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The Wigner Semi-Circle Law in Quantum Electro Dynamics

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Abstract: In the present paper, the basic ideas of the *stochastic limit of quantum theory* are applied to quantum electro-dynamics. This naturally leads to the study of a new type of quantum stochastic calculus on a *Hilbert module.* Our main result is that in the weak coupling limit of a system composed of a free particle (electron, atom,...) interacting, via the minimal coupling, with the quantum electromagnetic field, a new type of quantum noise arises, living on a Hilbert module rather than a Hilbert space. Moreover we prove that the vacuum distribution of the limiting field operator is not Gaussian, as usual, but a nonlinear deformation of the Wigner semi-circle law. A third new object arising from the present theory, is the so-called *interacting Fock space.* A kind of Fock space in which the n quanta, in the n-particle space, are not independent, but interact. The origin of all these new features is that we do not introduce the dipole approximation, but we keep the exponential response term, coupling the electron to the quantum electromagnetic field. This produces a nonlinear interaction among all the modes of the limit master field (quantum noise) whose explicit expression, that we find, can be considered as a nonlinear generalization of the *Fermi golden rule.*

0. Introduction

Quantum electro-dynamics (QED) studies the interaction between matter and radiation. Due to the nonlinearity of this interaction (cf. (1.2) below), an explicit solution of the equations of motion is not known and, for their study, several types of approximations have been introduced.

Probably the best known of these approximations is the *dipole approximation* in which the so-called *response term* ($e^{ik\cdot q}$ in (1.2)) which couples matter, represented by an electron in position q, to the k^{th} mode of the EM-field, is assumed to be 1. The dipole approximation has its physical motivations in the fact that, at optical frequencies and for atomic dimensions one estimates that $k \cdot q \approx 10^{-3}$ (cf. [10]) and therefore very small. Most of the concrete applications of QED (e.g. in quantum optics, laser theory, ...) have been obtained under the dipole approximation or by replacing the response term by the first few (at most 3) terms in its series expansion (the so-called multipole terms).

The first step towards the elimination of the dipole approximation was done in [2, 3], where the *quasi-dipole approximation* was introduced. In the present investigation this approximation is not assumed: we shall just keep the response term.

The most surprising consequence of this more precise analysis is that the limiting noise, which approximates the quantum electromagnetic field is no longer a quantum Brownian motion, but a nontrivial generalization of the *free noises*, introduced by Kümmerer and Speicher [9, 16] who were inspired by Voiculescu's free central limit theorems [17], and developed by Fagnola in [7]. This free noise has been generalized by Bozejko and Speicher [5, 6] and the quantum noise which, according to the present paper, arises canonically from QED is a further generalization in this direction.

The main difference between the usual Gaussian and the so-called *free Gaussian* fields is that the vacuum expectation values of products of creation and annihilation fields are obtained, in the free Gaussian case, by summing the products of pair correlation functions not over all the pair partitions, as in the usual (Boson or Fermion Gaussian) case, but only on the so-called **non-crossing pair partitions**, defined in Sect. 5 of the present paper. In terms of graphs this means that the summation does not run over all graphs, but only over the so-called **rainbow (or half-planar** – cf. [20]) **graphs.** However it seems that the connection between these graphs and the Wigner semi-circle law was not realized in the previous physical literature.

Another novel feature, arising from the present analysis (and which already emerged at the level of quasi-dipole approximation (cf. [3])) is that the noise does not live on a Hilbert space, but on a **Hilbert module** over the momentum algebra of the electron. This Hilbert module is described in Sect. 7 of the present article. The general notion of *Hilbert module* was introduced for purely mathematical reasons and up to now this notion had found its main applications in K-theory for operator algebras (we refer to [4, 8, 14, 15], for the general theory of Hilbert modules). This circumstance has required the development of a theory of *quantum stochastic calculus over a Hilbert module* (see [11, 12, 13]).

A third result of the present analysis is the emergence of a new type of Fock space, which we call the **interacting Fock space** because the quanta in the n-particle space are not independent but interact in a highly nonlinear way (cf. Sect. 7 below and in particular Theorem (7.6)). The vacuum distribution of the field operators in this space is not Gaussian but a nonlinear modification of the Wigner semicircle law to which it reduces exactly when the nonlinear factor arising in the interacting Fock space is put equal to zero. The Wigner semicircle law was discovered by Wigner [18] starting from a purely phenomenological model to mimic the behaviour of Hamiltonians of heavy nuclei. It is rather surprising that it arises naturally in QED and that its appearance is accompanied by the emergence of some new mathematical structures whose properties make them natural candidates for the description of those phenomena in which the self-interaction of quantum fields plays an essential role.

Since the dipole approximation is effective at low frequencies and small atomic dimensions, it is reasonable to hope that the results of the present paper might shed some light on a class of phenomena in which high frequencies or finite atomic dimensions play an essential role. The systematic investigation of this possibility seems to deserve further attention.

The outline of the present paper is as follows: in Sect. 1 we describe the Hamiltonian model: it is essentially a generalization of the Frölich *polaron Hamiltonian*, to which it reduces when one puts $p = 1$ and replaces the factor $1/\sqrt{|k|}$ by $1/|k|$ in formula (1.2) below, which describes the interaction Hamiltonian. In Sect. 2 we describe the collective vectors and determine the 2-point function of the quantum noise, to which the initial quantum field converges when $\lambda \to 0$. Section 3 is devoted to prove the non-Gaussianity of the limit noise. It describes, in the simplest possible case of the 4-point function, the mechanism through which only the *non-crossing pair partitions* (*rainbow graphs*) survive in the limit. This essentially results from a combined effect of the CCR plus the Riemann–Lebesgue Lemma. The full limit space of the quantum noise is obtained in Sect. 4. In Sect. 5 we introduce some combinatorial properties of the non-crossing pair partitions which shall be used in the following sections. The technical core of the paper is Sect. 6, where we obtain the limit of the joint correlations of arbitrary products of collective creation and annihilation operators, i.e. we prove the convergence, in the sense of mixed vacuum moments, of the collective process to the quantum noise which shall be described in Sect. 7.

This allows to obtain our main goal, i.e. to compute the limit of the matrix elements of the wave operator in arbitrary collective vectors.

In Sect. 7 we identify the limit noise space to a Hilbert module over the momentum C^*-algebra of the electron (the *small system* in our terminology). This is the *interacting Fock space* (more precisely – Fock module) mentioned above. In Sect. 8 we compute the vacuum distribution of the noise field and we show that, if the interaction among the field modes is neglected, it reduces to a convex combination of Wigner semi-circle laws parametrized by the momentum of the electron. Finally, in Sect. 9, we describe, without proof, the quantum stochastic differential equation (cf. (9.2)) satisfied by the weak coupling limit of the wave operator at time t (the unitary Markovian cocycle, in the language of quantum probability). This has to be meant in the sense of stochastic calculus on Hilbert modules, as developed in [11–13]. This section has been included for completeness. We did not include the proof because, although long and elaborated from the technical point of view, it does not need new ideas and techniques, being based on a procedure which has now become standard in the stochastic limit of quantum theory, namely: one considers the matrix elements, in some collective vectors, of the wave operator at time t and show that, in the limit $\lambda \to 0$, they satisfy the same ordinary differential equation, with the same initial condition, as the corresponding matrix elements of the solution of the stochastic equation, which is known to be unique from the general theory [11].

In the revised version of the present paper we have enlarged the introduction, added several comments and corrected several notational and linguistic misprints. No statement or proof has been changed with respect to the original version of the paper (submitted for publication in November 1992).

1. The QED Model

We consider a *free* particle, called, the **system**, and characterized by:

– its Hilbert space $L^2(\mathbf{R}^d)$ with $d \geqq 3$

– its Hamiltonian

$$H_S := -\Delta/2 = p^2/2\,,$$

where Δ is the Laplacian and $p = (p_1, p_2, \dots, p_d)$ the momentum operator.

This system interacts with a quantum field in Fock representation whose free Hamiltonian is informally written as:

$$H_R := \sum_{k \in \mathbf{Z}^d} |k| a_k^+ a_k\,, \tag{1.1}$$

where to each mode $k \in \mathbf{Z}^d$ is associated a representation of the CCR with creation and annihilation operators a_k^+, a_k respectively ($a_k = (a_{1,k}, a_{2,k}, \dots, a_{d,k})$). The interaction between the system and the field is informally written as:

$$\lambda H_I = \lambda \sum_k e^{ik\cdot q}\, p \otimes \frac{a_k}{\sqrt{|k|}} + \text{ h.c.} = \lambda(A(q)\cdot p + p \cdot A(q))\,, \tag{1.2}$$

where $A(q)$ is the vector potential, the tensor product $p \otimes a_k$ means

$$p \otimes a_k = \sum_{j=1}^{d} p_j \otimes a_{j,k} \tag{1.3}$$

and the factor λ is a small scalar (coupling constant). This interaction is obtained from the usual minimal coupling interaction by neglecting the term of order λ^2.

Notice that, dropping the λ^2-term, breaks the gauge invariance of the theory. The realization of the present program without dropping the λ^2-term is a nontrivial problem.

We conjecture that the limit should be the same. Even more difficult is the problem of realizing the present program for a *non-free particle.*

There are indications that this program should be realizable for some classes of potentials. These topics will be discussed elsewhere. They are mentioned here only to indicate some possible lines of development.

The total Hamiltonian we are going to consider is $H = H_S + H_R + \lambda H_I$. The most important object for such an interacting model is the wave operator at time t. In the interacting picture it can be written as

$$U_t^{(\lambda)} := e^{it(H_S+H_R)} e^{-it(H_S+H_R+\lambda H_I)}\,, \tag{1.4}$$

where $U_t^{(\lambda)}$ is the solution of the Schrödinger equation:

$$\frac{d}{dt} U_t^{(\lambda)} = -i\lambda H_I(t) U_t^{(\lambda)}, \qquad U_0^{(\lambda)} = 1 \tag{1.5}$$

and the *evolved interaction Hamiltonian* $H_I(t)$ is given by

$$H_I(t) := e^{it(H_S+H_R)} H_I e^{-it(H_S+H_R)}\,. \tag{1.6}$$

Replacing in (1.1), (1.2) the sum by a continuous integral and the factor $1/\sqrt{|k|}$ by a cut-off function $g(k)$, $H_I(t)$ is given by the expression:

$$H_I(t) = i \left[\int_{\mathbf{R}^d} dk e^{itp^2/2} e^{ik\cdot q} e^{-itp^2/2} (-ip) \otimes (S_t g)(k) a_k^+ - \text{h.c.} \right], \tag{1.7}$$

where g is a *good* function (e.g. a Schwartz function) and $\{S_t^0\}_{t\geqq 0}$ is the unitary group on $L^2(\mathbf{R}^d)$ given, in momentum representation, by

$$(S_t^0 g)(k) = e^{-it|k|}g(k) . \tag{1.8}$$

By the CCR, we have

$$e^{-itp^2/2}e^{-ik\cdot q}e^{itp^2/2} = e^{-ik\cdot q}e^{-itk\cdot p}e^{-it|k|^2/2} , \tag{1.9}$$

$$e^{-ik\cdot q}e^{-itk\cdot p} = e^{-itk\cdot p}e^{-ik\cdot q}e^{it|k|^2} . \tag{1.10}$$

Therefore, one can rewrite (1.7) in the form

$$H_I(t) = i\left[\int_{\mathbf{R}^d} dk e^{-ik\cdot q}e^{-itk\cdot p}e^{-it|k|^2/2}(-ip)\otimes (S_t^0 g)(k)a_k^+ - \text{h.c.}\right] . \tag{1.11}$$

Without loss of generality we can simply forget the factor $e^{-it|k|^2/2}$ by transfering it into S_t^0. More precisely, from now on, the 1-particle free evolution shall not be the original one (1.8), but the modified one

$$(S_t g)(k) := e^{-it(|k|+k^2/2)}g(k) . \tag{1.11a}$$

The integrals (1.7), (1.11), should be meant in the weak topology on the subspace $\mathscr{S}(\mathbf{R}^d)\otimes\mathscr{E}$ (of $L^2(\mathbf{R}^d)\otimes\Gamma(L^2(\mathbf{R}^d))$), algebraic tensor product of the Schwartz functions and the (algebraic) linear span of the exponential and number vectors. This means that the matrix elements of $H_I(t)$ in these vectors are well defined (and this is the only thing that shall be used in the following). More precisely, if f and h are in $\mathscr{S}(\mathbf{R}^d)$ and $t\in\mathbf{R}$, then the functions

$$T_{f,h,j,t} : k\in\mathbf{R}^d \mapsto \langle f, e^{itp^2/2}e^{ik\cdot q}e^{-itp^2/2}(-ip_j)h\rangle_{L^2(\mathbf{R}^d)};\quad j=1,2,\ldots,d \tag{1.12}$$

are also in $\mathscr{S}(\mathbf{R}^d)$ and the integrals

$$A_j^+(T_{f,h,j,t}S_t g) = \int_{\mathbf{R}^d} dk\langle f, e^{itp^2/2}e^{ik\cdot q}e^{-itp^2/2}(-ip_j)h\rangle (S_t g)(k)a_{j,k}^+ \tag{1.13}$$

