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The Weinstein Conjecture in Cotangent Bundles and Related Results

H. HOFER* - C. VITERBO

1. - Introduction and Results

Let $N = T^*M$ be the cotangent bundle of the smooth connected compact manifold M. N carries a canonical symplectic structure $\omega = -d\lambda$, where λ is the Liouville form on N, see [1]. We consider a Hamiltonian vectorfield

$$\dot{x} = X_H(x) \quad \text{on} \quad N \tag{1}$$

associated to a smooth function $H : N \to \mathbb{R}$ via

$$dH = X_H \neg\, \omega. \tag{2}$$

We are interested in finding periodic solutions on a given compact regular energy surface $S = \{H = \text{const.}\}$. Regular means $X_H(x) \neq 0$ on S. If S is the regular energy surface for another Hamiltonian vectorfield then periodic solutions for both vectorfields agree up to reparametrisation. This prompts the following abstraction. Denote by $l = l_S = \text{kern}\ (\omega/S)$ the line bundle on TS which is given by

$$l_x = \{v \in T_xS |\ \omega(v, w) = 0 \quad \text{for all} \quad w \in T_xS\}.$$

Then l determines the direction of every Hamiltonian vectorfield having S as a regular energy surface.

DEFINITION 1. A periodic Hamiltonian trajectory on S is a submanifold P of S such that

(i) P is diffeomorphic to S^1

(ii) $TP = l_S|P$

We denote by $\mathcal{P}(S)$ the collection of all periodic Hamiltonian trajectories on S.

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The breakthrough in the study of periodic solutions on compact energy surfaces in $\mathbb{R}^{2n}$ is due to P. Rabinowitz [18] and A. Weinstein [27], who proved the existence of periodic orbits on starshaped resp. convex energysurface in $\mathbb{R}^{2n}$. Quite recently the second author, [25], has been able to find periodic solutions on compact energysurfaces of contact type in $\mathbb{R}^{2n}$ and to prove a generalization of a conjecture by A. Weinstein [28] in the $\mathbb{R}^{2n}$-case. Motivated by Viterbo's result the first author and E. Zehnder, [12], simplified the arguments in [25] and proposed an existence mechanism based on a priori estimates for periodic solutions.

In this paper we shall prove, besides other results, the Weinstein conjecture for an interesting class of hypersurfaces in a cotangent bundle and answer some of the questions raised in [9]. We use and considerably refine the approach in [25] and [12] and combine it with the first author's critical point theory which already proved useful in the proof of a conjecture by V.I. Arnold [11]. Here we use some so far unpublished new results in [10]. We impose the following hypothesis:

(S) $N = T^*M$ is the cotangent bundle of a compact smooth connected manifold M of dimension at least 2. S is a compact connected hypersurface in N and the bounded component of $N \backslash S$ in N contains the zero section $M \subset T^*M$.

Some remarks concerning (S) are in order. If $N = T^*M$ and dim $M \geq 2$ then, as an easy consequence of Alexander duality [24, p. 296], we have the following situation: if S is a compact connected hypersurface in N not intersecting the zero section, then $N \backslash S$ has exactly two components, a bounded and an unbounded one. So (S) says that we assume that S "encloses" the zero section of N. As a consequence of (S) the canonical line bundle $l_S \to S$ is orientable. We can take the orientation given by a Hamiltonian vectorfield X_H having S as a regular energysurface such that $H(z) \to +\infty$ as z goes to infinity in the fibres of $N = T^*M$. If (S) holds the following definition given in [12] is useful.

DEFINITION 3. Let S satisfy (S). A parametrized family of compact hypersurfaces in $N = T^*M$ modelled on S is a diffeomorphism $\Psi : (-1, 1) \times S \to N$ onto an open neighborhood U of S with compact closure such that $\Psi(0, x) = x$ for all $s \in S$. We shall put $S_\varepsilon = \Psi(\{\varepsilon\} \times S)$ in the following.

If $P \in \mathcal{P}(S)$ we put $A(P) = \int \lambda | P$ where P inherits the orientation from $l_S \to S$ (recall $TP = l_S | P$).

Our main result is the following:

THEOREM 1. *Let S satisfy (S) and suppose Ψ is a parametrized family of compact hypersurfaces modelled on S. Then there exists a constant $d = d(\Psi) > 0$ such that for every $\delta > 0$ there is an $\varepsilon > 0$ in $|\varepsilon| < \delta$ for which the hypersurface S_ε carries a periodic orbit P_ε with*

$$0 < |A(P_\varepsilon)| < d. \tag{3}$$

Now arguing as in [12] we obtain a proof of the Weinstein conjecture for hypersurfaces satisfying (S) without the assumption $H^1(S;\mathbb{R}) = 0$:

COROLLARY 1. *If S satisfies (S) and S is of contact type then $\mathcal{P}(S) \neq \emptyset$.*

Moreover we obtain a result based on a priori estimates. Suppose $< \cdot, \cdot >$ is a Riemannian metric on N. If P is a one-dimensional submanifold of N we can define its length $L(P)$ in the obvious way.

COROLLARY 2. *Let S satisfy (S). Assume there exists a constant $c > 0$ such that*

$$\frac{1}{c}\, L(P) \leq |A(P)| \leq cL(P) \tag{4}$$

for all $P \in \mathcal{P}(S_\varepsilon)$ and all $\varepsilon \in (-1, 1)$ for some parametrized family Ψ modelled on S. Then $\mathcal{P}(S) \neq \emptyset$.

As in [12] Corollary 2 follows immediately from Theorem 1.

The proof of Theorem 1 is much more technical than the proofs in [12] or [25]. The basic references for the infinite dimensional set up are [13] and [14]. We shall use the same notation. Instead of working in cotangent bundles we will work in tangent bundles. For this we fix a Riemannian metric on M and pull back the symplectic structure in T^*M via the "Rieszmap" $TM \tilde{\rightarrow} T^*M : x \rightarrow < x, \cdot >$

2. - Basic set up and sketch of the proof of Theorem 1

We pick a smooth function $\varphi_1 : (0, +\infty) \times (-1, 1) \rightarrow \mathbb{R}$ having the following properties:

$$\begin{array}{lll} \varphi_1(b,s) = 0 & \text{for } s \in (-1, -\delta], & b \in (0, +\infty) \\ \varphi_1(b,s) = b & \text{for } s \in [\delta, 1), & b \in (0, +\infty) \\ D_2\varphi_1(b,s) > 0 & \text{for } s \in (-\delta, \delta), & b \in (0, +\infty) \\ D_1\varphi_1(b,s) \geq 0 & \text{for } s \in (-\delta, \delta), & b \in (0, +\infty). \end{array} \tag{1}$$

Further we pick a smooth function $\varphi_2 : \mathbb{R} \rightarrow \mathbb{R}$, such that

$$\begin{array}{ll} \varphi_2(s) = 0 & \text{for } s \in (-\infty, r] \\ \varphi_2'(s) > 0 & \text{for } s \in (r, +\infty) \\ \varphi_2(s) = \frac{1}{2}s^2 & \text{for } s \geq 2r \end{array} \tag{2}$$

where $\alpha = \mathrm{diam}(U)$, $U = \Psi((-1,1) \times S)$, and $\alpha < r < 2\alpha$. Using φ_1, φ_2 we define a smooth map $H : (0, +\infty) \times TM \rightarrow \mathbb{R}$ by

$$H_b(x) := H(b,x) := \begin{cases} 0 & \text{for} \quad x \in B \\ \varphi_1(b,\varepsilon) & \text{for} \quad x \in S_\varepsilon, \ \varepsilon \in [-\delta,\delta] \\ b & \text{for} \quad x \in A, \quad |x| \le r \\ \varphi_2(|x|) + b & \text{for} \quad |x| > r. \end{cases} \tag{3}$$

Here B is the bounded component of $TM \backslash \Psi([-\delta,\delta] \times S)$ and A the unbounded component. If we put

$$H_b(x) = \frac{1}{2}|x|^2 + \Delta_b(x) \tag{4}$$

then $\Delta : (b,x) \to \Delta_b(x)$ is a smooth realvalued function on $(0,+\infty) \times TM$. Moreover for $|x| > 2r$ we have $\Delta_b(x) = b$. Using this we obtain a continuous map $k : (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ defined by

$$k(b_1,b_2) = \sup_{x \in TM} |\Delta_{b_1}(x) - \Delta_{b_2}(x)| \tag{5}$$

with the property $k(b,b) = 0$. In the following we use some of the results in Klingenberg's books, [13] and [14]. Denote by $S^1 = \mathbb{R}/\mathbb{Z}$ the unit circle and by $\Lambda = H^1(S^1, M)$ the Hilbert manifold of absolutely continuous loops with square integrable derivative, i.e.,

$$E(q) := \frac{1}{2}\int_0^1 |\dot{q}(t)|^2 dt < \infty \quad \text{for} \quad q \in \Lambda.$$

We shall call $E(q)$ the energy of q. We define for $q \in \Lambda$

$$T_q\Lambda = H^1(q^*TM).$$

It is well known that

$$T\Lambda = \bigcup_{q \in \Lambda} T_q\Lambda$$

can be canonically identified with the tangent space of Λ, so that the notation $T\Lambda$ is justified. We can equip $T_q\Lambda$ with the inner product

$$(x,y) = \int_0^1 < x(t), y(t) > dt$$

and denote by $L_q\Lambda$ or L_q the completion. Then

$$L := L\Lambda := U_{q \in \Lambda} \ L_q\Lambda$$

carries the structure of a vectorbundle over Λ. We denote by $\pi : L \to \Lambda$ the canonical projection. Taking the tangent vectorfield along a H^1-curve defines a smooth section

$$\partial : \Lambda \to L : q \to \partial q := \dot{q}. \tag{6}$$

We define a smooth functional $\Phi \in C^\infty(L, \mathbb{R})$ by

$$\Phi(x) = (\partial \circ \pi(x), x) - \frac{1}{2}\|x\|^2 \tag{7}$$

where $\|x\|^2 = (x, x)$. The critical points of Φ are clearly the geodesics on M. In fact

$$\begin{aligned} 0 &= d\Phi(x)(h) \\ &= (\nabla(T\pi)h, x) + (\partial \circ \pi(x), \hat{K}h) - (x, \hat{K}h) \end{aligned} \tag{8}$$

for all $h \in T_xL$. Here ∇ denotes the covariant derivative along $q = \pi(x)$ associated to the Levi-Civita connection K on M. Moreover $\hat{K}$ is the Riemannian connection for the fibre metric $(\cdot,\cdot)$ on L which is induced by the Levi-Civita connection in the obvious way (see [13] or [14] for details). Hence (8) gives

$$\partial q = x, \ \nabla_q \cdot x = 0 \qquad \text{with } q = \pi(x)$$

or equivalently

$$\nabla_q \cdot \partial q = 0 \quad q \in \Lambda.$$

But this just means q is a critical point of $E \in C^\infty(\Lambda, \mathbb{R})$. Next we define for $b \in (0, +\infty)$ a C^1-functional $\Phi_b : L \to \mathbb{R}$ by

$$\Phi_b(x) = \Phi(x) - \int_0^1 \Delta_b(x(t))dt. \tag{9}$$

Unlike Φ the functional Φ_b is in general not smooth because of the nonzero Δ_b-part by known results on substitution operators in L^2-spaces. Moreover the critical points x of Φ_b are clearly the solutions of the Hamiltonian system

$$(HS)_b \qquad \begin{aligned} \dot{q} &= KH_b'(x) \\ -\nabla p &= (T\tau_M)H_b'(x). \end{aligned}$$

Here $q = \pi(x)$ and H_b' denotes the gradient of $H_b : TM \to \mathbb{R}$ with respect to the Riemannian metric $<\cdot,\cdot>_{TM} = <K\cdot, K\cdot> + <T\tau_M\cdot, T\tau_M\cdot>$ on TM. If now x is nonconstant and $x(t_0) \in S_\epsilon$ for some $t_0 \in \mathbb{R}/\mathbb{Z}$ then x parametrizes a $P_\epsilon \in \mathcal{P}(S_\epsilon)$.

