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The Weighted Mixed Curvature of a Foliated Manifold

Vladimir Rovenski[a]

[a] *Department of Mathematics, University of Haifa, Israel*

Abstract. We introduce the weighted mixed curvature of an almost product (e.g. foliated) Riemannian manifold equipped with a vector field. We define several qth Ricci type curvatures, which interpolate between the weighed sectional and Ricci curvatures. New concepts of the "mixed-curvature-dimension condition" and "synthetic dimension of a distribution" allow us to renew the estimate of the diameter of a compact Riemannian foliation and splitting results for almost product manifolds of nonnegative/nonpositive weighted mixed scalar curvature. We also study the Toponogov's type conjecture on dimension of a totally geodesic foliation with positive weighted mixed sectional curvature.

1. Introduction

The "weighted" curvature is defined for a Riemannian manifold (M, g) endowed with a vector field X, e.g., when X is a gradient of a density function $f : M \to \mathbb{R}_+$. The *weighted scalar curvature* appeared in Perelman's functionals for the Ricci flow. The *weighted Ricci curvature* was first studied by Lihnerovicz, and later by Bakry–Emery and many others. The study of *weighted Ricci tensor* (called also the N-Bakry–Emery–Ricci tensor) of the triple (M, g, X),

$$\mathrm{Ric}_X^N = \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X\, g - \frac{1}{N}\, X^\flat \otimes X^\flat \tag{1.1}$$

was motivated by the *curvature-dimension* condition $\mathbf{CD}(c, N)$: $\mathrm{Ric}_X^N \geqslant c$, for a brief overview see [2]. Here N is an upper bound of the "generalized dimension" of the weighted manifold, c is a lower bound of the Ricci tensor Ric, and $\mathcal{L}$ is the Lie derivative. The "musical" isomorphisms $\flat$ and $\sharp$ lower and raise indices of tensors. Definition (1.1) arises for the Ric of a warped product of M of dimension $N > 0$ with a manifold B, when the warping function $\phi = -(1/N)\log f$ and $X = \nabla f$.

Distributions on manifolds (i.e., subbundles of the tangent bundle) appear in various situations, e.g. as fields of tangent planes of foliations. Totally geodesic and Riemannian foliations, having the simplest *extrinsic geometry* (respectively, the tangent or orthogonal distribution has zero second fundamental form), are investigated in a number of works. There is interest of geometers to problems of existence of metrics on foliations and almost product manifolds with given curvature properties. One may consider three kinds of sectional curvature for a foliation: *tangential*, *transversal* and *mixed* (denoted by K_{mix}). The mixed curvature

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Email address: vrovenski@univ.haifa.ac.il (Vladimir Rovenski)

is encoded in the Riccati and Jacobi equations along the leaf geodesics. For constant $K_{\rm mix}$ the solutions of above equations (and the relative behavior of geodesics on nearby leaves) are well-known, e.g. [4]. Riemannian foliations with totally geodesic leaves have $K_{\rm mix} \geqslant 0$, and splitting of foliations with $K_{\rm mix} = 0$ is possible.

Our main object is $(M^{n+\nu}, g, X)$ equipped with complementary orthogonal distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ of ranks $\dim\widetilde{\mathcal{D}} = \nu > 0$ and $\dim\mathcal{D} = n > 0$. We study three kinds of *weighted mixed curvature* (sectional, qth Ricci and scalar) of almost product manifolds, introduce the notions of "synthetic dimension" of a distribution and the "mixed-curvature-dimension" condition, and obtain natural generalizations of several results known for the case of $X = 0$. Let $^\top$ and $^\perp$ denote orthogonal projections onto $\widetilde{\mathcal{D}}$ and $\mathcal{D}$, respectively. We define several functions on $(M, g, X, \widetilde{\mathcal{D}}, \mathcal{D})$, which "interpolate" between the weighed sectional and Ricci curvatures; such functions on (M, g) for $X = 0$ were introduced by H. Wu, and then studied by many geometers, see surveys in [4, 5]. Let W^q be a subspace of $\widetilde{\mathcal{D}}_m$ spanned by $q \leqslant \nu$ orthonormal vectors $\{x_1, \dots, x_q\}$ at a point $m \in M$, and $y \in \mathcal{D}_m$ a unit vector. Set

$$\widetilde{\rm Ric}_q(y, W) := \sum\nolimits_{i=1}^q K(y, x_i).$$

Riemannian manifolds of $\widetilde{\rm Ric}_q > 0$ form a lager class than ones of $K_{\rm mix} > 0$. The simplest curvature invariant of a pair $(\widetilde{\mathcal{D}}, \mathcal{D})$ is a function ${\rm S}_{\rm mix} : M \to \mathbb{R}$, see [4, 9],

$${\rm S}_{\rm mix}(m) := \sum\nolimits_{j=1}^n \widetilde{\rm Ric}_\nu(y_j, \widetilde{\mathcal{D}}_m), \tag{1.2}$$

called the *mixed scalar curvature*, i.e., an averaged mixed sectional curvature. Here $\{y_j\}$ is an orthonormal frame of $\mathcal{D}_m$. For example, ${\rm S}_{\rm mix} = {\rm Ric}_{y,y}$ when $n = 1$.

In contrast to scalar curvature, ${\rm S}_{\rm mix}$ is strongly related with the extrinsic geometry, and is involved in such research topics as prescribing ${\rm S}_{\rm mix}$ on pseudo-Riemannian manifolds [7] and the mixed Einstein–Hilbert action, see survey [6].