($j=1,2,\ldots,d$) define independent copies of the Boson Fock creation field over $L^2(\mathbf{R}^d)$ (notice that the Schwartz functions are an algebra, thus the product of two of them is still a test function) and therefore, for f,h as above and $\psi,\psi'\in\mathscr{E}$ the matrix element $\langle f\otimes\psi, H_I(t)h\otimes\psi'\rangle$ can be interpreted as

$$\sum_{j=1}^{d}\langle\psi, A_j^+(T_{f,h,j,t}S_t g)\psi'\rangle .$$

In conclusion we recall the basic strategy of the stochastic limit of quantum theory: the starting point is the formal solution of Eq. (1.5), given by the iterated series:

$$U_t^{(\lambda)} = \sum_{n=0}^{\infty}(-i)^n\lambda^n\int_0^t dt_1\cdots\int_0^{t_{n-1}} dt_n H_I(t_1)\cdots H_I(t_n) . \tag{1.14}$$

In usual perturbation theory one considers the first few terms of the series (1.14), in increasing powers of λ. For this procedure to make sense, it is required that the *coupling constant* be *small* in some sense. In the stochastic limit in quantum field theory one renormalizes time by the replacement $t \hookrightarrow t/\lambda^2$ and studies the limit, as $\lambda \to 0$, of matrix elements of the form:

$$\langle \Phi_\lambda(\xi), U^{(\lambda)}_{t/\lambda^2} \Phi'_\lambda(\eta)\rangle \tag{1.15}$$

where, $\xi, \eta \in L^2(\mathbf{R}^d)$ and the Φ_λ are the so-called *collective vectors*. They are chosen according to criteria which depend on the model and are suggested by the usual perturbation theory. The basic goal of the theory is to prove that, as $\lambda \longrightarrow 0$, the limit of (1.15) exists and has the form

$$\langle \Phi(\xi), U(t)\Phi'(\eta)\rangle\ , \tag{1.16}$$

where $\Phi(\xi), \Phi'(\eta)$ are vectors in the tensor product space of the system space with a certain limiting space $\mathscr{H}$, called *the noise space*, whose explicit form has to be determined, and the limit *process* $\{U(t)\}_{t\geqq 0}$ is unitary on the tensor product space for each $t \geqq 0$.

In the following, the subspace $\mathscr{S}(\mathbf{R}^d)$ of the Schwartz functions shall be often denoted $\mathscr{K} \subset L^2(\mathbf{R}^d)$. For any pair $f, g \in \mathscr{K}$, the condition

$$\int_{\mathbf{R}} |\langle f, S_t g\rangle| dt < \infty \tag{1.17}$$

which plays a crucial role in the theory is satisfied.

2. The Collective Operators and Collective Vectors

One of the basic heuristic rules of the stochastic limit of quantum field theory is that the choice of the collective vectors is suggested by first order analysis of the usual perturbation theory. Following this rule, in this section we shall introduce some preliminary considerations which give an intuitive idea on how to define the collective vectors.

For each $t \in \mathbf{R}_+$, define

$$A^+(S_t g) := \int_{\mathbf{R}^d} dk e^{-ik\cdot q} e^{itk\cdot p} (S_t g)(k) \otimes a^+_k \tag{2.1a}$$

as explained in formula (1.13) and define $A(S_t g)$ as the formal adjoint of $A^+(S_t g)$ i.e., recalling (1.11a)

$$A(S_t g) := \int_{\mathbf{R}^d} dk e^{-itk\cdot p} e^{ik\cdot q} (S_{-t}\bar{g})(k) \otimes a_k\ . \tag{2.1b}$$

Both $A^+(S_t g), A(S_t g)$ act on $L^2(\mathbf{R}^d) \otimes \Gamma(L^2(\mathbf{R}^d))$, where $\Gamma(L^2(\mathbf{R}^d))$ denotes the Boson Fock space over $L^2(\mathbf{R}^d)$, and behave like creation and annihilation operators respectively, more precisely, for each ξ, f in the Schwartz space $\mathscr{S}(\mathbf{R}^d)$:

$$A(S_t g)(\xi \otimes \Phi(f)) = \int_{\mathbf{R}^d} dk e^{-itk\cdot p} e^{ik\cdot q} \xi \otimes (S_{-t}\bar{g})(k) f(k) \Phi(f)\ , \tag{2.2}$$

where $\Phi(f)$ is the coherent or number vector with test function f and the integral in (2.2) is meant weakly on the domain $\mathscr{S} \otimes \mathscr{E}(\mathscr{S})$, where $\mathscr{E}(\mathscr{S})$ denotes the space algebraically generated by the coherent or number vectors with test functions in $\mathcal{S}$. In these notations, we have that

$$H(t) = i[A^+(S_t g)(-ip) - \text{h.c.}] \tag{2.3}$$

and the first order term of the interated series for $U_t^{(\lambda)}$ is:

$$-i\lambda \int_0^t dt_1 H(t_1) = \lambda \int_0^t dt_1 [A^+(S_{t_1} g)(-ip) - \text{h.c.}] = A^+\left(\lambda \int_0^t dt_1 S_{t_1} g\right)(-ip) - \text{h.c.}\,. \tag{2.4}$$

For simplicity, in (2.3), we write $(-ip)$ instead of $(-ip)\otimes 1_\Gamma$, where 1_Γ is the identity operator on $\Gamma(L^2(\mathbf{R}^d))$.

In particular, in the iterated series for U_{t/λ^2} the first order term is

$$A^+\left(\left(\lambda \int_0^{t/\lambda^2} dt_1 S_{t_1} g\right)(-ip)\right) - \text{h.c.}\,.$$

Following the above mentioned heuristic rule (see also [1]), we define *the collective annihilator process* by:

$$A_\lambda(0,t,g) := A\left(\lambda \int_0^{t/\lambda^2} dt_1 S_{t_1} g\right) = \lambda \int_0^{t/\lambda^2} dt_1 \int_{\mathbf{R}^d} dk e^{-itk\cdot p} e^{ik\cdot q} \otimes \overline{S_t g}(k) a_k; \quad t \in \mathbf{R} \tag{2.5a}$$

and its conjugate, *the collective creator process*, by:

$$A_\lambda^+(0,t,g) := A^+\left(\lambda \int_0^{t/\lambda^2} dt_1 S_{t_1} g\right) = \lambda \int_0^{t/\lambda^2} dt_1 \int_{\mathbf{R}^d} dk e^{-ik\cdot q} e^{itk\cdot p} \otimes S_t g(k) a_k^+; \quad t \in \mathbf{R}\,. \tag{2.5b}$$

More generally, for any bounded interval $[S,T] \subset \mathbf{R}$ and Schwartz function f, we shall define the collective creators and annihilators by:

$$A_\lambda^+(S,T,f) := \lambda \int_{S/\lambda^2}^{T/\lambda^2} dt \int_{\mathbf{R}^d} dk e^{-ik\cdot q} e^{itk\cdot p} \otimes S_t f(k) a_k^+\,. \tag{2.6}$$

Notice that, with these notations, the first order term of the iterated series (1.14) can be written in the form:

$$\lambda \int_0^{t/\lambda^2} dt_1 H_I(t_1) = i[pA_\lambda(0,t,g) - A_\lambda^+(0,t,g)p] \tag{2.7}$$

which has a formal similarity with the dipole approximation Hamiltonian (cf. [2]) except for the fact that in (2.7) the collective annihilators and creators also contain operators of the system space, hence they do not commute with p.

The first step of the program of the stochastic limit in quantum theory is to show that in a certain sense, as $\lambda \longrightarrow 0$, i.e. the collective operators converge to some kind of creation and annihilation operators $A^+(S,T,f)$ and $A(S,T,f)$ acting on some limit Hilbert space. Symbolically:

$$A_\lambda^+(S,T,f) \longrightarrow A^+(S,T,f); \qquad A_\lambda(S,T,f) \longrightarrow A(S,T,f)\,.$$

In order to determine the structure of this space, the consideration of the two point function is naturally suggested by the fact that the vacuum distribution of the creation and annihilation fields, before the limit, is Gaussian.

The limit of the two point function of the collective creation and annihilation operators can be obtained as follows: for each $s,t \geqq 0$, one has

$$\begin{aligned}&\lambda^2 \int_0^{t/\lambda^2} dt_1 \int_0^{s/\lambda^2} dt_2 \int_{\mathbf{R}^d} dk_1 \int_{\mathbf{R}^d} dk_2\ \langle 0|a_{k_1}a_{k_2}^+|0\rangle \\ &\times \langle \xi,(-ip)e^{-it_1k_1\cdot p}e^{ik_1\cdot q}\overline{(S_{t_1}g)}(k_1)e^{-ik_2\cdot q}e^{itk_2\cdot p}(-ip)(S_{t_2}g)(k_2)\eta\rangle \\ &= \lambda^2 \int_0^{t/\lambda^2} dt_1 \int_0^{s/\lambda^2} dt_2 \int_{\mathbf{R}^d} dk e^{-i(t_2-t_1)|k|^2/2}\langle (ip)\xi, e^{i(t_2-t_1)k\cdot p}(ip)\eta\rangle \bar{g}(k)(S_{t_2}g)(k)\end{aligned} \tag{2.8}$$

which, as $\lambda \to 0$, tends to:

$$t\wedge s \int_{-\infty}^{+\infty} d\tau \int_{\mathbf{R}^d} dk\langle (ip)\xi, e^{i\tau k\cdot p}(ip)\eta\rangle \bar{g}(k)(S_\tau g)(k)\,. \tag{2.9}$$

If, in (2.8), one replaces the zeros in the first two integrals by $T/\lambda^2, S/\lambda^2$ repectively, then the limit (2.9) is replaced by

$$\langle \chi_{[T,t]}, \chi_{[S,s]}\rangle_{L^2(\mathbf{R})} \cdot \int_{-\infty}^{+\infty} d\tau \int_{\mathbf{R}^d} dk\langle (ip)\xi, e^{i\tau k\cdot p}(ip)\eta\rangle \bar{g}(k)(S_\tau g)(k)\,. \tag{2.10}$$

Keeping in mind the definitions (2.5a) and (2.5b) of collective annihilator and creator, the above result can be rephrased as follows: the approximate *two point function*

$$\langle \xi\otimes\Phi, A_\lambda(0,t,g)A_\lambda^+(0,s,g)\ \eta\otimes\Phi\rangle \tag{2.11}$$

tends to an object which for some aspects, is very similar to a two point function. In Sect. 4 we show how to substantiate this analogy.

Remark. Although the collective creation and annihilation operators depend on the operator q, the limit of their 2-point function, i.e. (2.8), given by (2.9), is independent of q. However a remnant of the original q-dependence remains in the limit through the commutation relation (7.8).

The above considerations suggest to introduce **the collective number vectors**, which shall play a crucial role in the following.

Definition 2.1. *A collective number vector is a vector of $L^2(\mathbf{R}^d)\otimes\Gamma(L^2(\mathbf{R}^d))$ obtained by applying a (finite) product of collective creation operators to a vector of the form $\Phi\otimes\xi$, where Φ is the vacuum in $\Gamma(L^2(\mathbf{R}^d))$ and ξ is an arbitrary vector in $L^2(\mathbf{R}^d)$. Such a vector has the form*

$$A_\lambda^+(S_1,T_1,f_1)\cdot\dots\cdot A_\lambda^+(S_n,T_n,f_n)\xi\otimes\Phi\,. \tag{2.12}$$

In the following, when no confusion can arise, we shall use the simplified notation

$$\prod_{h=1}^{n} A^{+}_{\lambda,h}\xi \otimes \Phi \tag{2.13}$$

to denote a vector of the form (2.12). Notice, for future use, that **the indices of the creators in (2.13) are increasing from left to right**. The linear span of the collective vectors, for a given value of the coupling constant λ will be denoted $\mathscr{H}_\lambda$ and called **the space of collective vectors.**

It will be convenient to operate a change in notations with respect to the previous sections and to write the state space of the composite system in the form $\Gamma(L^2(\mathbf{R}^d)) \otimes L^2(\mathbf{R}^d)$, rather than $L^2(\mathbf{R}^d) \otimes \Gamma(L^2(\mathbf{R}^d))$.