DEFINITION. A *good critical point* of Φ_b is a point $x \in L$ such that

$$d\Phi_b(x) = 0 \quad \text{and} \quad x(0) \in S_\varepsilon$$

for some $\varepsilon \in (-\delta, \delta)$. All other critical points will be called *parasites*.

Note that for a function $x \in L$ the condition $x(0) \in S_\varepsilon$ does not make sense a priori. However if x is a critical point of Φ_b then $x \in C^\infty(S^1, TM)$ so that $x(0)$ is well defined.

If x is a parasite we have either

$$x \text{ is a constant} \tag{10}$$

or x is nonconstant and solves

$$\begin{aligned} \dot{q} &= KH_b'(x) = \frac{\varphi_2'(|x|)}{|x|}\, x \\ \nabla p &= 0. \end{aligned} \tag{11}$$

Consequently we have $\Phi_b(x) \leq 0$ if (10) holds or

$$\Phi_b(x) = \varphi_2'(|x(0)|)|x(0)| - \varphi_2(|x(0)|) - b \tag{12}$$

if (11) holds.

LEMMA 1. *There exists a $b_0 > 0$ not depending on δ (see Theorem 1) such that a nonconstant parasite solution x of $(HS)_b$ with $b \geq b_0$ and $\Phi_b(x) \geq 0$ satisfies*

$$q = \pi(x) \quad \textit{is a geodesic on} \quad M \tag{13}$$

and

$$\Phi_b(x) = E(q) - b. \tag{14}$$

PROOF. Since $x \neq$ const we infer that $|x(t)| = |x(0)| > r$ for all $t \in \mathbb{R}/\mathbb{Z}$. Hence

$$\dot{q} = \frac{\varphi_2'(|x(0)|)}{|x(0)|}\, x \text{ and } \nabla x = 0.$$

Therefore by (12) and our assumption on x

$$0 \leq \Phi_b(x) = \varphi_2'(|x(0)|)|x(0)| - \varphi_2(|x(0)|) - b.$$

If we define b_0 by

$$b_0 := 1 + \sup_{s \leq 2r} |\varphi_2'(s)s - \varphi_2(s)|$$

we infer for $b \geq b_0$

$$\Phi_b(x) \leq -1 \text{ if } |x(0)| \leq 2r.$$

Consequently our hypothesis on x implies $|x(0)| \geq 2r$. Hence using the property of H_b

$$\varphi_2'(|x(0)|) = |x(0)|.$$

This gives

$$\dot{q} = x \text{ and } \nabla x = 0.$$

So q is a geodesic. A straightforward calculation gives now

$$\Phi_b(x) = E(q) - b$$

as required. □

It is a standard fact from the theory of closed geodesics that there exists a Riemannian metric on M such that the set

$$\Gamma = \{c \in \mathbb{R} \,|\, c = E(q) \text{ and } q \text{ is a geodesic}\}$$

consists of isolated points (see [14]). We may assume that our Riemannian metric has this property. The following result will imply our main Theorem 1.

THEOREM 2. *There exists a continuous nonincreasing function* Θ : $(0, +\infty) \to [0, +\infty)$ *such that for every* $b \in (0, +\infty)$ *there exists a* 1*-periodic solution* x_b *of* $(HS)_b$ *such that*

$$\text{(15)} \qquad x_b \neq \text{const} \textit{ and } \Theta_b(x_b) = \Theta(b)$$
$$\Theta(b) \leq c(\Psi) \textit{ for a constant only depending on } \Psi.$$

The proof of Theorem 2, which we will sketch, will take the rest of the paper. Before that, let us show that Theorem 2 implies Theorem 1.

Let $b_0 > 0$ as given in Lemma 1. Then we may assume that all the x_b, $b \geq b_0$, are parasites (otherwise we are done). Hence

$$\Theta(b) = \Phi_b(x_b) = E(q_b) - b$$

where $q_b = \pi(x_b)$. Consequently $\Theta(b) + b \in \Gamma$, $\Theta(b) \leq c(\Psi)$, for all $b \geq b_0$. Since Γ is discrete we have for some constant c

$$\Theta(b) + b = c \text{ for all } b \geq b_0.$$

If now $b > c$ we obtain the contradiction

$$0 \leq \Theta(b) = c - b < 0.$$

Note that $\Theta(b) + b = \Theta(b_0) + b_0 \leq c(\Psi) + b_0$ for $b \geq b_0$. Hence $b \leq c(\Psi)+b_0-\Theta(b) \leq c(\psi)+b_0$. So we find a good critical point x_b with $b \leq c(\Psi)+b_0$. This gives

$$|(\partial \circ \pi(x_b), x_b)| = |H_b(x_b(0)) + \Phi_b(x_b)| \leq b + \Theta(b) \leq 2c(\Psi) + b_0 =: d(\Psi).$$

We define a Riemannian metric for Λ in the usual way by

$$(x, y)_\Lambda = (\nabla x, \nabla y) + (x, y)$$

for $(x, y) \in T\Lambda \oplus T\Lambda$. It is well known that $(\Lambda, (\cdot,\cdot)_\Lambda)$ is a complete Hilbert manifold for the metrics induced in the components of Λ by $(\cdot,\cdot)_\Lambda$.

For $q : \mathbb{R}/\mathbb{Z} \to M$ in Λ we denote by q again the induced map $\mathbb{R} \to M$. For $s, t \in \mathbb{R}$ we obtain an associated parallel transport along q

$$A^q_{s,t} : T_{q(t)}M \to T_{q(s)}M.$$

For $\tau \in \mathbb{R}$ we define an isometric vectorbundle isomorphism $Z(\tau) : L \to L$ by

$$(Z(\tau)x)(t) = A^q_{t.t+\tau}x(t + \tau) \text{ for a.e. } t \in \mathbb{R}. \tag{16}$$

We have

$$\begin{aligned} &Z(0) = Id \text{ and } Z(\tau_1)Z(\tau_2) = Z(\tau_1 + \tau_2) \\ &\lim_{\tau \to 0} \quad Z(\tau)x = x \text{ in } L. \end{aligned} \tag{17}$$

So $(Z(\tau))_{\tau\in\mathbb{R}}$ induces on each fibre L_q a strongly continuous representation of $\mathbb{R}$ by isometries. It turns out that the infinitesimal generator of this group on L_q is ∇ with domain $T_q\Lambda$.

Now we recall a concept introduced in [11].

DEFINITION 2. A bounded subset C of L is called *uniformly fibre precompact* if for given $\varepsilon > 0$ there exists $\tau_0 > 0$ such that

$$\begin{aligned} &\|Z(\tau)x - x\| \leq \varepsilon \quad \text{for all } x \in C \\ &\text{and all } |\tau| \leq \tau_0. \end{aligned}$$

If in addition C is closed we call it *uniformly fibre compact.*

Next we introduce a certain subset $\mathcal{H}$ of the homeomorphism group homeo(L) of L. Denote by $\delta : T\Lambda \to L$ the smooth vectorbundle map which is on each fibre the covariant derivative ∇ along the corresponding curve $q \in \Lambda$.

We say a smooth map $g \in C^\infty(L, \mathbb{R})$ has property (g) if the following holds

(g) The smooth fibre preserving maps $L \to L : x \to \hat{K} \circ g'(x)$ and $x \to \delta \circ (T\pi) \circ g'(x)$ map bounded sets into uniformly fibre precompact sets.

Here g' is the gradient of g with respect to the Riemannian metric $(\ ,\ )_L : TL \oplus TL \to \mathbb{R}$ defined by

$$(\cdot,\cdot)_L = (\hat{K}\cdot, \hat{K}\cdot) + ((T\pi)\cdot, (T\pi)\cdot)_\Lambda. \tag{18}$$

Now let $\varphi : L \to [-1, 0]$ be a smooth map. We call the pair (g, φ) admissible if g satisfies (g) and

$$\|\varphi(x)[\Phi'(x) - g'(x)]\| \le 1 \quad \text{for all } x \in L, \tag{19}$$

and moreover if there exists a continuous map $r : \Lambda \to \mathbb{R}$ mapping bounded sets into bounded sets such that

$$\varphi(x) = 0 \text{ if } \|x\| \ge r(\pi(x)). \tag{20}$$

Given an admissible pair (g, φ) we have a global flow $(\sigma_t)_{t\in\mathbb{R}} : L \to L$ associated to the differential equation

$$\dot{x} = \varphi(x)[\Phi'(x) - g'(x)].$$

Denote by $\sigma = \sigma(g, \varphi)$ the time one map, i.e., $\sigma = \sigma_1$. We put

$$\tilde{\mathcal{M}} = \{\sigma | \sigma = (g, \varphi) \text{ for } (g, \varphi) \text{ admissible}\}$$

and

$$\mathcal{M} = \{h | h = \sigma^1 o \sigma^2 \cdots o \sigma^n \text{ finite compositions of } \sigma^i \in \tilde{\mathcal{M}}\}.$$

Clearly $\mathcal{M}$ is a semigroup.

DEFINITION 3. $\mathcal{M}$ is called the *semigroup of basic deformations* in L. An element of $\mathcal{M}$ is called a basic deformation.

A key property of an $h \in \mathcal{M}$ is described in the following fundamental proposition which will be proved later.

PROPOSITION 1 (Intersection-Proposition). *Let L^0 be the zero section of L and let Σ be a compact subset of Λ. Denote by $\overline{H}^*$ Alexander Spanier cohomology with coefficients in a given commutative ring R. Then there exists for given $h \in \mathcal{M}$ an* injective (!) *ring homomorphism β_h such that the following diagram is commutative*

$$\begin{tikzcd} \overline{H}^*(h(L|\Sigma) \cap L^0) & \\ \overline{H}^*(\Lambda) \arrow[u, "(\pi|(h(L|\Sigma)\cap L^0))^*"] \arrow[r, "\mathrm{incl}^*"] & \overline{H}^*(\Sigma). \arrow[ul, "\beta_h"'] \end{tikzcd} \tag{21}$$

This has the following

COROLLARY 1. *$h(L|\Sigma) \cap L^0 \neq \emptyset$ for all $h \in \mathcal{M}$ if $\Sigma \neq \emptyset$.*

The proof of Proposition 1 is very technical and highly nontrivial. It is clearly false for $h \in \mathrm{homeo}(L)$.

Define for $n \in \mathbb{N}$ a smooth vectorbundle map $F_n : L \to T\Lambda$ by

$$\frac{1}{n}(\nabla F_n x, \nabla y) + (F_n x, y) = (x, y) \tag{22}$$

for all $y \in T_q\Lambda$ where $x \in L_q$. Now we define a family of smooth functionals by

$$\begin{gathered} \Phi_{b.n} \in C^\infty(L, \mathbb{R}) \\ \Phi_{b.n}(x) = \Phi(x) - \int_0^1 \Delta_b(F_n x)dt. \end{gathered} \tag{23}$$

The following result is the second key step in the proof of Theorem 2.

PROPOSITION 2 (Deformation Proposition). *There exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that the following holds.*

Given numbers $\varepsilon_0 > 0$, $d \in \mathbb{R}$ and an open neighborhood U of $Cr(\Phi_b, d) = \{x \in L | d\Phi_b(x) = 0, \Phi_b(x) = d\}$ with $dist(\partial U, Cr(\Phi(d)) > 0$ there exist numbers $\varepsilon \in (0, \varepsilon_0)$ and $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$ there is an $h_k \in \mathcal{M}(!)$ with

$$h_k(\Phi^{d+\varepsilon}_{b,n_k} \backslash U) \subset \Phi^{d-\varepsilon}_{b,n_k}. \tag{24}$$

Here $b > 0$ and $\Phi^c_{b,n_k} = \{x \in L \text{ with } \Phi_{b.n_k}(x) \leq c\}$.