Definition 1.1. We define the *weighted mixed qth Ricci curvature* of $\{y, W\}$ by

$$\widetilde{\mathcal{R}ic}_q^{\mathcal{N}}(y, W) := \widetilde{\rm Ric}_q(y, W) + \frac{q}{2}\,(\mathcal{L}_{X/\nu}\, g)(y, y) + \frac{q\,\nu}{\mathcal{N}}\,\langle X/\nu, y\rangle^2, \tag{1.3}$$

where $\mathcal{N} \in \mathbb{R}$ is called the *synthetic dimension of* $\widetilde{\mathcal{D}}$. By the *mixed-curvature-dimension* condition $\mathbf{CD}^\top(c, \mathcal{N}, q)$ for $\widetilde{\mathcal{D}}$, we mean the inequality

$$\widetilde{\mathcal{R}ic}_q^{\mathcal{N}} \geqslant c. \tag{1.4}$$

Similarly, $\mathcal{R}ic_q^N(x, W)$ for $W^q \subset \mathcal{D}_m$, $x \in \widetilde{\mathcal{D}}_m$, and $\mathbf{CD}^\perp(c, N, q)$ for $\mathcal{D}$ are defined.

Example 1.1. Let $M^{k+3} = S^3 \times \hat M^k$ $(k > 0)$ be the product of a unit 3-sphere and a Riemannian manifold. Suppose that Y is the lift of any unit vector field on S^3 and the Killing field X is the lift of Hopf field on S^3 (corresponding to standard complex structure on $\mathbb{R}^4 = \mathbb{C}^2$). Set $\mathcal{D} = {\rm span}(Y)$. Then $\widetilde{\mathcal{R}ic}_{k+1}^{\mathcal{N}}(Y, \cdot) \geqslant 1$.

Based on (1.2) and (1.3), define the *weighted mixed scalar curvature* by

$$\mathcal{S}^{N,\mathcal{N}} := \frac{1}{2}\,{\rm Tr}_g\Big(\mathcal{R}ic_n^N + \widetilde{\mathcal{R}ic}_\nu^{\mathcal{N}}\Big) = {\rm S}_{\rm mix} + \frac{1}{2}\,{\rm div}\, X + \frac{1}{2N}\,\|X^\top\|^2 + \frac{1}{2\mathcal{N}}\,\|X^\perp\|^2, \tag{1.5}$$

where $N, \mathcal{N} \in \mathbb{R}$ are called *synthetic dimensions of* $\mathcal{D}$ and $\widetilde{\mathcal{D}}$, respectively.

Let $\mathcal{F}^\nu$ be a totally geodesic foliation of $(M^{n+\nu}, g)$, and $R_x = (R_{\cdot,x}\,x)^\perp$ $(x \in T\mathcal{F})$ the Jacobi operator on $\mathcal{D}$. If the leaves are closed and $R_x > 0$ $(x \neq 0)$, then $\nu < n$; otherwise, any two of them will intersect.

D. Ferus [3] found a nice topological obstruction for existence of totally geodesic foliations: *if* $R_x \equiv k\,\mathrm{id}^\perp$ *for some* $k = \mathrm{const} > 0$ *and all unit* $x \in TL$ *on a complete leaf* L, *then* $\nu < \rho(n)$. Here $\rho(n) - 1$ is the number of linear independent vector fields on a sphere S^{n-1},

$$\rho((\mathrm{odd})\,2^{4b+c}) = 8b + 2^c \text{ for some } b \geqslant 0,\ 0 \leqslant c \leqslant 3,$$

and $\rho(n) \leqslant 2\log_2 n + 2 \leqslant n$. Among Toponogov's many contributions to Riemannian geometry is the following **conjecture** (see survey [4]): *The inequality* $\nu < \rho(n)$ *holds for totally geodesic foliations of closed Riemannian manifolds with* $K_{\rm mix} > 0$.

We introduce the *weighted* $\mathcal{D}$*-Jacobi operator* $\mathcal{R}_x$ $(x \in \widetilde{\mathcal{D}})$ by

$$\mathcal{R}_x := R_x + \Big(\frac{1}{2}\,(\mathcal{L}_{X/n}\, g)(x, x) + \langle X/n, x\rangle^2\Big)\,\mathrm{id}^\perp, \tag{1.6}$$

and similarly define the *weighted* $\widetilde{\mathcal{D}}$*-Jacobi operator* $\widetilde{\mathcal{R}}_y$ $(y \in \mathcal{D})$. Set

$$\mathcal{R}ic(x, x) := \mathrm{Tr}_g\,\mathcal{R}_x, \quad \widetilde{\mathcal{R}ic}(y, y) := \mathrm{Tr}_g\,\widetilde{\mathcal{R}}_y.$$

Thus, $\mathcal{R}ic(x, x) = \mathcal{R}ic^{\nu}_{n}(x, \mathcal{D}_m)$, see (1.3).

In Section 2, we use (1.4) to estimate the diameter of Riemannian foliations.

In Section 3 we use the weighted mixed scalar curvature (1.5) to prove new integral formulas and splitting theorems for almost product manifolds.

In Section 4 we study the Toponogov type **conjecture**: *The inequality* $\nu < \rho(n)$ *holds for a totally geodesic foliation* $\mathcal{F}^\nu$ *of a closed manifold* $(M^{n+\nu}, g, X)$ *under assumption* $\mathcal{R}_x > \|X^\top/n\|^2\,\mathrm{id}^\perp$ *for all unit vectors* $x \in T\mathcal{F}$.

Since $\mathcal{R}_x > 0$ $(x \neq 0)$ yields $R_x > -\frac{1}{2}\,(\mathcal{L}_{X/n}\, g)(x, x)\,\mathrm{id}^\perp$ then R_x (and, hence, $K_{\rm mix}$) in the above conjecture can be negative somewhere.

