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THE WEIGHTED HARDY'S INEQUALITY FOR NONINCREASING FUNCTIONS

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ABSTRACT. The purpose of this paper is to give an alternative proof of recent results of M. Arino and B. Muckenhoupt [1] and E. Sawyer [8], concerning Hardy's inequality for nonincreasing functions and related applications to the boundedness of some classical operators on general Lorentz spaces. Our approach will extend the results of [1, 8] to the values of the parameters which are inaccessible by the methods of these papers.

1. INTRODUCTION

The Hardy's inequality of the form

$$(1)\qquad \left(\int_0^\infty \left(\int_0^x f(t)\,dt\right)^q u(x)dx\right)^{1/q} \le C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p}$$

is well known. The problem of determining weights u and v for fixed p and q, $1 \le p,\ q \le \infty$, has been investigated by many authors [3, 5, 6, 12, 13]; furthermore, the important case $0 < q < 1 < p < \infty$ has recently been studied by G. Sinnamon [9, 10] (see , also [11]). The final form of these results may be summarized as follows.

Theorem 1. *Necessary and sufficient conditions for the validity of the inequality* (1) *with weights* $u, v \ge 0$ *for all* $f \ge 0$ *such that the right side of* (1) *is finite, and constant* C *independent of* f *are*

(a) *Let* $1 \le p \le q \le \infty$, $1/p + 1/p' = 1$. *Then* (1) *holds iff*

$$(2)\qquad A \equiv \sup_{t>0}\left(\int_t^\infty u(x)\,dx\right)^{1/q}\left(\int_0^t v^{1-p'}(x)\,dx\right)^{1/p'} < \infty.$$

Moreover $A \le C \le \alpha(p, q)A$.

(b) *Let* $0 < q < p < \infty$, $p > 1$, $1/r = 1/q - 1/p$, $1/q + 1/q' = 1$. *Then* (1) *holds iff*

(3)

$$B \equiv \left(\int_0^\infty \left[\left(\int_t^\infty u(x)\,dx\right)^{1/q}\left(\int_0^t v^{1-p'}(x)\,dx\right)^{1/q'}\right]^r v^{1-p'}(t)\,dt\right)^{1/r} < \infty$$

and $\beta_1(p, q)B \le C \le \beta_2(p, q)B$.

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The constants α, β_1, β_2 *depend only on* p, q.

Several authors [1, 7, 8] have recently considered inequality

$$\text{(4)} \qquad \left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^q w(x)\,dx\right)^{1/q} \le C\left(\int_0^\infty f^p(x)v(x)\,dx\right)^{1/p}$$

where f is a nonnegative and nonincreasing function.

Necessary and sufficient conditions are known for (4) if $1 \le p = q < \infty$, $w = v$. These can be derived using the so called "dilation function." Specifically, let $V(\tau) = \int_0^\tau v(x)\,dx$. Then the inequality

$$\text{(5)} \qquad \left(\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^p v(x)\,dx\right)^{1/p} \le C\left(\int_0^\infty g^p(x)v(x)\,dx\right)^{1/p}$$

is valid for nonincreasing functions $g \ge 0$ iff for some $\varepsilon > 0$

$$\text{(6)} \qquad \sup_{s>0} \frac{V(\tau s)}{V(s)} = O(\tau^{p-\varepsilon}), \qquad \tau \ge 1.$$

This was shown by D. W. Boyd [2]. S. G. Krein and E. M. Semenov [4, Chapter 2, Theorem 6.6].

Recently M. Arino and M. Muckenhoupt [1] have shown that (5) is equivalent to v satisfying the condition

$$\text{(7)} \qquad \int_t^\infty x^{-p}v(x)\,dx \le \frac{D}{t^p}\int_0^t v(x)\,dx$$

where the constant D does not depend on $t \in (0, \infty)$. They have also pointed out the application of this criterion to a similar problem for Hardy-Littlewood's maximal function on general Lorentz spaces.

E. Sawyer [8] has extended the above results to cases of different weights v, w and exponents $1 < p$, $q < \infty$. Sawyer's approach is quite general and reduces the problem to Hardy's inequality with arbitrary functions by means of duality. His method relies upon a proof of the reverse Hölder inequality of the form

$$\sup_{g\ge 0,\, g\downarrow} \frac{\int_0^\infty gf}{\left(\int_0^\infty g^p v\right)^{1/p}} \approx \left(\int_0^\infty \left(\int_0^x f\right)^{p'} \left(\frac{1}{V}\right)^{p'} v\,dx\right)^{1/p'} + \frac{\int_0^\infty f}{\left(\int_0^\infty v\right)^{1/p}}$$

where $g\downarrow$ means that the function g is nonincreasing. If T is the operator defined by $Tf(x) = \int_0^\infty k(x,y)f(y)\,dy$ where $k(x,y)$ is a nonnegative kernel and if for simplicity we assume that $\int_0^\infty v = \infty$, one can apply this result to obtain the equivalence

$$\left(\int_0^\infty (Tg)^q w\right)^{1/q} \le C\left(\int_0^\infty g^p v\right)^{1/p} \quad \forall 0 \le g\downarrow$$

$$\Leftrightarrow \left(\int_0^\infty \left(\int_0^x T^* f\right)^{p'} \left[\frac{1}{V}\right]^{p'} v\,dx\right)^{1/p'} \le C\left(\int_0^\infty f^{q'} w^{1-q'}\right)^{1/q'} \quad \forall f \ge 0.$$

For instance, if $Tf(x) = \frac{1}{x}\int_0^x f(t)\,dt$, then

$$\text{(8)} \qquad \int_0^x T^* f = \int_0^x f + x\int_x^\infty f(t)\frac{dt}{t}.$$

Sawyer then proved the following theorem via the corresponding Hardy's inequalities for the operators on the right side of (8) using Theorem 1 and its dual.

