in v_0, i.e. $\tilde{\mu}(v) \gtrsim 1$ for all $v \in B_1(T^{-1}v_0)$. Moreover, for v' in this nondegeneracy cone $\tilde{\Xi}(v)$, we have

$$\begin{aligned}\tilde{K}(v,v') &= (1+|v_0|)^{-2} K(Tv,Tv'),\\ &\geqslant \lambda(1+|v_0|)^{-1+\gamma+2s}|Tv-Tv'|^{-d-2s}.\end{aligned}$$

Therefore, from the computation in the proof of Lemma A.6, for some universal constant $r > 0$, and $D_r = T(B_r(T^{-1}v_0))$, we get

$$\begin{aligned}\iint_{D_1\times D_1} (g(v)-g(v'))^2 K(v,v')\,\mathrm{d}v\,\mathrm{d}v' &= \iint_{B_1(T^{-1}v_0)\times B_1(T^{-1}v_0)} (\tilde{g}(v)-\tilde{g}(v'))^2\tilde{K}(v,v')\,\mathrm{d}v\,\mathrm{d}v',\\ &\geqslant c(1+|v_0|)^{-2}\iint_{B_r(T^{-1}v_0)\times B_r(T^{-1}v_0)} \frac{(\tilde{g}(v)-\tilde{g}(v'))^2}{|v-v'|^{d+2s}}\,\mathrm{d}v\,\mathrm{d}v',\\ &= c(1+|v_0|)^{-1+\gamma+2s}\iint_{B_r(T^{-1}v_0)\times B_r(T^{-1}v_0)} \frac{|\tilde{g}(v)-\tilde{g}(v')|^2}{|v-v'|^{d+2s}}\,\mathrm{d}v\,\mathrm{d}v',\\ &= c(1+|v_0|)^{1+\gamma+2s}\iint_{D_r\times D_r} \frac{|g(v)-g(v')|^2}{|T^{-1}v-T^{-1}v'|^{d+2s}}\,\mathrm{d}v\,\mathrm{d}v'.\end{aligned}$$

Note that $|T^{-1}v - T^{-1}v'|$ is equivalent to the metric $d(v,v')$ introduced in [37]. The set D_r is exactly the ball of radius r centered at v_0. Therefore, covering $\mathbb{R}^d$ with these balls $D_r(v_0)$ and adding up, we get

$$\iint_{d(v,v')<1} (g(v)-g(v'))^2 K(v,v')\,\mathrm{d}v\,\mathrm{d}v' \geqslant c(1+|v_0|)^{1+\gamma+2s}\iint_{d(v,v')<r} \frac{|g(v)-g(v')|^2}{d(v,v')^{d+2s}}\,\mathrm{d}v\,\mathrm{d}v'.$$

The right hand side is the same as the norm $\|g\|^2_{N^{s,\gamma}}$ introduced in [37] minus a lower order correction corresponding to the tails of the integral.

Remark A.8. It is interesting to notice that the estimate of Lemma A.6 depends only on the structure of the kernel described in Lemma A.3. It would be interesting to see whether the result of Lemma A.6 holds for general kernels K (not necessarily arising from the Boltzmann equation) under less restrictive conditions on the cones $A(v)$. There is an interesting conjecture mentioned in [33] which is related to our condition (1.4).

A.2. **Technical lemmas.**

A.2.1. *Change of variables.* We recall here a change of variables from [59].

Lemma A.9 (Change of variables [59, Lemma A.1])**.** *For any non-negative function of* (v, v_*, v', v'_*),

$$\int_{\mathbb{R}^d}\int_{\mathbb{S}^{d-1}} F\,\mathrm{d}\sigma dv_* = 2^{d-1}\int_{\mathbb{R}^d}\frac{1}{|v'-v|}\int_{w\perp v'-v} F\frac{1}{r^{d-2}}\,\mathrm{d}w\,\mathrm{d}v' \tag{A.3}$$

$$= \int_{\mathbb{R}^d}\frac{1}{|v'_*-v|}\int_{w\perp v'_*-v} F\frac{1}{r^{d-2}}\,\mathrm{d}w\,\mathrm{d}v'_*. \tag{A.4}$$

Other changes of variables were used in proofs.