2. The diameter of a compact Riemannian foliation

The *weighted mixed sectional curvature* is the weighted sectional curvature of the planes that non-trivially intersect each of the distributions,

$$\widetilde{\mathcal{K}}^N(y, x) := K(y, x) + (\frac{1}{2}(\mathcal{L}_{X/\nu}\, g)(y, y) + \frac{\nu}{N}\,\langle X/\nu, y\rangle^2)\|x\|^2, \tag{2.1}$$

see (1.3) for $q = 1$ and $W = \{x\}$. Similarly, we define $\mathcal{K}^N(x, y)$. The x and y in (2.1) are placed in asymmetric way; generally, we have $\mathcal{K}^n(x, y) \neq \widetilde{\mathcal{K}}^\nu(y, x)$. Observe that $\widetilde{\mathcal{R}ic}^{N}_{q}(y, W) = \sum_{i=1}^q \widetilde{\mathcal{K}}^N(y, x_i)$. The weighted sectional curvature appears in the formula for the second variation of energy of a path.

Lemma 2.1. *Let* $\mathcal{F}^\nu$ *be a Riemannian foliation of* (M, g, X), *and* $\bar\gamma : [a, b] \times (-\varepsilon, \varepsilon) \to M$ *a variation of the geodesic* $\gamma(t) = \bar\gamma(t, 0)$ *and the variation field on* γ, $x(t) = \partial_s\bar\gamma|_{s=0}$ *belongs to* $T\mathcal{F}$. *Then the index form on a geodesic* γ *is*

$$\mathcal{I}(x, x) = \int_a^b \Big(\|\dot x - \langle\dot\gamma, X\rangle\, x\|^2 - \widetilde{\mathcal{K}}^\nu(\dot\gamma, x)\,\|x\|^2\Big)\,dt + \langle\dot\gamma, X\rangle\|x\|^2\,\Big|_a^b. \tag{2.2}$$

Proof. It is known that for a Riemannian foliation, a geodesic started orthogonally to a leaf remains to be orthogonal to the leaves. Thus, the proof is similar to the proof of [11, Proposition 5.1]. □

The next lemma concerns the geometry of subspaces in Euclidean vector space.

Lemma 2.2 (see [4]). *Let* V_1, V_2 *are subspaces in* $\mathbb{R}^l$, $\dim V_1 = \dim V_2$. *Then there exist orthonormal bases* $\{a_i\} \subset V_1$, $\{b_i\} \subset V_2$ *(which correspond to extremal values of angle between given subspaces) with the property* $a_i \perp b_j$ $(i \neq j)$.

Let $\operatorname{diam}\mathcal{F}$ be the maximal distance between the leaves of a foliation $\mathcal{F}$ of (M, g, X).

Theorem 2.1. *Let $(M^{n+\nu}, g, X)$ be endowed with a Riemannian foliation $\mathcal{F}^\nu$ ($T\mathcal{F} = \widetilde{\mathcal{D}}$) with compact leaves of the second fundamental form h. If $\boldsymbol{CD}^\top(c, \mathcal{N}, q)$ holds for some $\mathcal{N} \geqslant \nu$, $c > 0$ and $1 \leqslant q \leqslant \nu$, then*

$$(\operatorname{diam}\mathcal{F})^2 \leqslant \frac{2\,q\|X^\perp\|}{c + q\|X^\perp\|^2} + \begin{cases} \frac{2q}{c}\,\|h\| + \frac{\pi^2}{4} & \text{if } \nu \leqslant n-1, \\ \frac{2q}{c}\,\|h\| + (q-\nu+n-1)\frac{\pi^2}{4c} & \text{if } n-1 < \nu < n+q-1, \\ \frac{2q}{c}\,\|h\| & \text{if } \nu \geqslant n+q-1. \end{cases}$$

Proof. Consider two leaves L_1, L_2 with distance $l = \operatorname{dist}(L_1, L_2)$, which is reached at points $m_1 \in L_1$ and $m_2 \in L_2$. The shortest geodesic $\gamma(t)$ $(0 \leqslant t \leqslant 1)$ with length l between m_1, m_2 is orthogonal to L_1 and L_2. Since $\mathcal{F}$ is a Riemannian foliation, γ intersect the leaves orthogonally for all $t \in (0,1)$.

Assume the second case: $n-1 < \nu < n-1+q$ (the other two cases are similar). Then the parallel displacement of $T_{m_1}L_1$ along γ will intersect $T_{m_2}L_2$ by q'−dimensional subspace V_2, where $\nu - n + 1 \leqslant q' < q$. The inverse image of V_2 in $T_{m_1}L_1$ we denote by V_1. For small l, let $T_{m_1}L_1 = V_1 \oplus V_1'$ be the orthogonal decomposition where the parallel image of V_1' is uniquely projected onto $T_{m_2}L_2$ (denote its orthogonal projection in $T_{m_2}L_2$ by V_2'). Let vectors $e_1, \ldots, e_{q'}$ form an orthonormal basis of V_1 and continue them to parallel vector fields $\bar e_1, \ldots, \bar e_{q'}$ along γ. Obviously, $\bar e_1(m_2), \ldots, \bar e_{q'}(m_2)$ belong to V_2. Let vectors $a_1, \ldots, a_s$ (where $s = q - q' = \dim V_1'$) form an orthonormal basis of V_1' and vectors $b_1, \ldots, b_s$ form an orthonormal basis of V_2', and continue them to parallel vector fields $\bar a_1, \ldots, \bar a_s$ and $\bar b_1, \ldots, \bar b_s$ along γ. Consider the field of parallel planes $\sigma_i(t)$ along γ, spanned by vectors $\bar a_i(t)$, $\bar b_i(t)$. Assume, that $\{a_i\}$, $\{b_i\}$ correspond to extremal angles between V_1' and parallel image of V_2', see Lemma 2.2. Then $\sigma_i(t) \perp \sigma_j(t)$ for $i \neq j$. We take the unit vector $\tilde b_i(t) \in \sigma_i(t)$ such that $\langle \bar a_i, \tilde b_i(t)\rangle = 0$. One may choose b_i and $\tilde b_i(t)$ with the properties $\langle \bar a_i, \bar b_i\rangle \geqslant 0$ and $\langle \bar b_i, \tilde b_i(t)\rangle \geqslant 0$. Let us introduce the unit vector fields $x_i(t) = (\cos\theta_i t)\,\bar a_i + (\sin\theta_i t)\,\tilde b_i(t)$ along γ, where $\theta_i = \arccos(\bar a_i, \bar b_i) \in [0, \frac{\pi}{2}]$. Note that $\langle x_i(t), x_j(t)\rangle = 0$ $(i \neq j)$, and $\langle \dot x_i(t), x_i(t)\rangle = 0$. We have $q' + s = q$. Using the 2nd variation of $\mathcal{E}$ of γ, (2.2), along $x_i(t)$ and $\bar e_j$, we obtain