Theorem 2. *Necessary and sufficient conditions for the validity of* (4) *with the weights* $w \geq 0$ *and* $v \geq 0$ *for all nonincreasing functions* $f \geq 0$ *with finite right side of* (4) *and constant* C *independent of* f *are*

(a) *Let* $1 < p \leq q < \infty$. *Then* (4) *is valid iff*

$$A_0 \equiv \sup_{t>0} \left(\int_0^t w(x)\,dx\right)^{1/q} \left(\int_0^t v(x)\,dx\right)^{-1/p} < \infty, \tag{9}$$

$$A_1 \equiv \sup_{t>0} \left(\int_t^\infty x^{-q} w(x)\,dx\right)^{1/q} \left(\int_0^t x^{p'} V^{-p'}(x) v(x)\,dx\right)^{1/p'} < \infty \tag{10}$$

and $\alpha_1(p, q)(A_0 + A_1) \leq C \leq \alpha_2(p, q)(A_0 + A_1)$.

(b) *Let* $1 < q < p < \infty$, $1/r = 1/q - 1/p$. *Then* (4) *is valid iff*

$$B_0 \equiv \left(\int_0^\infty \left[\left(\int_0^t w(x)\,dx\right)^{1/p} \left(\int_0^t v(x)\,dx\right)^{-1/p}\right]^r w(t)\,dt\right)^{1/r} < \infty, \tag{11}$$

$$B_1 = \left(\int_0^\infty \left[\left(\int_t^\infty x^{-q} w(x)\,dx\right)^{1/q} \cdot \left(\int_0^t x^{p'} V^{-p'}(x) v(x)\,dx\right)^{1/q'}\right]^r t^{p'} V^{-p'}(t) v(t)\,dt\right)^{1/r} < \infty \tag{12}$$

and also $\beta_3(p, q)(B_0 + B_1) \leq C \leq \beta_4(p, q)(B_0 + B_1)$. *The constants* α_1, α_2, β_3, β_4 *depend only on* p, q.

In §2 we shall give an alternative proof of Theorem 2. The main new result of the paper is the following extension of Theorem 2 to the case $0 < q < 1 < p < \infty$, and $0 < p \leq q < \infty$, $0 < p < 1$.

Theorem 3. *Let the assumptions of Theorem* 2 *hold. Then*

(a) *Assertion* (b) *of Theorem* 2 *holds if* $0 < q < 1 < p < \infty$.

(b) *Let* $0 < p \leq q < \infty$, $0 < p < 1$. *Then* (4) *is valid iff* $A_0 < \infty$ *and*

$$\mathscr{A}_1 = \sup_{t>0} t \left(\int_t^\infty x^{-q} w(x)\,dx\right)^{1/q} \left(\int_0^t v(x)\,dx\right)^{-1/p} < \infty \tag{13}$$

and $\gamma_1(p, q)(A_0 + \mathscr{A}_1) \leq C \leq \gamma_2(p, q)(A_0 + \mathscr{A}_1)$. *The constants* γ_1, γ_2 *depend only on* p, q.

The proof of Theorem 3 is given in §3; at the end of this section we state and prove a new sufficient condition (Proposition 2) for (4) when $0 < q < p < 1$.

The following notations will be employed in the paper.

$$V(\tau) \equiv \int_0^\tau v(x)dx; \qquad W(\tau) \equiv \int_0^\tau w(x)\,dx; \qquad \Psi_0^{1-p'}(x) \equiv x^{p'} V^{-p'}(x) v(x).$$

$A \ll B$ means the inequality $A \leq cB$ with the constant c depending on p, q only. Products and quotients of the forms $0 \cdot \infty$, $\frac{\infty}{\infty}$, $\frac{0}{0}$ are taken to be 0.

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2. Proof of Theorem 2

2.1. First we consider the characterization problem for the inequality

$$\left(\int_0^\infty f^q w\right)^{1/q} \leq C\left(\int_0^\infty f^p v\right)^{1/p}. \tag{14}$$

We need the following preliminary results.

Lemma. *If* $0 < q < p < \infty$, $1/r = 1/q - 1/p$, *then* $A_0 \ll B_0$ *and*

$$B_0 \equiv \left(\int_0^\infty W^{r/p} V^{-r/p} w\right)^{q/r} < \infty \Leftrightarrow \mathscr{B}_0 \equiv \left(\int_0^\infty W^{r/q} V^{-r/q} v\right)^{1/r} < \infty.$$

Moreover, $B_0 \ll \mathscr{B}_0 \ll B_0$.