Lemma A.10 (Change of variables – II). *Let $F : \mathbb{R}^d \to \mathbb{R}$ be any integrable function. Then, the following identities hold.*

$$\int_{\partial B_r} \int_{\{w : w \perp \sigma\}} F(w) \, \mathrm{d}w \, \mathrm{d}\sigma = \omega_{d-2} r^{d-1} \int_{\mathbb{R}^d} \frac{F(z)}{|z|} \, \mathrm{d}z, \tag{A.5}$$

$$\int_{\partial B_r} \int_{\{w : w \perp \sigma\}} F(\sigma + w) \, \mathrm{d}w \, \mathrm{d}\sigma = \omega_{d-2} r^{d-1} \int_{\mathbb{R}^d \setminus B_r} F(z) \frac{(|z|^2 - r^2)^{\frac{d-3}{2}}}{|z|^{d-2}} \, \mathrm{d}z, \tag{A.6}$$

$$\int_{\partial B_r} \int_{\{w : w \perp \sigma\}} \sigma F(\sigma + w) \, \mathrm{d}w \, \mathrm{d}\sigma = \omega_{d-2} r^{d+1} \int_{\mathbb{R}^d \setminus B_r} z F(z) \frac{(|z|^2 - r^2)^{\frac{d-3}{2}}}{|z|^{d}} \, \mathrm{d}z. \tag{A.7}$$

Here the constant ω_{d-2} stands for the surface area of $\mathcal{S}^{d-2}$. Note that the integrals on the left hand side are on spheres and hyperplanes, thus $\mathrm{d}w$ and $\mathrm{d}\sigma$ stand for differential of surface.

A.2.2. *Positive part of subsolutions.*

Lemma A.11 (Positive part of subsolutions). *Let f be a subsolution of (1.2) in a cylinder Q. Then $f_+ = \max(f, 0)$ is still a subsolution of (1.2) (where h is replaced with $h\mathbf{1}_{f \geqslant 0}$) in Q.*

Proof. Since we assume that $f_t + v \cdot \nabla_x f \in L^2$, then

$$\partial_t f_+ + v \cdot \nabla_x f_+ = \begin{cases} f_t + v \cdot \nabla_x f & \text{where } f > 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{A.8}$$

The equality holds in the sense of distributions.

In order to conclude that f_+ is a subsolution of (1.2), we need to prove the following inequality in the sense of distributions.

$$L_v f_+ \geqslant \begin{cases} L_v f & \text{where } f > 0, \\ 0 & \text{elsewhere.} \end{cases} \tag{A.9}$$

Let $\gamma(r) = r_+$ and let $\{\gamma_\delta\}_\delta$ be a smooth approximation of γ such that $|\gamma_\delta'| \leqslant 1$ and γ_δ is convex. Let ρ_ε be an even mollifier and $f_\varepsilon = \rho_\varepsilon * f$. Here the mollification is done with respect to the variable v only.

Since $f \in L^2([0,T], \mathbb{R}^d, H^s(\mathbb{R}^d))$, it is not hard to see that

$$\lim_{\delta \to 0} \lim_{\varepsilon \to 0} \gamma_\delta(f_\varepsilon) \rightharpoonup f_+ \qquad \text{weakly in } L^2([0,T], \mathbb{R}^d, H^s(\mathbb{R}^d)).$$

Therefore, because of Theorem 4.1, for any smooth test function φ, we have

$$\begin{aligned} \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \iiint L_v[\gamma_\delta(f_\varepsilon)] \varphi \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t &= \lim_{\delta \to 0} \lim_{\varepsilon \to 0} \iint \mathcal{E}(\gamma_\delta(f_\varepsilon), \varphi) \, \mathrm{d}x \, \mathrm{d}t \\ &= \iint \mathcal{E}(f_+, \varphi) \, \mathrm{d}x \, \mathrm{d}t, \\ &= \iiint L_v[f_+] \varphi \, \mathrm{d}v \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

This proves that $L_v[\gamma_\delta(f_\varepsilon)]$ converges to $L_v[f_+]$ in the sense of distributions.