The proof will be given later. We would like to remark that one can actually show that $Cr(\Phi_b, d)$ is compact so that the distance hypothesis is not needed. Now we can construct $\Theta : (0, +\infty) \to [0, +\infty)$ with the desired properties.

Fix a compact subset Σ of Λ. The properties of Σ will be specified later. We only assume for the moment that $\Sigma \neq \emptyset$. Define for $n \in \mathbb{N}$

$$\Theta_n : (0, +\infty) \to \mathbb{R} \cup \{-\infty, +\infty\}$$

by

$$\Theta_n(b) = \inf_{h\in\mathcal{M}} \sup\ \Phi_{b.n}(h(L|\Sigma)). \tag{25}$$

Since $h = Id \in \mathcal{M}$ we have

$$\Theta_n(b) \leq \sup\ \Phi_{b.n}(L|\Sigma) \leq \sup\ \Phi(L|\Sigma) < \infty. \tag{26}$$

Moreover by Corollary 1 $h(L|\Sigma) \cap L^0 \neq \emptyset$, which implies

$$\begin{aligned} &\sup\ \Phi_{b.n}(h(L|\Sigma)) \\ &\geq \inf\ \Phi_{b.n}(L^0) \\ &= 0. \end{aligned} \tag{27}$$

Summing up (26) and (27) we see that Θ_n is real valued.

Using the function k introduced in (5) we infer

$$|\Theta_n(b_1) - \Theta_n(b_2)| \leq k(b_1, b_2) \tag{28}$$

which gives the continuity of Θ_n. Finally by the fourth property of φ_1 in (1) we see that $b \to \Theta_n(b)$ is non increasing.

This proves the following.

LEMMA 2. *$(\Theta_n)_{n\in\mathbb{N}}$ is a continuous family of nonincreasing functions on $(0, +\infty)$ with values in $[0, +\infty)$. Moreover the family is equicontinuous at every point $b \in (0, +\infty)$ and $\{\Theta_n(1)\}$ is bounded by* (26) *and* (27).

From the Ascoli-Arzela Theorem we find a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that $(\Theta_{n_k})_{k\in\mathbb{N}}$ converges uniformly on compact subsets of $(0, +\infty)$ to a continuous nonincreasing function $\Theta \in C((0, +\infty),\ [0, +\infty))$.

The sequence (n_k) we get here is actually the sequence which one can take in Proposition 2.

Now we can prove Theorem 2. We have to distinguish between the cases $\pi_1(M) = 0$ and $\pi_1(M) \neq 0$. Here the first case is technically more involved than the second. For this reason we shall only sketch the second case. So assume for the following $\pi_1(M) = 0$. Then Sullivan's theory of minimal models for rational homotopy type, [22] or [23], guarantees the non-triviality of rational Alexander Spanier cohomology groups of Λ for an infinite number of dimensions. By a remark in [29] we can find a compact set Σ in Λ such that the inclusion $i : \Sigma \to \Lambda$ induces a nontrivial isomorphism in cohomology $i^* : \overline{H}^*(\Lambda) \to \overline{H}^*(\Sigma)$ up to a dimension $k_0 > \dim(M)$. We apply the procedure just described to this compact set Σ in order to obtain the continuous function Θ.

PROOF OF THEOREM 2. We prove Theorem 2 in two parts, namely

Claim 1. There exists a critical point x_b of Φ_b with $\Phi_b(x_b) = \Theta(b)$,

and

Claim 2. One can take x_b in Claim 1 to be nonconstant.

PROOF OF CLAIM 1. Arguing indirectly we may assume $Cr(\Phi_b, \Theta(b)) = \emptyset$. Now we employ Proposition 2. Without loss of generality we may assume $n_k = k$ in order not to complicate the notation. Taking $U = \emptyset$ we find for $\varepsilon_0 = 1$ numbers $n_0 \in \mathbb{N}$ and $\varepsilon \in (0, 1)$ such that there is a $h_n \in \mathcal{M}$ for every

$n \geq n_0$ with

$$h_n(\Phi_{b,n}^{\Theta(b)+\varepsilon}) \subset \Phi_{b.n}^{\Theta(b)-\varepsilon}. \tag{29}$$

Now we have for $n \geq n_1 \geq n_0$, for some $n_1 \geq n_0$, that $|\Theta_n(b) - \Theta(b)| < \frac{\varepsilon}{2}$. By the definition of Θ_n we find for $n \geq n_1$ a homeomorphism $\tilde{h}_n \in \mathcal{M}$ with

$$\sup \ \Phi_{b,n}(\tilde{h}_n(L|\Sigma)) \leq \Theta_n(b) + \frac{\varepsilon}{2} < \Theta(b) + \varepsilon. \tag{30}$$

Hence

$$\Theta_n(b) \leq \sup \ \Phi_{b,n}(h_n \circ \tilde{h}_n(L|\Sigma)) \leq \Theta(b) - \varepsilon$$

giving a contradiction since $|\Theta_n(b) - \Theta(b)| < \frac{\varepsilon}{2}$ for $n \geq n_1$. This contradiction shows that $\Theta(b)$ is a critical level for Φ_b.

PROOF OF CLAIM 2. If $\Theta(b) > 0$ then $x_b \in Cr(\Phi_b, \Theta(b))$ cannot be constant since for any constant critical point we have automatically $\Phi_b(x_b) \leq 0$. Now assume $\Theta(b) = 0$ and suppose all critical points of Φ_b on level zero are constant. We find a number $\sigma > 0$ such that $\Lambda^\sigma = \{q \in \Lambda | E(q) \leq \sigma\}$ has $\Lambda^0 \simeq M$ as a deformation retract (see [13] or [14]). Then $U = L|\Lambda^\sigma$ is a neighborhood of $Cr(\Phi_b, 0)$ such that ∂U has positive distance. We find $\varepsilon > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ there exists a homeomorphism $h_n \in \mathcal{M}$ with the property

$$h_n(\Phi_{b.n}^{\varepsilon} \backslash U) \subset \Phi_{b.n}^{-\varepsilon}. \tag{31}$$

We can pick for $n \geq n_1 \geq n_0$ an homeomorphism $\tilde{h}_n \in \mathcal{M}$ with

$$\sup \ \Phi_{b,n}(\tilde{h}_n(L|\Sigma)) \leq \frac{\varepsilon}{2} \tag{32}$$

for some suitable integer $n_1 \geq n_0$. Hence we must have for $n \geq n_1$

$$h_n(\tilde{h}_n(L|\Sigma) \backslash U) \subset \Phi_{b.n}^{-\varepsilon}. \tag{33}$$

Consequently $h_n(\tilde{h}_n(L|\Sigma)\backslash U) \cap L^0 = \emptyset$ which implies

$$h_n \circ \tilde{h}_n(L|\Sigma) \cap L^0 \subset h_n(U). \tag{34}$$

Since $\overline{h}_n := h_n \circ \tilde{h}_n \in \mathcal{M}$ we obtain for $n \geq n_1$ the following commutative diagram by Proposition 1

$$\begin{tikzcd}
\overline{H}^*(h_n(U)) \arrow[r, "j^*"] & \overline{H}^*(\overline{h}_n(L|\Sigma) \cap L^0) & \\
& \overline{H}^*(\Lambda) \arrow[ul, "(\pi|h_n(U))^*"] \arrow[u, "(\pi|(\overline{h}_n(L|\Sigma) \cap L^0))^*"'] \arrow[r, "i^*"] & \overline{H}^*(\Sigma). \arrow[ul, "\beta_{h_n}"']
\end{tikzcd} \tag{35}$$

Here i and j are inclusions. By our assumption on Σ, i_k^* is an isomorphism from $\overline{H}^k(\Lambda) \neq 0$ onto $\overline{H}^k(\Sigma) \neq 0$ for some $k >$ dim M. Hence $\beta_{h_n,k} \circ i_k^*$ is nontrivial and injective. Consequently $j_k^* \circ (\pi|h_n(U))^*$ is nonzero and injective. On the other hand since $U = L|\Lambda^\sigma$ we see that

$$\overline{H}^k(U) \doteq \overline{H}^k(\Lambda^\sigma) \doteq \overline{H}^k(\Lambda^0) \doteq \overline{H}^k(M) = 0 \tag{36}$$

which shows that $j_k^* \circ (\pi|h_n(U))^* = 0$ contradicting the properties of Σ. This contradiction proves Claim 2 and the proof of Theorem 2 is complete. So it remains to prove Proposition 1 and 2.

If $\pi_1(M) \neq 0$ one can take $\Sigma = \{q\}$ where q is noncontractible to a point and the above arguments work as well, in fact they are considerably simpler because we can work in a component of Λ containing only homologically nontrivial curves.

3. - Some results on uniform fibre compactness

We denote by D the covariant derivative in $L \xrightarrow{\pi} \Lambda$ associated to the Riemannian connection $\hat{K}$. Assume $\varphi : \mathbb{R} \to \Lambda$ is a smooth curve. For numbers $s_0 \leq s_1$ we define

$$l(\varphi, s_0, s_1) = \int_{s_0}^{s_1} \|\dot{\varphi}(s)\|_\Lambda \; ds \tag{1}$$

and

$$m(\varphi, s_0, s_1) = \sup \{\|\partial \circ \varphi(s)\| \mid s \in [s_0, s_1]\}. \tag{2}$$

LEMMA 3. *There exists a constant $d > 0$ only depending on M such that the following estimate holds*

$$\begin{aligned} &\|Z(\tau)C(s_1, s_0)x - C(s_1, s_0)x\| \\ &\leq \|Z(\tau)x - x\| + 2d\|x\| \; |\tau|^{\frac{1}{2}} \; l(\varphi, s_0, s_1)m(\varphi, s_0, s_1) \end{aligned} \tag{3}$$

where $C(s_1, s_0) : L_{\varphi(s_0)} \to L_{\varphi(s_1)}$ is the parallel transport along φ.

This has been proved for a similar case in [11]. However the proof here is a little bit simpler and we give it for the convenience of the reader.