$$\begin{aligned}
\mathcal{E}''_{x_i}(0) &= \langle h(b_i, b_i), \dot\gamma(1)/l\rangle - \langle h(a_i, a_i), \dot\gamma(0)/l\rangle + \theta_i^2 \\
&\quad - l^2 \int_0^1 \Big(\widetilde{\mathcal{K}}^\nu(\dot\gamma, x_i(t)) + \langle \dot\gamma/l, X\rangle^2\Big)\,dt + 2\langle \dot\gamma/l, X\rangle\,|_0^1 \geqslant 0, \\
\mathcal{E}''_{\bar e_j}(0) &= \langle h(\bar e_j, \bar e_j), \dot\gamma(1)/l\rangle - \langle h(e_j, e_j), \dot\gamma(0)/l\rangle \\
&\quad - l^2 \int_0^1 \Big(\widetilde{\mathcal{K}}^\nu(\dot\gamma, \bar e_j) + \langle \dot\gamma/l, X\rangle^2\Big)\,dt + 2\langle \dot\gamma/l, X\rangle\,|_0^1 \geqslant 0.
\end{aligned} \tag{2.3}$$

Since $s = q - q' \leqslant q - \nu + n - 1$, $\sum_i \theta_i^2 \leqslant \frac{\pi^2}{4}s$, we have $\sum_{i=1}^{q'} \|\langle h(b_i, b_i), \dot\gamma(1)/l\rangle - \langle h(a_i, a_i), \dot\gamma(0)/l\rangle\| \leqslant 2q'\|h\|$ and $\sum_{j=1}^{s} \|\langle h(\bar e_j, \bar e_j), \dot\gamma(1)/l\rangle - \langle h(e_j, e_j), \dot\gamma(0)/l\rangle\| \leqslant 2s\,\|h\|$. By (1.4) and condition $\mathcal{N} \geqslant \nu$, we get $\widetilde{\mathcal{R}ic}_q^{\,\nu}(\dot\gamma, W) \geqslant c$ for W spanned by $x_i(t)$ and $\bar e_j$; hence, $\sum_{i=1}^{q'} \widetilde{\mathcal{K}}^\nu(\dot\gamma, x_i(t)) + \sum_{j=1}^{q-q'} \widetilde{\mathcal{K}}^\nu(\dot\gamma, \bar e_j) \geqslant c$. Then by (2.3), $l^2(c + q\|X^\perp\|^2) \leqslant 2q\|h\| + (q - \nu + n - 1)\,\pi^2/4 + 2q\,\|X^\perp\|$. $\square$

By the third case of Theorem 2.1 we have the following.

Corollary 2.1. *Let $(M^{n+\nu}, g)$ be endowed with a compact totally geodesic foliation $\mathcal{F}^\nu$ and a vector field X tangent to the leaves. If condition $\widetilde{\mathcal{R}ic}_q^{\,\mathcal{N}} > 0$ is satisfied for some $\mathcal{N} \geqslant \nu$ and $1 \leqslant q \leqslant \nu$, then $\nu < n + q - 1$.*

3. Around the weighted mixed scalar curvature

Define tensors for one of distributions, $\widetilde{\mathcal{D}}$; similar tensors for $\mathcal{D}$ are defined using $\widetilde{\ }$ notation. Let $T, h : \widetilde{\mathcal{D}} \times \widetilde{\mathcal{D}} \to \mathcal{D}$ be the integrability tensor and the 2nd fundamental form of $\widetilde{\mathcal{D}}$, $T(u,v) := (1/2)\,[u,\,v]^\perp$, $h(u,v) := (1/2)\,(\nabla_u v + \nabla_v u)^\perp$. The mean curvature vector of $\widetilde{\mathcal{D}}$ is $H = \operatorname{Tr}_g h$. A distribution $\widetilde{\mathcal{D}}$ is called *totally umbilical*, *harmonic*, or *totally geodesic*, if $h = \frac{1}{v} H \cdot g^\top$, $H = 0$ or $h = 0$, respectively. The shape operator A of $\widetilde{\mathcal{D}}$ and the operator $T^\sharp$ are $\langle A_Z u, v\rangle = \langle h(u^\top, v^\top), w^\perp\rangle$ and $g(T^\sharp_w u, v) = \langle T(u^\top, v^\top), w^\perp\rangle$. The local adapted orthonormal frame $\{E_a, \mathcal{E}_i\}$, where $\{E_a\} \subset \widetilde{\mathcal{D}}$, always exists on M. We use inner products of tensors, e.g. $\|h\|^2 = \sum_{i,j} \|h(\mathcal{E}_i, \mathcal{E}_j)\|^2$ and $\|T\|^2 = \sum_{i,j} \|T(\mathcal{E}_i, \mathcal{E}_j\|^2$.

3.1. *Integral formulas*

Integral formulae for foliations relate extrinsic geometry of the leaves with curvature and provide obstructions for existence of foliations with given geometry. In Section 3.1, we consider singular distributions, that is those defined outside a "singularity set" Σ, a finite union of pairwise disjoint closed submanifolds of codimension $\geqslant k$ under assumption that improper integrals $\int_M \|\xi\|^s\, {\rm dV_g}$ converge for suitable vector fields ξ defined on $M \setminus \Sigma$.