Thus, in order to obtain (A.9) and finish the proof, we need to prove that for all $\delta > 0$ and $\varepsilon > 0$,

$$L_v[\gamma_\delta(f_\varepsilon)] \geqslant \gamma_\delta'(f_\varepsilon) L_v[f_\varepsilon]. \tag{A.10}$$

In fact, we can check by a direct computation that the inequality holds pointwise. Indeed,

$$\begin{aligned} L_v[\gamma_\delta(f_\varepsilon)](v) &= \int (\gamma_\delta(f_\varepsilon(v')) - \gamma_\delta(f_\varepsilon(v))) K(v, v') \, \mathrm{d}v', \\ &\geqslant \int \gamma_\delta'(f_\varepsilon(v))(f_\varepsilon(v') - f_\varepsilon(v)) K(v, v') \, \mathrm{d}v' \qquad \text{using the convexity of } \gamma_\delta, \\ &= \gamma_\delta'(f_\varepsilon(v)) L_v f_\varepsilon(v). \end{aligned}$$

Taking the limit as $\delta \to 0$ and $\varepsilon \to 0$ in (A.10) we obtain (A.9). Combining it with (A.8) we finish the proof. $\square$

Lemma A.12 (Maximum principle). *If f is a weak subsolution of (1.2), with $h = 0$, in $Q = (a, b] \times \Omega_x \times \Omega_v$, then*

$$\operatorname*{esssup}_{Q} f \leqslant \operatorname{esssup} \left\{ f(t, x, v) : t \in ([a, b] \times \overline{\Omega_x} \times \mathbb{R}^d) \backslash (a, b] \times \Omega_x \times \Omega_v \right\}.$$

Proof. Let $m = \operatorname{esssup}\{f(t, x, v) : t \in ([a, b] \times \overline{\Omega_x} \times \mathbb{R}^d) \backslash (a, b] \times \Omega_x \times \Omega_v\}$. The proof follows using $(f - m)_+$ as a test function in (5.5). □

References

[1] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg. Entropy dissipation and long-range interactions. *Arch. Ration. Mech. Anal.*, 152(4):327–355, 2000.

[2] R. Alexandre, Y. Morimoto, S. Ukai, C. J. Xu, and T. Yang. Global existence and full regularity of the Boltzmann equation without angular cutoff. *Communications in Mathematical Physics*, 304(2):513–581, 2011.

[3] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang. The Boltzmann equation without angular cutoff in the whole space: I, global existence for soft potential. *Journal of Functional Analysis*, 262(3):915 – 1010, 2012.

[4] Radjesvarane Alexandre. A review of Boltzmann equation with singular kernels. *Kinet. Relat. Models*, 2(4):551–646, 2009.

[5] Radjesvarane Alexandre and Mouhamad El Safadi. Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. I. Non-cutoff case and Maxwellian molecules. *Math. Models Methods Appl. Sci.*, 15(6):907–920, 2005.

[6] Radjesvarane Alexandre and Mouhamad Elsafadi. Littlewood-Paley theory and regularity issues in Boltzmann homogeneous equations. II. Non cutoff case and non Maxwellian molecules. *Discrete Contin. Dyn. Syst.*, 24(1):1–11, 2009.

[7] Radjesvarane Alexandre, Yoshinore Morimoto, Seiji Ukai, Chao-Jiang Xu, and Tong Yang. Regularity of solutions for the Boltzmann equation without angular cutoff. *C. R. Math. Acad. Sci. Paris*, 347(13-14):747–752, 2009.