PROOF. Define $\hat{\varphi} : \mathbb{R} \times \mathbb{R} \to M : \hat{\varphi}(s, t) = \varphi(s)(t)$. By a density argument we may assume that $\hat{\varphi}$ is smooth and we get the general formula (3) by a simple approximation argument. For t_0, t_1, s_0, s_1 in $\mathbb{R}$ denote by

$$A(s_0; t_1, t_0) : T_{\hat{\varphi}(s_0, t_0)}M \to T_{\hat{\varphi}(s_0, t_1)}M$$
$$B(t_0; s_1, s_0) : T_{\hat{\varphi}(s_0, t_0)}M \to T_{\hat{\varphi}(s_1, t_0)}M$$

the parallel transport maps in M along $t \to \hat{\varphi}(s_0, t)$ and $s \to \hat{\varphi}(s, t_0)$ respectively. We have obviously

$$(C(s_1, s_0)x)(t) = B(t; s_1, s_0)x(t) \text{ for a.a.t.}$$

For $x \in T_{\hat{\varphi}(s_0, t)}M, y \in T_{\hat{\varphi}(s_0, t_0)}M$ we put

$$a(s,t) = B(t; s, s_0)x - A(s; t, t_0)B(t_0; s, s_0)y$$
$$b(s,t) = A(s; t, t_0)B(t_0; s, s_0)y.$$

We compute

$$\begin{aligned}(4) \qquad (\frac{\partial}{\partial s}|a|^2)(s,t) &= 2 < a(s,t), (\nabla_s a)(s,t) > \\ &= 2 < a(s,t), -(\nabla_s b)(s,t) > .\end{aligned}$$

Now

$$\begin{aligned}(5) \qquad \nabla_t \nabla_s b(s,t) &= \nabla_s \nabla_t b(s,t) + R(\hat{\varphi}_t(s,t), \hat{\varphi}_s(s,t))b(s,t) \\ &= R(\hat{\varphi}_t(s,t), \hat{\varphi}_s(s,t))b(s,t).\end{aligned}$$

Here $R : TM \oplus TM \to TM$ is the curvature and $\hat{\varphi}_t, \hat{\varphi}_s$ denote the partial derivatives. From (5)

$$\begin{aligned}(6) \quad \nabla_s b(s,t) &= A(s; t, t_0)(\nabla_s b)(s, t_0) \\ &+ (\int_{t_0}^{t} A(s; t, h)R(\hat{\varphi}_t(s,h), \hat{\varphi}_s(s,h))A(s; h, t_0)dh)(B(t_0; s, s_0)y) \\ &=: K(s; t, t_0)B(t_0; s, s_0)y\end{aligned}$$

since $(\nabla_s b)(s, t_0) = 0$. Now we combine (4) and (6) to obtain

$$(\frac{\partial}{\partial s}|a|^2)(s,t) = -2 \ < a(s,t), K(s; t, t_0)B(t_0; s, s_0)y > .$$

Let $d > 0$ be a constant such that $|R(a,b,c)| \leq d|a| \cdot |b| \cdot |c|$ for $(a,b,c) \in TM \oplus TM \oplus TM$. Then we estimate

$$(7) \qquad (\frac{\partial}{\partial s}|a|^2)(s,t) \leq 2|a(s,t)| \ |\int_{t_0}^{t} d|\hat{\varphi}_t(s,h)| \ |\hat{\varphi}_s(s,h)|dh| \ |y|.$$

Now we find for $x \in L_q$ using (7) $(q = \varphi(s_0))$

$$\begin{aligned}
&\frac{d}{ds}\|Z(\tau)C(s,s_0)x - C(s,s_0)x\|^2 \\
&\leq 2d \int_0^1 \Big[|(Z(\tau)C(s,s_0)x)(t) - (C(s,s_0)x)(t)| \cdot \Big| \int_{t+\tau}^{t} |\hat{\varphi}_t(s,h)| |\hat{\varphi}_s(s,h)| dh \Big| \\
&\qquad \cdot |x(t+\tau)| \Big] \, dt \\
&\leq 2d \|Z(\tau)C(s,s_0)x - C(s,s_0)x\| \; \|x\| \cdot \max_{t \in \mathbb{R}} \int_{t+\tau}^{t} |\hat{\varphi}_t(s,h)| \; \hat{\varphi}_s(s,h)|dh| \\
&\leq 2d \|Z(\tau)C(s,s_0)x - C(s,s_0)x\| \; \|x\| \; \|\varphi'(s)\|_\infty \; \|\partial\varphi(s)\| \; |\tau|^{\frac{1}{2}}.
\end{aligned}$$

Integrating this inequality yields

$$\begin{aligned}
&\|Z(\tau)C(s,s_0)x - C(s,s_0)x\| \\
&\leq \|Z(\tau)x - x\| + \left| \int_{s_0}^{s_1} 2d \| \varphi'(s)\|_\infty \; \|\partial\varphi(s)\| ds \right| \; \|x\| \; |\tau|^{\frac{1}{2}} \\
&\leq \|Z(\tau)x - x\| + 2d\|x\| \; |\tau|^{\frac{1}{2}} \; l(\varphi, s_0, s_1) m(\varphi, s_0, s_1).
\end{aligned}$$

□

Next we need a variant of a representation formula similar to that used in [11]. We take however a form which is adapted to the Intersection Proposition which we have to prove. Consider the differential equation

$$\dot{x} = \varphi(x)[\Phi'(x) - g'(x)] =: \xi(x) \tag{8}$$

where φ, g have the properties as described in §2. So in particular g satisfies (g) and $\|\xi(x)\|_L \leq 1$ for all $x \in L$. Denote the flow associated to (8) by

$$\mathbb{R} \times L \to L : (t, x) \to x * t.$$

We define a map $\sigma_\xi : \mathbb{R} \times L \to \Lambda$ by

$$\sigma_\xi(s, x) = \pi(x * s). \tag{9}$$

Then $\sigma_\xi(0, x) = \pi(x)$. For fixed $x \in L$ and $t, s \in \mathbb{R}$ we obtain a parallel transport in L along $\sigma_\xi(\cdot, x)$ associated to the Riemannian connection $\hat{K}$ previously introduced. Hence we obtain induced maps

$$U_x^\xi(t,s) : L_{\sigma_\xi(s.x)} \to L_{\sigma_\xi(t.x)}.$$

Denote by $\mathcal{L} \to \Lambda \times \Lambda$ the Banach space bundle over $\Lambda \times \Lambda$ where the fibre over $(q_1, q_2) \in \Lambda \times \Lambda$ consists of all bounded linear operators $L_{q_1} \to L_{q_2}$. By our assumptions which imply that ξ is a smooth vectorfield on L we can consider the map $(x, t, s) \to U_x^\xi(t, s)$ as a smooth map from $L \times \mathbb{R} \times \mathbb{R}$ into $\mathcal{L}$. Now suppose $D : L \to L$ is a continuous fibre preserving map. We associate to D and ξ a new map

$$D_\xi : \mathbb{R} \times L \to L : D_\xi(s, x) = U_x^\xi(0, s)D(x * s).$$

Clearly D_ξ is continuous and

$$\pi \circ D_\xi(t, x) = \pi(x) \quad \text{for all} \quad (t, x) \in \mathbb{R} \times L. \tag{10}$$

Hence D_ξ is a homotopy of fibre preserving maps. We need the following.

LEMMA 4. *Assume ξ is as described above and $D : L \to L$ is a continuous fibre preserving map mapping bounded sets into uniformly fibre precompact sets. Then D_ξ maps bounded sets in $\mathbb{R} \times L$ into uniformly fibre precompact sets in L.*

PROOF. Since $\|\xi(x)\|_L \leq 1$ for all $x \in L$ we obtain the estimate

$$l(\sigma_\xi(\cdot, x), s_0, s_1) \leq |s_1 - s_0|. \tag{11}$$

Moreover we have the standard estimate (see [13], [14])

$$\begin{aligned} &|E(\sigma_\xi(s_1, x))^{\frac{1}{2}} - E(\sigma_\xi(s_0, x))^{\frac{1}{2}}| \\ &\leq \frac{1}{\sqrt{2}} d_\Lambda(\sigma_\xi(s_1, x), \sigma_\xi(s_0, x)) \leq \frac{1}{\sqrt{2}}|s_1 - s_0| \end{aligned} \tag{12}$$

where d_Λ is the metric on (a component of) Λ. (12) of course implies that for given bounded set W of L there exists a constant $c(W) > 0$ such that

$$m(\sigma_\xi(\cdot, x), s_0, s_1) \leq c(W)[1 + |s_0| + |s_1|]. \tag{13}$$

We estimate now using Lemma 3

$$\begin{aligned} &\|Z(\tau)D_\xi(s, x) - D_\xi(s, x)\| \\ &= \|Z(\tau)U_x^\xi(0, s)D(x * s) - U_x^\xi(0, s)D(x * s)\| \\ &\leq \|Z(\tau)D(x * s) - D(x * s)\| + 2d\|D(x * s)\| \; |\tau|^{\frac{1}{2}} \\ &\qquad\qquad l(\sigma_\xi(\cdot, x), s_0, s_1)m(\sigma_\xi(\cdot, x), s_0, s_1). \end{aligned} \tag{14}$$

Since the flow maps bounded sets into bounded sets the previous estimates and our assumption on D implies the desired result. □

The next estimate is very useful

LEMMA 5. *Let $x \in L_q$ and assume there is a constant $d > 0$ such that*

$$|(x, \delta h)| \leq d \|h\|_\infty \qquad \text{for all } h \in T_q\Lambda$$

where $\| \ \|_\infty$ is the maximum norm. Then

$$\|Z(\tau)x - x\| \leq d \ |\tau|^{\frac{1}{2}} \text{ for } |\tau| \leq 1.$$

PROOF. The strongly continuous group $\tau \to Z(\tau)$ on L_q has the infinitesimal generator δ with domain $T_q\Lambda \subset L_q$. Hence we have for $h \in T_q\Lambda$

$$\frac{d}{d\tau} Z(\tau)h = \delta Z(\tau)h.$$

We compute therefore for $x \in L_q$ and $h \in T_q\Lambda$

$$\frac{d}{d\tau}(Z(\tau)x, h) = \frac{d}{d\tau}(x, Z(-\tau)h) = (x, -\delta Z(-\tau)h).$$

This implies

$$(Z(\tau)x - x, h) = (-x, \delta(\int_0^\tau Z(-s)h \ ds)). \tag{15}$$

Now if $|\tau| \leq 1$, $h \in T_q\Lambda$ and $t \in \mathbb{R}$

$$\begin{aligned} |(\int_0^\tau Z(-s)h \ ds)(t)| &= |\int_0^\tau A_{t.t-s}h(t-s)ds| \\ &\leq (\int_0^1 |h(t)|^2 dt)^{\frac{1}{2}} \ |\tau|^{\frac{1}{2}}. \end{aligned} \tag{16}$$

Combining (16) and our assumptions we find

$$\begin{aligned} &\|Z(\tau)x - x\| \\ &\leq \sup_{\|h\| \leq 1} |(-x, \delta(\int_0^\tau Z(-s)h \ ds))| \\ &\leq \sup_{\|h\| \leq 1} d \|\int_0^\tau Z(-s)h \ ds\|_\infty \\ &\leq d \ |\tau|^{\frac{1}{2}}. \end{aligned}$$

Next define a smooth vectorbundle map $\gamma : L \to L$ by

$$\gamma = Id - \delta\delta^*.$$

Here $\delta^* : L \to T\Lambda$ is the adjoint of $\delta : T\Lambda \to L$.

LEMMA 6. *We have the estimates*

$$\|Z(\tau)\gamma x - \gamma x\| \leq \|x\| \; |\tau|^{\frac{1}{2}} \qquad |\tau| \leq 1.$$
$$\|\gamma x\| \leq 2\|x\|$$

PROOF. We have for $x \in L_q$ and $h \in T_q\Lambda$

$$\begin{aligned} &(\gamma x, \delta h) \\ &= (x, \delta h) - (\delta\delta^* x, \delta h) \\ &= (x, \delta h) - (\delta^* x, h)_\Lambda + (\delta^* x, h) \\ &= (\delta^* x, h). \end{aligned}$$

Since $\|\delta^*\| = \|\delta\| \leq 1$ we infer

$$\begin{aligned} |(\gamma x, \delta h)| &\leq \|x\| \; \|h\| \\ &\leq \|x\| \; \|h\|_\infty. \end{aligned}$$

By Lemma 5 this implies

$$\|Z(\tau)\gamma x - \gamma x\| \leq \|x\| \; |\tau|^{\frac{1}{2}} \quad \text{for} \quad |\tau| \leq 1.$$

Moreover

$$\|\gamma x\| \leq 2\|x\|.$$

4. - The Intersection Result

First we give a representation result for the maps in $\tilde{\mathcal{M}}$ and $\mathcal{M}$ respectively similar to a result in [11].

