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THE WIGNER TRANSFORM AND SOME EXACT PROPERTIES OF LINEAR OPERATORS*

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Abstract. The Wigner transform of an integral kernel on the full line generalizes the Fourier transform of a translation kernel. The eigenvalue spectra of Hermitian kernels are related to the topographic features of their Wigner transforms. Two kernels whose Wigner transforms are equivalent under the unimodular affine group have the same spectrum of eigenvalues and have eigenfunctions related by an explicit linear transformation. Any kernel whose Wigner transform has concentric ellipses as contour lines, yields an eigenvalue problem which may be solved exactly.

1. Introduction. The integral kernel $K\{x, y\}$ which defines the functional linear transformation

$$\int_{-\infty}^{\infty} dy\, K\{x, y\} f(y) = g(x) \quad \text{or} \quad Kf = g \tag{1.1}$$

on the full line may be reexpressed as

$$K\{x, y\} = K\left(x-y, \frac{x+y}{2}\right). \tag{1.2}$$

If K should prove independent of its second argument $(x+y)/2$, then (1.1) may be reduced to an elementary form by Fourier transformation. More generally, if the dependence of K upon $(x+y)/2$ is *slow* then Fourier transformation upon only the fast variable $u = x - y$

$$\tilde{K}(p, q) = \int du \exp(-ipu) K(u, q) \tag{1.3}$$

leads to a very detailed approximate description of the structure of K as a linear operator, through the use of a two-scale analysis [1], [2] (this issue, pp. 356–377). The transformation (1.2) followed by (1.3) was used by Wigner [3] in another context (Wigner states that the transform was "found by L. Szilard and the present author some years ago for another purpose") and will be called the Wigner transformation, while $\tilde{K}$ will be called the Wigner transform of $K\{x, y\}$, and the (p, q)-plane will be called the Wigner plane. By (1.2), if $K\{x, y\}$ is Hermitian then $K(u, q)$ undergoes complex conjugation if u is replaced by $-u$, and it follows from (1.3) that $\tilde{K}(p, q)$ is real. If (1.2) and (1.3) are merged, the Wigner transformation becomes

$$WK = \tilde{K}(p, q) = \int du\, K\left\{q + \frac{u}{2}, q - \frac{u}{2}\right\} \exp(-ipu), \tag{1.4}$$

whose inverse is

$$W^{-1}\tilde{K} = K\{x, y\} = \frac{1}{2\pi} \int dp\, \tilde{K}\left(p, \frac{x+y}{2}\right) \exp[ip(x-y)]. \tag{1.5}$$

Thus any real $\tilde{K}$, for which (1.5) is integrable in some sense, yields a Hermitian kernel.

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In (1.1) we allow generalized functions which are equivalent to differential (or more generally pseudo-differential) operators, which still will yield a well-defined Wigner transform from (1.4); for example, if K represents a Sturm–Liouville operator, the two-scale analysis referred to gives the familiar results of the WKB procedure [1].

In particular, the two-scale analysis of the Wigner transform yields a generic asymptotic result for the eigenvalues of kernels whose dependence upon $(x+y)/2$ is slow: Asymptotically a particular value of the Wigner transform, constant on a contour line which encloses an area $(2n+1)\pi$ on the Wigner plane, will be an eigenvalue of the kernel. Thus asymptotically λ_n is an eigenvalue if the closed curve (referred to as a λ-curve)

$$\tilde{K}(p,q)=\lambda_n \text{ encloses area } \mathscr{A}(\lambda_n)=(2n+1)\pi. \tag{1.6}$$

Relation (1.6), which we term the "area rule", was demonstrated in [1]. Under wide circumstances the area rule remains valid if instead of slowness we consider $n\uparrow\infty$; examples of this will be shown in the present paper.

Consider the following mapping of the Wigner plane:

$$\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} P_0 \\ Q_0 \end{pmatrix}+M\begin{pmatrix} p' \\ q' \end{pmatrix}, \qquad \det M=1. \tag{1.7}$$

Such a transformation, which carries straight lines to straight lines and triangles to triangles with the same area, we will call *unimodular affine*. When a member of the unimodular affine group of transformations acts upon the arguments of a Wigner transform $\tilde{K}(p,q)$, a new kernel transform results, $\tilde{K}'=\tilde{K}'(p',q')$, for which the area rule (1.6) yields the same spectrum of approximate eigenvalues. We will show that $\tilde{K}'$ is the transform of a kernel which in fact has *exactly* the same eigenvalue spectrum as the kernel K. In addition the eigenfunctions of K and K' may be explicitly related.