3. Non-Gaussianity of the Quantum Noise

From the previous two sections we know that our original physical model is Boson Gaussian, i.e. the creation and annihilation fields satisfy the CCR and their vacuum distributions are Gaussian.

In this section we want to give an intuitive idea of the mechanism through which, by applying to the present model the procedure of the stochastic limit of quantum theory, in the limit $\lambda \to 0$ of the vacuum correlation functions of the collective operators all the terms, corresponding to crossing partitions, vanish and the only nontrivial contributions come from the non-crossing pair partitions (cf. Sect. 5 below for a quick review and references on this notion). This phenomenon corresponds to *the breaking of the Gaussianity* (because in the Gaussian case all pair partitions, and not only the non-crossing ones, contribute to the correlations). The lowest order correlation function where the distinction between crossing and non-crossing pair partitions (and therefore between Gaussian and non-Gaussian correlations) can become apparent corresponds to the **four point function.**Therefore in this section we shall study the limit of

$$\langle A^{+}_{\lambda}(T_1,f_1)A^{+}_{\lambda}(T_2,f_2)\Phi \otimes \xi,\ A^{+}_{\lambda}(T'_1,f'_1)A^{+}_{\lambda}(T'_2,f'_2)\Phi \otimes \eta\rangle \tag{3.1}$$

(the general case is dealt with in Sects. 7 and 8).

By the definition of collective creation and annihilation operators, (3.1) is equal to

$$\Big\langle \lambda^2 \int_0^{T_1/\lambda^2} dt_1 \int_0^{T_2/\lambda^2} dt_2 \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} dk_1 dk_2 e^{-ik_1\cdot q} e^{it_1 k_1\cdot p} e^{-ik_2\cdot q} e^{it_2 k_2 \cdot p}$$

$$\times (S_{t_1}f_1)(k_1)(S_{t_2}f_2)(k_2) a^{+}_{k_1} a^{+}_{k_2}|0\Big\rangle \otimes \xi\,,$$

$$\Big\langle \lambda^2 \int_0^{T'_1/\lambda^2} ds_1 \int_0^{T'_2/\lambda^2} ds_2 \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} e^{-ik'_1\cdot q} e^{is_1 k'_1\cdot p} e^{-ik'_2\cdot q} e^{is_2 k_2 \cdot p}$$

$$\times (S_{s_1}f'_1)(k'_1)(S_{s_2}f'_2)(k_2) a^{+}_{k'_1} a^{+}_{k'_2}|0 \otimes \eta\Big\rangle$$

$$= \lambda^4 \int_0^{T_1/\lambda^2} dt_1 \int_0^{T_2/\lambda^2} dt_2 \int_0^{T'_1/\lambda^2} ds_1 \int_0^{T'_2/\lambda^2} ds_2 \int_{\mathbf{R}^{4d}} dk_1 dk_2 dk'_1 dk'_2$$

$$\times \langle \xi, e^{-it_2 k_2 \cdot p} e^{ik_2 \cdot q} e^{-it_1 k_1 \cdot p} e^{ik_1 \cdot q} e^{-ik_1' \cdot q} e^{is_1 k_1' \cdot p} e^{-ik_2' \cdot q} e^{is_2 k_2' \cdot p} \eta \rangle$$

$$\times \overline{(S_{t_1} f_1)}(k_1) \overline{(S_{t_2} f_2)(k_2)} (S_{s_1} f_1')(k_1')(S_{s_2} f_2')(k_2') \langle 0 | a_{k_2} a_{k_1} a^+_{k_1'} a^+_{k_2'} | 0 \rangle$$

$$= \lambda^4 \int_0^{T_1/\lambda^2} dt_1 \int_0^{T_2/\lambda^2} dt_2 \int_0^{T_1'/\lambda^2} ds_1 \int_0^{T_2'/\lambda^2} ds_2 \int_{\mathbf{R}^{2d}} dk_1 dk_2 \tag{3.2}$$

$$\times [\langle \xi, e^{-it_2 k_2 \cdot p} e^{ik_2 \cdot q} e^{-it_1 k_1 \cdot p} e^{is_1 k_1 \cdot p} e^{-ik_2 \cdot q} e^{is_2 k_2 \cdot p} \eta \rangle$$

$$\times \bar{f}_1(k_1)(S_{s_1 - t_1} f_1')(k_1) \bar{f}_2(k_2)(S_{s_2 - t_2} f_2')(k_2)$$

$$+ \langle \xi, e^{-it_2 k_2 \cdot p} e^{ik_2 \cdot q} e^{-it_1 k_1 \cdot p} e^{ik_1 \cdot q} e^{-ik_2 \cdot q} e^{is_1 k_2 \cdot p} e^{-ik_1 \cdot q} e^{is_2 k_1 \cdot p} \eta \rangle$$

$$\times \bar{f}_1(k_1)(S_{s_2 - t_1} f_2')(k_1) \bar{f}_2(k_2)(S_{s_1 - t_2} f_1')(k_2)] \,. \tag{3.3}$$

Using the CCR, one can move the q-factors together in the right-hand side of (3.2). Thus one can rewrite (3.1) as

$$\lambda^4 \int_0^{T_1/\lambda^2} dt_1 \int_0^{T_2/\lambda^2} dt_2 \int_0^{T_1/\lambda^2} ds_1 \int_0^{T_2/\lambda^2} ds_2 \int_{\mathbf{R}^{2d}} dk_1 dk_2$$

$$\times \left[\langle \xi, e^{-it_2 k_2 \cdot p} . e^{is_2 k_2 \cdot p} e^{-it_1 k_1 \cdot p} e^{-is_1 k_1 \cdot p} e^{i(s_1 - t_1) k_2 \cdot k_1} \eta \rangle \right.$$

$$\times \bar{f}_1(k_1)(S_{s_1 - t_1} f_1')(k_1) \bar{f}_2(k_2)(S_{s_2 - t_2} f_2')(k_2)$$

$$+ \langle \xi, e^{-it_2 k_2 \cdot p} e^{is_1 k_2 \cdot p} e^{-it_1 k_1 \cdot p} e^{-is_2 k_1 \cdot p} e^{i(s_1 - t_1) k_2 \cdot k_1} \eta \rangle$$

$$\left. \times \bar{f}_1(k_1)(S_{s_2 - t_1} f_2')(k_1) \bar{f}_2(k_2)(S_{s_1 - t_2} f_1')(k_2) \right] \,. \tag{3.4}$$

Notice that, in the first term in (3.4), f_1 is paired with f_1' and f_2 with f_2' (non-crossing); while in the second one, f_1 is paired with f_2' and f_2 with f_1' (crossing).

With the change of variables

$$\tau_1 = \lambda^2 t_1, \qquad \tau_2 = \lambda^2 t_2, \qquad s_1 - \tau_1/\lambda^2 = u, \qquad s_2 - \tau_2/\lambda^2 = v \,, \tag{3.5}$$

the first term of (3.4) becomes

$$\int_0^{T_1} d\tau_1 \int_0^{T_2} d\tau_2 \int_{-\tau_1/\lambda^2}^{(T_1' - \tau_1)/\lambda^2} du \int_{-\tau_2/\lambda^2}^{(T_2' - \tau_2)/\lambda^2} dv \int_{\mathbf{R}^{2d}} dk_1 dk_2$$

$$\times \langle \xi, e^{ivk_2 \cdot p} e^{iuk_1 \cdot p} \eta \rangle e^{iuk_1 \cdot k_2} \bar{f}_1(k_1) \bar{f}_2(k_2)(S_u f_1')(k_1)(S_v f_2')(k_2) \tag{3.6}$$

and as λ tends to zero, this goes to

$$\langle \chi_{[0,T_1]}, \chi_{[0,T_2']} \rangle_{L^2(\mathbf{R})} \cdot \int_{-\infty}^{+\infty} du \int_{-\infty}^{+\infty} dv \int_{\mathbf{R}^{2d}} dk_1 dk_2 \bar{f}_1(k_1) \bar{f}_2(k_2) e^{iuk_1 \cdot k_2}$$

$$\times (S_u f_1')(S_v f_2')(k_2) \cdot \langle \xi, e^{i(uk_1 + vk_2) \cdot p} \eta \rangle \,. \tag{3.7}$$

With the change of variables

$$\tau_1 = \lambda^2 t_1, \qquad \tau_2 = \lambda^2 t_2, \qquad u = s_1 - \tau_2/\lambda^2, \qquad v = s_2 - \tau/\lambda^2 \,, \tag{3.8}$$

the second term of (3.4) becomes

$$\int_0^{T_1} d\tau_1 \int_0^{T_2} d\tau_2 \int_{-\tau_2/\lambda^2}^{(T_1'-\tau_2)/\lambda^2} du \int_{-\tau_1/\lambda^2}^{(T_2'-\tau_1/\lambda_2)} dv \int_{\mathbf{R}^{2d}} dk_1 dk_2$$
$$\times \left[\langle \xi, e^{i(uk_2+vk_1)\cdot p}\eta\rangle \cdot (S_u f_1')(k_2)(S_v f_2')(k_1)\bar{f}_1(k_1)\bar{f}_2(k_2)\cdot e^{iuk_1\cdot k_2}\right]$$
$$\times e^{ik_1\cdot k_2(\tau_2-\tau_1)/\lambda^2}\,. \tag{3.9}$$

By the Riemann–Lebesgue lemma, (3.9) tends to zero as $\lambda \to 0$.

We shall now proceed to the proof of the fact that the vanishing, in the limit, of all the contributions coming from crossing partitions is a general feature of QED.

4. The Limit of the Spaces of the Collective Vectors

Our next step will be to try to determine the limit

$$\lim_{\lambda\to 0}\langle \Phi_\lambda, \Phi_\lambda'\rangle =: \langle \Phi, \Phi'\rangle$$

of the scalar product of two collective number vectors. This will give the Hilbert space where the limits of the collective creation and annihilation operators act.

According to our choice of these vectors, this amounts to study the limit, as $\lambda \to 0$, of scalar products of the form:

$$\left\langle \prod_{h=1}^{n} A_{\lambda,h}^{+}\Phi\otimes\xi,\ \prod_{h=1}^{n} A_{\lambda,h}^{+}\Phi\otimes\eta\right\rangle .$$

The present section shall be devoted to the proof of the following result:

Theorem 4.1 *For any $N, n \in N$, the limit, as $\lambda \to 0$, of the scalar product of two collective number vectors*:

$$\left\langle \prod_{h=1}^{N} A_{\lambda,h}^{+}\Phi\otimes\xi,\ \prod_{h=1}^{n} A_{\lambda,h}^{+}\Phi\otimes\eta\right\rangle \tag{4.1}$$

exists and is equal to:

$$\delta_{N,n}\prod_{h=1}^{n}\langle \chi_{[S_h,T_h]},\ \chi_{[S_h',T_h']}\rangle\cdot\int_{-\infty}^{+\infty} du_1\cdots\int_{-\infty}^{+\infty} du_n \int_{\mathbf{R}^{nd}} dk_1\cdots dk_n$$
$$\times\left\langle \xi, \prod_{h=1}^{n} e^{iu_hk_h\cdot p}\eta\right\rangle \prod_{h=1}^{n}(S_{u_h}f_h')(k_h)\bar{f}_h(k_h)\exp\left(i\sum_{r=1}^{n-1}\sum_{h=r}^{n-1} u_r k_r\cdot k_{h+1}\right)\,. \tag{4.2}$$

Remark. If the factor

$$\left\langle \xi, \prod_{h=1}^{n} e^{iu_hk_h\cdot p}\eta\right\rangle\cdot\exp\left(i\sum_{r=1}^{n-1}\sum_{h=r}^{n-1} u_r k_{h+1}\cdot k_r\right) \tag{4.3}$$

were absent in (4.2), then the expression (4.2) would coincide with the scalar product

$$\left\langle \bigotimes_{j=1}^{n}(\chi_{[S_j,T_j]}\otimes f_j),\ \bigotimes_{j=1}^{n}(\chi_{[S_j',T_j']}\otimes f_j')\right\rangle \tag{4.4}$$

defined on the tensor product of n copies of the space:

$$L^2(\mathbf{R}) \otimes \mathscr{K} ,$$

where $L^2(\mathbf{R})$ has the usual scalar product, while $\mathscr{K} \subset L^2(\mathbf{R}^d)$ has the scalar product

$$(f \mid g) := \int_{\mathbf{R}} dt \langle f, S_t g \rangle_{L^2(\mathbf{R}^d)} . \tag{4.5}$$

Notice that the scalar product (4.4) can also be written in the form

$$\left\langle \prod_{j=1}^{n} A^+(\chi_{[S_j,T_j]} \otimes f_j)\Psi, \ \prod_{j=1}^{n} A^+(\chi_{[S'_j,T'_j]} \otimes f'_j)\Psi \right\rangle , \tag{4.6}$$

where $A^+(\cdot)$ denotes the creation operator on the full Fock space over $L^2(\mathbf{R}) \otimes \mathscr{K}$, with the scalar product (4.5), and Ψ is the vacuum vector in this space.