Proof. First we show "$\Rightarrow$". Let $B_0 < \infty$, then

$$V^{-r/p}(x) W^{r/q}(x) = V^{-r/p} \int_0^x dW^{r/q} \leq \int_0^x V^{-r/p} dW^{r/q} \to 0, \qquad x \to 0.$$

This implies that $A_0 \ll B_0$ and integration by parts yields

$$\infty > B_0^r = \frac{q}{r}\int_0^\infty V^{-r/p} dW^{r/q} \geq \frac{q}{r}\int_0^\infty W^{r/q} d(-V^{-r/p}) = \frac{q}{p}\mathscr{B}_0^r.$$

The proof of "$\Leftarrow$" is similar.

Proposition 1. *Necessary and sufficient conditions for the validity of* (14) *for all nonincreasing* $f \geq 0$ *are as follows.*

(a) *Let* $0 < p \leq q < \infty$. *Then* (14) *holds iff* $A_0 < \infty$.

(b) *Let* $0 < q \leq p < \infty$. *Then* (14) *holds iff* $B_0 < \infty$.

Remark. For $1 < p$, $q < \infty$ Proposition 1 is contained in Remark (i) of [8].

Proof. (a) The necessary part is trivial by substituting in the inequality $f = f_t = \chi_{[0,t]}$, where $\chi_E(x) = 1$ for $x \in E$ and $\chi_E(x) = 0$ for $x \notin E$. To prove sufficiency we proceed as follows. Let $h \geq 0$, $\operatorname{supp} h \subset (0, \infty)$, $f(x) = \int_x^\infty h$ and $\int_0^\infty f^p v < \infty$. Integration by parts and Minkowski's inequality yields

$$\int_0^\infty f^q w = \int_0^\infty W d(-f^q) \leq A_0^q \int_0^\infty V^{q/p} d(-f^q)$$
$$\leq A_0^q \left(\int_0^\infty \left(\int_x^\infty d(-f^q)\right)^{p/q} v\right)^{q/p} = A_0^q \left(\int_0^\infty f^p v\right)^{q/p}.$$

The proof is complete if $\int_0^\infty v = \infty$. If $\int_0^\infty v < \infty$, then we modify the argument by taking $f(x) = c + \int_x^\infty h$.

We next consider Proposition 1(b). Let us assume $\int_0^\infty v = \infty$. The necessary part proceeds as follows.

$$\mathscr{B}_0^r = \int_0^\infty W^{r/q} V^{-r/q} v = \int_0^\infty v(x)dx \int_x^\infty W^{r/q} V^{-(r/q)-1} v \equiv \int_0^\infty f_0^p v$$

where f_0 is defined by the left side. Let us assume temporarily that $B_0 < \infty$, then using Lemma (14) with $f = f_0$ yields

$$C^q B_0^{qr/p} \geq C^q \left(\frac{q}{p}\right)^{q/p} \mathscr{B}_0^{qr/p} \geq \int_0^\infty w(x) \left(\int_x^\infty W^{r/q} V^{-(r/q)-1} v\right)^{q/p} dx$$
$$\gg \int_0^\infty w W^{r/p} V^{-r/p} = B_0^r.$$

This implies that $C \gg B_0$ and the temporary assumption $B_0 < \infty$ can be removed in the usual way.

To prove sufficiency, let $h \geq 0$, $\operatorname{supp} h \subset (0, \infty)$, $f(x) = \int_x^\infty h$ and $\int_0^\infty f^p v < \infty$. We have

$$\int_0^\infty f^q w = \int_0^\infty \left(\int_s^\infty h\right)^q w(s)\,ds$$
$$= \int_0^\infty \left(\left(\int_s^\infty h\right)^q \frac{w^{q/p}(s)\,ds}{\left(\int_s^\infty W^{r/p} V^{-r/p} v\right)^{q/r}} \right)$$
$$\cdot \left(w^{1-q/p}(s) \left(\int_s^\infty W^{r/p} V^{-r/q} v\right)^{q/r} \right)$$

(apply Hölder's inequality with p/q, r/q),

$$\leq \left(\int_0^\infty \left(\int_s^\infty h\right)^p \frac{w(s)\,ds}{\left(\int_s^\infty W^{r/p} V^{-r/q} v\right)^{p/r}}\right)^{q/p} \left(\int_0^\infty w(s)\,ds \int_s^\infty W^{r/p} V^{-r/q} v\right)^{q/r}$$
$$= \mathscr{B}_0^q \left(\int_0^\infty \left(\int_s^\infty h\right)^p \left(\int_s^\infty W^{r/p} V^{-r/q} v\right)^{-p/r} dW(s)\right)^{q/p}$$

(integrating by parts)

$$\ll \mathscr{B}_0^q \left(\int_0^\infty \left(\int_s^\infty h\right)^{p-1} h(s) \left(\int_s^\infty W^{r/p} V^{-r/q} v\right)^{-p/r} W(s)\,ds\right)^{q/p}$$
$$\ll \mathscr{B}_0^q \left(\int_0^\infty \left(\int_s^\infty h\right)^{p-1} h(s) V(s)\,ds\right)^{q/p},$$

because

$$\left(\int_s^\infty W^{r/p} V^{-r/q} v\right)^{-p/r} \ll \frac{V(s)}{W(s)}.$$

Consequently

$$\int_0^\infty f^q w \ll \mathscr{B}_0^q \left(\int_0^\infty \left(\int_s^\infty h\right)^p v(s)\,ds\right)^{q/p} \ll B_0^q \left(\int_0^\infty f^p v\right)^{q/p}$$

and Proposition 1(b) is proved under the assumption $\int_0^\infty v = \infty$.