LEMMA 7. *Let ξ be as described in the definitions of $\mathcal{M}$ and $\tilde{\mathcal{M}}$. Then there exist continuous maps $a_{11}, a_{22} : \mathbb{R} \times L \to (0, +\infty)$, and $a_{12}, a_{21} : \mathbb{R} \times L \to (-\infty, 0]$ mapping bounded sets into compact sets of $(0, +\infty)$ and $(-\infty, 0]$ respectively, and continuous maps $B_i : \mathbb{R} \times L \to L$ mapping bounded sets into uniformly fibre precompact sets such that they satisfy*

(1) $\qquad a_{11}(0, x) = 1 = a_{22}(0, x)$ *for all* $x \in L$

$$a_{21}(0,x) = 0 = a_{12}(0,x) \text{ for all } x \in L$$
$$B_1(0,x) = B_2(0,x) = 0 \text{ for all } x \in L$$
$$\pi \circ B_1(s,x) = \pi \circ B_2(s,x) = \pi(x) \text{ for all } x \in L \text{ and } s \geq 0$$

and in matrix notation we have the following identity:

$$\begin{bmatrix} Id_\xi(s,x) \\ (\partial \circ \pi)_\xi(s,x) \end{bmatrix} = \begin{bmatrix} a_{11}(s,x) & a_{12}(s,x) \\ a_{21}(s,x) & a_{22}(x,x) \end{bmatrix} \begin{bmatrix} x \\ \partial \circ \pi(x) \end{bmatrix} + \begin{bmatrix} B_1(s,x) \\ B_2(s,x) \end{bmatrix}. \tag{2}$$

Here we use the notation introduced in §3, for example $Id_\xi : \mathbb{R} \times L \to L : Id_\xi(s,x) = U_x^\xi(0,s)(x * s)$ etc.

PROOF. We have

$$(T\pi)\Phi' = \delta^* \text{ and } \hat{K}\Phi' = \partial \circ \pi - Id. \tag{3}$$

Now put $\alpha^\pm = \frac{1}{2}(-1 \pm \sqrt{5})$ and denote by D_z the covariant derivative in L associated to $\hat{K}$ above $\tau_\Lambda(z)$ in the direction z. We compute with $\gamma = Id - \delta\delta^*$:

$$\begin{aligned}
(4) \qquad & D_{\frac{\partial}{\partial s}\sigma_\xi(s,x)}(\partial \circ \pi(x * s) + \alpha^\pm x * s) \\
&= \delta \circ (T\pi)(\frac{d}{ds}(x * s)) + \alpha^\pm \hat{K}(\frac{d}{ds}(x * s)) \\
&= \varphi(x * s)[\delta\delta^*(x * s) + \alpha^\pm(\partial \circ \pi(x * s) - x * s)] \\
&- \varphi(x * s)[\delta \circ (T\pi)g'(x * s) + \alpha^\pm \hat{K}g'(x * s)] \\
&= \varphi(x * s)[(1 - \alpha^\pm)x * s + \alpha^\pm \partial \circ \pi(x * s)] \\
&- \varphi(x * s)[\delta \circ (T\pi)g'(x * s) + \alpha^\pm \partial \circ \pi(x * s) + \gamma(x * s)] \\
&= \alpha^\pm \varphi(x * s)[\frac{1 - \alpha^\pm}{\alpha^\pm} x * s + \partial \circ \pi(x * s)] \\
&- \varphi(x * s)[\delta \circ (T\pi)g'(x * s) + \alpha^\pm \partial \circ \pi(x * s) + \gamma(x * s)] \\
&=: \alpha^\pm \varphi(x * s)[\partial \circ \pi(x * s) + \alpha^\pm x * s] + B_1^\pm(x,x).
\end{aligned}$$

For $B_1^\pm$ we derive the following estimate using the estimates for γ in Lemma 6.

$$\begin{aligned}
(5) \qquad & \|Z(\tau)B_1^\pm(s,x) - B_1^\pm(s,x)\| \\
&\leq |\varphi(s,x)| \; \|Z(\tau)\delta \circ (T\pi) \circ g'(x * s) - \delta \circ (T\pi)g'(x * s)\| \\
&+ |\varphi(s,x)| \; |\alpha^\pm| \; \|Z(\tau)\hat{K}g'(x * s) - \hat{K}g'(x * s)\| \\
&+ |\varphi(s,x)| \; \|x * s\| \; |\tau|^{\frac{1}{2}}.
\end{aligned}$$

Since $\|\xi(x)\|_L \le 1$ and g satisfies (g) we infer also using Lemma 6 that $B_1^{\pm}$ map bounded sets of $\mathbb{R} \times L$ into uniformly fibre precompact subsets of L. From (4) we obtain via the variation of constant formula

$$
\begin{aligned}
(6) \qquad & \partial \circ \pi(x * s) + \alpha^{\pm}(x * s) \\
&= \exp\Big(\int_0^s \alpha^{\pm}\varphi(\tau, x)d\tau\Big)U_x^{\xi}(s,0)[\partial \circ \pi(x) + \alpha^{\pm}x] \\
&+ \int_0^s [\exp\Big(\int_t^s \alpha^{\pm}\varphi(\tau, x)d\tau\Big)U_x^{\xi}(s,t)B_1^{\pm}(t,x)]dt \\
&=: \varphi^{\pm}(s,x)U_x^{\xi}(s,0)[\partial\ \pi(x) + \alpha^{\pm}x] + B_2^{\pm}(s,x).
\end{aligned}
$$

It is an immediate consequence of Lemma 3 and (5) that $B_2^{\pm}$ map bounded subsets of $\mathbb{R} \times L$ into uniformly fibre precompact subsets of L. From (6) we obtain now the following formulas

$$
\begin{aligned}
(7) \qquad (\alpha^+ - \alpha^-)x * s = {} & U_x^{\xi}(s,0)[(\varphi^+(s,x) - \varphi^-(s,x))\partial \circ \pi(x) \\
& + (\alpha^+\varphi^+(s,x) - \alpha^-\varphi^-(s,x))x] \\
& + (B_2^+(s,x) - B_2^-(s,x))
\end{aligned}
$$

and

$$
\begin{aligned}
(8) \qquad (\alpha^- - \alpha^+)\partial \circ \pi(x\ s) = {} & U_x^{\xi}(s,0)[(\alpha^-\varphi^+(s,x) - \alpha^+\varphi^-(s,x))\partial \circ \pi(x) \\
& + (\alpha^-\alpha^+\varphi^+(s,x) - \alpha^-\alpha^+\varphi^-(s,x))x] \\
& + (\alpha^- B_2^+(x,x) - \alpha^+ B_2^-(s,x)).
\end{aligned}
$$

Multiplying (7) by $\frac{1}{\alpha^+ - \alpha^-}\ U_x^{\xi}(0,s)$ and (8) by $\frac{1}{\alpha^- - \alpha^+}\ U_x^{\xi}(0,s)$ and using Lemma 3 again for the expressions involving $U_x^{\xi}(0,s)B^{\pm}(s,x)$ we find the desired representation. □

LEMMA 8. *Let $h = \sigma^1 \circ \cdots \circ \sigma^n$ be an element of $\mathcal{M}$ where $\sigma^i \in \tilde{\mathcal{M}}$. Assume σ^i is the time one map for the flow associated to the differential equation $\dot{x} = \xi_i(x)$ where ξ_i has the usual properties as specified in the definition of $\tilde{\mathcal{M}}$. Denote by σ_s^i the time-s-map for $s \in [0,1]$ and put $h_s = \sigma_s^1 \circ \cdots \circ \sigma_s^n$. Then there exist continuous maps $a, b : [0,1] \times L \to \mathbb{R}$ and $B : [0,1] \times L \to L$ such that*

$$
\begin{aligned}
(9) \qquad & h_s(x) = U^{\xi_1}_{\sigma_s^2 \cdots \circ \sigma_s^n(x)}(s,0) \cdots \circ U_x^{\xi_n}(s,0) \\
& [a(s,x)x + b(s,x)\partial \circ \pi(x) + B(s,x)] \\
& a(0,x) = 1 \textit{ for all } x \in L,\ b(0,x) = 0 \textit{ for all } x \in L \\
& a(s,x) > 0 \textit{ and } b(s,x) \le 0 \textit{ for all } s \in [0,1] \textit{ and } x \in L
\end{aligned}
$$

$$B(0,x)=0,\ \ \pi\circ B(s,x)=\pi(x) \textit{ for all } s\in[0,1] \textit{ and } x\in L.$$

Moreover a, b map bounded sets into compact sets of $(0,+\infty)$, $(-\infty,0]$ respectively, and B maps bounded sets into uniformly fibre precompact sets.

PROOF. We proof the representation by induction. If $h_s=\sigma_s^1$ we have by Lemma 7

$$\begin{bmatrix} h_s(x) \\ \partial\circ\pi(h_s(x)) \end{bmatrix} = \begin{bmatrix} U_x^{\xi_1}(s,0) & 0 \\ 0 & U_x^{\xi_1}(s,0) \end{bmatrix} \left[\begin{bmatrix} a_{11}(s,x) & a_{22}(s,x) \\ a_{21}(s,x) & a_{22}(s,x)\end{bmatrix} \begin{bmatrix} x \\ \partial\circ\pi(x)\end{bmatrix} + \begin{bmatrix} B_1(s,x) \\ B_2(s,x)\end{bmatrix}\right] \tag{10}$$

which shows that Lemma 8 holds for $n=1$. Now assume we have proved for $k\le n-1$ that $\tilde h_s=\sigma_s^1\circ\cdots\circ\sigma_s^k$ can be represented by

$$\begin{bmatrix} \tilde h_s(x) \\ \partial\circ\pi(h_s(x)) \end{bmatrix} = \begin{bmatrix} U^{\xi_1}_{\sigma_s^2\circ\cdots\circ\sigma_s^k(x)}(s,0)\circ\cdots\circ U_x^{\xi_k}(s,0) & 0 \\ 0 & U^{\xi_1}_{\sigma_s^2\circ\cdots\circ\sigma_s^k}(s,0)\circ\cdots\circ U_x^{\xi_k}(s,0) \end{bmatrix} \left[\begin{bmatrix} \tilde a_{11}(s,x) & \tilde a_{12}(s,x) \\ \tilde a_{21}(s,x) & \tilde a_{22}(s,x)\end{bmatrix} \begin{bmatrix} x \\ \partial\circ\pi(x)\end{bmatrix} + \begin{bmatrix} \tilde B_1(s,x) \\ \tilde B_2(s,x)\end{bmatrix}\right] \tag{11}$$

where the data has the required properties. We show now that (11) also holds for the case $k=n$ which implies (9). Let $h_s=\sigma_s^1\circ\cdots\circ\sigma_s^n=\tilde h_s\circ\sigma_s^n$. We find with the obvious abbreviations

$$\begin{aligned}\begin{bmatrix} h_s(x) \\ \partial\circ\pi(h_s(x)) \end{bmatrix} &= \tilde U(s,\sigma_s^n(x))\left[\tilde A(s,\sigma_s^n(x))\begin{bmatrix} \sigma_s^n(x) \\ \partial\circ\pi(\sigma_s^n(x))\end{bmatrix} + \begin{bmatrix} \tilde B_1(s,\sigma_s^n(x)) \\ \tilde B_2(s,\sigma_s^n(x))\end{bmatrix}\right] \\ &= \tilde U(s,\sigma_s^n(x))\left[\tilde A(s,\sigma_s^n(x))\hat U(s,x)\left[\hat A(s,x)\begin{bmatrix} x \\ \partial\circ\pi(x)\end{bmatrix} + \begin{bmatrix} \hat B_1(s,x) \\ \hat B_2(s,x)\end{bmatrix}\right] + \begin{bmatrix} \tilde B_1(s,\sigma_s^n(x)) \\ \tilde B_2(s,\sigma_s^n(x))\end{bmatrix}\right] \\ &= \tilde U(s,\sigma_s^n(x))\hat U(s,x)\ \left[\tilde A(s,\sigma_s^n(x))\hat A(s,x)\begin{bmatrix} x \\ \partial\circ\pi(x)\end{bmatrix}\right.\end{aligned} \tag{12}$$

$$+\tilde{A}(s,\sigma_s^n(x))\hat{U}(s,x)\begin{bmatrix}\hat{B}_1(s,x)\\ \hat{B}_2(s,x)\end{bmatrix}+\begin{bmatrix}\tilde{B}_1(s,\sigma_s^n(x))\\ \tilde{B}_2(s,\sigma_s^n(x))\end{bmatrix}\Bigg]\Bigg]$$

$$=: \check{U}(s,x)\left[\check{A}(s,x)\begin{bmatrix}x\\ \partial\circ\pi(x)\end{bmatrix}+\begin{bmatrix}\check{B}_1(s,x)\\ \check{B}_2(s,x)\end{bmatrix}\right].$$

Now one easily verifies that $\check{U}(s,x)$ is the desired 2×2 matrix and that $\check{A}$ has also the desired form. An application of Lemma 3 to (12) shows that $\check{B}_1$ and $\check{B}_2$ map bounded sets into uniformly fibre precompact sets. The other properties are also easily verified. □

Now we are in the position to prove the intersection result.