Lemma 3.1 (see [10]). *If* $(k-1)(s-1) \geqslant 1$ *and* ξ *is a vector field on* $M \setminus \Sigma$ *such that* $\|\xi\| \in L^s(M, g)$ *then* $\int_M (\operatorname{div} \xi)\, {\rm dV_g} = 0$ *holds.*

The divergence of the vector field $H + \tilde H$ on a Riemannian almost product manifold was calculated explicitly in [9]:

$$\operatorname{div}(H + \tilde H) = {\rm S}_{\rm mix} - \|T\|^2 - \|\tilde T\|^2 + \|h\|^2 + \|\tilde h\|^2 - \|H\| - \|\tilde H\|^2. \tag{3.1}$$

The $\mathcal{D}$-*divergence* of a vector field ξ is given by $\operatorname{div}^\perp \xi = \sum_i \langle \nabla_i\, \xi, \mathcal{E}_i\rangle$, and we have

$$\operatorname{div}^\perp(\xi^\perp) = \operatorname{div}(\xi^\perp) + \langle \xi, H\rangle, \quad \operatorname{div}^\top(\xi^\top) = \operatorname{div}(\xi^\top) + \langle \xi, \tilde H\rangle. \tag{3.2}$$

By (3.1) and using $\operatorname{div} \tilde H = \operatorname{div}^\top \tilde H - \|\tilde H\|^2$ and $\operatorname{div} H = \operatorname{div}^\perp H - \|H\|^2$, we get

$$\operatorname{div}^\top \tilde H + \operatorname{div}^\perp H = {\rm S}_{\rm mix} + \|\tilde h\|^2 + \|h\|^2 - \|\tilde T\|^2 - \|T\|^2. \tag{3.3}$$

Next propositions are based on (3.1) and (3.3) and extend results in [9].

Proposition 3.1. *Let* (M, g) *be a closed Riemannian manifold endowed with complementary orthogonal distributions* $\widetilde{\mathcal{D}}$ *and* $\mathcal{D}$ *defined on* $M \setminus \Sigma$ *with* $\operatorname{codim} \Sigma \geqslant k$, *and a vector field* X *such that* $\|\xi\|_g \in L^s(M, g)$, *where* $\xi = \tilde H + H + \frac{1}{2} X$ *and* $(k-1)(s-1) \geqslant 1$. *Then for all* $N, \mathcal{N} \neq 0$ *the following integral formula holds:*

$$\int_M \Big\{ \mathcal{S}^{N,\mathcal{N}} - \|T\|^2 - \|\tilde T\|^2 + \|h\|^2 + \|\tilde h\|^2 - \|H\|^2 - \|\tilde H\|^2 - \frac{\|X^\top\|^2}{2N} - \frac{\|X^\perp\|^2}{2\mathcal{N}} \Big\}\, {\rm dV_g} = 0.$$

Proof. This follows from (3.1), (1.5) and Lemma 3.1. □

We say that (M', g') is a *leaf of a distribution* $\widetilde{\mathcal{D}}$ on (M, g) if M' is a submanifold of M with induced metric g' and $T_m M' = \widetilde{\mathcal{D}}_m$ for any $m \in M'$.

Proposition 3.2. *Let* (M, g) *be a closed Riemannian manifold endowed with complementary orthogonal distributions* $\widetilde{\mathcal{D}}$ *and* $\mathcal{D}$ *with* $\operatorname{div}^\perp H = 0$, *defined on* $M \setminus \Sigma$ *with* $\operatorname{codim} \Sigma \geqslant k$, *and a vector field* $X \in \mathfrak{X}^\top$ *such that* $\|\xi_{\,|M'}\|_g \in L^s(M', g')$ *for all leaves* (M', g') *of* $\widetilde{\mathcal{D}}$, *where* $\xi = \tilde H + \frac{1}{2} X$ *and* $(k-1)(s-1) \geqslant 1$. *Then for all* $N \neq 0$ *and* $\mathcal{N} \in \mathbb{R}$ *the following integral formula holds:*

$$\int_{M'} \Big\{ \mathcal{S}^{N,\mathcal{N}} - \|\tilde T\|^2 + \|h\|^2 + \|\tilde h\|^2 + \frac{1}{2} \langle X, \tilde H\rangle - \frac{1}{2N} \|X\|^2 \Big\}\, {\rm dV_{g'}} = 0.$$

Proof. This follows from (3.3), (1.5) and Lemma 3.1. □

3.2. Splitting of almost product manifolds

Applying S.T.Yau version of Stokes' theorem on a complete open (M,g) yields the following.

Lemma 3.2 (see Proposition 1 in [1]). *Let (M,g) be a complete open Riemannian manifold endowed with a vector field ξ such that $\operatorname{div}\xi \geqslant 0$. If the norm $\|\xi\|_g \in L^1(M,g)$ then $\operatorname{div}\xi \equiv 0$.*

In Section 3.2 we extend some splitting theorems [4, 8] to the case of almost product manifolds of nonnegative/nonpositive weighted mixed scalar curvature.

In next two theorems we consider harmonic distributions with $\mathcal{S}^{N,\mathcal{N}} \geqslant 0$.