In the case $\int_0^\infty v < \infty$ we have to change the weight v to $v + \varepsilon$, $\varepsilon > 0$. The result follows from the previous case, by obtaining uniform estimates with respect to ε and then applying Fatou's lemma. For instance, if (14) holds and $\int_0^\infty v < \infty$ then the inequality $(\int_0^\infty f^q w)^{1/q} \le C(\int_0^\infty f^p (v+\varepsilon))^{1/p}$ is also valid with the same constant C. It implies $B_{0,\varepsilon} \ll C$, where

$$B_{0,\varepsilon} = \left[\int_0^\infty W^{r/p}(t)\left(\int_0^t v + \varepsilon\right)^{-r/p} w(t)dt\right]^{1/r}.$$

By Fatou's lemma, we get $B_0 \ll C$. Conversely, if $B_0 < \infty$ and $\int_0^\infty v < \infty$, then $B_{0,\varepsilon} \le B_0$ and we get $C \ll B_{0,\varepsilon} \le B_0$ by the above arguments.

2.2. Next we prove the necessary part of Theorem 2. Let us assume that $\int_0^\infty v = \infty$ and start with case (a), $1 < p \le q < \infty$. Let $0 < \tau < \infty$ and

$$f_\tau(s) = \left(\int_s^\tau x^{p'} V^{-p'-1}(x) v(x)\, dx\right)^{1/p} \chi_{[0,\tau]}(s),$$

then f_τ is nonincreasing and

$$\int_0^\infty f_\tau^p v = \int_0^\tau x^{p'} V^{-p'}(x) v(x) dx.$$

Inequality (4) with $f = f_\tau$ yields

$$\begin{aligned}
&C\left(\int_0^\tau x^{p'} V^{-p'}(x) v(x)\, dx\right)^{1/p} \\
&\quad \ge \left(\int_0^\infty \left(\frac{1}{x}\int_0^x \left(\int_s^\tau y^{p'} V^{-p'-1}(y) v(y)\, dy\right)^{1/p} \chi_{[0,\tau]}(s)\, ds\right)^q w(x)\, dx\right)^{1/q} \\
&\quad \ge \left(\int_\tau^\infty x^{-q} w(x)\, dx\right)^{1/q} \int_0^\tau \left(\int_s^\tau y^{p'} V^{-p'-1}(y) v(y)\, dy\right)^{1/p} ds.
\end{aligned}$$

Changing the order of integration we have

$$\begin{aligned}
\int_0^\tau &\left(\int_s^\tau y^{p'} V^{-p'-1}(y) v(y)\, dy\right)^{1/p} ds \ge \int_0^\tau s^{p'/p} \left(\int_s^\tau V^{-p'-1}(y) v(y) dy\right)^{1/p} ds \\
&= p \int_0^\tau s^{p'/p} \int_s^\tau \left(\int_y^\tau V^{-p'-1} v\right)^{-1/p'} V^{-p'-1}(y) v(y)\, dy\, ds \\
&= p \int_0^\tau V^{-p'-1}(y) v(y) \left(\int_0^y s^{p'/p}\, ds\right) \left(\int_y^\tau V^{-p'-1} v\right)^{-1/p'} dy \\
&\ge (p-1) \int_0^\tau V^{-p'-1}(y) v(y) y^{p'} \left(\int_y^\infty V^{-p'-1} v\right)^{-1/p'} dy \\
&= (p')^{-1/p} p \int_0^\tau y^{p'} V^{-p'}(y) v(y)\, dy.
\end{aligned}$$

This shows that $C \gg A_1$ when $\int_0^\infty v = \infty$. This case $\int_0^\infty v < \infty$ can be easily proved by the arguments at the end of the previous section.

2.3. Let us note before proving the necessary part of case (b) that the following conditions are equivalent $B_1 < \infty \Leftrightarrow$

$$\mathscr{B}_1 \equiv \left(\int_0^\infty \left[\left(\int_t^\infty x^{-q} w(x)\,dx \right)^{1/p} \left(\int_0^t \Psi_0^{1-p'}(x)\,dx \right)^{1/p'} \right]^r t^{-q} w(t)\,dt \right)^{1/r} < \infty$$

and, in particular, $\mathscr{B}_1 \geq (q/p')^{1/r} B_1$.

We write

$$\begin{aligned} B_1^r &= \int_0^\infty \left(\int_t^\infty x^{-q} w(x)\,dx \right)^{r/q} \left(\int_0^t \Psi_0^{1-p'} \right)^{r/q'} \Psi_0^{1-p'}(t) V^{-1}(t) V(t)\,dt \\ &= \int_0^\infty \left(\int_y^\infty \left(\int_t^\infty x^{-q} w(x)\,dx \right)^{r/q} \right. \\ &\qquad \cdot \left. \left(\int_0^t \Psi_0^{1-p'} \right)^{r/q'} \Psi_0^{1-p'}(t) V^{-1}(t)\,dt \right) v(y)\,dy \\ &\equiv \int_0^\infty f_0^p v \end{aligned}$$

where the function f_0 is again determined by the left side. Then f_0 is nonincreasing and (4) yields

$$\begin{aligned} C B_1^{r/p} = C \left(\int_0^\infty f_0^p v \right)^{1/p} &\geq \left(\int_0^\infty \left(\frac{1}{s} \int_0^s f_0 \right)^q w(s)\,ds \right)^{1/q} \\ &\equiv \left(\int_0^\infty \Phi^q(s) s^{-q} w(s)\,ds \right)^{1/q}. \end{aligned}$$