PROOF OF PROPOSITION 1. Let $h\in\mathcal{M}$ and put $h=\sigma^1\circ\cdots\circ\sigma^n$ where $\sigma^i\in\tilde{\mathcal{M}}$. σ^i is the time one map of a flow associated to a differential equation $\dot{x}=\xi_i(x)$ where ξ_i has the properties specified in the definition of $\tilde{\mathcal{M}}$. Put as before

$$h_s=\sigma_s^1\circ\cdots\circ\sigma_s^n \text{ for } s\in[0,1].$$

We find a constant $c>0$ such that

$$x=\sigma_s^i(x) \qquad \text{for all } x\in L|\Sigma \text{ with } \|x\|\geq c \text{ and } s\in[0,1]. \tag{13}$$

Denote by $\mathrm{Int}(s)$ for $s\in[0,1]$ the set

$$\mathrm{Int}(s)=h_s(L|\Sigma)\cap L^0 \tag{14}$$

and

$$\mathrm{Fix}\ (s)=h_s^{-1}(\mathrm{Int}(s)). \tag{15}$$

By (13) we have $h_s(x)=x$ for all $s\in[0,1]$ provided $x\in L|\Sigma$ and $\|x\|\geq c$. The problem $h_s(L|\Sigma)\cap L^0\neq\emptyset$ is, using Lemma 8, equivalent to

$$0=a(s,x)x+b(s,x)\ \partial\circ\pi(x)+B(s,x) \qquad x\in L|\Sigma. \tag{16}$$

Dividing by $a(s,x)>0$ (16) takes the form

$$0=x+\tilde{b}(s,x)\ \partial\circ\pi(x)+\tilde{B}(s,x) \qquad s\in L|\Sigma \tag{17}$$

where $\tilde{b}:[0,1]\times L\to(-\infty,0]$ maps bounded sets into compact sets, $\tilde{b}(0,x)=0$, and $\tilde{B}:[0,1]\times L\to L$ maps bounded sets into uniformly fibre precompact sets and $\tilde{B}(0, x) = 0$. Moreover $\tilde{b}(s, x) = 0$ and $\tilde{B}(s, x) = 0$ for all $s \in [0, 1]$ and $x \in L|\Sigma$ with $\|x\| \geq c$. (17) is a fixed point problem for a fibre preserving map in $L|\Sigma \to \Sigma$ of the form

$$x = T_s(x) \tag{18}$$

where $T : [0, 1] \times L|\Sigma \to L|\Sigma$ maps bounded sets into precompact sets (recall that Σ is compact and $\tilde{B}$ maps bounded sets into uniformly fibre precompact sets). Moreover

$$\begin{aligned} &T_0(x) = 0, \ \ T_s(x) = 0 \text{ for } \|x\| \geq c \\ &x \in L|\Sigma. \end{aligned} \tag{19}$$

Further $\mathrm{Fix}(s) = \{x \in L|\Sigma | T_s(x) = x\}$. $L|\Sigma$ is an ANR_Σ in the sense of Dold [7]. Dold's fixed point transfer for ENR_Σ generalizes trivially to ANR_Σ if the base space Σ is compact (see [2] or [11]). (It also generalizes if Σ is only paracompact as it was shown in a thesis of one of Professor Dold's students which is unfortunately not published (B. Schäfer, Ph.D Thesis 1981)). Using the properties of this slight generalization of the fixed point transfer we find a group homomorphism $tr(s)$ for $s \in [0, 1]$ making the following diagram commutative

$$\begin{tikzcd} \overline{H}^*(\Sigma) \arrow[r, "\pi_s^*"] \arrow[rr, bend right, "id"'] & \overline{H}^*(\mathrm{Fix}(s)) \arrow[r, "tr(s)"] & \overline{H}^*(\Sigma) \end{tikzcd} \tag{20}$$

for all $s \in [0, 1]$. Here $\overline{H}$ denotes Alexander-Spanier-Cohomology with coefficients in a commutative ring R and $\pi_s : \mathrm{Fix}(s) \to \Sigma$ is induced by the projection $\pi : L \to \Lambda$. Further we have the cohomology-commutative diagram

$$\begin{tikzcd} \Sigma \arrow[d, hook, "\mathrm{incl}"'] & \mathrm{Fix}(s) \arrow[l, "\pi_s"'] \arrow[r, "\tilde{h}_s", "\sim"'] \arrow[dr, "h_s|\mathrm{Fix}(s)"'] & \mathrm{Int}(s) \arrow[d, hook, "\mathrm{incl}"] \\ \Lambda & & L \arrow[ll, "\pi"] \end{tikzcd} \tag{21}$$

Passing in (21) to cohomology and using (20) we finally obtain for $s = 1$

$$\begin{tikzcd} \overline{H}^*(\Sigma) \arrow[d, "\pi_1^*"'] & & \overline{H}^*(\Lambda) \arrow[ll, "\mathrm{incl}^*"'] \arrow[d, "\pi^*", "\wr"'] \arrow[ddll, "(\pi|\mathrm{Int}(1))^*"] \\ \overline{H}^*(\mathrm{Fix}(1)) \arrow[d, "(\tilde{h}_1^*)^{-1}"'] & & \overline{H}^*(L) \arrow[dll, "\mathrm{incl}^*"] \\ \overline{H}^*(\mathrm{Int}(1)) & & \end{tikzcd} \tag{22}$$

Put now $\beta_h = (h_1^*)^{-1}\pi_1^*$. Then β_h is injective since π_1^* is injective by (20). Observing that $\mathrm{Int}(1) = h(L|\Sigma) \cap L^0$ the proof is complete. □

5. - The Deformation Result

In order to prove Proposition 2 we need some more estimates.

LEMMA 9. *For a bounded subset S of L_q, $q \in \Lambda$ the following statements are equivalent*

(i) *S is precompact.*

(ii) *Given $\varepsilon > 0$ there exists $\tau_0 > 0$ such that $\|Z(\tau)x - x\| \leq \varepsilon$ for all $x \in S$ and $|\tau| \leq \tau_0$.*

This is a straightforward variant of a classical result of Riesz, see [30] for a proof.

The following result is crucial.

LEMMA 10. *Assume $(q_n) \subset \Lambda$ is a bounded sequence. Then the following statements are equivalent*

(i) *(q_n) is precompact in Λ.*

(ii) *Given $\varepsilon > 0$ there exists $\tau_0 > 0$ such that $\|Z(\tau)\partial q_n - \partial q_n\| \leq \varepsilon$ for all $n \in \mathbb{N}$ and $|\tau| \leq \tau_0$.*

PROOF. (i) $\Longrightarrow$ (ii) is trivial.
(ii) $\Longrightarrow$ (i) We have the estimate

$$\begin{aligned} d_M(q(t), q(s)) &\leq \left| \int_s^t |\partial q(\tau)| d\tau \right| \\ &\leq \|\partial q\| \; |t-s|^{\frac{1}{2}}. \end{aligned}$$

By our hypothesis using the Ascoli-Arzela-Theorem we may assume that $q_n \to q$ uniformly for some continuous loop q (after passing to some subsequence). Taking some exponential chart based at some smooth loop "a" close to q we have with

$$\begin{aligned} q_n &= \exp_a(\xi_n) \qquad & \xi_n \in H^1(a^*TM) \\ q &= \exp_a(\xi) \qquad & \xi \in C(a^*TM) \end{aligned}$$

the following representations of ∂q_n

$$\partial_a \xi_n = \nabla \xi_n + \Theta_a(\xi_n)$$

where ∇ is the covariant derivative along "a" and Θ_a is the "twist map" based at "a". By our assumption we have

$$\|G(\xi_n)^{\frac{1}{2}}(\nabla \xi_n + \Theta_a(\xi_n))\| \leq c.$$

Since $\xi_n \to \xi$ uniformly and $\Theta_a(\xi_n) \to \Theta_a(\xi)$ uniformly we see that (ξ_n) is bounded in $H^1(a^*TM)$. Using the weak completeness of a Hilbert space we must have $\xi \in H^1(a^*TM)$ and $\xi_n \to \xi$ weakly and $q \in \Lambda$. Consider the family of paths $\varphi_n : [0,1] \to \Lambda$ defined by

$$\varphi_n(s) = \exp_a((1-s)\xi_n + s\xi).$$

(They are clearly well-defined for large n). We have

$$\|\partial\varphi_n(s)\| \leq c_1 \text{ for all } n \in \mathbb{N} \text{ and all } s \in [0,1].$$

Moreover the length of the paths is bounded by a constant $c_2 > 0$ independent of $n \in \mathbb{N}$. Denote by $C_n : L_{q_n} \to L_q$ the paralleltransport along φ_n. As a consequence of Lemma 3 we find a constant $c_2 > 0$ such that

$$\|Z(\tau)C_n\partial q_n - C_n\partial q_n\| \leq \|Z(\tau)\partial q_n - \partial q_n\| + c_2|\tau|^{\frac{1}{2}}$$

for all $n \in \mathbb{N}$. Hence by our assumption (ii) $(C_n\partial q_n)$ is precompact in L_a. Eventually taking a subsequence we may assume

$$C_n\partial q_n \to y \text{ in } L_q \tag{1}$$

for some $y \in L_q$. Now we pass to local coordinates in order to study (1)

$$\tilde{C}_n(\nabla\xi_n + \Theta_a(\xi_n)) \to \eta \qquad \text{in } L_a \tag{2}$$

where $y = \nabla_2 \exp_a(\xi)\eta$. One verifies easily that $\tilde{C}_n^{-1}\eta \to \eta$ in L_a. Since $\Theta_a(\xi_n) \to \Theta_a(\xi)$ uniformly we see that $(\nabla\xi_n)$ is a Cauchy sequence. So (ξ_n) is convergent in $H^1(a^*TM)$ and the limit must be $\xi \in H^1(a^*TM)$. Consequently

$$q_n \to q \text{ in } \Lambda$$

as required. □

Next we need some estimates on the vectorbundle maps F_n introduced in §2.