Our goal in the following sections will be to write the limit (4.1) in the form (4.6) where the $A^+(\cdot)$ are *some sort of creation operators* acting on *some sort of Fock space* in which Ψ plays the role of vacuum vector.

The obstruction to the use of the usual Fock or full Fock space comes precisely from the scalar factor (4.3) which is itself the product of two factors, each of which is related to the other two new features arising from our construction, namely:

(i) the $\xi - \eta$-scalar product is related to the fact that the noise lives on a Hilbert module rather than a Hilbert space.

(ii) The exp-factor is related to the fact that the space (more precisely, the module) over the test function space is not the usual Fock, or full Fock space, but the *interacting Fock space.*

These features will be better understood in the following.

Without loss of generality, in the following we shall suppose that $S_j = S'_j = 0$ and rewrite T'_j as S_j for all $j = 1, \ldots, n$. Moreover, it is obvious that we have to prove Theorem 4.1 only in the case $n = N$.

In order to prove Theorem 4.1 let us first notice that the explicit form of the scalar product (4.1) is:

$$\lambda^{2n} \int_0^{T_1/\lambda^2} dt_1 \int_0^{S_1/\lambda^2} ds_1 \cdots \int_0^{T_n/\lambda^2} dt_n \int_0^{S_n/\lambda^2} ds_n \int_{\mathbf{R}^{2nd}} dk_1 \cdots dk_n dk'_1 \cdots dk'_n$$

$$\times \langle \xi, e^{-it_nk_n \cdot p} e^{ik_n \cdot q} \ldots e^{-it_1k_1 \cdot p} e^{ik_1 q} \cdot e^{-ik'_1 q} e^{is_1 k'_1 \cdot p} \ldots e^{-ik'_n \cdot q} e^{is_n k'_n \cdot p} \eta \rangle$$

$$\times \prod_{h=1}^{n} \overline{(S_{t_h} f_h)}(k_h) \cdot (S_{s_h} f'_h)(k'_h) \langle 0 \mid a_{k_n} \cdots a_{k_1} a^+_{k'_1} \cdots a^+_{k'_n} \mid 0 \rangle . \tag{4.7}$$

By explicitly performing the vacuum expectation, (4.7) can be written in the form:

$$\sum_{\sigma \in S_n} \lambda^{2n} \int_0^{T_1/\lambda^2} dt_1 \int_0^{S_1/\lambda^2} ds_1 \cdots \int_0^{T_n/\lambda^2} dt_n \int_0^{S_n/\lambda^2} ds_n \int_{\mathbf{R}^{nd}} dk_1 \cdots dk_n$$

$$\times \langle \xi, e^{-it_nk_n \cdot p} e^{ik_n \cdot q} \ldots e^{-it_1k_1 \cdot p} e^{ik_1 \cdot q} e^{-ik_{\sigma(1)} \cdot q} e^{is_1 k_{\sigma(1)} \cdot p} \ldots e^{-ik_{\sigma(n)} \cdot q} e^{is_n k_{\sigma(n)} \cdot p} \eta \rangle$$

$$\times \prod_{h=1}^{n} (S_{s_h - t_{\sigma(h)}} f'_h)(k_{\sigma(h)}) \bar{f}_h(k_h) , \tag{4.8}$$

where $\mathscr{S}_n$ denotes the group of permutations over the symbols $\{1, 2, \ldots, n\}$.

Lemma 4.2. *In the limit $\lambda \to 0$, only the identity permutation in the sum in* (4.8), *gives a non zero contribution.*

Proof. The proof is an extension, to arbitrary n, of the argument used in Sect. 3 for the case $n = 2$. Notice that the factors $\exp(\pm k_j q)$ are independent of the times s_j, t_j and each of them appears twice: once with the "+" and another with the "−" sign. If $\sigma(j) = h$, by permuting the factor $\exp(ik_h q)$ with all the factors

$$\exp(it_m k_m p), \quad m = h-1,\ h-2, \ldots, 1\ ,$$

and then with the factors

$$\exp(is_r k_{\sigma(r)} p), \quad r = 1, \ldots, \sigma^{-1}(h) - 1 = j - 1\ ,$$

we shall erase this factor by multiplication with $\exp(-ik_h q) = \exp(-ik_{\sigma(j)} q)$. Because of the CCR, each of the first $h-1$ exchanges gives rise to a scalar factor of the form

$$\exp(-it_m k_m \cdot k_h), \quad m = 1, \ldots, h-1\ ,$$

and each of the other $\sigma^{-1}(h) - 1 = j - 1$ exchanges gives rise to a scalar factor of the form

$$\exp(is_r k_{\sigma(r)} \cdot k_h), \quad r = 1, \ldots, \sigma^{-1}(h) - 1\ .$$

When all these exchanges have been performed, the scalar product in the expression (4.8) becomes equal to:

$$\begin{aligned}
&\exp\left(\frac{i}{2}\left(\left[\sum_{r=1}^{\sigma^{-1}(1)-1} s_r k_1 \cdot k_{\sigma(r)} - t_1 k_1 \cdot k_2\right.\right.\right.\\
&\quad + \sum_{r=1}^{\sigma^{-1}(2)-1} s_r k_2 \cdot k_{\sigma(r)} - (t_1 k_1 \cdot k_3 + t_2 k_2 \cdot k_3)\\
&\quad + \sum_{r=1}^{\sigma^{-1}(3)-1} s_r k_3 \cdot k_{\sigma(r)} - \cdots - (t_1 k_1 \cdot k_n + \cdots + t_{n-1} k_{n-1} \cdot k_n)\\
&\quad \left.\left.\left. + \sum_{r=1}^{\sigma^{-1}(n)-1} s_r k_n \cdot k_{\sigma(r)}\right)\right]\right) \xi, \prod_{h=1}^{n} e^{i(s_h - t_{\sigma(h)}) k_{\sigma(h)} \cdot p} \eta \Big\rangle\\
&= \left\langle \xi, \prod_{h=1}^{n} e^{i(s_h - t_{\sigma(h)}) k_{\sigma(h)} \cdot p} \eta \right\rangle\\
&\quad \times \exp\left(i\left[\sum_{h=1}^{n} \sum_{r=1}^{\sigma^{-1}(h)-1} s_r k_h \cdot k_{\sigma(r)} - \sum_{h=1}^{n-1} \sum_{r=1}^{h} t_r k_r \cdot k_{h+1}\right]\right)\ . \qquad (4.9)
\end{aligned}$$

We shall now rewrite the scalar exponential in (4.9) so to make more transparent the difference between negligible and non negligible terms. First of all notice that, for any permutation $\sigma \in \mathscr{S}_n$:

$$\begin{aligned}
\sum_{h=1}^{n-1} \sum_{r=1}^{h} t_r k_r \cdot k_{h+1} &= t_1 k_1 \cdot k_2 + t_1 k_1 \cdot k_3 + t_2 k_2 \cdot k_3 + \cdots\\
&\quad + t_1 k_1 \cdot k_n + \cdots + t_{n-1} k_{n-1} \cdot k_n\\
&= \sum_{h=1}^{n-1} t_h k_h \cdot \sum_{m=h+1}^{n} k_m = \sum_{h=1}^{n} t_{\sigma(h)} k_{\sigma(h)} \cdot \sum_{m=\sigma(h)+1}^{n} k_m\ . \qquad (4.10)
\end{aligned}$$

Notice that to say that $h_1,\dots,h_n \in \{1,\dots,n\}$ are such that the sequence $\sigma^{-1}(h_1)$, $\sigma^{-1}(h_2),\dots,\sigma^{-1}(h_n)$ is ordered in increasing order, is equivalent to say that

$$\sigma(m) = h_m; \quad m = 1,\dots,n .$$

With these notations we also have

$$\sum_{h=1}^{n} \sum_{r=1}^{\sigma^{-1}(h)-1} s_r k_{\sigma(r)} \cdot k_h = \sum_{m=1}^{n} \sum_{r=1}^{\sigma^{-1}(h_m)-1} s_r k_{\sigma(r)} \cdot k_{h_m}$$

$$= \sum_{m=1}^{n} \sum_{r=1}^{m-1} s_r k_{\sigma(r)} \cdot k_{\sigma(m)} = \sum_{r=1}^{n-1} s_r k_{\sigma(r)} \cdot \sum_{m=r+1}^{n} k_{\sigma(m)} . \tag{4.11}$$

Thus the difference between (4.11) and (4.10), i.e. the factor appearing in the scalar exponential in (4.9), can be written in the form:

$$\sum_{h=1}^{n-1} (s_h - t_{\sigma(h)}) k_{\sigma(h)} \sum_{m=h+1}^{n} k_{\sigma(m)} + \sum_{h=1}^{n-1} t_{\sigma(h)} k_{\sigma(h)} \left[\sum_{m=h+1}^{n} k_{\sigma(m)} - \sum_{m=\sigma(h)+1}^{n} k_m \right] . \tag{4.12}$$

Therefore, the expression (4.8) can be written as

$$\sum_{\sigma\in\mathscr{S}_n} \lambda^{2n} \int_0^{T_1/\lambda^2} dt_1 \int_0^{S_1/\lambda^2} ds_1 \cdots \int_0^{T_n/\lambda^2} dt_n \int_0^{S_n/\lambda^2} ds_n \int_{\mathbf{R}^{nd}} dk_1 \cdots dk_n$$

$$\times \left\langle \xi, \prod_{h=1}^{n} e^{i(s_h - t_{\sigma(h)}) k_{\sigma(h)} \cdot p} \eta \right\rangle \exp\left[i \sum_{h=1}^{n-1} (s_h - t_{\sigma(h)}) k_{\sigma(h)} \sum_{m=h+1}^{n} k_{\sigma(h)} \right]$$

$$\times \exp\left[i \sum_{h=1}^{n-1} t_{\sigma(h)} k_{\sigma(h)} \cdot \left(\sum_{m=h+1}^{n} k_{\sigma(m)} - \sum_{m=\sigma(h)+1}^{n} k_m \right) \right] . \tag{4.13}$$

Now we replace the scalar product in (4.8) by the right-hand side of (4.9) and make the following change of variables in (4.8):

$$\tau_h = \lambda^2 t_h, \qquad u_h = s_h - \tau_{\sigma(h)}/\lambda^2, \quad h = 1,\dots,n .$$

From the Riemann–Lebesgue Lemma it follows that, as $\lambda \to 0$, the only terms (in (4.8)) that can give a nonzero contribution are those coming from the factors such that the identity

$$0 = \sum_{h=1}^{n-1} t_{\sigma(h)} k_{\sigma(h)} \cdot \left[\sum_{m=h+1}^{n} k_{\sigma(m)} - \sum_{m=\sigma(h)+1}^{n} k_h \right] \tag{4.14}$$

is satisfied for almost all $(k_1,\dots,k_n) \in \mathbf{R}^{nd}$ and almost all $(t_1,\dots,t_n) \in \mathbf{R}^n$. So for almost all $(k_1,\dots,k_n) \in \mathbf{R}^{nd}$,

$$\sum_{m=h+1}^{n} k_{\sigma(m)} = \sum_{m=\sigma(h)+1}^{n} k_m; \quad \forall h = 1,\dots,n . \tag{4.15}$$

Letting $h = n-1$ in (4.15), the fact that (4.15) is true for almost all $(k_1,\dots,k_n) \in \mathbf{R}^{nd}$ implies that in the sum $\sum_{m=\sigma(h)+1}^{n} k_m$, there is only one term, i.e. the cardinality of the set $\{\sigma(h)+1,\dots,n\}$ is one. This is equivalent to say that

$$\sigma(n-1) + 1 = n . \tag{4.16a}$$

Letting $h = n-2$ in (4.15) and using (4.16a) one finds

$$k_{\sigma(n-1)} + k_{\sigma(n)} = k_{n-1} + k_n = k_{\sigma(n-2)+1} + k_n ,$$

or equivalently

$$\sigma(n-1) = \sigma(n-2) + 1 = n-1 . \tag{4.16b}$$

Iterating one finds

$$\sigma = id , \tag{4.17}$$

i.e. σ must be the identity permutation.

