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The Wrinkling of Thin Membranes: Part I—Theory

A method to describe the stress situation in a wrinkled membrane is presented. In this paper it will be shown that a special deformation tensor can be chosen which leads to the correct stress state of a membrane after wrinkling when it is substituted in the constitutive equation. The method can be used for anisotropic membranes in geometrically and physically nonlinear analysis. The case of simple shear and stretching of a membrane is considered to illustrate the potency of the method.

Introduction

In modern technology thin membranes are often used as construction elements. Examples can be found in aircraft and spacecraft applications. The building industry also uses membranes (fabric constructions). The objective of our research is to study the force transmission from muscle to bone near the elbow joint. The connective tissue structures which connect contractile elements from the muscle to the bones often consist of thin membrane-like structures. Because of their geometry the membrane-like tissues will wrinkle easily. This wrinkling has much influence on the stress state and the force transmission.

Models describing the mechanical behavior of membranes are usually based on the assumption that membranes have zero flexural stiffness. In normal membrane theory, however, negative stresses are possible. A membrane theory which accounts for wrinkling does not allow any negative stress to appear. When a negative stress is about to appear the membrane will wrinkle. A model for the stress field after wrinkling is a so-called tension field. By definition a tension field is uniaxial in the sense that it has only one nonzero principal stress component. In the direction perpendicular to the lines of tension the membrane is wrinkled.

The modelling was apparently started by Wagner (1929). He tried to explain the behavior of thin metal webs and spars carrying a shear load well in excess of the initial buckling value. Many authors (for example Reissner, 1938; Kondo et al., 1955; Mansfield, 1970, 1977) contributed to the geometrically linear analysis. This modelling is based on the theorem that the lines of tension in a wrinkled membrane are exactly in a direction for which the strain energy of the membrane is at a maximum. In this situation the wrinkling of isotropic and anisotropic membranes can be described (Mansfield, 1977). In geometrically nonlinear theory the analysis is more complex. Wu (1981) presented a model describing the wrinkling of membranes in finite plane-stress theory. He modified the deformation tensor by introducing an extra parameter. The value of this parameter was determined by the condition that the stress in wrinkling direction is zero. The modification of the deformation tensor was chosen in a way that the principal Cauchy directions did not change because of the wrinkling, which is only true when the material is isotropic.

Since the connective tissue structures we study may show large deformations and anisotropy, it was necessary to develop a new model capable of dealing with these phenomena. In this paper a detailed discussion of the theory is given.

Theory

Let us assume that: (*a*) Plane-stress theory can be used; (*b*) Flexure of the membrane does not introduce stresses in the membrane; (*c*) The membrane is not able to support any negative stress. If a negative stress is about to appear the membrane will wrinkle at once.

Although not essential for the theory we restrict ourselves to materials which behave "Cauchy-elastic". Thus, if it is taken into account that constitutive equations have to be objective, we are allowed to write for the Cauchy stress tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\sigma} = 1/J\mathbf{F}.\mathbf{H}(\mathbf{E}).\mathbf{F}^c \quad (1)$$

with: $\mathbf{F}$ the deformation tensor; $J = \det(\mathbf{F})$ the Jacobian of the deformation tensor; $\mathbf{E}$ the Green-Lagrange strain tensor; and $\mathbf{H}$ a tensor function of $\mathbf{E}$.

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Notice that the deformation tensor **F** can only be used in equation (1) if it contains the real deformations of a membrane. If however, a theoretical model does not account for wrinkling, deformations **F** may occur which result in negative Cauchy stresses. In reality these deformations will not occur because the membrane will wrinkle. The exact shape of the membrane after wrinkling is not definable with our theory. However, it is possible to use a special modified deformation tensor in the constitutive equation which results in the real Cauchy stresses.

Consider a small part of a membrane (Fig. 1) in the neighborhood of position $\vec{x}$. Vector $\vec{a}$ is a vector tangent to the midsurface of the membrane. In mathematical terms the assumption that no negative stresses occur can be represented by:

$$\vec{a}\cdot\sigma\cdot\vec{a}\geq 0 \tag{2}$$

The tensor σ is the real Cauchy stress tensor.