Theorem 3.1. *Let (M,g) be a complete open (or closed) Riemannian manifold endowed with complementary orthogonal integrable distributions $(\widetilde{\mathcal{D}},\mathcal{D})$ and a vector field $X \in \mathfrak{X}^\top$ obeying conditions $\langle X,\tilde H\rangle = 0$ and $\|X_{|M'}\|_{g'} \in L^1(M',g')$ for all leaves (M',g') of $\widetilde{\mathcal{D}}$. Suppose that $\widetilde{\mathcal{D}}$ is harmonic and $\mathcal{S}^{N,\mathcal{N}} \geqslant 0$ for some $N<0$ and $\mathcal{N}\neq 0$. Then $X=0$ and M splits.*

Proof. By conditions and (1.5), (3.2) and (3.3), we have

$$\operatorname{div}^\top(\tilde H + \frac{1}{2}\,X) = \mathcal{S}^{N,\mathcal{N}} + \|\tilde h\|^2 + \|h\|^2 - \frac{1}{2N}\,\|X\|^2.$$

By Lemma 3.2 for each leaf, $\operatorname{div}^\top(\tilde H + \frac{1}{2}\,X) = 0$ when $\mathcal{S}^{N,\mathcal{N}} \geqslant 0$ and $N<0$; thus, $h = 0 = \tilde h$ and $X = 0$. By de Rham decomposition theorem, (M,g) splits. □

Corollary 3.1. *Let (M,g,X) be endowed with two complementary orthogonal integrable distributions $(\widetilde{\mathcal{D}},\mathcal{D})$ and a vector field $X\in\mathfrak{X}^\top$. Suppose that $\mathcal{D}$ is harmonic. Then $\widetilde{\mathcal{D}}$ has no compact harmonic leaves M' with positive $\mathcal{S}^{N,\mathcal{N}}|_{M'}$ for some $N<0$ and $\mathcal{N}\neq 0$.*

Theorem 3.2. *Let (M,g) be a closed or a complete open Riemannian manifold endowed with complementary orthogonal harmonic foliations and a vector field X such that $\|X\|_g\in L^1(M,g)$. If $\mathcal{S}^{N,\mathcal{N}} \geqslant 0$ for some $N,\mathcal{N}<0$ then $X=0$ and M splits.*

Proof. Under conditions, from (3.1) we obtain

$$\frac{1}{2}\operatorname{div} X = \mathcal{S}^{N,\mathcal{N}} + \|\tilde h\|^2 + \|h\|^2 - \frac{1}{2N}\,\|X^\top\|^2 - \frac{1}{2\mathcal{N}}\,\|X^\perp\|^2.$$

By Lemma 3.2, we get $\operatorname{div} X = 0$ when $\mathcal{S}^{N,\mathcal{N}} \geqslant 0$ and $N,\mathcal{N}<0$. Thus, $h = 0 = \tilde h$ and $X = 0$. By de Rham decomposition theorem, (M,g) splits. □

If $\mathcal{D}$ is totally umbilical then $\|\tilde h\|^2 - \|\tilde H\|^2 = -\frac{n-1}{n}\,\|\tilde H\|^2$, and similarly, for $\widetilde{\mathcal{D}}$.

Theorem 3.3. *Let (M,g) be a closed (or a complete open) Riemannian manifold endowed with complementary orthogonal totally umbilical distributions $\widetilde{\mathcal{D}}$ and $\mathcal{D}$ and a vector field X obeying $\|\xi_{|M}\|_g\in L^1(M,g)$, where $\xi = \tilde H + H + \frac{1}{2}\,X$. Suppose that $\mathcal{S}^{N,\mathcal{N}} \leqslant 0$ for some $N,\mathcal{N}>0$. Then $X=0$ and M splits.*

Proof. Under conditions, from (3.1) we get

$$\operatorname{div}\xi = \mathcal{S}^{N,\mathcal{N}} - \|T\|^2 - \|\tilde T\|^2 - \frac{n-1}{n}\|\tilde H\|^2 - \frac{\nu-1}{\nu}\|H\|^2 - \frac{1}{2N}\,\|X^\top\|^2 - \frac{1}{2\mathcal{N}}\,\|X^\perp\|^2.$$

From this and Lemma 3.2 and since $\mathcal{S}^{N,\mathcal{N}} \leqslant 0$ for $N,\mathcal{N}>0$, we get $\operatorname{div}\xi = 0$. Thus $T,\tilde T,H,\tilde H$ and X vanish. By de Rham theorem, (M,g) splits. □

4. The Toponogov conjecture

Much is known about foliations with $K_{\rm mix}$ = const. Examples are 1) *k-nullity* foliations on manifolds with degenerate curvature tensor (the certain metrics are called partially hyperbolic, parabolic or elliptic); 2) *relative nullity* foliations of curvature-invariant submanifolds M (e.g. of space forms). Submanifolds with positive relative nullity index $\mu(M) = \dim\ker h(m)$ (introduced by Chern and Kuiper) have a structure of ruled developable submanifolds.

The proof of Ferus's result (mentioned in the Introduction) is based on analysis of the matrix Riccati equation $\dot B_{\dot\gamma} + (B_{\dot\gamma})^2 + R_{\dot\gamma} = 0$ on a leaf geodesic γ, where $B_x = B(x,\cdot)$ for short, and the *co-nullity tensor* $B(x,y) = -(\nabla_y x)^\perp$ $(x\in T\mathcal{F},\ y\in T^\perp\mathcal{F})$ of $\mathcal{F}$. In the paper we define weighted modification of the co-nullity tensor of a totally geodesic foliated manifold (M,g) equipped with a vector field X,

$$\mathcal{B}_x = B_x - \langle X/n, x\rangle\,{\rm id}^\perp .$$

The following "weighted" Riccati equation holds along leaf geodesics:

$$\dot{\mathcal{B}}_{\dot\gamma} + (\mathcal{B}_{\dot\gamma})^2 + 2\,\langle X/n,\dot\gamma\rangle\,\mathcal{B}_{\dot\gamma} + \mathcal{R}_{\dot\gamma} = 0. \tag{4.1}$$

Next theorem and corollary with constant $\mathcal{R}_x > 0$ generalize Ferus's results [3].