As usual we assume that $\int_0^\infty v = \infty$ and $B_1 < \infty$ and will prove that $C \gg B_1$. We have

$$\begin{aligned} \Phi(s) &\geq \int_0^s \left(\int_y^s \left(\int_t^\infty x^{-q} w(x)\,dx \right)^{r/q} \left(\int_0^t \Psi_0^{1-p'} \right)^{r/q'} \Psi_0^{1-p'}(t) V^{-1}(t)\,dt \right)^{1/p} dy \\ &\geq \left(\int_s^\infty x^{-q} w(x)\,dx \right)^{r/qp} J(s), \end{aligned}$$

where

$$J(s) \equiv \int_0^s \left(\int_y^s \left(\int_0^t \Psi_0^{1-p'} \right)^{r/q'} \Psi_0^{1-p'}(t) V^{-1}(t)\,dt \right)^{1/p} dy.$$

Also we have for $1 < q < p < \infty$ the lower bound.

$$\begin{aligned} J(s) &\geq \int_0^s y^{p'/p} \left(\int_y^s \left(\int_0^t \Psi_0^{1-p'} \right)^{r/q'} V^{-p'-1}(t) v(t)\,dt \right)^{1/p} dy \\ &= p \int_0^s y^{p'/p} dy \int_y^s \left(\int_\tau^s \left(\int_0^t \Psi_0^{1-p'} \right)^{r/q'} V^{-p'-1}(t) v(t)\,dt \right)^{-1/p'} \\ &\quad \cdot \left(\int_0^\tau \Psi_0^{1-p'} \right)^{r/q'} V^{-p'-1}(\tau) v(\tau)\,d\tau \geq p \left(\int_0^s \Psi_0^{1-p'} \right)^{-r/q'p'} J_1(s), \end{aligned}$$

where

$$J_1(s) \equiv \int_0^s y^{p'/p} dy \int_y^s \left(\int_\tau^s V^{-p'-1}(t)v(t)\,dt\right)^{-1/p'} \times \left(\int_0^\tau \Psi_0^{1-p'}\right)^{r/q'} V^{-p'-1}(\tau)v(\tau)\,d\tau.$$

Changing the order of integration we get

$$\begin{aligned} J_1(s) &= \int_0^s \left(\int_0^\tau \Psi_0^{1-p'}\right)^{r/q'} V^{-p'-1}(\tau)v(\tau) \\ &\quad \times \left(\int_0^\tau y^{p'/p}dy\right)\left(\int_\tau^s V^{-p'-1}(t)v(t)\,dt\right)^{-1/p'} d\tau \\ &\geq \frac{1}{p'}\int_0^s \tau^{p'}\left(\int_0^\tau \Psi_0^{1-p'}\right)^{r/q'} V^{-p'-1}(\tau)v(\tau)\left(\int_\tau^\infty V^{-p'-1}(t)v(t)\,dt\right)^{-1/p'} d\tau \\ &= (p')^{-1/p}\int_0^s \left(\int_0^\tau \Psi_0^{1-p'}\right)^{r/q'} \Psi_0^{1-p'}(\tau)\,d\tau = (p')^{1/p'}(1/r)\left(\int_0^s \Psi_0^{1-p'}\right)^{r/p'}. \end{aligned}$$

Putting these lower bounds on $J_1(s)$ and $J(s)$ into the estimate for $\Phi(s)$ we have

$$\Phi(s) \gg \left(\int_s^\infty x^{-q}w(x)\,dx\right)^{r/qp}\left(\int_0^s \Psi_0^{1-p'}\right)^{r/p'q}.$$

This implies

$$\begin{aligned} [CB_1^{r/p}]^q &\gg \int_0^\infty \left(\int_s^\infty x^{-q}w(x)dx\right)^{r/p}\left(\int_0^s \Psi_0^{1-p'}\right)^{r/p'} s^{-q}w(s)ds \\ &= \mathscr{B}_1^r \geq (q/p')B_1^r. \end{aligned}$$

Hence $C \gg B_1$ in the case $1 < q < p < \infty$ and $\int_0^\infty v = \infty$. As before the case $\int_0^\infty v < \infty$ follows by the arguments used at the end of §2.1.