[8] Radjesvarane Alexandre, Yoshinori Morimoto, Seiji Ukai, Chao-Jiang Xu, and Tong Yang. Global well-posedness theory for the spatially inhomogeneous Boltzmann equation without angular cutoff. *Comptes Rendus Mathematique*, 348(15):867–871, 2010.

[9] Radjesvarane Alexandre, Yoshinori Morimoto, Seiji Ukai, Chao-Jiang Xu, and Tong Yang. Regularizing effect and local existence for the non-cutoff Boltzmann equation. *Archive for Rational Mechanics and Analysis*, 198(1):39–123, 2010.

[10] Radjesvarane Alexandre and Cédric Villani. On the Boltzmann equation for long-range interactions. *Communications on pure and applied mathematics*, 55(1):30–70, 2002.

[11] Claude Bardos, François Golse, and David Levermore. Fluid dynamic limits of kinetic equations. I. Formal derivations. *J. Statist. Phys.*, 63(1-2):323–344, 1991.

[12] Guy Barles, Emmanuel Chasseigne, Adina Ciomaga, and Cyril Imbert. Lipschitz regularity of solutions for mixed integro-differential equations. *J. Differential Equations*, 252(11):6012–6060, 2012.

[13] Guy Barles, Emmanuel Chasseigne, and Cyril Imbert. Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. *J. Eur. Math. Soc. (JEMS)*, 13(1):1–26, 2011.

[14] Martin T. Barlow, Richard F. Bass, Zhen-Qing Chen, and Moritz Kassmann. Non-local Dirichlet forms and symmetric jump processes. *Trans. Amer. Math. Soc.*, 361(4):1963–1999, 2009.

[15] Richard F. Bass and Moritz Kassmann. Harnack inequalities for non-local operators of variable order. *Trans. Amer. Math. Soc.*, 357(2):837–850, 2005.

[16] Richard F. Bass and Moritz Kassmann. Hölder continuity of harmonic functions with respect to operators of variable order. *Comm. Partial Differential Equations*, 30(7-9):1249–1259, 2005.

[17] Richard F. Bass and David A. Levin. Transition probabilities for symmetric jump processes. *Trans. Amer. Math. Soc.*, 354(7):2933–2953 (electronic), 2002.

[18] S. Bernstein. Sur la nature analytique des solutions des équations aux dérivées partielles du second ordre. *Math. Ann.*, 59(1-2):20–76, 1904.

[19] C. Bjorland, L. Caffarelli, and A. Figalli. Non-local gradient dependent operators. *Adv. Math.*, 230(4-6):1859–1894, 2012.

[20] Krzysztof Bogdan and Paweł Sztonyk. Harnack's inequality for stable Lévy processes. *Potential Anal.*, 22(2):133–150, 2005.

[21] Luis Caffarelli, Chi Hin Chan, and Alexis Vasseur. Regularity theory for parabolic nonlinear integral operators. *Journal of the American Mathematical Society*, 24(3):849–869, 2011.

[22] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.

[23] Luis Caffarelli and Luis Silvestre. Hölder regularity for generalized master equations with rough kernels. In *Advances in analysis: the legacy of Elias M. Stein*, volume 50 of *Princeton Math. Ser.*, pages 63–83. Princeton Univ. Press, Princeton, NJ, 2014.

[24] Torsten Carleman. Sur la théorie de l'équation intégrodifférentielle de Boltzmann. *Acta Math.*, 60(1):91–146, 1933.

[25] Héctor Chang Lara and Gonzalo Dávila. Regularity for solutions of nonlocal, nonsymmetric equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(6):833–859, 2012.

[26] Héctor A. Chang-Lara and Gonzalo Dávila. Hölder estimates for non-local parabolic equations with critical drift. *J. Differential Equations*, 260(5):4237–4284, 2016.

[27] Yemin Chen and Lingbing He. Smoothing estimates for Boltzmann equation with full-range interactions: Spatially homogeneous case. *Archive for rational mechanics and analysis*, 201(2):501–548, 2011.