Proof of Theorem 4.1. For $\sigma = I$, the last term of (4.9) is equal to

$$\begin{aligned}
&\exp\left(i\left[\sum_{h=1}^{n}\sum_{r=1}^{h-1} s_r k_r \cdot k_h - \sum_{h=1}^{n-1}\sum_{r=1}^{h} t_r k_r \cdot k_{h+1}\right]\right)\\
&= \exp\left(i\left[\sum_{r=1}^{n-1}\sum_{h=r+1}^{n} s_r k_r \cdot k_h - \sum_{r=1}^{n-1}\sum_{h=r}^{n-1} t_r k_r \cdot k_{h+1}\right]\right)\\
&= \exp\left(i\sum_{r=1}^{n-1}\sum_{h=r+1}^{n} (s_r - t_r) k_r \cdot k_h\right) .
\end{aligned} \tag{4.18}$$

Therefore (4.8) is equal to the limit of

$$\begin{aligned}
&\lambda^{2n}\int_0^{T_1/\lambda^2} dt_1 \int_0^{S_1/\lambda^2} ds_1 \cdots \int_0^{T_n/\lambda^2} dt_n \int_0^{S_n/\lambda^2} ds_n \int_{\mathbf{R}^{nd}} dk_1 \cdots dk_n\\
&\times \left\langle \xi, \prod_{h=1}^{n} e^{i(s_h - t_h)p}\eta \right\rangle \cdot \prod_{h=1}^{n} (S_{s_h - t_h} f_h')(k_h)\bar{f}_h(k_h)\\
&\times \exp\left(i\sum_{r=1}^{n-1}\sum_{h=r}^{n-1} (s_r - t_r) k_r \cdot k_{h+1}\right) + o(1) ,
\end{aligned} \tag{4.19}$$

where $o(1)$ denotes the sum over all the permutations $\delta \in \mathcal{S}_n$, different from the identity, which, according to Lemma (4.2), tend to zero as $\lambda \longrightarrow 0$. With the change of variables:

$$\tau_h := \lambda^2 t_h, \qquad u_h = s_h - \tau_h/\lambda^2, \quad h = 1,\ldots,n ,$$

(4.18) tends to (4.2), and this ends the proof.

5. Non-Crossing Pair Partitions

Since the notion of *non-crossing pair partition* will play a crucial role in the following, we devote the present section to introduce some basic properties of these partitions (for more complete information we refer to [5, 6, 16]).

Non-crossing pair partitions play, for Wigner processes, the role played by the pair partitions for the usual Gaussian processes. More precisely: it is known that the mixed moments of order $2n$ (n is a natural integer) of a mean zero Gaussian process, are given by the sum, over all pair partitions of $\{1,\ldots,2n\}$, of the products of the 2-point functions (covariance) of the process computed over all the pairs of the partition.

If the sum over *all* pair partitions is replaced by the sum over *all non-crossing* pair partitions, in the sense specified below, one obtains the notion of *Wigner process.*

To each pair of natural integers (m_h, m_h), we associate the closed interval:

$$[m_h', m_h] := \{x \in \mathbf{N} : m_h' \leqq x \leqq m_h\}, \tag{5.1}$$

and we shall say that the two pairs (m_h', m_h), (m_k', m_k) are non-crossing if

$$[m_h', m_h] \cap [m_k', m_k] = \begin{cases} \emptyset \\ [m_h', m_h] \\ [m_k', m_k] \end{cases}. \tag{5.2}$$

Condition (5.2) means that either the two intervals associated to the pairs do not intersect, or one of them is contained in the other one.

Definition 5.1. *Let n be a natural integer. A non-crossing pair partition of the set* $\{1, 2, \ldots, 2n\}$ *is a pair partition*

$$(m_1', m_1), \ldots, (m_n', m_n) \tag{5.3}$$

such that any two pairs (m_h', m_h), (m_k', m_k) *of the partition are non-crossing.*

Lemma 5.2. *Given n numbers*

$$m_1 < m_2 < \cdots < m_n$$

in the set $\{1, \ldots, 2n\}$. *If there exists a non-crossing pair partition*

$$(m_1', m_1), \ldots, (m_n', m_n)$$

of the set $\{1, \ldots, 2n\}$ *such that*

$$m_h' < m_h; \quad h = 1, \ldots, n, \tag{5.4}$$

then it is unique. Moreover, for each $h_0 = 1, \ldots, n$, *the number* m_{h_0}' *is uniquely determined by the condition*:

$$m_{h_0}' := \max\{x \in \{1, \ldots, 2n\} \backslash \{m_h\}_{h=1}^n : x < m_{h_0} \text{ and } |\{x+1, \ldots, m_{h_0} - 1\} \cap \{m_1, \ldots, m_n\}| = |\{x+1, \ldots, m_{h_0} - 1\} \cap (\{1, \ldots, 2n\} \backslash \{m_h\}_{h=1}^n)|\}, \tag{5.5}$$

where, $|\{\cdots\}|$ *denotes the cardinality of set* $\{\cdots\}$

Remark. By construction

$$\{m_h'\}_{h=1}^n = \{1, \ldots, 2n\} \backslash \{m_h\}_{h=1}^n, \tag{5.6}$$

and the sequence $\{m_h'\}_{h=1}^n$ is not necessarily increasing in h.

Proof. The proof can be done by the induction using the following facts:

- in the non-crossing pair partition (5.3), m_1' is surely equal to $m_1 - 1$;
- given (m_1', m_1) as above, $\{(m_1', m_1), \ldots, (m_n', m_n)\}$ is a non-crossing pair partition of $\{1, 2, \ldots, 2n\}$ if and only if $\{(m_2', m_2), \ldots, (m_n', m_n)\}$ is a non-crossing pair partition of $\{1, 2, \ldots, 2n\} \setminus \{m_1', m_1\}$.

– since, by the previous two arguments, given the (m_j), the pair (m_1', m_1) is uniquely determined, the partition (5.3) is unique if and only if the corresponding partition of the set $\{1,2,\dots,2n\} \setminus \{m_1', m_1\}$ is unique.

In the following we shall use the notation

$$n.c.p.p.\{1,\dots,2n\} \tag{5.7}$$

to denote the family of all non-crossing pair partitions of the set $\{1,\dots,2n\}$.

6. Limits of Matrix Elements of Arbitrary Products of Collective Operators

In order to calculate the limit (1.15), the knowledge of the limits of scalar products of collective number vectors is not sufficient: one has to identify the limit of matrix elements of arbitrary products of collective creation and annihilation operators.

The goal of the present section is to describe these limits. More precisely, we shall prove the following theorem:

Theorem 6.1. *For any natural integer n and with the notations*:

$$A_\lambda^0 := A_\lambda, \qquad A_\lambda^1 := A_\lambda^+ , \tag{6.1}$$

the limit

$$\lim_{\lambda\to 0} \left\langle \Phi \otimes \xi,\ \sum_{\varepsilon\in\{0,1\}^{2n}} A_\lambda^{\varepsilon(1)}(S_1, T_1, f_1) \cdots A_\lambda^{\varepsilon(2n)}(S_{2n}, T_{2n}, f_{2n}) \Phi \otimes \eta \right\rangle \tag{6.2}$$

exists and is equal to

$$\sum_{\{m_h', m_h\}_{h=1}^n \in n\,c\,p\,p\,\{1,\ ,2n\}} \prod_{h=1}^n \langle \chi_{[S_{m_h}, T_{m_h}]}, \chi_{[S_{m_h'}, T_{m_h'}]} \rangle_{L^2(\mathbf{R})}$$

$$\times \int_{-\infty}^{+\infty} du_1 \cdots \int_{-\infty}^{+\infty} du_n \int_{\mathbf{R}^{nd}} dk_1 \cdots dk_n \prod_{h=1}^n (S_{u_h} f_{m_h})(k_h) \bar{f}_{m_h'}(k_h)$$

$$\times \left\langle \xi, \prod_{h=1}^n e^{iu_h k_h \cdot p} \eta \right\rangle \cdot \exp\left(i \sum_{h=1}^{n-1} \sum_{r=h+1}^{n} u_h k_h \cdot k_r \chi_{(m_r', m_r)}(m_h) \right) . \tag{6.3}$$

In order to prove this theorem we must do some preparation. Without loss of generality we shall assume that $S_h = 0$ for all $h = 1,\dots,2n$.

Let us consider the matrix elements of arbitrary products of collective creation and annihilation operators (in contrast with (4.7) where only anti-Wick-ordered products were considered):

$$\langle \Phi \otimes \xi,\ A_\lambda(0, T_1, f_1) \cdots A_\lambda^+(0, T_1', f_1') \cdots A_\lambda^+(0, T_n', f_n'), \dots, A_\lambda(0, T_n, f_n) \Phi \otimes \eta \rangle . \tag{6.4}$$

Counting from left to right, let

$$1 \leqq m_1 < m_2 < \cdots < m_n \leqq 2n \tag{6.5}$$

denote the places where the creation operators appear. Since Φ is the vacuum vector, the expression (6.4) is clearly equal to zero if either $m_n < 2n$ or $m_1 = 1$.

Remark. The scalar product (6.4) is $\neq 0$ only if for any $r = 1, 2, \ldots, n$, the cardinality of $\{1, \ldots, 2r\} \backslash \{m_1, \ldots, m_n\}$ is less than or equal to $\max\{h = 1, \ldots, n;\ m_h \leqq 2r\}$. That is, if it happens that, for any $r = 1, \ldots, n$, in the first $2r$ operators in the operator product in (6.4), there are more creators than annihilators.

Using the definition (2.5) of A_λ and A_λ^+, (6.4) becomes

$$\lambda^{2n} \int_0^{T_1/\lambda^2} dt_1 \cdots \int_0^{T_{m_1}/\lambda^2} dt_{m_1} \cdots \int_0^{T_{2n}/\lambda^2} dt_{m_n} \int_{\mathbf{R}^{2nd}} d\vec{k}$$
$$\times \langle \xi, e^{-itk_1 \cdot p} e^{ik_1 \cdot q} \ldots e^{-ik_{m_1} q} e^{is_{m_1} k_{m_1} \cdot p} \ldots e^{-it_n \cdot k_n \cdot p} e^{ik_n \cdot q} \ldots e^{-ik_{m_n} \cdot q} e^{it_{m_n} k_{m_n} \cdot p} \eta \rangle$$
$$\times \prod_{h=1}^{n} (S_{t_{m_h}} f'_{m_h})(k_{m_h}) \cdot \prod_{\alpha \in \{1, \ ,2n\} \backslash \{m_1, \ ,m_n\}} \overline{(S_{t_\alpha} f_\alpha)}(k_\alpha)$$
$$\times \langle 0 \mid a_{k_1} \cdots a^+_{k_{m_1}} \cdots a_{k_n} \cdots a^+_{k'_{m_n}} \mid 0 \rangle \, . \tag{6.7}$$

It is clear that the vacuum expectation value in (6.7) is of the form

$$\sum_{\substack{(\overline{m}_1, \ ,\overline{m}_n) = \{1, \ ,2n\} \backslash \{m_h\}_{h=1}^n \\ \overline{m}_h < m_h, h=1, \ ,n}} \prod_{h=1}^{n} \delta_{k_{\overline{m}_h}, \ k_{m_h}}$$
$$\times \lambda^{2n} \int_0^{T_1/\lambda^2} dt_1 \cdots \int_0^{T_{2n}/\lambda^2} dt_{2n} \int_{\mathbf{R}^{nd}} d\vec{k} \sum_{\substack{(\overline{m}_1, \ ,\overline{m}_n) = \{1, \ ,2n\} \backslash \{m_n\}_{h=1}^n \\ \overline{m}_h < m_h, h=1, \ ,n}}$$
$$\times \langle \xi, e^{-it_{\overline{m}_1} k_1 \cdot p} e^{ik_1 \cdot q} \ldots e^{-im_1 \cdot q} e^{it_{m_1} k_1 \cdot p} e^{-it_{\overline{m}_n} k_n \cdot p} e^{ik_n \cdot q} \ldots e^{-ik_n \cdot q} e^{it_{m_n} k_n \cdot p} \eta \rangle$$
$$\times \prod_{h=1}^{n} \bar{f}_{\overline{m}_h}(k_h)(S_{t_{m_h} - t_{\overline{m}_h}} f_{m_h})(k_h) \, . \tag{6.8}$$