Under this same group, ellipses transform to ellipses. The special nature of elliptical λ-curves is underlined by the following result, also proven here. If $\tilde{K}(p,q)=\lambda$ are a family of concentric ellipses then the eigenvalues and eigenfunctions are explicit.

2. Some elementary relations. If $A\{x,y\}$ and $B\{x,y\}$ are two Hermitian kernels, then their product trace (or natural inner product) is related to their Wigner transform by

$$\begin{aligned}\operatorname{Tr} AB=\int dx\,dy\,A\{x,y\}B\{y,x\}&=\frac{1}{2\pi}\int dp\,dq\,\tilde{A}(p,q)\tilde{B}(p,q)\\ &=\frac{1}{2\pi}\int d\boldsymbol{\xi}\,\tilde{A}(\boldsymbol{\xi})\tilde{B}(\boldsymbol{\xi}),\end{aligned} \tag{2.1}$$

where in the last form we use the 2-dimensional variable

$$\boldsymbol{\xi}=\begin{pmatrix} p \\ q \end{pmatrix}. \tag{2.2}$$

An informal proof is immediate if we substitute (1.5) for both A and B into (2.1) and recognize the Fourier representation of the δ-function.

The Wigner transform of the identity kernel is immediate from (1.3):

$$\text{If } A\{x,y\}=\delta(x-y) \text{ then } \tilde{A}(\boldsymbol{\xi})=1. \tag{2.3}$$

If one member of the operator product (2.1) is the identity, and the other is K, (2.3) gives

(2.4) $$\operatorname{Tr} K = \int dx\, K\{x, x\} = \frac{1}{2\pi}\int d\xi\, \tilde{K}(\xi)$$

(which may be divergent).

Commonly, a Hermitian kernel has a complete orthonormal set of eigenfunctions $\psi_n(x)$ which satisfy

(2.5) $$\int dy\, K\{x, y\}\psi_n(y) = \lambda_n \psi_n(x),$$

in which case K has the spectral representation

(2.6) $$K = \sum_n \lambda_n E_n, \quad \text{where } E_n\{x, y\} = \psi_n(x)\psi_n^*(y).$$

Under the operator product

(2.7) $$(AB)\{x, y\} = \int dz\, A\{x, z\}B\{z, y\},$$

the projection operators E_n satisfy

(2.8) $$E_n^2 = E_n, \qquad E_m E_n = 0 \quad \text{if } m \neq n.$$

From (2.6), (2.4),

(2.9) $$1 = \int dx\, E_n\{x, x\} = \frac{1}{2\pi}\int d\xi\, \tilde{E}_n(\xi),$$

while from (2.8), (2.1),

(2.10) $$\frac{1}{2\pi}\int d\xi\, \tilde{E}_m(\xi)\tilde{E}_n(\xi) = \delta_{mn},$$

where if $m \neq n$, $\delta_{mn} = 0$, and $\delta_{mm} = 1$.

By (2.6), (2.8), the eigenvalue equation (2.5) has its counterpart for projection operators

(2.11) $$KE_n = \lambda_n E_n.$$

In (2.11) we may regard the argument "y" as a constant in $E_n\{x, y\}$, so that E_n as a function of x solves the eigenvalue equation (2.5).

3. The image of operator multiplication on the Wigner plane. The operator multiplication law (2.7) above implies a corresponding bilinear rule upon transformation to the Wigner plane:

(3.1) $$\widetilde{AB} = \tilde{A} \otimes \tilde{B},$$

which may be evaluated by expressing A and B in (2.7) in terms of their Wigner transforms by (1.5) and then transforming the operator product by (1.4). The necessary algebra is simplified if we note that

(3.2) $$\tilde{A}(\xi) = \int d\xi_A\, \delta(\xi - \xi_A)\tilde{A}(\xi_A),$$

and similarly for $\tilde{B}(\xi)$, which reduces the job to evaluating the composition law for