2.4. Now we prove the sufficient part of Theorem 2. Let us assume at first that $\int_0^\infty v = \infty$; we take as before

$$f(x) = \int_x^\infty h, \quad h \geq 0, \quad \operatorname{supp} h \subset (0, \infty), \quad \int_0^\infty f^p v < \infty.$$

We will get an upper bound on the expression

$$F = \left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^q w(x)\,dx\right)^{1/q}.$$

We have

$$\frac{1}{x}\int_0^x f(t)\,dt = \frac{1}{x}\int_0^x \left(\int_t^\infty h\right)dt = \int_x^\infty h + \frac{1}{x}\int_0^x sh(s)\,ds,$$

consequently

$$F \leq \left(\int_0^\infty f^q w\right)^{1/q} + F_1.$$

The first term on the right side has been bounded in the proof of the Proposition 1. Let us bound the second term. We have, setting $H = hV$,

$$F_1^q \equiv \int_0^\infty \left(\frac{1}{x}\int_0^x sh(s)\,ds\right)^q w(x)\,dx = \int_0^\infty \left(\frac{1}{x}\int_0^x \frac{sH(s)}{V(s)}\,ds\right)^q w(x)\,dx.$$

Integrating by parts and setting $\int_0^s H = G$, we get

$$\begin{aligned}\int_0^x \frac{sH(s)}{V(s)}\,ds = \int_0^x \frac{s}{V(s)}dG(s) &= \frac{xG(x)}{V(x)} - \int_0^x \frac{G(s)}{V(s)}\,ds + \int_0^x \frac{sv(s)}{V^2(s)}G(s)\,ds \\ &\le \frac{xG(x)}{V(x)} + \int_0^x \frac{sv(s)}{V^2(s)}G(s)\,ds.\end{aligned}$$

This implies

$$\begin{aligned}F_1^q &\ll \int_0^\infty \left(\frac{G}{V}\right)^q w + \int_0^\infty \left(\frac{1}{x}\int_0^x \frac{sv(s)}{V^2(s)}G(s)\,ds\right)^q w(x)\,dx \\ &\equiv F_2^q + F_3^q.\end{aligned} \tag{16}$$

Integrating by parts we have

$$F_2^q = \int_0^\infty \left(\frac{G}{V}\right)^q w = \int_0^\infty \left(\frac{G}{V}\right)^q dW \le q\int_0^\infty \frac{G^q v}{V^{q+1}}W. \tag{17}$$

Now we have to consider separately the cases $1 < p \le q < \infty$ and $1 < q < p < \infty$. If $1 < p \le q < \infty$ then (9) yields

$$F_2^q \le qA_0^q \int_0^\infty \frac{G^q v}{V^{q+1-q/p}}.$$

We have

$$G(s) = \int_0^s hV = \int_0^s v(x)\left(\int_x^s h\right)dx \le \int_0^s fv.$$

Consequently

$$F_2^q \le qA_0^q \int_0^\infty \left(\int_0^s fv\right)^q \frac{v(s)ds}{V^{q+1-q/p}(s)}.$$

It is easy to show that

$$\left(\int_t^\infty \frac{v(s)ds}{V^{q+1-q/p}(s)}\right)^{1/q}\left(\int_0^t v\right)^{1/p'} \le (p'/q)^{1/q},$$

therefore applying Theorem 1 we get

$$F_2^q \le \alpha^q(p, q)p'A_0^q\left(\int_0^\infty f^p v\right)^{q/p}. \tag{18}$$

Denoting $sv(s)G(s)/V^2(s) = \Phi(s)$ and again applying Theorem 1 and (10) we find that

$$
\begin{aligned}
F_3^q &= \int_0^\infty \left(\frac{1}{x}\int_0^x \Phi(t)\,dt\right)^q w(x)dx \le \alpha^q(p\,,\,q)A_1^q\left(\int_0^\infty \Phi^p\Psi_0\right)^{q/p} \\
&= \alpha^q(p\,,\,q)A_1^q\left(\int_0^\infty \frac{G^p v}{V^p}\right)^{q/p} \\
&\le \alpha^q(p\,,\,q)A_1^q\left(\int_0^\infty \left(\int_0^s fv\right)^p \frac{v(s)ds}{V^p(s)}\right)^{q/p} \\
&\le \alpha^q(p\,,\,q)\alpha^q(p\,,\,p)(p'/p)^{q/p}A_1^q\left(\int_0^\infty f^p v\right)^{q/p}.
\end{aligned} \tag{19}
$$

Combining the estimates (15), (16), (18), and (19) we get the sufficient part of Theorem 2(a).

The proof of the sufficient part of Theorem 2(b) is similar. Let $0 < q < p < \infty$, then using (17) and Hölder's inequality with exponents p/q and r/q we get

$$
\begin{aligned}
F_2^q &\le q\int_0^\infty \frac{G^q v}{V^{q+1}}W = q\int_0^\infty \left\{\left(\frac{G}{V}\right)^q v^{1-q/r}\right\}\left\{WV^{-1}v^{q/r}\right\} \\
&\le q\left(\int_0^\infty W^{r/q}V^{-r/q}v\right)^{q/r}\left(\int_0^\infty \left(\frac{G}{V}\right)^p v\right)^{q/p}.
\end{aligned}
$$

Applying the lemma and the last part of (19) we find that

$$
F_2^q \le q\alpha^q(p\,,\,p)(p'/p)^{q/p}(p/q)^{q/r}B_0^q\left(\int_0^\infty f^p v\right)^{q/p}.
$$

The estimate of F_3^q in the case $0 < q < p < \infty$, $p > 1$ is produced using Theorem 1(b) exactly as above; we get

$$
F_3^q \le \beta_2^q(p\,,\,q)\alpha^q(p\,,\,p)(p'/p)^{q/p}B_1^q\left(\int_0^\infty f^p v\right)^{q/p}.
$$

Thus the sufficient parts of Theorem 2(b) and Theorem 3 are proved if $\int_0^\infty v = \infty$.