Lemma 6.2. *In the limit $\lambda \to 0$ the only terms of the summation* (6.8) *which give a non-zero contribution are those corresponding to those n-uples $(\overline{m}_1, \ldots, \overline{m}_n)$ which satisfy the equation*

$$\chi_{[\overline{m}_r+1, 2n-1]}(m_h) - \chi_{[\overline{m}_r+1, 2n-1]}(\overline{m}_h) + \chi_{[m_r+1, 2n-1]}(\overline{m}_h) - \chi_{[m_r, 2n-1]}(m_h) = 0 \, . \tag{6.9}$$

Proof. Remember that the indices $\overline{m}_j$ in (6.8) correspond to annihilators in (6.4) and are associated to exponential factors of the form

$$e^{-it_{\overline{m}_j} k_j \cdot p} e^{ik_j \cdot q} \tag{6.10}$$

in (6.8), while the indices m_j correspond to creators and are associated to exponential factors of the form

$$e^{-ik_j \cdot q} e^{it_{m_j} k_j \cdot p} \, . \tag{6.11}$$

Also in this case, as in the proof of Theorem 4.1, we shall permute the exponentials in the scalar product (6.8) in order to cancel the factors $\exp(\pm ik_j \cdot q)$. However in this case we shall employ a different grouping strategy, namely, instead of permuting each factor $\exp(\pm ik_j \cdot q)$ until it is cancelled by its conjugate factor, we bring all the factors $\exp(\pm ik_j \cdot q)$ to the right of all the factors $\exp(\pm t_{\alpha_j} k_j \cdot p)$ $(\alpha_j = m_j$ or $\overline{m}_j)$ with the exception of the factor $\exp(it_{m_n} k_n \cdot p)$, which remains on the extreme right.

The scalar phase factor which arises when all the permutations have been performed has the form $\exp(i\Sigma)$, where Σ is a real number which can be expressed as the sum of four different types of terms:

$$-\sum_{r=1}^{n-1} \sum_{h=r}^{n-1} t_{m_h} k_r \cdot k_h \,, \tag{6.12a}$$

$$\sum_{h=1}^{n-1} \sum_{r=1}^{n} \chi_{[m_h+1,2n-1]}(\overline{m}_r)\, t_{\overline{m}_r} k_h \cdot k_r \,, \tag{6.12b}$$

$$\sum_{r=1}^{n-1} \sum_{h=1}^{n} \chi_{[\overline{m}_h+1,2n-1]}(m_r)\, t_{m_r} k_h \cdot k_r \,, \tag{6.12c}$$

$$-\sum_{r=1}^{n} \sum_{h=1}^{n} \chi_{[\overline{m}_h+1,2n-1]}(\overline{m}_r)\, t_{\overline{m}_r} k_r \cdot k_h \,. \tag{6.12d}$$

The term (6.12a) arises from the commutation of the factors $\exp(-ik_r \cdot q)$, $(r = 1,\dots,n-1)$ with the factors $\exp(it_{m_h} k_h \cdot p)$, for $h = r,\dots,n-1$.

The term (6.12b) arises from the commutation of the factors $\exp(-ik_h \cdot q)$, $(h = 1,\dots,n)$ with the factors $\exp(it_{\overline{m}_r} k_r \cdot p)$ for $r = 1,\dots,n-1$. The characteristic function in this term is motivated by the fact that the term $\exp(-ik_h \cdot q)$ first appears on the right of $\exp(-it_{\overline{m}_h} k_h \cdot p)$ because of the convention on the indices of the collective vectors (cf. the end of Sect. 2) and the fact that $\overline{m}_h < m_h$, this implies that among all the factors $\exp -it_{\overline{m}} \cdot p$, it will have to commute only with those for which

$$m_h < \overline{m}_r < 2n$$

(recall that the $\overline{m}_j$ are not increasing in general). Notice that, since $m_n = 2n$ the n^{th} term of the second sum in (6.12b) is zero.

The term (6.12c) arises from the commutation of the factors $\exp(ik_h \cdot q)$, $(h = 1,\dots,n)$ with the factors $\exp(it_{m_r} k_r \cdot p)$ for $r = 1,\dots,n-1$. The characteristic function in this term is motivated by the fact that the term $\exp(ik_h \cdot q)$ first appears on the right of $\exp(-it_{\overline{m}_h} k_h \cdot p)$ for the above mentioned convention. Therefore, among the factors $\exp(it_{m_r} k_r \cdot p)$, it will have to commute only with those for which

$$\overline{m}_h < m_r < 2n \,.$$

Finally the term (6.12d) arises from the commutation of the factors $\exp(ik_h \cdot q)$, $(h = 1,\dots,n)$ with the factors $\exp(-it_{\overline{m}_r} k_r \cdot p)$ for $r = 1,\dots,n$. The characteristic function in this term has the same origin as in the previous one.

Now notice that the sum of the four expressions (6.12a,b,c,d) can be rewritten as

$$\sum_{1 \leq h,r \leq n} [\chi_{[m_r+1,2n-1]}(\overline{m}_h) t_{\overline{m}_h} + \chi_{[\overline{m}_r+1,2n-1]}(m_h) t_{m_h} - \chi_{[m_r,2n-1]}(m_h) t_{m_h} - \chi_{[\overline{m}_r+1,2n-1]}(\overline{m}_h) t_{\overline{m}_h}] k_h \cdot k_r \,, \tag{6.13}$$

where, in (6.13), the first term in square brackets corresponds to (6.12b), the second one to (6.12c), the third one to (6.12a) (recall that the map $j \mapsto m_j$ is strictly increasing) and the fourth one to (6.12d).

By adding and subtracting into the square bracket of (6.13), the terms

$$\pm t_{\overline{m}_h}\chi_{[\overline{m}_r+1,2n-1]}(m_h)k_h\cdot k_r\,; \qquad \pm t_{\overline{m}_h}\chi_{[\overline{m}_r+1,2n-1]}(\overline{m}_h)k_h\cdot k_r,$$

the double summation (6.13) can be rewritten as

$$\sum_{1\leqq h,r\leqq n} k_h\cdot k_r(t_{m_h}-t_{\overline{m}_h})\chi_{[\overline{m}_r+1,2n-1]}(m_h)$$
$$+\sum_{1\leqq h,r\leqq n} t_{\overline{m}_h}k_h\cdot k_r(\chi_{[\overline{m}_r+1,2n-1]}(m_h)-\chi_{[\overline{m}_r+1,2n-1]}(\overline{m}_h))$$
$$-\sum_{1\leqq h,r\leqq n} (t_{m_h}-t_{\overline{m}_h})k_h\cdot k_r\chi_{[m_r,2n-1]}(m_h)$$
$$+\sum_{1\leqq h,r\leqq n} t_{\overline{m}_h}k_n\cdot k_r(\chi_{[m_r+1,2n-1]}(\overline{m}_h)-\chi_{[m_r,2n-1]}(m_h))\,.$$

By the same argument used in the proof of Theorem 4.1, we conclude that, in the limit $\lambda\to 0$, only those terms of the summation (6.8) can survive, for which

$$\sum_{1\leqq h,r\leqq n} t_{\overline{m}_h}k_h\cdot k_r[\chi_{[\overline{m}_r+1,2n-1]}(m_h)-\chi_{[\overline{m}_r+1,2n-1]}(\overline{m}_h)$$
$$+\chi_{[m_r+1,2n-1]}(\overline{m}_h)-\chi_{[m_r,2n-1]}(m_h)]=0$$

for almost all $(t_1,\dots,t_{2n})\in\mathbf{R}^{2n}$ and almost all $(k_1,\dots,k_n)\in\mathbf{R}^{nd}$ and this condition is equivalent to Eq. (6.9).

Lemma 6.3. *For any natural number n and any choice of* $1<m_1<\cdots<m_n=2n$ *in the set* $\{1,\dots,2n\}$, *Eq.* (6.9), *in the unknowns* $\overline{m}_1,\dots,\overline{m}_n$ *has a unique solution satisfying*

$$\overline{m}_h<m_h;\quad h=1,\dots,n\,. \tag{6.14}$$

Moreover the solution $(\overline{m}_1,\dots,\overline{m}_h)$ *is characterized by the property that*

$$(\overline{m}_1,m_1),\ (\overline{m}_2,m_2),\dots,(\overline{m}_n,m_n) \tag{6.15}$$

is the unique non-crossing partition of the set $\{1,\dots,2n\}$ *associated to the set* $\{m_h\}_{h=1}^n$ *in the sense specified by Lemma* 5.1.

Proof. We distinguish two cases:

1) $$\chi_{[\overline{m}_r+1,2n-1]}(m_h)=0\,.$$

In this case, the non-crossing property implies that one must also have

$$\chi_{[m_r,2n-1]}(m_h)=0\,,$$

because $\overline{m}_r<m_r$. Hence

$$\chi_{[m_r+1,2n-1]}(\overline{m}_h)=\chi_{[\overline{m}_r+1,2n-1]}(\overline{m}_h)\,,$$

and therefore

$$\overline{m}_h>\overline{m}_r\Leftrightarrow\overline{m}_h>m_r\,.$$

2) $$\chi_{[\overline{m}_r+1,2n-1]}(m_h)=1.$$

In this case, if $\chi_{[m_r,2n-1]}(m_h)=1$, by the same arguments as above, we have

$$\overline{m}_h>\overline{m}_r\Leftrightarrow\overline{m}_h>m_r\qquad\forall\,h,r=1,\dots,n\,.$$

Notice that this is equivalent to saying that

$$m_h > \overline{m}_r \Leftrightarrow \overline{m}_h > \overline{m}_r \qquad \forall\, h, r = 1, \ldots, n$$

and clearly this can be reformulated as

$$m_h > \overline{m}_r \Rightarrow \overline{m}_h > \overline{m}_r \qquad \forall\, h, r = 1, \ldots, n\,.$$

Now if $\chi_{[m_r, 2n-1]}(m_h) = 0$, one must have

$$\chi_{[\overline{m}_r+1, 2n-1]}(\overline{m}_h) = 1\,.$$

That is, in any case, $\chi_{[\overline{m}_r+1, 2n-1]}(m_h) = 1$ implies that

$$\chi_{[\overline{m}_r+1, 2n-1]}(\overline{m}_h) = 1\,,$$

which is equivalent to

$$m_h > \overline{m}_r \Rightarrow \overline{m}_h > \overline{m}_r\,.$$

Summing up, we have that either $[\overline{m}_h, m_h] \cap [\overline{m}_r, m_r] = \emptyset$ or $[\overline{m}_h, m_h] \subseteq [\overline{m}_r, m_r]$. Hence the partition (6.14) is non-crossing.

7. Why Hilbert Modules?

In order to interpret the expression (6.3), obtained in Theorem (6.1), for the limit of matrix elements of arbitrary products of collective creators and annihilators, we describe in this section the Hilbert module on which the limits of the collective creation and annihilation processes live. By analogy with the Fock and the free Brownian motions, we shall refer to this process as *the interacting free module Brownian motion.* The first example of a quantum noise living on a (nontrivial) Hilbert module was considered in [3], the theory of stochastic integration and stochastic differential equations on Hilbert modules was developed in [11–13].