$\delta(\xi-\xi_A)\otimes\delta(\xi-\xi_B)$. We easily calculate

(3.3) $$W^{-1}\delta(\xi-\xi_A)=\frac{\delta(q-q_A)}{2\pi}\exp(iup_A),$$

from which we calculate

(3.4) $$\begin{aligned}\delta(\xi-\xi_A)\otimes\delta(\xi-\xi_B)&=W((W^{-1}\delta(\xi-\xi_A))(W^{-1}\delta(\xi-\xi_B)))\\&=\frac{1}{\pi^2}\exp\{4i\Delta(\xi_A,\xi_B,\xi)\},\end{aligned}$$

where

(3.5) $$\Delta(\xi_A,\xi_B,\xi)=\frac{1}{2}\begin{vmatrix}q_A-q & q_B-q\\ p_A-p & p_B-p\end{vmatrix}$$

is the area on the Wigner plane contained within the triangle with vertices at ξ_A, ξ_B, ξ. It therefore follows that

(3.6) $$\tilde{A}\otimes\tilde{B}=\frac{1}{\pi^2}\int d\xi_A\,d\xi_B\,\tilde{A}(\xi_A)\tilde{B}(\xi_B)\exp\{4i\Delta(\xi_A,\xi_B,\xi)\}.$$

4. An invariance law for eigenvalues. Under unimodular affine transformation (1.7)

(4.1) $$\xi=\xi_0+\mathbf{M}\xi'=T(\xi'),$$

the area of any triangle, and in particular (3.5), is preserved. In addition, the Jacobian of the transformation (4.1) is unity. Thus if we write

(4.2) $$\tilde{A}(\xi)=\tilde{A}(T(\xi'))=\tilde{A}'(\xi')=(T\tilde{A})(\xi'),$$

and similarly for $\tilde{B}$, it follows that

(4.3) $$T(\tilde{A}\otimes\tilde{B})=\tilde{A}'\otimes\tilde{B}'.$$

Now, by (3.1), the eigenvalue equation (2.11) becomes

(4.4) $$\tilde{K}\otimes\tilde{E}_n=\lambda_n\tilde{E}_n.$$

Under unimodular affine transformation this becomes, by (4.3),

(4.5) $$\tilde{K}'\otimes\tilde{E}'_n=\lambda_n\tilde{E}'_n$$

with the same eigenvalue. Under inverse Wigner transformation, (4.5) yields

(4.6) $$K'E'_n=\lambda_nE'_n,$$

another kernel and another set of eigenfunctions but the same spectrum. Thus there are whole classes of operator kernels with identical eigenvalue spectra, whose Wigner transforms map to one another under unimodular affine transformation of the Wigner plane.

5. The unimodular affine action upon eigenfunctions. Evidently there is a relation between the eigenfunctions $\psi_n(x)$ of a given kernel K, and the eigenfunctions $\psi'_n(x')$ of the related kernel $K'\{x',y'\}$ whose Wigner transform is obtained from $\tilde{K}(\xi)$ by unimodular affine transformation on the Wigner plane. In the projective kernel $E'_n\{x',y'\}$ of (2.6) we may regard y' as a fixed parameter, whence E'_n regarded as a function of x' is proportional to $\psi'(x')$. Thus the transformation from $\psi(x)$ to $\psi'(x')$

may be evaluated through explicit calculation of

$$E'_n = W^{-1}TWE_n, \tag{5.1}$$

by (1.4), (4.2), (1.5). The calculation is straightforward and the result is most conveniently stated as two separate partial results for the translational and unimodular parts of T.

Case 1. *Translation.* If M in (1.7) is the identity then

$$p = p' + P_0, \qquad q = q' + Q_0. \tag{5.2}$$

From (1.5) we can virtually read off the result

$$\psi'(x') = e^{iP_0x}\psi(x' + Q_0). \tag{5.3}$$

Case 2. *Unimodular transformation.* $P_0 = 0$, $Q_0 = 0$ in (1.7) and

$$M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{with } \alpha\delta - \beta\gamma = 1. \tag{5.4}$$