[28] Yemin Chen and Lingbing He. Smoothing estimates for Boltzmann equation with full-range interactions: Spatially inhomogeneous case. *Archive for Rational Mechanics and Analysis*, 203(2):343–377, 2012.

[29] Zhen-Qing Chen, Panki Kim, and Takashi Kumagai. Global heat kernel estimates for symmetric jump processes. *Trans. Amer. Math. Soc.*, 363(9):5021–5055, 2011.

[30] Ennio De Giorgi. Sulla differenziabilità e lanaliticità delle estremali degli integrali multipli regolari, mem. *Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat*, 3:25–43, 1957.

[31] L. Desvillettes and C. Villani. On the trend to global equilibrium for spatially inhomogeneous kinetic systems: the Boltzmann equation. *Invent. Math.*, 159(2):245–316, 2005.

[32] Laurent Desvillettes and Bernt Wennberg. Smoothness of the solution of the spatially homogeneous Boltzmann equation without cutoff. *Comm. Partial Differential Equations*, 29(1-2):133–155, 2004.

[33] Bartlomiej Dyda and Moritz Kassmann. Regularity estimates for elliptic nonlocal operators. *arXiv preprint arXiv:1509.08320*, 2015.

[34] Matthieu Felsinger and Moritz Kassmann. Local regularity for parabolic nonlocal operators. *Communications in Partial Differential Equations*, 38(9):1539–1573, 2013.

[35] I. Gamba, N. Pavlović, and M. Tasković. On the pointwise propagation of exponential tails for the Boltzmann equation without cutoff,. Preprint.

[36] F Golse, Cyril Imbert, Clément Mouhot, and Alexis Vasseur. Harnack inequality for kinetic fokker-planck equations with rough coefficients and application to the landau equation. *arXiv preprint arXiv:1607.08068*, 2016.

[37] Philip Gressman and Robert Strain. Global classical solutions of the Boltzmann equation without angular cut-off. *Journal of the American Mathematical Society*, 24(3):771–847, 2011.

[38] Philip T Gressman and Robert M Strain. Global classical solutions of the Boltzmann equation with long-range interactions. *Proceedings of the National Academy of Sciences*, 107(13):5744–5749, 2010.

[39] Zhaohui Huo, Yoshinori Morimoto, Seiji Ukai, and Tong Yang. Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff. *Kinet. Relat. Models*, 1(3):453–489, 2008.

[40] Cyril Imbert and Luis Silvestre. An introduction to fully nonlinear parabolic equations. In *An introduction to the Kähler-Ricci flow*, volume 2086 of *Lecture Notes in Math.*, pages 7–88. Springer, Cham, 2013.

[41] Moritz Kassmann. A priori estimates for integro-differential operators with measurable kernels. *Calc. Var. Partial Differential Equations*, 34(1):1–21, 2009.

[42] Moritz Kassmann, Marcus Rang, and Russell W. Schwab. Integro-differential equations with nonlinear directional dependence. *Indiana Univ. Math. J.*, 63(5):1467–1498, 2014.

[43] Moritz Kassmann and Russell W Schwab. Regularity results for nonlocal parabolic equations. *arXiv preprint arXiv:1305.5418*, 2013.

[44] Takashi Komatsu. Continuity estimates for solutions of parabolic equations associated with jump type Dirichlet forms. *Osaka J. Math.*, 25(3):697–728, 1988.

[45] N. V. Krylov. *Nonlinear elliptic and parabolic equations of the second order*, volume 7 of *Mathematics and its Applications (Soviet Series)*. D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian by P. L. Buzytsky [P. L. Buzytskiĭ].

[46] N. V. Krylov and M. V. Safonov. A property of the solutions of parabolic equations with measurable coefficients. *Izv. Akad. Nauk SSSR Ser. Mat.*, 44(1):161–175, 239, 1980.

[47] Héctor Chang Lara and Gonzalo Dávila. Regularity for solutions of non local parabolic equations. *Calc. Var. Partial Differential Equations*, 49(1-2):139–172, 2014.