For each f in Schwartz space, define

$$\tilde{f}(t) := \int_{\mathbf{R}^d} (S_t f)(k) e^{-ikq} e^{ikp} dk \tag{7.1}$$

then, $\tilde{f}$ is a map from $\mathbf{R}$ to the bounded operators on $L^2(\mathbf{R}^d)$. Denote by $\mathscr{P}$ the W^*- algebra generated by $\{e^{ikp}; k \in \mathbf{R}^d\}$ (i.e. *the momentum W^*-algebra* of the system) and by $\mathscr{F}$ the $\mathscr{P}$-right-linear span of $\{\tilde{f}; f \in \mathscr{K}\}$. Then $\mathscr{F}$ is a $\mathscr{P}$ - right module and therefore the algebraic tensor product between $L^2(\mathbf{R})$ and $\mathscr{F}$ (denoted by $L^2(\mathbf{R}) \odot \mathscr{F}$) is a $\mathscr{P}$ (in fact $1 \odot \mathscr{P}$)-right module. On the $\mathscr{P}$-right module $L^2(\mathbf{R}) \odot \mathscr{F}$, we introduce a $\mathscr{P}$-right bi-linear, $\mathscr{P}$ valued form:

$$(\,\cdot\mid\cdot\,) \;: L^2(\mathbf{R}) \odot \mathscr{F} \times L^2(\mathbf{R}) \odot \mathscr{F} \longrightarrow \mathscr{P} \tag{7.2}$$

by

$$(\alpha \otimes \tilde{f} \mid \beta \otimes \tilde{g}) := \langle \alpha, \beta \rangle_{L^2(\mathbf{R})} \cdot \int_{\mathbf{R}} du \int_{\mathbf{R}^d} dk e^{-iukp} \bar{f}(k)(S_u g)(k)\,. \tag{7.3}$$

In the following we shall identify f with its equivalence class with respect to the $\mathscr{P}$-valued inner product defined by (7.3). Thus $L^2(\mathbf{R}) \odot \mathscr{F}$ becomes a $\mathscr{P}$-pre-Hilbert module. The positivity of the right-hand side of (7.3) is not obvious by inspection but it follows from Theorem (4.1). For each $n \in N$, we denote by $(L^2(\mathbf{R}) \odot \mathscr{F})^{\odot n}$ the algebraic tensor product of n-copies of $L^2(\mathbf{R}) \odot \mathscr{F}$ and define the $\mathscr{P}$-right sesqui-linear, $\mathscr{P}$ valued form:

$$(\cdot \mid \cdot) : (L^2(\mathbf{R}) \odot \mathscr{F})^{\odot n} \times (L^2(\mathbf{R}) \odot \mathscr{F})^{\odot n} \longrightarrow \mathscr{P} \tag{7.4}$$

by

$$\begin{aligned}((\alpha_1 \otimes \tilde{f}_1) \otimes \cdots \otimes (\alpha_n \otimes \tilde{f}_n) \mid (\beta_1 \otimes \tilde{g}_1) \otimes \cdots \otimes (\beta_n \otimes \tilde{g}_n))\\ := \prod_{h=1}^{n} \langle \alpha_h, \beta_h \rangle_{L^2(\mathbf{R})} \cdot \int_{\mathbf{R}^n}^{d} u_1 \cdots du_n \int_{\mathbf{R}^{nd}}^{d} k_1 \cdots dk_n \prod_{h=1}^{n} [e^{-iu_h k_h p} \bar{f}_h(k_h)(S_{u_h} g_h)(k_h)\\ \times \exp\left(i \sum_{1 \leqq r \leqq h \leqq n-1} u_r k_r k_{h+1} \right)\end{aligned} \tag{7.5}$$

and we still identify the $\tilde{f}_j$'s with their equivalence classes with respect to the equivalence relation stated before. Thus for each $n \in N$, with the $\mathscr{P}$-right bi-linear, $\mathscr{P}$ valued form given by (7.5), $(L^2(\mathbf{R}) \odot \mathscr{F})^{\odot n}$ becomes a $\mathscr{P}$-pre-Hilbert module and the notion $(L^2(\mathbf{R}) \odot \mathscr{F})^{\otimes n}$ will be used to denote it.

Since for each $n \in N$, $(L^2(\mathbf{R}) \otimes \mathscr{F})^{\odot n}$ is a $\mathscr{P}$-pre-Hilbert module, (again the positivity of (7.5) follows from Theorem 4.1), the direct sum $\mathbf{C} \oplus \bigoplus_{n=1}^{\infty} (L^2(\mathbf{R}) \odot \mathscr{F})^{\odot n}$ makes sense and will be denoted by $\Gamma(L^2(\mathbf{R}) \odot \mathscr{F})$ and called *the Fock module* over $L^2(\mathbf{R}) \odot \mathscr{F}$. In this pre-Hilbert module, the vector $\Psi := 1 \oplus 0 \oplus 0 \cdots$ is called the **vacuum vector**. One can easily show that

Lemma 7.1. *The number vector subset*

$$\Gamma := \{A^+(\alpha_1 \otimes \tilde{f}_1) \cdots A^+(\alpha_n \otimes \tilde{f}_n)\Phi;\ n \in N,\ \alpha_j \in L^2(\mathbf{R}), \tilde{f}_j \in \mathscr{F},\ j = 1, \ldots, n\} \tag{7.6}$$

is a $\mathscr{P}$-total subset of $\Gamma(L^2(\mathbf{R}) \odot \mathscr{F})$.

Definition 7.2. *For each element of $L^2(\mathbf{R}) \odot \mathscr{F}$, the creator with respect to this element, denoted by $A^+(\cdot)$, is defined on the $\mathscr{P}$-right linear span of Γ by $\mathscr{P}$-right linearity and*

$$\begin{aligned}A^+(\alpha \otimes \tilde{f})[(\alpha_1 \otimes \tilde{f}_1) \otimes \cdots \otimes (\alpha_n \odot \tilde{f}_n)\Psi]\\ := (\alpha \otimes \tilde{f}) \otimes (\alpha_1 \otimes \tilde{f}_1) \otimes \cdots \otimes (\alpha_n \otimes \tilde{f}_n)\Phi\,,\end{aligned} \tag{7.7}$$

where $n \in N$, α, $\alpha_j \in L^2(\mathbf{R})$, $\tilde{f}, \tilde{f}_j \in \mathscr{F}$, $j = 1, \ldots, n$. The formal adjoint is called annihilator and denoted by $A(\cdot)$.

Remark. In general, $A^+(\alpha_1 \otimes \tilde{f}_1)A^+(\alpha_2 \otimes \tilde{f}_2)$ is not necessarily equal to $A^+(\alpha_2 \otimes \tilde{f}_2)A^+(\alpha_1 \otimes \tilde{f}_1)$.

Definition 7.3. *For each $k_0 \in \mathbf{R}^d$, the left action of $e^{ik_0 p_0}$ on $\Gamma(L^2(\mathbf{R}) \odot \mathscr{F}) \otimes \mathscr{H}_0$ is defined by*

$$e^{ik_0 p_0} A^+(\alpha \otimes \tilde{f}) := A^+(\alpha \otimes \widetilde{e^{ik_0 p_1} f})e^{ik_0 p_0} \tag{7.8}$$

for all $\alpha \in L^2(\mathbf{R})$, $\tilde{f} \in \mathscr{F}$, where p_0 is the momentum on $\mathscr{H}_0$ and p_1 is the momentum on the one-particle space of $\mathscr{F}$.

It is easy to show that if * denotes the adjoint with respect to the $\mathscr{P}$-valued scalar product given by (7.3), then $(e^{ik_0 p_0})^* = e^{-ik_0 p_0}$

Theorem 7.4. *For each $n \in N$, $\alpha, \alpha_j \in L^2(\mathbf{R}), \tilde{f}, \tilde{f}_j \in \mathscr{F}$ $(j = 1, \ldots, n)$,*

$$A(\alpha \otimes \tilde{f})A^+(\alpha_1 \otimes \tilde{f}_1) \cdots A^+(\alpha_n \otimes \tilde{f}_n)\Psi \tag{7.9}$$

is equal to

$$(\alpha \otimes \tilde{f} \mid \alpha_1 \otimes \tilde{f}_1)A^+(\alpha_2 \otimes \tilde{f}_2) \cdots A^+(\alpha_n \otimes \tilde{f}_n)\Psi \, . \tag{7.10}$$

Proof. For each $n \in N$, $\alpha, \alpha_j \in L^2(\mathbf{R})$, $\tilde{f}, \tilde{f}_j \in \mathscr{F}$ $(j = 1, \ldots, n)$, $\beta_k \in L^2(\mathbf{R}), \tilde{g}_k \in \mathscr{F}$ $(k = 0, 1, \ldots, n)$, since $A = (A^+)^+$, we have

$$\begin{aligned}
&\langle A^+(\alpha_1 \otimes \tilde{f}_1) \cdots A^+(\alpha_n \otimes \tilde{f}_n)\Phi, A(\alpha \otimes \tilde{f})A^+(\beta_0 \otimes \tilde{g}_0)A^+(\beta_1 \otimes \tilde{g}_1) \cdots A^+(\beta_n \otimes \tilde{g}_n)\Phi\rangle \\
&\quad = \langle A^+(\alpha \otimes \tilde{f})A^+(\alpha_1 \otimes \tilde{f}_1) \cdots A^+(\alpha_n \otimes \tilde{f}_n)\Phi, \\
&\qquad A^+(\beta_0 \otimes \tilde{g}_0)A^+(\beta_1 \otimes \tilde{g}_1) \cdots A^+(\beta_n \otimes \tilde{g}_n)\Phi\rangle \, .
\end{aligned} \tag{7.11}$$

Denote $\alpha := \alpha_0, \tilde{f} := \tilde{f}_0$. By Definition (7.11) is equal to

$$\begin{aligned}
&\prod_{h=0}^{n} \langle \alpha_h, \, \beta_h\rangle \cdot \int_{-\infty}^{+\infty} du_0 \cdots \int_{-\infty}^{+\infty} du_n \int_{\mathbf{R}^{(n+1)d}} dk_0 \cdots dk_n \\
&\quad \times \prod_{h=0}^{n} e^{iu_h k_h \cdot p} \cdot \prod_{h=0}^{n} (S_{u_h} g_h)(k_h)\bar{f}_h(k_h) \cdot \exp\left(i \sum_{r=0}^{n-1} \sum_{h=r}^{n-1} u_r k_r \cdot k_{h+1}\right) .
\end{aligned} \tag{7.12}$$

Rewriting the expression (7.12) in the form:

$$\begin{aligned}
&\prod_{h=1}^{n} \langle \alpha_h, \, \beta_h\rangle \cdot \int_{-\infty}^{+\infty} du_1 \cdots \int_{-\infty}^{+\infty} du_n \int_{\mathbf{R}^{nd}} dk_1 \cdots dk_n \\
&\quad \times \prod_{h=1}^{n} e^{iu_h k_h \cdot p} \cdot \prod_{h=1}^{n} (S_{u_h} g_h)(k_h)\bar{f}_h(k_h) \cdot \exp\left(i \sum_{r=1}^{n-1} \sum_{h=r}^{n-1} u_r k_r \cdot k_{h+1}\right) \\
&\quad \times \langle \alpha, \, \beta_0\rangle \cdot \int_{-\infty}^{+\infty} du_0 \int_{\mathbf{R}^d} dk_0 \; e^{iu_0 k_0 \cdot p} \cdot (S_{u_0} g_0)(k_0)\bar{f}(k_0) \cdot \exp\left(i \sum_{h=1}^{n} u_0 k_0 \cdot k_h\right) ,
\end{aligned} \tag{7.13}$$

one finishes the proof.