In this case the two integrations contributed by W and W^{-1} in (5.1) may be separated, by a change of variable, into a product of functionally decoupled integrations upon the two factors of $E_n\{x, y\} = \psi_n(x)\psi_n^*(y)$, and we find

$$\psi'_n(x') = \frac{1}{\sqrt{2\pi\omega\gamma}} \int dx \exp\left[-\frac{i}{2}\left\{\frac{\delta}{\gamma}x'^2 - \frac{2}{\gamma}xx' + \frac{\alpha}{\gamma}x^2\right\}\right]\psi_n(x), \tag{5.5}$$

which is the integral transformation induced by the unimodular action on the Wigner plane. ω represents a constant of unit magnitude up to which (5.5) is undetermined. We note that both (5.3) and (5.5) are unitary transformations, a property guaranteed by (2.9), (4.6) and the area-preserving property of the transformation T.

If, in particular, the unimodular matrix M defines a rigid rotation of the Wigner plane

$$M = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}, \tag{5.6}$$

then (5.5) reduces to

$$\psi'(x') = \frac{1}{\sqrt{2\pi i \sin t}} \int dx \exp\left[\frac{i}{2}\left\{(\cot t)(x^2 + x'^2) - \frac{2xx'}{\sin t}\right\}\right]\psi(x). \tag{5.7}$$

This integral transformation may be recognized as the action of the Green's function for the time dependent normalized Schrodinger harmonic oscillator equation, upon the arbitrary initial function $\psi(x)$. The transformation (5.5) is far more general than (5.7); however, as unimodular matrices fall into three types, (i) "elliptic", (ii) "hyperbolic", (iii) "parabolic", depending on whether the eigenvalues of M are (i) complex conjugates on the unit circle, (ii) real and reciprocals, (iii) both unity. The form (5.6) is representative only of the elliptic type. In the limiting case where $\gamma = \beta = 0$ and $\delta = 1/\alpha$, it is easily shown that

$$K'\{x', y'\} = \frac{1}{\alpha}K\{\alpha x', \alpha y'\}, \tag{5.8}$$

whence $\psi'_n(x') = \psi_n(\alpha x')/\sqrt{\alpha}$.

6. An illustration. A useful example is the quantum mechanical harmonic oscillator equation

$$(6.1)\qquad \left(-\frac{d^2}{dx^2}+x^2\right)\psi_n=\lambda_n\psi_n.$$

Then, as is well known, the eigenvalues and eigenfunctions are given by

$$(6.2)\qquad \lambda_n=2n+1,$$

$$(6.3)\qquad \psi_n(x)=\mathscr{H}_n(x)=\frac{H_n(x)\exp(-x^2/2)}{(\pi^{1/2}2^n n!)^{1/2}}.$$

In (6.3), $H_n(x)$ represents the Hermite polynomial [4] of order n and the set $\{\mathscr{H}_n(x)\}$ are orthonormal. According to definition (2.6),

$$(6.4)\qquad E_n\{x, y\}=\frac{H_n(x)H_n(y)\exp(-(x^2+y^2)/2)}{\pi^{1/2}2^n n!}=\mathscr{H}_n(x)\mathscr{H}_n(y),$$

and as shown in § 8,

$$(6.5)\qquad WE_n=\tilde{E}_n=2(-)^n L_n(2(p^2+q^2))\exp\left(-\frac{p^2+q^2}{2}\right),$$

where L_n is the Laguerre polynomial [4] of order n. Observe that the λ-curves for (6.1) are concentric circles,

$$(6.6)\qquad p^2+q^2=\lambda.$$

In a more general vein we consider the Hermitian eigenvalue problem,

$$(6.7)\qquad \begin{aligned}&\left\{-A\frac{d^2}{dx^2}-iB\left(x\frac{d}{dx}+\frac{d}{dx}x\right)+Cx^2+2i(BQ_0+AP_0)\frac{d}{dx}\right.\\ &\qquad \left.-2(CQ_0+BP_0)x+(AP_0^2+CQ_0^2+2BP_0Q_0)\right\}\psi=\lambda\psi,\end{aligned}$$

where the coefficients are real and

$$(6.8)\qquad D^2=AC-B^2>0.$$

Except for (6.8) there is no restriction on the constants of (6.7); the particular arrangement of coefficients is taken for convenience. In fact, the Wigner transform of the kernel corresponding to the operator in (6.7) leads to the λ-curves

$$(6.9)\qquad A(p-P_0)^2+2B(p-P_0)(q-Q_0)+C(q-Q_0)^2=\lambda.$$

Under the restriction (6.8), this is a family of concentric ellipses, centered at (P_0, Q_0). The quantum mechanical harmonic oscillator is just a special case of (6.7) or (6.9).