Inequality (2) means that, in equilibrium, there can be no direction with negative Cauchy stress. There is an infinite number of inequality conditions because $\vec{a}$ is arbitrary. It can be proved, however, that the following finite number of inequality conditions are necessary and sufficient in order to satisfy equation (2):

$$\vec{n}_1\cdot\sigma\cdot\vec{n}_1\geq 0 \tag{3}$$

$$\vec{n}_2\cdot\sigma\cdot\vec{n}_2\geq 0 \tag{4}$$

$$\vec{n}_1\cdot\sigma\cdot\vec{n}_2=0 \tag{5}$$

where $\vec{n}_1$ and $\vec{n}_2$ are orthonormal vectors denoting the principal directions of the real Cauchy stress tensor. So, if the membrane wrinkles, these vectors give the a priori unknown directions of the principal Cauchy stresses in the wrinkled membrane. The two inequalities represent the condition that neither of the principal Cauchy stresses can be negative. These conditions are necessary because a negative principal stress would contradict the assumption that negative stresses are not possible. These conditions are sufficient because determination of the stress in an arbitrary direction $\vec{a}$:

$$\vec{a}\cdot\sigma\cdot\vec{a}=[(\vec{a}\cdot\vec{n}_1)\vec{n}_1+(\vec{a}\cdot\vec{n}_2)\vec{n}_2]\cdot\sigma\cdot[(\vec{a}\cdot\vec{n}_1)\vec{n}_1 + (\vec{a}\cdot\vec{n}_2)\vec{n}_2]=(\vec{a}\cdot\vec{n}_1)^2\vec{n}_1\cdot\sigma\cdot\vec{n}_1+(\vec{a}\cdot\vec{n}_2)^2\vec{n}_2\cdot\sigma\cdot\vec{n}_2 \tag{6}$$

always leads to a positive stress in direction $\vec{a}$ if conditions (3) and (4) are true.

Considering conditions (3), (4), and (5), the following situations are possible. If both the principle stresses are positive the membrane is taut. If both the principle stresses are zero the membrane is slack. The only possibility left is the situation in which one principal stress is zero (for example, in the $\vec{n}_1$ direction) and the other principal stress (in the $\vec{n}_2$ direction) is positive. In the latter case conditions (3), (4), and (5) are transformed into:

$$\vec{n}_1\cdot\sigma\cdot\vec{n}_1=0 \tag{7}$$

$$\vec{n}_2\cdot\sigma\cdot\vec{n}_2>0 \tag{8}$$

$$\vec{n}_1\cdot\sigma\cdot\vec{n}_2=0 \tag{9}$$

In this situation the membrane may be wrinkled. In Fig. 2 a wrinkled membrane part is shown. The deformed configuration of this membrane part, if it would not have wrinkled, is indicated by the dotted lines. The dotted membrane part would be the result of the deformation tensor **F** corresponding with a theory which does allow for negative principal stresses. In this figure $\vec{n}_1$ again is the a priori unknown direction in which the real principal Cauchy stress is zero. The problem to be solved is the determination of the principal stress in the direction of the unit vector $\vec{n}_2$. Because of the second assumption, the stresses in the membrane part stay the same if we

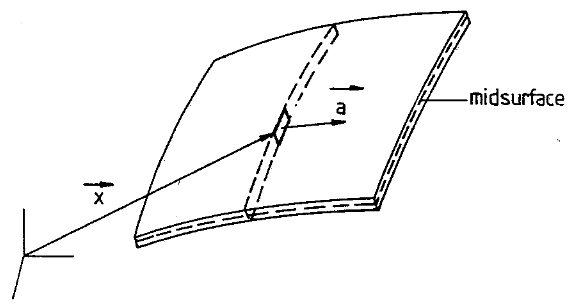

Fig. 1 Vector $\vec{a}$ is touching the midsurface of the membrane

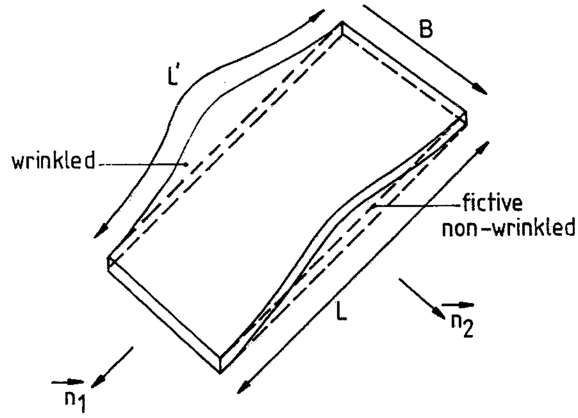

Fig. 2 A wrinkled membrane part with deformed length *L'* and deformed width *B*. Also the fictive nonwrinkled membrane part is shown (dotted lines) with length *L*<*L'* and width *B*.