LEMMA 11. *For all $n \in \mathbb{N}$ F_n is a smooth vectorbundle map. Moreover we have*

$$(F_n x, y) = (x, F_n y) \text{ for all } (x,y) \in L \oplus L. \tag{3}$$

Further we have the following estimates

$$\begin{aligned} &\|Z(\tau)F_n x - F_n x\| < \sqrt{n}\ \|x\|\ |\tau| \\ &\|F_n x\| \leq \|x\| \\ &\|Z(\tau)F_n x - F_n x\|^2 \leq 2\|x\|\ \|Z(\tau)x - x\|. \end{aligned} \tag{4}$$

PROOF. The first is clear. For the second part replace in

$$\frac{1}{n}(\delta F_n x, \delta y) + (F_n x, y) = (x, y) \text{ for all } (x, y) \in L_q \oplus T_q\Lambda$$

y by $F_n x$. Then we obtain

$$(5) \qquad \begin{aligned} &\|F_n x\| \leq \|x\| \\ &\|\delta F_n x\| \leq \sqrt{n}\ \|x\|. \end{aligned}$$

Now let $h \in T_q\Lambda$. Using the fact that δ restricted to the fibre over q is the infinitesimal generator of the group $(Z(\tau))_{\tau\in\mathbb{R}}$ we compute

$$\begin{aligned} \frac{d}{d\tau}\|Z(\tau)h - h\|^2 &= 2(Z(\tau)h - h, \delta h) \\ &\leq 2\|Z(\tau)h - h\|\ \|\delta h\|. \end{aligned}$$

This implies

$$(6) \qquad \|Z(\tau)h - h\| \leq \|\nabla h\|\ |\tau| \qquad \text{for } h \in T_q\Lambda.$$

Now combining (5) and (6) gives

$$\|Z(\tau)F_n x - F_n x\| \leq \sqrt{n}\|x\|\ |\tau|.$$

For the last inequality we compute using $Z(\tau)\delta = \delta Z(\tau)$ since δ is the infinitesimal generator for $(Z(\tau))_{\tau\in\mathbb{R}}$

$$\begin{aligned} &\|Z(\tau)F_n x - F_n x\|^2 \\ \leq\ & \frac{1}{n}\|\delta(Z(\tau)F_n x - F_n x)\|^2 + \|Z(\tau)F_n x - F_n x\|^2 \\ =\ & \frac{1}{n}[2\|\delta F_n x\|^2 - 2(\delta Z(\tau)F_n x, \delta F_n x)] \\ & + [2\|F_n x\|^2 - 2(Z(\tau)F_n x, F_n x)] \\ =\ & 2[\frac{1}{n}\ \|\delta F_n x\|^2 + \|F_n x\|^2] \\ & - 2[\frac{1}{n}(\delta F_n x, \delta Z(\tau)F_n x) + (F_n x, Z(\tau)F_n x)] \\ =\ & 2[(x, F_n x) - (x, Z(\tau)F_n x)] \\ =\ & 2(x - Z(-\tau)x, F_n x) \\ \leq\ & 2\|x - Z(-\tau)x\|\ \|F_n x\| \\ \leq\ & 2\|x\|\ \|Z(\tau)x - x\| \end{aligned}$$

as required. □

LEMMA 12. *Let* $(x_k) \subset L$, $x \in L$, *and* $(n_k) \subset \mathbb{N}$ *such that* $x_k \to x$ *in* L *and* $n_k \to \infty$. *Then*

$$F_{n_k} x_k \to x \text{ in } L.$$

PROOF. Since by Lemma 11

$$\|F_{n_k} x_k\| \leq \|x_k\| \tag{7}$$

we infer $\limsup_{k\to\infty} \|F_{n_k} x_k\| \leq \|x\|$. Moreover we have by Lemma 11 $\|\delta F_{n_k} x_k\| \leq \sqrt{n_k}\|x_k\|$. Hence taking a sequence $(y_k) \subset T\Lambda$ such that $y_k \to y$ and $\tau_\Lambda(y_k) = \pi(x_k)$ we have since

$$\frac{1}{n_k}(\delta F_{n_k} x_k, \delta y_k) + (F_{n_k} x_k, y_k) = (x_k, y_k)$$

taking the limit

$$\lim(F_{n_k} x_k, y_k) = (x, y). \tag{8}$$

Now (7) and (8) imply immediately $F_{n_k} x_k \to x$ in L. □

Finally we need

LEMMA 13. *For fixed* $x \in L_q$, $y, z \in T_q\Lambda$, *with* $R : TM \oplus TM \oplus TM \to TM$ *being the curvature, we have the following identity*

$$\begin{aligned}
&\frac{1}{n}(R(y,\dot{q})F_n x, \delta z) + \frac{1}{n}(\delta F_n x, R(y,\dot{q})z) \\
&+\frac{1}{n}(\delta(D_y F_n)x, \delta z) + ((D_y F_n)x, z) = 0.
\end{aligned}$$

Moreover for some constant $c > 0$ *only depending on* M

$$\left(\frac{1}{n}\|(D_y F_n)x\|_\Lambda^2 + \left(1 - \frac{1}{n}\right)\|(D_y F_n)x\|^2\right)^{\frac{1}{2}} \leq c\frac{1}{n^{\frac{1}{4}}}\|x\| \ \|\partial q\| \ \|y\|_\infty.$$

For a proof see [11].

Recall the definition of $\Phi_{b.n}$

$$\begin{aligned}
\Phi_{b,n} &= \Phi - \int_0^1 \Delta_b(F_n x)dt \\
&=: \Phi - g_{b.n}
\end{aligned}$$

LEMMA 14. $g_{b.n}$ *satisfies* (g). *In particular we have the following estimates for some constant* $c(b)$ *depending continuously on* b, *but being independent of* $n \in \mathbb{N}$

(i) $\|\hat{K} g'_{b.n}(x)\| \leq c(b)$ *for all* $x \in L$

(ii) $\|g'_{b.n}(x)\|_L \leq c(b)(1 + \frac{1}{n^{\frac{1}{4}}}\|x\| \ \|\partial q\|)$ *for all* $q \in \Lambda$ *and* $x \in L_q$

(iii) $\|Z(\tau)\delta(T\pi)g'_{b.n}(x) - \delta(T\pi)g'_{b.n}(x)\| \leq c(b)(1+\|x\| \ \|\partial q\|) \ |\tau|^{\frac{1}{2}}$ *for all* $q \in \Lambda$ *and* $x \in L_q$ $\tau \in [-1, 1]$

(iv) $\|Z(\tau)\hat{K} g'_{b,n}(x) - \hat{K} g'_{b,n}(x)\| \leq c(b) \ \sqrt{n} \ |\tau|$ *for all* $x \in L$ *and* $\tau \in \mathbb{R}$.

PROOF. Denote by Δ'_b the gradient of Δ_b with respect to the inner product $< \cdot, \cdot >_{TM} = < K\cdot, K\cdot >_M + < T\tau_M \cdot, T\tau_M \cdot >_M$
We have with $H_b(x)(t) = \Delta'_b(x(t))$

$$
\begin{aligned}
(9) \quad & (g'_{b.n}(x), a)_L \\
&= (\hat{K} H_b F_n x, \hat{K}(T F_n)a) + ((T\pi) H_b F_n a, (T\pi)(T F_n)a) \\
&= (\hat{K} H_b F_n x, F_n \hat{K} a) + (\hat{K} H_b F_n x, (D_{(T\pi)a} F_n)x) + ((T\pi) H_b F_n x, (T\pi)a).
\end{aligned}
$$

Hence

$$
\hat{K} g'_{b.n}(x) = F_n \hat{K} H_b F_n x \tag{10}
$$

which implies (i) since $\|F_n x\| \leq \|x\|$ and $\|H_b(x)\| \leq \text{const}(b)$ for all $x \in L$. Moreover using Lemma 13

$$
\begin{aligned}
& |(g'_{b.n}(x), a)_L| \\
\leq \ & c_2(b)\|\hat{K} a\| + c_2(b)\frac{1}{n^{\frac{1}{4}}} \ \|x\| \ \|\partial q\| \ \|(T\pi)a\|_\infty + c_2(b)\|(T\pi)a\| \\
\leq \ & c_3(b)(1 + n^{-\frac{1}{4}}\|x\| \ \|\partial q\|) \ \|a\|_L.
\end{aligned}
$$

Consequently for a suitable $c(b)$

$$
\|g'_{b,n}(x)\|_L \leq c(b)(1 + n^{-\frac{1}{4}}\|x\| \ \|\partial q\|). \tag{11}
$$

This proves (ii). Next we have

$$
\begin{aligned}
(12) \quad & (\delta(T\pi)g'_{b,n}(x), \delta h) \\
&= -((T\pi)g'_{b.n}(x), h) + ((T\pi)H_b F_n x, h) + (\hat{K} H_b F_n x, (D_h F_n)x).
\end{aligned}
$$

Applying Lemma 13 to (12) gives, for a suitable $c(b)$ depending continuously on b,

$$
\begin{aligned}
(13) \quad & |(\delta(T\pi)g'_{b.n}(x), \ h)| \\
&\leq c_1(b)(1 + \|x\| \ \|\partial q\|) \ \|h\| + c_1(b) \ \|h\| + c_1(b)\|(D_h F_n)x\| \\
&\leq c(b)(1 + \|x\| \ \|\partial q\|)\|h\|_\infty.
\end{aligned}
$$

From Lemma 5 and (13) we deduce

$$\begin{aligned}&\|Z(\tau)\ \delta(T\pi)g'_{b.n}(x) - \delta(T\pi)g'_{b.n}(x)\|\\ &\leq c(b)(1+\|x\|\ \|\partial q\|)\ |\tau|^{\frac{1}{2}}\end{aligned}$$

which proves (iii). In order to prove (iv) observe that $\hat{K}g'_{b.n}(x) = F_n\hat{K}H_bF_nx$ (see (10)). Now using Lemma 11 (4) we find

$$\begin{aligned}&\|Z(\tau)\hat{K}g'_{b,n}(x) - \hat{K}g'_{b.n}(x)\|\\ &\leq \sqrt{n}\ \|\hat{K}H_bF_nx\|\ |\tau|\\ &\leq c(b)\ \sqrt{n}\ |\tau|\end{aligned}$$

which establishes (iv). □

Recall the definition of Θ as limit of a sequence Θ_{n_k} where the convergence is uniformly on compact subsets of $(0,+\infty)$. Without loss of generality we may assume $n_k = k$ in order to simplify the notation. We need the following

LEMMA 15. *Let $f : TM \to TM$ be a continuous fibre preserving map satisfying the estimate*

$$|f(x)| \leq c(1+|x|) \textit{ for all } x \in TM \tag{13}$$

for some positive constant c. Then the induced map $\hat{f} : L \to L : x \to \hat{f}(x)$, $\hat{f}(x)(t) = f(x(t))$ a.c., is continuous and maps uniformly fibre precompact sets into uniformly fibre precompact sets.

PROOF. It is well known that the assumption (13) implies the continuity of $\hat{f}$. Let B be a uniformly fibre precompact subset of L. Since B is bounded $\pi(B)$ is bounded and by the compact embedding $\Lambda \hookrightarrow C(S^1, M)$ a finite number of exponential charts, say $(\exp_{q_i}(\frac{1}{2}V_i))_{i+1\cdots k}$ cover $\pi(B)$. Here $q_i \in C^\infty(S, M)$ and $\exp_{q_i} : V_i \to \Lambda$ is a diffeomorphism onto an open neighborhood of q_i. We have $V_i \subset H^1(q_i^*TM)$ and may assume that V_i is a convex open zero neighborhood. In order to prove the lemma we may assume without loss of generality that $\pi(B) \subset \exp_q(\frac{1}{2}V)$ where $q = q_1$ and $V = V_1$. Now arguing indirectly we find a sequence $(x_n) \subset B$, a sequence $(\tau_n) \subset \mathbb{R}$, $\tau_n \to 0$, and an $\varepsilon > 0$ such that

$$\|Z(\tau_n)\hat{f}(x_n) - \hat{f}(x_n)\| \geq \varepsilon. \tag{14}$$

Let $q_n = \pi(x_n)$. Introducing local coordinates we have $q_n = \exp_q(\xi_n)$, $\xi_n \in \frac{1}{2}V$, where the (ξ_n) are bounded in $H^1(q^*TM)$. Eventually taking a subsequence we may assume

$$\begin{aligned}&\xi_n \to \xi_0 \in cl(\frac{1}{2}V) \text{ weakly in } H^1(q^*TM)\\ &\xi_n \to \xi_0 \text{ uniformly.}\end{aligned} \tag{15}$$

Using the usual trivializations, see [13, 14], we have the commutative diagram with $U = \exp_q(\frac{1}{2}V)$

$$\begin{array}{ccc} L|cl(U) & \xrightarrow{\quad \hat{f} \quad} & L|cl(U) \\ \nabla_2 \exp_q \uparrow\wr & & \uparrow\wr\ \nabla_2 \exp_q \\ cl(\frac{1}{2}V) \times L^2(q^*TM) & \xrightarrow{\hat{h}} & cl(\frac{1}{2}V) \times L^2(q^*TM) \end{array} \tag{16}$$

where $\hat{h}$ is induced by a map $h : q^*cl(U) \oplus q^*TM \to q^*cl(U) \oplus q^*TM$ that has the same properties as f.