Now let us compute the matrix elements of arbitrary products of creation and annihilation operators on the limit Hilbert module, which we interpret as limit noise space. For simplicity we shall not distinguish f from $\tilde{f}$. What we must compute is the following object:

$$\langle \Psi, A^{\varepsilon(1)}(\alpha_1 \otimes f_1) \cdots A^{\varepsilon(2n)}(\alpha_{2n} \otimes f_{2n})\Psi\rangle \, , \tag{7.14}$$

where, $n \in N$, $\alpha_j \in L^2(\mathbf{R}), f_j \in \mathscr{F}$ $(j = 1, \ldots, 2n)$, $\varepsilon \in \{0, 1\}^{2n}$ and

$$A^0 := A, \qquad A^1 := A^+ \, .$$

It is clearly sufficient to compute (7.14) only in the case

$$\varepsilon(1) = 0, \qquad \varepsilon(2n) = 1 \, . \tag{7.15}$$

Lemma 7.5. *Equation* (7.14) *is not equal to zero only if*

$$\sum_{h=1}^{2n} \varepsilon(h) = n\,. \tag{7.16}$$

Proof. The lemma can be easily verified in the case $n = 1$. Suppose by induction that $\sum_{h=1}^{2n} \varepsilon(h) \neq n$ implies that (7.14) is equal to zero and consider

$$\langle \Psi, A^{\varepsilon(1)}(\alpha_1 \otimes f_1) \cdots A^{\varepsilon(2(n+1))}(\alpha_{n+1} \otimes f_{2(n+1)})\Psi\rangle\,. \tag{7.17}$$

Denote

$$h := \min\{x \in \{1,\dots,2n\};\ \varepsilon(x) = 1\}\,, \tag{7.18}$$

the position where the first creator is. By Theorem (7.4), (7.17) is equal to

$$\langle \Phi, A^{\varepsilon(1)}(\alpha_1 \otimes f_1) \cdots A^{\varepsilon(h-2)}(\alpha_{h-2} \otimes f_{h-2})(\alpha_{h-1} \otimes f_{h-1} \mid \alpha_h \odot f_h)$$

$$A^{\varepsilon(h+1)}(\alpha_{h+1} \otimes f_{h+1}) \cdots A^{\varepsilon(2(n+1))}(\alpha_{n+1} \otimes f_{2(n+1)})\Phi\rangle\,. \tag{7.19}$$

Now by applying Lemma (7.3), one can move the inner product $(\alpha_{h-1} \otimes f_{h-1} \mid \alpha_h \otimes f_h)$ in (7.19) out from the inner product $\langle \Phi, \dots \Phi\rangle$. Since in order to produce this inner product we have used one creator and one annihilator, it follows that

$$\sum_{r=1}^{2(n+1)} \varepsilon(r) \neq n+1 \Longleftrightarrow \sum_{1 \leqq r \leqq 2(n+1),\ r \notin \{h-1,h\}} \varepsilon(r) \neq n\,. \tag{7.20}$$

By the induction assumption we complete the proof.

The same technique can be used to prove the following

Theorem 7.6. *Denote*

$$\{m_h\}_{h=1}^{n} := \{r \in \{1,\dots,2n\};\ \varepsilon(r) = 1\}, \quad 1 < m_1 < \cdots < m_n = 2n\,. \tag{7.21}$$

The inner product (7.14) is equal to zero if $\{m_h\}_{h=1}^{n}$ does not define a non-crossing pair partition of $\{1,\dots,2n\}$. If it does (and in this case we know from Lemma 5.2 that it is unique) (7.14) is equal to

$$\prod_{h=1}^{n} \langle \alpha_{m_h}, \alpha_{m'_h}\rangle_{L^2(\mathbf{R})}$$
$$\times \int_{-\infty}^{+\infty} du_1 \dots \int_{-\infty}^{+\infty} du_n \int_{\mathbf{R}^{nd}} dk_1 \dots dk_n \prod_{h=1}^{n} (S_{u_h} f_{m_h})(k_h) \bar{f}_{m'_h}(k_h)$$
$$\times \left\langle \xi, \prod_{h=1}^{n} e^{iu_h k_h \cdot p} \eta \right\rangle \cdot \exp\left(i \sum_{h=1}^{n-1} \sum_{r=h+1}^{n} u_h k_h \cdot k_r \chi_{(m'_r, m_r)}(m_h) \right), \tag{7.22}$$

where, $\{m'_h, m_h\}_{h=1}^{n}$ is the unique pair partition of $\{1,\dots,2n\}$ determined by $\{m_h\}_{h=1}^{n}$.

8. The Wigner Semicircle Law

This section is devoted to describe how the Wigner semicircle law arises from the above considerations.

It is well known that the distinctive characteristic of the Wigner semicircle law is the role of the non-crossing pair partition, in the expression of its momenta. But in our case, the situation becomes more complicated since the inner product of the *n-particle space* is not the product of n copies of the inner product of the *one-particle space*. Thus even if we obtain only non-crossing pair partitions, in the sum each of them is weighted by a factor depending on the partition. We shall see that the Wigner semicircle law corresponds to the case in which these weighting factors are put equal to zero.

In order to evidentiate the above mentioned connection between the vacuum distribution of the field operator and the Wigner semi-circle law, we introduce the probability spaces $(\Omega_n, \mathcal{A}_n, \mathbf{P}_n)$, $n \in N$:

- $\Omega_n := \{\{m'_h, m_h\}_{h=1}^n$: non-crossing pair partition of $\{1, \ldots, 2n\}\}$;
- $\omega_{m',m} := \{m'_h, m_h\}_{h=1}^n \in \Omega_n$;
- $\mathcal{A}_n$ is the discrete σ-algebra on Ω_n;
- $\mathbf{P}_n \ : \ \Omega_n \longrightarrow \ [0,1]$ is the probability defined by

$$\mathbf{P}_n(\omega_{m',m}) := \frac{1}{\mid \Omega_n \mid} . \tag{8.1}$$

For each $1 \leqq h, r \leqq n$, define a random variable:

$$X_{h,r}(\omega_{m',m}) := \begin{cases} 0, & \text{if } h \geqq r, \\ \chi_{(m'_r, m_r)}(m_h), & \text{if } h < r . \end{cases} \tag{8.2}$$

Then

Lemma 8.1. *For each non-crossing pair partition* $\{m'_h, m_h\}_{h=1}^n$,

$$\sum_{\{m'_h, m_h\}_{h=1}^n \in \Omega_n} \{1, \ldots, 2n\} \exp\left(\frac{i}{2}\sum_{h=1}^{n-1}\sum_{r=h+1}^{n} u_h k_h \cdot k_r \chi_{(m'_r, m_r)}(m_h)\right)$$
$$= \mid \Omega_n \mid_n \cdot \mathbf{E}_n \exp\left(i\sum_{h=1}^{n-1}\sum_{r=h+1}^{n} u_h k_h \cdot k_r X_{h,r}\right) , \tag{8.3}$$

where $\mathbf{E}_n$ denotes expectation with respect to the random variables $X_{h,r}$, defined by (8.2), with respect to the probability measure $\mathbf{P}_n$ and $\mid \Omega_n \mid_n$ denotes the cardinality of the set Ω_n of all the non-crossing pair partitions of $\{1, 2, \ldots, \ 2n\}$.

For each $n \in N$, $\alpha \in L^2(\mathbf{R})$, $f \in K$, define the field operator by $B_{\alpha,f} = A(\alpha \otimes \tilde{f}) + A^+(\alpha \otimes \tilde{f})$. Then

$$\langle \Phi, B_{\alpha,f}^{2n+1} \Phi \rangle = 0 . \tag{8.4}$$

Moreover, Theorem 7.6 shows that for each $\xi \in L^2(\mathbf{R}^d)$,

$$\langle \xi, \langle \Phi, B_{\alpha,f}^{2n} \Phi \rangle \xi \rangle = \sum_{\{m'_h, m_h\}_{h=1}^n \in \Omega_n} \|\alpha\|_{L^2(\mathbf{R})}^{2n} \int_{-\infty}^{+\infty} du_1 \ldots \int_{-\infty}^{+\infty} du_n$$
$$\times \int_{\mathbf{R}^{nd}} dk_1 \cdots dk_n \int_{\mathbf{R}^d} d\mu_\xi(k) \prod_{h=1}^{n} (S_{u_h} f)(k_h) \bar{f}(k_h) \cdot e^{iu_h k_h \cdot k}$$
$$\times \exp\left(i\sum_{h=1}^{n-1}\sum_{r=h+1}^{n} u_h k_h \cdot k_r \chi_{(m'_r, m_r)}(m_h)\right) , \tag{8.5}$$

where μ_ξ is the spectral measure of p with respect to the fixed vector ξ.

By applying Lemma 8.1, we have that

$$\langle \Phi \otimes \xi, B^{2n}_{\alpha,f} \Phi \otimes \xi \rangle = | \Omega_n |_n \|\alpha\|^{2n}_{L^2(\mathbf{R})} \int_{-\infty}^{+\infty} du_1 \dots \int_{-\infty}^{+\infty} du_n$$

$$\times \int_{\mathbf{R}^{nd}} dk_1 \cdots dk_n \int_{\mathbf{R}^d} d\mu_\xi(k) \prod_{h=1}^{n} (S_{u_h} f)(k_h) \bar{f}(k_h) \cdot e^{iu_h k_h \cdot k}$$

$$\times \mathbf{E}_n \exp\left(i \sum_{h=1}^{n-1} \sum_{r=h+1}^{n} u_h k_h \cdot k_r X_{h,r} \right) \tag{8.6}$$

and in the following the right-hand side of (8.6) will be denoted by

$$| \Omega_n |_n \|\alpha\|^{2n}_{L^2(\mathbf{R})} \int_{\mathbf{R}^d} d\mu_\xi(k) \, M^f_n(k) \, . \tag{8.7}$$

Lemma 8.2. *For each $n \in N$, the function $M^f_n(\cdot)$ defined by* (8.6), (8.7), *has the following properties*:

i) $M^f_n(\cdot) \geqq 0$ *and continuous*;
ii) $M^f_n(\cdot)$ *satisfies the bound, uniform in* k:

$$M^f_n(k) \leqq \left[\int_{\mathbf{R}} du \int_{\mathbf{R}^d} dx \, | \bar{f}(x)(S_u f)(x) | \right]^n ; \quad \forall k \in \mathbf{R}^d \, . \tag{8.8}$$

Notice that the vacuum odd moments of $B_{\alpha,f}$ are zero and, if each factor $M^f_n(k)$ were the $2n^{\text{th}}$ power of some function $c_f(k)$, independent on n, then the expression (8.6) would be the moment of order $2n$ of a random variable with distribution given by a convex combination of Wigner semi-circle laws with parameter

$$c_f(k)\|\alpha\|_{L^2(\mathbf{R})}$$

and mixing measure given by the spectral measure of the momentum operator in the state ξ. This is not the case because of the interaction term in (8.6), i.e. the factor under E_n-expectation. Neglecting this term, i.e. putting $X_{h,r} = 0$, $\forall h,r$, the right-hand side of (8.6) reduces to

$$| \Omega_n |_n \|\alpha\|^{2n}_{L^2(\mathbf{R})} \int_{\mathbf{R}^d} d\mu_\xi(k) [(f \mid f)(k)]^n \, , \tag{8.10}$$

where

$$(f \mid f)(k) := \int_{\mathbf{R}} du \int_{\mathbf{R}^d} dy (S_u f)(y) \bar{f}(y) e^{iuy \cdot k} \, , \tag{8.11}$$

which is precisely of the type discussed above with

$$c_f(k) = (f \mid f)(k) \, .$$

In this sense we have claimed in the introduction that the vacuum distribution of the limit field operator is a nonlinear modification of (a convex combination of) Wigner semi-circle laws.

9. The Limit Stochastic Process

Up to now we have discussed the convergence, in the sense of mixed moments, of the collective creation and annihilation processes to a new type of quantum noise. The results proved in the previous section, combined with techniques now standard in the stochastic limit of quantum theory, allow to deduce the explicit form of the stochastic equation for the limit of the time-rescaled wave operator. The full proof, which is unfortunately rather long, shall be published elsewhere [19]. We state here however the final result because the explicit form of the equation is particularly simple and easy to use.

Theorem 9.1. *For each* $t \geqq 0$, $\{S_h, T_h\}_{h=1}^{N}$, $\{S_h', T_h'\}_{h=1}^{N'} \subset \mathbf{R}$, $\{f_h\}_{h=1}^{N}$, $\{f_h'\}_{h=1}^{N'} \subset K$ *and* $\xi, \eta \in L^2(\mathbf{R}^d)$, *the limit*

$$\left\langle \prod_{h=1}^{N} A_\lambda^+(S_h, T_h, f_h)\Phi \otimes \xi, U_{t/\lambda^2} \prod_{h=1}^{N'} A_\lambda^+(S_h', T_h', f_h')\Phi \otimes \eta \right\rangle \tag{9.1}$$

exists and is equal to the solution of the quantum stochastic differential equation with respect to the free module Brownian Motion:

$$U(t) = 1 + \int_0^t (dA_s^+(g)(-ip) - (-ip)^+ dA_s(g) - (-ip)^+(g \mid g)_-(-ip)ds)U(s) \tag{9.2}$$

on the full Fock $\mathscr{P}$*-module described in Sect.* 7, *where the half-inner product* $(\cdot \mid \cdot)_-$ *is defined by*

$$(f \mid g)_- := \int_{-\infty}^{0} dt \int_{\mathbf{R}^d} dk e^{-itkp} \bar{f}(k)(S_t g)(k)\,. \tag{9.3}$$

The proof that the solution of the quantum stochastic differential equation (9.2) is effectively a unitary operator and in fact the very meaning of this equation, depends on the theory of the **free stochastic calculus over a Hilbert module**, which has been recently developed in [13].

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