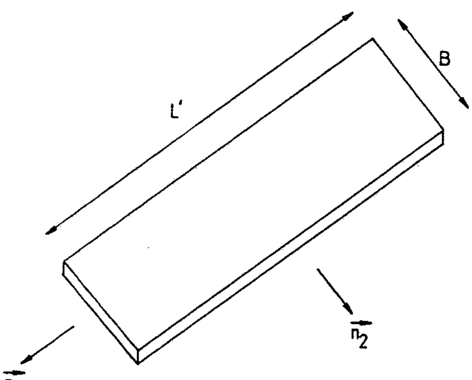

Fig. 3 The wrinkled membrane part straight in the plane determined by $\vec{n}_1$ and $\vec{n}_2$

straighten it (by flexure only) in the plane determined by $\vec{n}_1$ and $\vec{n}_2$ (Fig. 3).

The deformation tensor which would have given the membrane part of Fig. 3 is a deformation tensor which corresponds with the real stresses, because that membrane part contains the real stresses as we argued above. Since the membrane part of Fig. 3 has the same shape, but is only longer in the $\vec{n}_1$ direction in comparison with the fictive nonwrinkled part, that deformation tensor has to be of the form:

$$\mathbf{F}'=(\mathbf{I}+\beta\vec{n}_1\vec{n}_1)\cdot\mathbf{F} \tag{10}$$

with **I** denoting the unit tensor. The tensor $(\mathbf{I}+\beta\vec{n}_1\vec{n}_1)$ lengthens the fictive nonwrinkled membrane part to become just as long as the real wrinkled membrane part.

It should be noticed that, when the material is anisotropic, the principal directions after wrinkling in general differ from the principal directions in the fictive nonwrinkled situation.

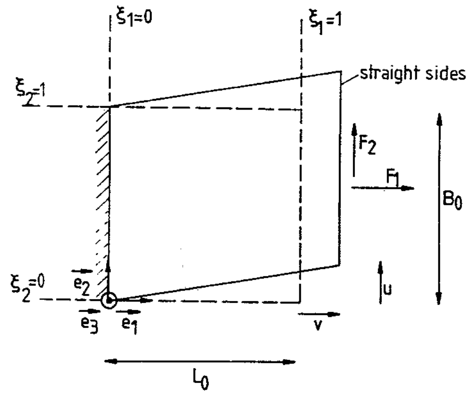

Fig. 4 Simple shear and stretching of a membrane

The parameter β and the direction of the principal frame have to be determined by using the coupled nonlinear conditions (7) and (9). The parameter β, which is never negative, is a measure of the wrinkliness of the membrane.

Summarizing we may state that the real stress state in a wrinkled membrane can be determined by using the modified deformation tensor $\mathbf{F}'$ in the constitutive equation:

$$\sigma(\mathbf{F}') = 1/J \ \ \mathbf{F}' \cdot \mathbf{H}(\mathbf{E}') \cdot \mathbf{F}'^{c} \tag{11}$$

where:

$$\mathbf{E}' = 1/2(\mathbf{F}'^{c} \cdot \mathbf{F}' - \mathbf{I}); \quad J = \det(\mathbf{F}') \tag{12}$$

and where β and the direction of the principal frame have to be determined by making use of the equations (7) and (9).

At this point there is only one problem left. Suppose a deformation tensor $\mathbf{F}$ is given which, using constitutive equation (1), generates one or two negative principal Cauchy stresses. It is not immediately clear what the condition of the membrane shall be. To be able to determine whether the membrane is wrinkled or slack, the Green-Lagrange strain tensor based on $\mathbf{F}$ should be considered. When both the principal values of this strain tensor are negative, there are no directions at which material is stretched, so the membrane part is slack. If, however, in this situation, one of the principal values of this strain tensor is positive, there are directions at which material is stretched, so there are positive stresses, thus the membrane part is wrinkled.

Notice that no assumptions, such as geometrical linearity or isotropic material have been made. The preceding formulation is generally applicable.

Example: Simple Shear and Stretching of a Membrane

In this example a membrane is deformed by simple shear and stretching (Fig. 4). F_1 and F_2 are the forces required to shear the membrane over a distance u and stretch it over a distance v. Material coordinates are denoted by ξ_i $(0 \le \xi_i \le 1;$ $i = 1,2,3)$. The initial position vector of a material point is given by:

$$\vec{x}_0 = \xi_2 B_0 \vec{e}_2 + \xi_1 L_0 \vec{e}_1 + \xi_3 D_0 \vec{e}_3 \tag{13}$$

where B_0 is the initial width, L_0 is the initial length and D_0 is the initial thickness of the membrane.