Theorem 4.1. *Let $(M^{n+\nu}, g, X)$ be endowed with a totally geodesic foliation $\mathcal{F}^\nu$. Suppose that there exist k = const >0 and a point $m\in M$ such that $\mathcal{R}_{\dot\gamma} = k\,{\rm id}^\perp$ holds along any leaf geodesic $\gamma : [0,\pi/\sqrt{k}]\to M$ with $\gamma(0)=m$, and*

$$\langle X/n,\dot\gamma\rangle^2 \leqslant k. \tag{4.2}$$

Then $\nu < \rho(n)$.

Proof. Assume the contrary, then there are unit vectors $x\in T_m\mathcal{F}$ and $y\in T^\perp_m\mathcal{F}$ and $\lambda_0\leqslant 0$ such that $\mathcal{B}_x\, y = \lambda_0 y$ for a geodesic $\gamma(t)$ with initial velocity $\dot\gamma(0)=x$. Let $\bar y(t)$ ($\bar y(0)=y$) be a parallel vector field along γ. The eigenvectors of the solution $\mathcal{B}_{\dot\gamma}$ of (4.1) with $\mathcal{R}_{\dot\gamma} = k\,{\rm id}^\perp$ do not depend on t. Then $\mathcal{B}_{\dot\gamma}\,\bar y(t) = \lambda(t)\,\bar y(t)$ for certain eigenfunction $\lambda(t)$, which satisfies the scalar Riccati equation

$$\dot\lambda + \lambda^2 + 2\,\lambda\,\langle X/n,\dot\gamma\rangle + k = 0. \tag{4.3}$$

By (4.2), solution $\lambda(t)$ of (4.3) cannot be extended to $[0,\pi/\sqrt{k}]$, a contradiction. □

The *relative nullity space* of the second fundamental form h of a submanifold $M\subset\bar M$ at $m\in M$ is $\ker h(m) = \{x\in T_mM : h(x,y)=0 \text{ for all } y\in T_mM\}$. A submanifold $M\subset\bar M$ is *curvature-invariant* if the curvature tensor of $\bar M$ obeys $(\bar R_{x,y}\,z)^\perp = 0$, $(x,y,z\in TM)$. Such submanifold with positive index of relative nullity $\mu(M) = \min_{m\in M}\dim\ker h(m)$ has a ruled developable structure.

The *extrinsic qth Ricci curvature* is defined by

$$\mathrm{Ric}^q_h(x_0,x_1,\ldots,x_q) = \sum\nolimits_{i=1}^q \Big(\langle h(x_0,x_0), h(x_i,x_i)\rangle - \langle h(x_0,x_i), h(x_0,x_i)\rangle\Big).$$

For $q=1$ it is called an *extrinsic sectional curvature*.

Corollary 4.1 (for $X=0$ see [4]). *Let M^n be a complete curvature-invariant submanifold in $(\bar M^{n+p},\bar g, X)$. Suppose that there exists real $k>0$ such that along any geodesic $\gamma:\mathbb{R}\to M$ starting at $\dot\gamma(0)\in\ker h$, the weighted Jacobi operator of $\bar M$ obeys* (4.2) *and $\bar{\mathcal{R}}_{\dot\gamma} = k\,{\rm id}^\perp$. Then M is a totally geodesic submanifold if any of the requirements a), b) are satisfied:*

a) $\mu(M) > \nu(n) := \max\{t : t<\rho(n-t)\}$, *b)* $\mathrm{Ric}^q_h \leqslant 0$ *and* $2p < n-\nu(n) - q + \delta_{1q}$.

In [4], we studied Toponogov's conjecture for a foliation given near a complete leaf: the necessity of additional assumptions in local case was shown, while the conjecture was confirmed for ruled submanifolds of spherical space forms. For foliated Riemannian manifolds, our geometric construction is based on estimates of the length and the volume of associated Jacobi field of "extremal geodesics", and it examines conditions when co-nullity tensor of a foliation has no real eigenvectors, providing $\nu < \rho(n)$. Here we extend the above methods to study the Toponogov type conjecture. A smooth (1,1)-tensor field $Y(t) : T^\perp_{\gamma(t)}\mathcal{F} \to T^\perp_{\gamma(t)}\mathcal{F}$ on a leaf geodesic γ is called a *Jacobi tensor* if it satisfies the equation $\ddot Y + R_{\dot\gamma} Y = 0$, and $\ker Y(t) \cap \ker Y'(t) = \{0\}$ for all t; hence, the action of Y on linearly independent parallel sections of $T^\perp_\gamma\mathcal{F}$ gives rise to linearly independent Jacobi vector fields. We have $B_{\dot\gamma} = \dot Y Y^{-1}$. A solution $y(t) \subset T^\perp_{\gamma(t)}\mathcal{F}$ of the equation $\ddot y + R_{\dot\gamma}\, y = 0$ with constant operator $R_{\dot\gamma} = k\,\mathrm{id}^\perp > 0$ is $y(t) = y(0)\cos(\sqrt{k}\,t) + \frac{y'(0)}{\sqrt{k}}\sin(\sqrt{k}\,t)$. If $y(0)$ and $y'(0)$ are linearly independent, then $y(t)$ parameterizes an ellipse in the plane $y(0)\wedge y'(0)$, and the area of the parallelogram $y(t)\wedge y'(t)$ is constant.