Under the unimodular affine transformation

$$(6.10)\qquad \begin{pmatrix}p'\\ q'\end{pmatrix}=\begin{pmatrix}\dfrac{\mu\nu}{\sqrt{2}}, & -\dfrac{\mu}{\nu\sqrt{2}}\\ \dfrac{\nu}{\mu\sqrt{2}}, & \dfrac{1}{\mu\nu\sqrt{2}}\end{pmatrix}\begin{pmatrix}p-P_0\\ q-Q_0\end{pmatrix}$$

with

(6.11) $$\mu=\left(\frac{A}{C}\right)^{1/4},\qquad \nu=\left(\frac{\sqrt{AC}-B}{\sqrt{AC}+B}\right)^{1/4},$$

(6.9) reduces to

(6.12) $$p'^2+q'^2=\frac{\lambda}{D}.$$

Hence the eigenvalues of (6.7) are

(6.13) $$\lambda_n=(2n+1)D.$$

We can also explicitly represent the eigenfunctions of (6.7). Taking this in two stages, the translation portion of the transformation (6.10) is accounted for by expressing the eigenfunction corresponding to (6.13) as (see (5.3))

(6.14) $$\psi_n(x)=\Psi_n(x-Q_0)\exp(-iP_0x).$$

Then, from (5.5),

(6.15) $$\Psi_n(y)=\frac{1}{\sqrt{2\pi\gamma\omega}}\int dx\,\mathcal{H}_n(x)\exp\left[-\frac{i}{2\gamma}(\delta y^2-2xy+\alpha x^2)\right],$$

where the values of α, β, γ, δ follow from

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}=\begin{pmatrix}\mu\nu/\sqrt{2}, & -\mu/\nu\sqrt{2}\\ \nu/\mu\sqrt{2}, & 1/\mu\nu\sqrt{2}\end{pmatrix}^{-1}=\begin{pmatrix}1/\mu\nu\sqrt{2}, & \mu/\nu\sqrt{2}\\ -\nu/\mu\sqrt{2}, & \mu\nu/\sqrt{2}\end{pmatrix}$$

and (6.11). The integral in (6.15) can be directly evaluated by means of the generating function

(6.16) $$G=\exp\left[2xz-z^2-\frac{x^2}{2}\right]=\sum z^n\left(\frac{\pi^{1/2}2^n}{n!}\right)^{1/2}\mathcal{H}_n(x).$$

If (6.16) is applied to (6.15), a straightforward analysis then leads to

(6.17) $$\Psi_n(y)=(-i)^n\left(\frac{\gamma-i\alpha}{\gamma+i\alpha}\right)^{n/2}\frac{\mathcal{H}_n(y/\sqrt{\gamma^2+\alpha^2})}{\sqrt{\mu}\sqrt{\gamma+i\alpha}}\exp\left[-\frac{iy^2}{2}(\alpha\beta+\gamma\delta)\right].$$

The Ψ_n are orthonormal by construction, a fact which is also obvious by inspection. We eliminate unnecessary constants by choosing ω so that, instead of (6.17),

(6.18) $$\begin{aligned}\Psi_n&=\frac{\mathcal{H}_n(y/\sqrt{\gamma^2+\alpha^2})}{\sqrt{\gamma^2+\alpha^2}}\exp\left[-\frac{iy^2}{2}(\alpha\beta+\gamma\delta)\right]\\ &=\left(\frac{D}{C}\right)^{1/2}\mathcal{H}_n\left(y\left(\frac{D}{C}\right)^{1/2}\right)\exp\left[-\frac{iy^2}{2}\frac{B}{D}\right],\end{aligned}$$

where in the last form we have substituted in terms of the original constants of (6.7) and (6.8).

These exact results find immediate application in approximately determining those eigenfunctions and eigenvalues, of a more general integral kernel, which arise from the presence of a summit or valley in its Wigner transform, which gives a leading dependence as in (6.9). This matter is discussed further in [2].