[48] Pierre-Louis Lions. Régularité et compacité pour des noyaux de collision de Boltzmann sans troncature angulaire. *Comptes Rendus de l'Académie des Sciences-Series I-Mathematics*, 326(1):37–41, 1998.

[49] Zhang Liqun. The C^α regularity of a class of ultraparabolic equations. In *Third International Congress of Chinese Mathematicians. Part 1, 2*, volume 2 of *AMS/IP Stud. Adv. Math., 42, pt. 1*, pages 619–622. Amer. Math. Soc., Providence, RI, 2008.

[50] Ralf Metzler and Joseph Klafter. The restaurant at the end of the random walk: recent developments in the description of anomalous transport by fractional dynamics. *Journal of Physics A: Mathematical and General*, 37(31):R161, 2004.

[51] Yoshinori Morimoto, Seiji Ukai, Chao-Jiang Xu, Tong Yang, et al. Regularity of solutions to the spatially homogeneous boltzmann equation without angular cutoff. *Discrete and Continuous Dynamical Systems-Series A*, 24(1):187–212, 2009.

[52] Jürgen Moser. A harnack inequality for parabolic differential equations. *Communications on pure and applied mathematics*, 17(1):101–134, 1964.

[53] Clément Mouhot. Quantitative lower bounds for the full Boltzmann equation. I. Periodic boundary conditions. *Comm. Partial Differential Equations*, 30(4-6):881–917, 2005.

[54] John Nash. Continuity of solutions of parabolic and elliptic equations. *American Journal of Mathematics*, 80(4):931–954, 1958.

[55] Andrea Pascucci and Sergio Polidoro. The Moser's iterative method for a class of ultraparabolic equations. *Commun. Contemp. Math.*, 6(3):395–417, 2004.

[56] Russell W Schwab and Luis Silvestre. Regularity for parabolic integro-differential equations with very irregular kernels. *arXiv preprint arXiv:1412.3790*, 2014.

[57] Luis Silvestre. Hölder estimates for solutions of integro-differential equations like the fractional Laplace. *Indiana Univ. Math. J.*, 55(3):1155–1174, 2006.

[58] Luis Silvestre. On the differentiability of the solution to the Hamilton-Jacobi equation with critical fractional diffusion. *Adv. Math.*, 226(2):2020–2039, 2011.

[59] Luis Silvestre. A new regularization mechanism for the boltzmann equation without cut-off. *arXiv preprint arXiv:1412.4706*, 2014.

[60] Renming Song and Zoran Vondraček. Harnack inequality for some classes of Markov processes. *Math. Z.*, 246(1-2):177–202, 2004.

[61] Peter Tankov. *Financial modelling with jump processes*, volume 2. CRC press, 2003.
[62] Cédric Villani. Regularity estimates via the entropy dissipation for the spatially homogeneous Boltzmann equation without cut-off. *Rev. Mat. Iberoamericana*, 15(2):335–352, 1999.
[63] Cédric Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 71–305. North-Holland, Amsterdam, 2002.
[64] WenDong Wang and LiQun Zhang. The C^α regularity of a class of non-homogeneous ultraparabolic equations. *Sci. China Ser. A*, 52(8):1589–1606, 2009.
[65] Wendong Wang and Liqun Zhang. The C^α regularity of weak solutions of ultraparabolic equations. *Discrete Contin. Dyn. Syst.*, 29(3):1261–1275, 2011.
[66] Wikipedia. Hilbert's nineteenth problem, 2016. [Online; accessed 23-June-2016].

(C. Imbert) CNRS & Department of Mathematics and Applications, École Normale Supérieure (Paris), 45 rue d'Ulm, 75005 Paris, France
E-mail address: Cyril.Imbert@ens.fr

(L. Silvestre) Mathematics Department, University of Chicago, Chicago, Illinois 60637, USA
E-mail address: luis@math.uchicago.edu