Lemma 4.1 (see [4]). *Let a solution $y(t) \subset \mathbb{R}^n$ of the Jacobi ODE*

$$\ddot y + R(t) y = 0 \quad (0 \leqslant t \leqslant \pi/\sqrt{k}), \tag{4.4}$$

be written in the form $y(t) = \bar y(t) + u(t)$, where $\bar y(t) = y(0)\cos(\sqrt{k}\,t) + \frac{y'(0)}{\sqrt{k}}\sin(\sqrt{k}\,t)$ and the norm $\|R(t) - k\,\mathrm{id}\,\| \leqslant \varepsilon_1 < k/2$. Then

$$\|u(t)\| \leqslant \frac{\varepsilon_1}{k - (1-\cos(\sqrt{k}\,t))\,\varepsilon_1}\int_0^t \sqrt{k}\,|\bar y(s)|\sin(\sqrt{k}\,(t-s))\,ds\,.$$

The *turbulence of a leaf L* of a totally geodesic foliation is the rotational component of the co-nullity tensor, see [4],

$$a(L) = \sup\{\langle B_x(y), z\rangle :\ x \in TL,\ y,z \in TL^\perp,\ y \perp z,\ \|x\| = \|y\| = \|z\| = 1\}.$$

If $a(L) = 0$ for all leaves then $T^\perp\mathcal{F}$ is tangent to a totally umbilical foliation.

Theorem 4.2 (Local). *Let $\mathcal{F}^\nu$ be a totally geodesic foliation of $(M^{n+\nu}, g, X)$, and there exists a point $m \in M$ such that along any leaf geodesic $\gamma : [0, \pi/\sqrt{k}] \to L$ $(\gamma(0) = m)$ we have* (4.2) *and*

$$0 < k_1\,\mathrm{id}^\perp \leqslant \mathcal{R}_{\dot\gamma} \leqslant k_2\,\mathrm{id}^\perp, \tag{4.5}$$

$$(k_2 - k_1 + 2\,\varepsilon)\max\{a(L)^2, k\} \leqslant 0.3\,k(k_2+\varepsilon), \tag{4.6}$$

where $k = (k_1+k_2)/2$ and $\varepsilon := \|\langle\nabla_{\dot\gamma}(X/n), \dot\gamma\rangle + \langle X/n, \dot\gamma\rangle^2\| < k_1$. Then $\nu < \rho(n)$.

Proof. Notice that $\varepsilon = 0$ is provided by $X^\top = 0$. It is sufficient to show that linear operators $\mathcal{B}_x : T^\perp_m\mathcal{F} \to T^\perp_m\mathcal{F}$, $(x \neq 0)$, have no real eigenvalues. Suppose the opposite, i.e., there exist unit vectors $x_0 \in T_m\mathcal{F}$, $y_0 \in T^\perp_m\mathcal{F}$ and $\lambda \leqslant 0$ with the property $\mathcal{B}_{x_0}(y_0) = \lambda y_0$. Let $\gamma(t) : [0,\pi/\sqrt{k}] \to M$, $\dot\gamma(0) = x_0$ be a leaf geodesic, and $y(t) : \gamma \to T^\perp_\gamma\mathcal{F}$ a Jacobi vector field on γ through the vector y_0. Hence (4.4) holds with $\|R(t) - k\,\mathrm{id}\,\| \leqslant \frac{k_2-k_1}{2} + \varepsilon$, see (1.6), where $\dot y = \nabla_{\dot\gamma}\, y$ and $\ddot y = \nabla_{\dot\gamma}\nabla_{\dot\gamma}\, y$.

The Jacobi vector field $y(t)$ may be written in a form $y(t) = (\cos(\sqrt{k}\,t) + (\lambda/\sqrt{k})\sin(\sqrt{k}\,t))y_0 + u(t)$, where $u(0) = u'(0) = 0$. (For $k_2 = k_1$, we have $u(t) = 0$, hence $y(t)$ vanishes at $t_0 = \mathrm{arcctg}(-\lambda/\sqrt{k})/\sqrt{k}$. This contradiction completes the alternative proof of Theorem 4.1). We show for (4.5) and $k_1 - \varepsilon \geqslant 0.582(k_2+\varepsilon)$, that the function $|y(t)|$ – the length of the Jacobi vector field $y(t)$ – has a local minimum at t_m in the interval $(0, \pi/\sqrt{k})$.

Our second observation is that the function $V(t)$ – the area of a parallelogram, whose sides are the vectors $y(t)$ and $y'(t)$, varies "slowly" along a geodesic γ. (This function is constant when $k_2 = k_1$.) By Lemma 4.1 and (4.6), these will yield a contradiction, because $V(t)$ cannot increase from zero value $V(0)$ to a "large" value $V(t_m)$ on a given interval with length $t_m < \pi/\sqrt{k}$. The rest of proof repeats the steps 2-4 in the proof of Theorem 4.10, case (a) in [4] with ε of different sense. □

Theorem 4.3 (Decomposition). *Let $\mathcal{F}^\nu$ be a compact totally geodesic foliation of* $(M^{n+\nu}, g, X)$. *Suppose that* (4.2) *and the following hold:*

$$0 \leqslant k_1\,\mathrm{id}^\perp \leqslant \mathcal{R}_x \leqslant k_2\,\mathrm{id}^\perp \quad (x \in T\mathcal{F},\ \|x\| = 1),$$

$$(k_2 - k_1 + 2\,\varepsilon)\cdot \max\{a(L)^2, k\} \leqslant 0.3\,(k_2 + \varepsilon)\,k, \tag{4.7}$$

where L is some leaf, $k = \frac{1}{2}(k_1 + k_2)$ and $\varepsilon := \big\| \langle \nabla_x (X/n), x\rangle + \langle X/n, x\rangle^2 \big\| < k_1$. If $\nu \geqslant \rho(n)$ then $k_1 = k_2 = 0$ and M splits along $\mathcal{F}$.

Proof. By the proof of Theorem 4.2, we get $K_{\rm mix} = 0$. Hence, our compact totally geodesic foliation splits, see [4, Lemma 4.14]. □

Theorem 4.2 (with $\mathcal{R}_x > 0$) generalizes [4, Theorem 4.10] when $X = 0$. Theorem 4.3 (with $\mathcal{R}_x \geqslant 0$) generalizes [4, Theorem 4.16] when $X = 0$. Theorems 4.2 and 4.3 are not true without conditions (4.6) and (4.7), but their coefficient 0.3 is obtained by the method for proving and can presumably be increased.

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