The observation, after (5.7), that there is a natural sense in which the operator $(-d^2/dx^2+x^2)$ "generates" a rigid rotation (5.6) of the Wigner plane, has a natural generalization in the operator (6.7) which in the same sense generates on the Wigner plane a "flow" that is area preserving, carries straight lines to straight lines and has as invariant streamlines the conic sections (6.9). If $D^2 > 0$ then the flow is on ellipses and is similar, under a unimodular transformation, to a rigid rotation. If $D^2 = 0$, the flow is along parabolas, and if $D^2 < 0$, the flow leaves invariant a set of hyperbolas given by (6.9).

7. Representation of operator composition as a series. The image on the Wigner plane of a scalar multiple of the identity operator is simply the same scalar, according to (2.3). In this case the composition (3.1) with another operator image reduces to ordinary numerical multiplication. This suggests asymptotic simplifications in operator composition if one of the two operators in (3.6) is "slow on the Wigner plane." With this in mind we rewrite (3.6) as

$$\tilde{A} \otimes \tilde{B} = \frac{1}{\pi^2} \int d\xi_1\, d\xi_2\, \tilde{A}(\xi + \xi_1)\tilde{B}(\xi + \xi_2) \exp[4i\Delta(\xi_1, \xi_2, 0)], \tag{7.1}$$

and formally expand $\tilde{A}$ in the Taylor series;

$$\tilde{A}(\xi + \xi_1) = \sum \frac{1}{n!}\left(\xi_1 \cdot \frac{\partial}{\partial \xi}\right)^n \tilde{A}(\xi).$$

If this is substituted into (7.1), each term may be evaluated with the use of δ-functions and their derivatives. The result is

$$\begin{aligned} \tilde{A} \otimes \tilde{B} &= \sum \frac{(-i)^n}{2^n} \sum_{k=0}^{n} \frac{(-)^k}{n!} \binom{n}{k} \frac{\partial^n \tilde{B}}{\partial p^k\, \partial q^{n-k}} \cdot \frac{\partial^n \tilde{A}}{\partial q^k\, \partial p^{n-k}} \\ &= \tilde{A}\tilde{B} - \frac{i}{2}(\tilde{B}_q\tilde{A}_p - \tilde{B}_p\tilde{A}_q) - \frac{1}{8}(\tilde{B}_{qq}\tilde{A}_{pp} - 2\tilde{B}_{pq}\tilde{A}_{pq} + \tilde{B}_{pp}\tilde{A}_{qq}) \pm \cdots, \end{aligned} \tag{7.2}$$

which gives the desired expansion.

An alternate, formal, representation is gotten by inspection:

$$\tilde{A} \otimes \tilde{B} = \left\{ \exp\left[-\frac{i}{2}\left(\frac{\partial}{\partial q_B} \frac{\partial}{\partial p_A} - \frac{\partial}{\partial p_B} \frac{\partial}{\partial q_A} \right) \right] \tilde{A}(\xi_A)\tilde{B}(\xi_B) \right\}_{\xi_A = \xi_B = \xi} \tag{7.3}$$

A related form of this operator was presented by Moyal [5]. On comparison with (3.6) we can also formally write,

$$\frac{1}{\pi^2} \exp[4i\Delta(\xi_A, \xi_B, \xi)] = \delta(\xi - \xi_A)\delta(\xi - \xi_B) \exp\left[-\frac{i}{2}\left(\frac{\partial^2}{\partial q_B\, \partial p_A} - \frac{\partial^2}{\partial p_B\, \partial q_A} \right) \right]. \tag{7.4}$$

If either $\tilde{A}$ or $\tilde{B}$ is polynomial the series in (7.2) terminates after a finite number of terms. Thus, for example, if

$$\tilde{A} = \tilde{B} = p^2 + q^2,$$

then

$$\tilde{A} \otimes \tilde{B} + 1 = (p^2 + q^2)^2. \tag{7.5}$$

Simple inspection then reveals that

$$W^{-1}\xi^4 \to \left(-\frac{d^2}{dx^2} + x^2 \right)^2 + 1. \tag{7.6}$$

Thus (7.6) leads to Hermite functions, (6.3), for eigenfunctions; and eigenvalues given by $(2n+1)^2+1$. Further, it is clear that we can treat any polynomial in ξ^2 by the same considerations, and thus that any polynomial in ξ^2 corresponds to the Wigner transform of a differential operator which has $\{\mathcal{H}_n(x)\}$ as eigenfunctions.