If $0 < \int_0^\infty v < \infty$, then the estimate must be slightly corrected. In this case we have instead of (17)

$$
F_2^q \le \left[\int_0^\infty hV\right]^q \left(\int_0^\infty v\right)^{-q}\int_0^\infty w + q\int_0^\infty \frac{G^q v}{V^{q+1}}W.
$$

The first term on the right side of this inequality can be estimated trivially since

$$
\int_0^\infty hV = \int_0^\infty fv \le \left(\int_0^\infty f^p v\right)^{1/p}\left(\int_0^\infty v\right)^{1/p'};
$$

hence we get again that

$$
F_2^q \ll A_0^q\left(\int_0^\infty f^p v\right)^{q/p} \quad \text{for } 1 < p \le q < \infty
$$

or

$$
F_2^q \ll B_0^q\left(\int_0^\infty f^p v\right)^{q/p} \quad \text{for } 0 < q < p < \infty,\ p > 1,
$$

because $A_0 \ll B_0$.

3. Proof of Theorem 3

3.1. The sufficient part of Theorem 3(a) has been proved in §2.4 above. We will show the necessary part exactly by the method of §2.3. We only need to modify the estimate of the term denoting $J(s)$. If $0 < q < 1 < p < \infty$, then $q' < 0$ and we have

$$J(s) \equiv \int_0^s \left(\int_y^s \left(\int_0^t \Psi_0^{1-p'} \right)^{r/q'} \Psi_0^{1-p'}(t) V^{-1}(t)\, dt \right)^{1/p} dy$$
$$\geq \left(\int_0^s \Psi_0^{1-p'} \right)^{r/q'p} J_2(s).$$

It has already been shown in §2.2 that

$$J_2(s) = \int_0^s \left[\int_y^s \Psi_0^{1-p'}(\tau) V^{-1}(\tau)\, d\tau \right]^{1/p} dy \geq p(p')^{-1/p} \int_0^s \Psi_0^{1-p'}.$$

Thus we get

$$J(s) \geq p(p')^{-1/p} \left(\int_0^s \Psi_0^{1-p'} \right)^{r/p'q}$$

and the rest of the reasoning is exactly the same as above in §2.3.

3.2. Now we consider Theorem 3(b), i.e., $0 < p \leq q < \infty$, $0 < p < 1$. The necessary part is trivial. The sufficient part is demonstrated as follows. As above we use the notation

$$f(x) = \int_x^\infty h, \quad h \geq 0, \quad \operatorname{supp} h \subset (0, \infty), \quad \int_0^\infty f^p v < \infty.$$

If $0 < p < 1$ then applying Minkowski's inequality we get

$$\int_0^x f(s)\, ds = \int_0^x \left(\int_s^\infty h \right) ds = p^{1/p} \int_0^x \left[\int_s^\infty \left(\int_y^\infty h \right)^{p-1} h(y)\, dy \right]^{1/p} ds$$
$$\leq 2^{-1/p'} p^{1/p} \left\{ \int_0^x \left[\int_s^x \left(\int_y^\infty h \right)^{p-1} h(y)\, dy \right]^{1/p} ds \right.$$
$$\left. + \int_0^x \left[\int_x^\infty \left(\int_y^\infty h \right)^{p-1} h(y)\, dy \right]^{1/p} ds \right\}$$
$$\leq 2^{-1/p'} p^{1/p} \left(\int_0^x \left(\int_y^\infty h \right)^{p-1} h(y) y^p\, dy \right)^{1/p} + 2^{-1/p'} x f(x).$$

Now we have

$$I = \left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^q w(x)\, dx \right)^{1/q} \ll \left(\int_0^\infty f^q w \right)^{1/q}$$
$$+ \left(\int_0^\infty \left(\int_0^x \left(\int_y^\infty h \right)^{p-1} h(y) y^p\, dy \right)^{q/p} x^{-q} w(x)\, dx \right)^{1/q}$$
$$\equiv I_1 + I_2.$$

Proposition 1(a) yields

$$I_1 \ll A_0 \left(\int_0^\infty f^p(x) v(x)\, dx \right)^{1/p}.$$

Applying Minkowski's inequality we find that

$$I_2 \le \left(\int_0^\infty \left(\int_y^\infty h \right)^{p-1} h(y) y^p \, dy \left(\int_y^\infty x^{-q} w(x)\, dx \right)^{p/q} dy \right)^{1/p}$$

$$\le \mathscr{A}_1 \left(\int_0^\infty \left(\int_y^\infty h \right)^{p-1} h(y) V(y)\, dy \right)^{1/p}$$

$$= p^{-1/p} \mathscr{A}_1 \left(\int_0^\infty f^p(x) v(x)\, dx \right)^{1/p}.$$

Combination of these upper bounds on I_1 and I_2 completes the proof.