We write in local coordinates

$$x_n = \nabla_2 \exp_q(\xi_n, \eta_n) \quad \xi_n \in \frac{1}{2}V,\ \eta_n \in L^2(q^*TM).$$

If $C_n : L_{q_n} \to L_{q_0},\ q_0 = \exp_q(\xi_0)$ denote the parallel transport along the curve $s \to \exp_q((1-s)\xi_n + s\xi_0)$ then by Lemma 3 we find for every $\delta > 0$ a number $\tau_0 > 0$ such that

$$\|Z(\tau)C_n x_n - C_n x_n\| \le \delta \qquad \text{for } |\tau| < \tau_0.$$

In local coordinates let us write $\tilde{\eta}_n$ for the representative of $C_n x_n$. Hence if $\tilde{C}_n$ represents C_n we have

$$\tilde{C}_n(\xi_n, \eta_n) = (\xi_0, \tilde{\eta}_n).$$

Writing down the equations for the parallel transport, using (15), we infer that $\eta_n - \tilde{\eta}_n \to 0$ in $L^2(q^*TM)$. Again by Lemma 3 we may assume that $\tilde{\eta}_n$ is convergent (eventually we have to take a subsequence). Hence we find

$$\eta_n \to \eta_0 \text{ in } L^2(q^*TM).$$

Consequently

$$(\xi_n, \eta_n) \to (\xi_0, \eta_0)$$

where the first coordinate converges weakly in $H^1(q^*TM)$ and the second strongly in $L^2(q^*TM)$. Hence in diagram (16) we see that the principal part of h (the second component) converges strongly in $L^2(q^*TM)$. It is now straightforward that

$$C_n \hat{f}(x_n) - \nabla_2 \exp_q \hat{h}(\xi_0, \eta_n) \to 0 \text{ in } L_{q_0}.$$

Hence $(C_n \hat{f}(x_n))$ is precompact in L_{q_0}. By Lemma 3 $(\hat{f}(x_n))$ is uniformly fibre precompact. (Apply C_n^{-1} to $C_n \hat{f}(x_n)$.)

LEMMA 16. *If $(x_n) \subset L$ is a uniformly fibre precompact sequence then $(\hat{K} g'_{b.n}(x_n))$ is uniformly fibre precompact.*

PROOF. If (x_n) is $(ufpc)$ then by Lemma 11 $(F_n x_n)$ is $(ufpc)$. Since

$$\hat{K} g'_{b.n}(x_n) = F_n \hat{K} H_b F_n x_n := F_n \circ \hat{f} \circ F_n x_n$$

where $\hat{f}$ is as in Lemma 15, we see that $(\hat{f}(F_n x_n))$ is $(ufpc)$.

Again by Lemma 11 $(\hat{K} g'_{b,n}(x_n))$ is $(ufpc)$. □

We may assume $\Theta_k \to \Theta$ uniformly on compact subsets of $(0, +\infty)$.

We shall prove now Proposition 2 for $n_k = k$.

PROOF OF PROPOSITION 2.

Step 1. Given a sequence $(x_k) \subset L$ such that $\Phi_{b.k}(x_k) \to d$ and $\|\Phi'_{b.k}(x_k)\|_L \to 0$ the sequence (x_k) is precompact. Moreover any limit point x of (x_k) satisfies $\Phi_b(x) = d$, $\Phi'_b(x) = 0$.

We have to show that every subsequence of (x_k) has a convergent subsequence. Without loss of generality we show that (x_k) has a convergent subsequence. Since $\|\Phi'_{b.k}(x_k)\|_L \to 0$ we find

$$\tag{17} \begin{aligned} &\|\partial q_k - x_k - \hat{K} g'_{b.k}(x_k)\| \to 0 \\ &\|x_k - \gamma(x_k) - \delta \circ (T\pi) g'_{b.k}(x_k)\| \to 0. \end{aligned}$$

(For the second part in (17) we used that $\|(T\pi)\Phi'_{b,k}(x_k)\|_\Lambda \to 0$ implies $\|\delta \circ (T\pi)\Phi'_{b,k}(x_k)\| \to 0$). From (17) we infer, for some sequence $\varepsilon_k \to 0$ and constant $c(b) > 0$,

$$\begin{aligned} 0 &= (\partial q_k, x_k) - \frac{1}{2}\|x_k\|^2 - \frac{1}{2}\|x_k\|^2 - (\hat{K} g'_{b.k}(x_k), x_k) - \varepsilon_k \|x_k\| \\ &\leq d_k - \frac{1}{2}\|x_k\|^2 + c(b) + |\varepsilon_k|\ \|x_k\| \end{aligned}$$

where $d_k \to d$. Hence $(\|x_k\|)$ is bounded. Using (17) again we see that $(\|\partial q_k\|)$ is bounded. Now using (17) (the second part) we see that

$$\tag{18} \begin{aligned} &\|Z(\tau)x_k - x_k\| \\ &= \|Z(\tau)(\gamma(x_k) + \delta \circ (T\pi) g'_{b.k}(x_k) - y_k) \\ &- (\gamma(x_k) + \delta \circ (T\pi) g'_{b.k}(x_k) - y_k)\| \\ &\leq \|x_k\|\ |\tau|^{\frac{1}{2}} + c(b)(1 + \|x_k\|\ \|\partial q_k\|)\ |\tau|^{\frac{1}{2}} + \|Z(\tau) y_k - y_k\|. \end{aligned}$$

Here we used Lemma 6 and 14. Moreover (y_k) is a suitable sequence in L such that $\|y_k\| \to 0$. Since $(\|\partial q_k\|)$ is bounded (q_k) is bounded in Λ. Consequently (18) implies that (x_k) is $(ufpc)$. Using the first part of (17) and Lemmas 10 and 16 we see that (q_k) is precompact. Since (q_k) is precompact and (x_k) is

$(ufpc)$ we infer that (x_k) is precompact. So we may assume $x_k \to x$ for some x in L. Now taking the limit in $\|\Phi'_{b.k}(x_k)\| \to 0$ using Lemma 12 we obtain

$$\Phi'_b(x) = 0 \tag{19}$$

and similarly

$$\Phi_b(x) = d. \tag{20}$$

Step 2. Given numbers $\varepsilon_0 > 0$, $d \in \mathbb{R}$ and an open neighborhood U of $Cr(\Phi_b, d)$ there exist numbers $\varepsilon \in (0, \varepsilon_0)$ and $k_0 \in \mathbb{N}$ such that

$$\|\Phi'_{b.k}(x)\|_L \geq \varepsilon \text{ if } x \in (\Phi_{b.k})_{d-\varepsilon}^{d+\varepsilon} \backslash U \text{ and } k \geq k_0.$$

Here $\Phi^r = \{x \in L | \Phi(x) \leq r\}$, $\Phi_r = \{x \in L | \Phi(x) \geq r\}$ and $\Phi_{r_2}^{r_1} = \Phi^{r_1} \cap \Phi_{r_2}$.

PROOF. Arguing indirectly we find a sequence n_k which is monotonic and $n_k \to \infty$ such that for suitable x_k we have

$$\begin{aligned} &\|\Phi'_{b.n_k}(x_k)\| \leq \frac{1}{k} \\ &x_k \notin U \\ &\Phi_{b.n_k}(x_k) \to d; \end{aligned}$$

by Step 1 (x_k) is precompact and a limit point, say x, satisfies $\Phi_b(x) = d$, $\Phi'_b(x) = 0$. But we must have $x \notin U$ contradicting the fact that $Cr(\Phi_b, d)$ is contained in U.

Step 3. Completion of the proof.

Let U be a neighborhood of $Cr(\Phi_b, d)$ with $2\rho := \operatorname{dist}(\partial U,\ Cr(\Phi_b, d)) > 0$. Pick an open neighborhood V of $Cr(\Phi_b, d)$ with $V \subset U$ and $\operatorname{dist}(\partial U, V) \geq \rho$. By Step 2 we find $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0]$ we have

$$\|\Phi'_{b.k}(x)\|_L \geq \varepsilon \text{ for } x \in (\Phi_{b.k})_{d-\varepsilon}^{d+\varepsilon} \backslash V$$

provided $k \geq k_0$ for a suitable integer k_0. Given $x \in L$, $x \neq 0$ we find for $t \in \mathbb{R}$

$$\begin{aligned} \Phi_{b.k}(tx) &= t(\partial q, x) - t^2\ \frac{1}{2}\ \|x\|^2 - g_{b.k}(tx) \\ &\leq t(\partial q, x) - t^2\ \frac{1}{2}\ \|x\|^2 \\ &\leq |t|\ \|\partial q\|\ \|x\| - |t|^2\ \frac{1}{2}\ \|x\|^2. \end{aligned}$$

Hence there exists a smooth function $q \to t(q)$, $t(q) > 0$, mapping bounded sets into bounded sets such that for $\|x\| \geq t(q)$ if $x \in L_q$ we have

$$\Phi_{b.k}(x) \leq d - 1.$$

Hence we can find a smooth function $\varphi : L \to [-1, 0]$ such that

$$\varphi(x) = 0 \text{ if } \|x\| \geq r(q) \tag{21}$$

for some $r : \Lambda \to \mathbb{R}$ mapping bounded sets smoothly into bounded sets and

$$\|\varphi(x)\Phi'_{b.k}(x)\| \leq 1 \text{ for all } k \in \mathbb{N} \text{ and } x \in L \tag{22}$$

and

$$\begin{aligned} &\|\varphi(x)\Phi'_{b.k}(x)\| \geq \varepsilon_1 && \text{for some } \varepsilon_1 \in (0, \min\{\varepsilon_0, 1\}) \\ &\text{if } x \in (\Phi_{b.k})_{d-\varepsilon_1}^{d+\varepsilon_1} \backslash V && \text{and } k \geq k_0. \end{aligned} \tag{23}$$

Now one estimates using standard idea (see [11]) for some $\varepsilon \in (0, \varepsilon_1)$ (suitably small) that

$$(\Phi_{b.d})^{d+\varepsilon} \backslash U *_k 1 \subset \Phi_{b.k}^{d+\varepsilon}.$$

Here $(x, t) \to x *_k t$ is the flow associated to $\xi_k = \varphi\Phi'_{b.k}$ for all $k \geq 1$. Now define $h_k : L \to L$ by

$$h_k(x) = x *_k 1.$$

Since $\Phi'_{b.k} = \Phi'_b - g'_{b.k}$ and $g_{b.k}$ satisfies (g) we find that $h_k \in \mathcal{M}$, actually in $\tilde{\mathcal{M}}$. This completes the proof of Proposition 2.

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