8. The case of concentric circular contour lines. If the Wigner transform of a linear operator is constant on contour lines which are similar ellipses, concentric about an arbitrary point and with principal axes at arbitrary inclination, the unimodular affine transformation (6.10) will carry that Wigner transform to one whose contours are concentric circles about $\boldsymbol{\xi}=0$. Thus, as suggested in § 6, an exact solution to the eigenvalue problem for operators which yield concentric circular contours will also solve the eigenvalue problem for the wider class of operators just described. We will now develop the exact solution for any operator which yields concentric circular contours.

We begin with two generating function identities, of which the first defines the normalized Laguerre functions $\mathcal{L}_n$:

$$\frac{1}{1+z}\exp\left\{-\frac{1}{2}\left(\frac{1-z}{1+z}\right)J\right\}=\sum_{n=0}^{\infty} z^n\mathcal{L}_n(J)=U(J,z). \tag{8.1}$$

Here $\mathcal{L}_n$ is related to the corresponding Laguerre polynomial L_n by

$$\mathcal{L}_n(J)=(-1)^nL_n(J)\exp\left(-\frac{J}{2}\right). \tag{8.2}$$

Equation (8.1) results from (8.2) and the standard generating function for Laguerre polynomials [6].

In terms of the normalized Hermite functions, (6.3), Mehler's formula [7] is

$$G=\frac{1}{\sqrt{\pi(1-z^2)}}\exp\left\{-\frac{(1/2)(z^2+1)(x^2+y^2)-2zxy}{1-z^2}\right\}=\sum_{n=0}^{\infty} z^n\mathcal{H}_n(x)\mathcal{H}_n(y). \tag{8.3}$$

If z is a real number with magnitude less than unity, (8.3) defines a Hermitian operator and expresses it in terms of a sum of projection operators. Thus, as a special case of (2.6):

$$G\{x,y;z\}=\sum_{n=0}^{\infty} z^nE_n\{x,y\}. \tag{8.4}$$

In this sense $G\{x,y;z\}$ in Mehler's formula (8.3) is the generating function for the projection operators

$$E_n\{x,y\}=\mathcal{H}_n(x)\mathcal{H}_n(y).$$

The Wigner transformation (1.4) upon (8.3) is a standard Gaussian integration easily accomplished by "completing the square", and yields

$$\tilde{G}(p,q;z)=\frac{2}{1+z}\exp\left\{-\frac{1-z}{1+z}(p^2+q^2)\right\}=\sum_{n=0}^{\infty} z^n\tilde{E}_n(p,q), \tag{8.5}$$

whence (8.5) is a generating function for the Wigner transforms $\tilde{E}_n$ of the projection operators E_n in (8.4). If we compare (8.5) with (8.1), it follows that

$$\tilde{G}(p,q;z)=2\sum z^n\mathcal{L}_n(2J),\quad \text{where } J=p^2+q^2, \tag{8.6}$$

whence

$$\tilde{E}_n=2\mathcal{L}_n(2J). \tag{8.7}$$

(Equation (8.7) is a special case of a more general result given by Groenewald [8], who essentially Wigner transforms $\mathcal{H}_m(x)\mathcal{H}_n(y)$ and finds a radial dependence which is an associated Laguerre function and a sinusoidal angular dependence. See also Bruer [9, Appendix].)

Two notable facts bear mention. The first is that *every* projection operator image is constant on the *same* set of concentric circles. Hence this same set of contours will be inherited by any linear combination of them of the form

$$\tilde{K}(p^2+q^2)=\sum_{n=0}^{\infty}\lambda_n\tilde{E}_n(2(p^2+q^2)) \tag{8.8}$$

for arbitrary λ_n. Secondly, the normalized Laguerre functions form a *complete orthonormal basis* for a wide class of functions on the half line, so that *any* reasonable $\tilde{K}$ which is constant on concentric circles may be expanded as in (8.8), with

$$\lambda_n=\int_0^{\infty} dJ\,\tilde{K}(J)\mathcal{L}_n(2J). \tag{8.9}$$

We note that this is a special case of (2.1), as

$$\lambda_n=\mathrm{Tr}\,KE_n=\frac{1}{2\pi}\int d\xi\,\tilde{K}\tilde{E}_n=\int_0^{\infty} dJ\,\tilde{K}(J)\tilde{E}_n(J). \tag{8.10}$$

Thus any kernel $K\{x, y\}$ whose Wigner transform is constant on concentric circles is an inverse transform of (8.8), and has eigenfunctions $\mathcal{H}_n(x)$ as in (6.3) and eigenvalues given by (8.9).