3.3. The following result is a sufficient condition for (4) if $0 < q < p < 1$.

Proposition 2. *Let* $0 < q < p < 1$. *Then the inequality* (4) *is valid for all nonincreasing functions* $f \ge 0$, *if* $B_0 < \infty$ *and*

$$D_1 \equiv \left(\int_0^\infty \left(\int_t^\infty x^{-q} w(x)\, dx \right)^{r/q} V^{-r/p}(t) t^{r-1}\, dt \right)^{1/r} < \infty.$$

Proof. Let us denote $\int_t^\infty x^{-q} w(x)\, dx$ by $\mathscr{W}_0(t)$ and suppose that $f(x) = \int_x^\infty h$ as above. It has already been shown that

$$\int_0^x f(s) dx \le 2^{-1/p'} p^{1/p} \left(\int_0^x \left(\int_y^\infty h \right)^{p-1} h(y) y^p \, dy \right)^{1/p} + 2^{-1/p'} x f(x).$$

Then we have as above

$$I = \left(\int_0^\infty \left(\frac{1}{x} \int_0^x f(t)\, dt \right)^q w(x)\, dx \right)^{1/q} \ll I_1 + I_2.$$

Applying Proposition 1(b) we get

$$I_1 \ll B_0 \left(\int_0^\infty f^p(x) v(x)\, dx \right)^{1/p}.$$

An upper bound on I_2 is obtained as follows.

$$I_2^q \ll \int_0^\infty \left(\int_0^x \left(\int_y^\infty h \right)^{p-1} h(y) y^p \, dy \right)^{q/p} x^{-q} w(x)\, dx$$

$$= \int_0^\infty \left[\left(\int_0^x \left(\int_y^\infty h \right)^{p-1} h(y) y^p \, dy \right)^{q/p} x^{-q^2/p} w^{q/p}(x) \right.$$

$$\left. \cdot \left(\int_0^x \mathscr{W}_0^{r/p}(t) V^{-r/p}(t) t^{r-1} dt \right)^{-q/r} \right]$$

$$\cdot \left[x^{-q(1-q/p)} w^{1-q/p}(x) \left(\int_0^x \mathscr{W}_0^{r/p}(t) V^{-r/p}(t) t^{r-1}\, dt \right)^{q/r} \right] dx$$

(applying Hölder's inequality with exponents p/q, r/q),

$$\leq \left(\int_0^\infty \left(\int_0^x \left(\int_y^\infty h\right)^{p-1} h(y)y^p dy\right) x^{-q}w(x)\right.$$

$$\cdot \left.\left(\int_0^x \mathscr{W}_0^{r/p}(t)V^{-r/p}(t)t^{r-1}dt\right)^{-p/r} dx\right)^{q/p}$$

$$\cdot \left(\int_0^\infty x^{-q}w(x)\int_0^x \mathscr{W}_0^{r/p}(t)V^{-r/p}(t)t^{r-1}dt\right)^{q/r}$$

$$= D_1^q \left(\int_0^\infty \left(\int_y^\infty h\right)^{p-1} h(y)y^p \int_y^\infty x^{-q}w(x)\right.$$

$$\cdot \left.\left(\int_0^x \mathscr{W}_0^{r/p}(t)V^{-r/p}(t)t^{r-1}dt\right)^{-p/r} dxdy\right)^{q/p}.$$

Let us note that

$$y^p \int_y^\infty x^{-q}w(x)\left(\int_0^x \mathscr{W}_0^{r/p}(t)V^{-r/p}(t)t^{r-1}dt\right)^{-p/r} dx$$

$$\leq y^p \left(\int_0^y \mathscr{W}_0^{r/p}(t)V^{-r/p}(t)t^{r-1}dt\right)^{-p/r} \mathscr{W}_0(y) \leq y^p V(y)\left(\int_0^y t^{r-1}dt\right)^{-p/r}$$

$$= r^{p/r}V(y).$$

This implies the estimate

$$I_2^q \ll D_1^q \left(\int_0^\infty f^p v\right)^{q/p},$$

and ends the proof.

References

1. M. Arino and B. Muckenhoupt, *Maximal function on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.
2. D. W. Boyd, *The Hilbert transform on rearrangement-invariant spaces*, Canad. J. Math. **19** (1967), 599–616.
3. J. S. Bradley, *Hardy inequalities with mixed norms*, Canad. Math. Bull. **21** (1978), 405–408.
4. S. G. Krein, Yu. I. Petunin and E. M. Semenov, *Interpolation of linear operators*, Transl. Math. Monographs, vol. 54, Amer. Math. Soc., Providence, R.I., 1982.
5. V. G. Maźja, *Sobolev spaces*, Springer-Verlag, Berlin, 1985.
6. B. Muckenhoupt, *Hardy's inequality with weights*, Studia Math. **44** (1972), 31–38.
7. C. J. Neugebauer, *Weighted norm inequalities for averaging operators of monotone functions*, Publ. Math. (to appear).
8. E. T. Sawyer, *Boundedness of classical operators on classical Lorentz spaces*, Studia Math. **96** (1990), 145–158.
9. G. Sinnamon, *A weighted gradient inequality*, Proc. Roy. Soc. Edinburgh Sect. A **111** (1989), 329–335.
10. ——, *Weighted Hardy and Opial-type inequalities*, J. Math. Anal. Appl. **160** (1991), 434–445.

11. V. D. Stepanov, *Weighted inequalities for a class of Volterra convolution operators*, J. London Math. Soc. (2) **45** (1992), 232–242.

12. G. Talenti, *Osservazione sopra una classe di disuguaglianze*, Rend. Sem. Mat. Fis. Milano **39** (1969), 171–185.

13. G. Tomaselli, *A class of inequalities*, Boll. Un. Mat. Ital. Ser. **4** (1969), 622–631.

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