As special cases we observe:

$$\int \mathcal{L}_n(2J)\,dJ=1, \tag{8.11}$$

$$\int \mathcal{L}_n(2J)J\,dJ=(2n+1), \tag{8.12}$$

$$\int \mathcal{L}_n(2J)J^2\,dJ=(2n+1)^2+1. \tag{8.13}$$

Equation (8.11) follows from the fact that a delta function has unit eigenvalue, (8.12) is a restatement of (6.1, 2) and (8.13) follows the example at the close of § 7. (All three results also follow easily from the generating function (8.1).)

9. Comparison with area rule. For cases reducible to concentric circular λ-curves, the area rule calculation of eigenvalues may be compared with the exact form for the eigenvalue (8.9). This we now discuss under the limit $n\uparrow\infty$, but give up the requirement of a slow variation in the underlying operator.

An asymptotic analysis demonstrates that the Laguerre function $\mathcal{L}_n$ (8.2) has a peaking form when $J=O(n)$ [4]. We can avoid the details of such an asymptotic analysis with the observation that

$$\int \mathcal{L}_n(2J)[J-(2n+1)]\,dJ=0. \tag{9.1}$$

This follows directly from (8.11) and (8.12). Thus, if we write

$$\lambda_n = \int_0^\infty \mathscr{L}_n(2J)\tilde{K}(J)\,dJ$$

$$\approx \int_0^\infty \mathscr{L}_n(2J)\Big\{\tilde{K}(2n+1)+[J-(2n+1)]\tilde{K}'(2n+1) + \frac{1}{2!}[J-(2n+1)]^2\tilde{K}''(2n+1)\Big\}\,dJ+\cdots, \tag{9.2}$$

it follows from (8.13) that

$$\lambda_n \sim \tilde{K}(2n+1)+\tfrac{1}{2}\tilde{K}''(2n+1). \tag{9.3}$$

The first term of (9.3) is the prediction of area rule and the second term has been carried as an error estimate for purposes of delimiting the range over which area rule is correct. In fact, it is clear from (9.3) that if

$$\ln \tilde{K}(J) = o(J), \tag{9.4}$$

then the area rule, which in this case gives

$$\lambda_n \sim \tilde{K}(2n+1), \tag{9.5}$$

is asymptotically valid.

As this argument demonstrates, the area rule for an operator $\tilde{K}$, which behaves exponentially for $J\uparrow\infty$, is in general incorrect. As an illustration of this phenomenon we recall an example given in [1]. The kernel

$$K\{x,y\} = \frac{1+\beta}{2}\exp\left\{-\left(\frac{x-y}{2}\right)^2 - \left(\beta\frac{x+y}{2}\right)^2\right\}$$

has the Wigner transform

$$\tilde{K} = \sqrt{\pi}(1+\beta)\exp\left(-(p^2+q^2)\right) = \sqrt{\pi}(1+\beta)\exp(-J)$$

(with $q=\beta(x+y)/2$). Use of the area rule gives

$$\lambda_n \sim \sqrt{\pi}(1+\beta)\exp[-(2n+1)\beta],$$

in contrast with the exact value

$$\Lambda_n = \sqrt{\pi}\left(\frac{1-\beta}{1+\beta}\right)^n.$$

In this case $\ln\tilde{K} = O(J)$, so that the criterion (9.4) is violated; thus, although λ_n and Λ_n agree in the limit $\beta\to 0$, n fixed, the agreement is not uniformly valid in n. Another case

$$\tilde{K} = \frac{1}{1+J}$$

is treated in [2]. In this case $\ln\tilde{K} = O(\ln J) = o(J)$ so that (9.4) is fulfilled and the area rule is valid for $n\uparrow\infty$.

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