If no wrinkling takes place, the displacement field is given by:

$$\vec{u} = \xi_2 u \vec{e}_2 + \xi_1 v \vec{e}_1 + \xi_3 w \vec{e}_3 \tag{14}$$

where u is the shearing distance, v is the stretching distance, and w is the variation in the thickness of the membrane. This leads to the deformation tensor:

$$\mathbf{F} = \mathbf{I} + u/L_0 \vec{e}_2 \vec{e}_1 + v/L_0 \vec{e}_1 \vec{e}_1 + w/D_0 \vec{e}_3 \vec{e}_3 \tag{15}$$

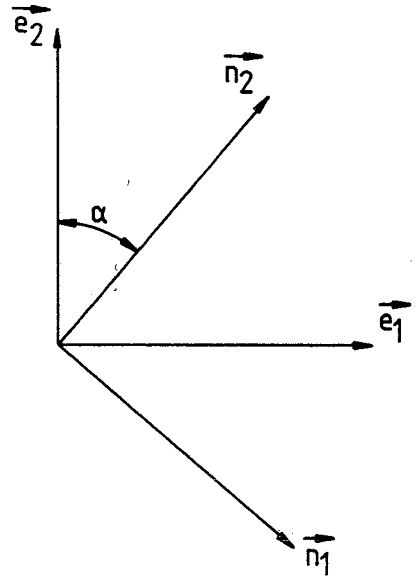

Fig. 5 The direction of the principal frame is indicated by the angle α

In simulating transversely isotropic material, the following representation of the tensor function $\mathbf{H}(\mathbf{E})$ with respect to the frame $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ is chosen:

$$H^e_{11} = \frac{E}{(1+v)(1-2v)}[(1-v)E^e_{11} + v(E^e_{22} + E^e_{33})] + \frac{(f-1)E}{(1-v^2)} E^e_{11} \tag{16}$$

$$H^e_{22} = \frac{E}{(1+v)(1-2v)}[(1-v)E^e_{22} + v(E^e_{11} + E^e_{33})] \tag{17}$$

$$H^e_{33} = \frac{E}{(1+v)(1-2v)}[(1-v)E^e_{33} + v(E^e_{11} + E^e_{22})] \tag{18}$$

$$H^e_{ij} = \frac{E}{1+v} E^e_{ij}, \quad i \text{ not equal } j \tag{19}$$

E^e_{ij} are the components of the Green-Lagrange strain tensor with respect to the frame $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$. The factor f determines the measure of anisotropy of the membrane (if $f = 1$ then the membrane is isotropic, if $f > 1$ then $\vec{e}_1$ is the stiffest direction).

If the membrane wrinkles the real Cauchy stresses have to be determined by using the following set of equations:

$$\mathbf{F}' = (\mathbf{I} + \beta \vec{n}_1 \vec{n}_1) \cdot (\mathbf{I} + u/L_0 \vec{e}_2 \vec{e}_1 + v/L_0 \vec{e}_1 \vec{e}_1 + w/D_0 \vec{e}_3 \vec{e}_3) \tag{20}$$

$$\sigma(\mathbf{F}') = 1/J \mathbf{F}' \cdot \mathbf{H}(\mathbf{E}') \cdot \mathbf{F}'^{c} \tag{21}$$

where:

$$J = \det(\mathbf{F}'); \quad \mathbf{E}' = 1/2(\mathbf{F}'^{c} \cdot \mathbf{F}' - \mathbf{I}) \tag{22}$$

and in the equations (16) to (19) E^e_{ij} is replaced by E'^e_{ij}.

Denoting the direction of the principal frame by the angle α (Fig. 5), the parameters α, β, and w have to be determined by using conditions (23) and (24):

$$\vec{n}_1 \cdot \sigma(\mathbf{F}') \cdot \vec{n}_1 = 0 \tag{23}$$

$$\vec{n}_1 \cdot \sigma(\mathbf{F}') \cdot \vec{n}_2 = 0 \tag{24}$$

and the plane-stress condition:

$$\vec{e}_3 \cdot \sigma(\mathbf{F}') \cdot \vec{e}_3 = 0 \tag{25}$$

Using equations (20) to (22) with the conditions (23) to (25) leads to three nonlinear coupled equations in the unknowns α,

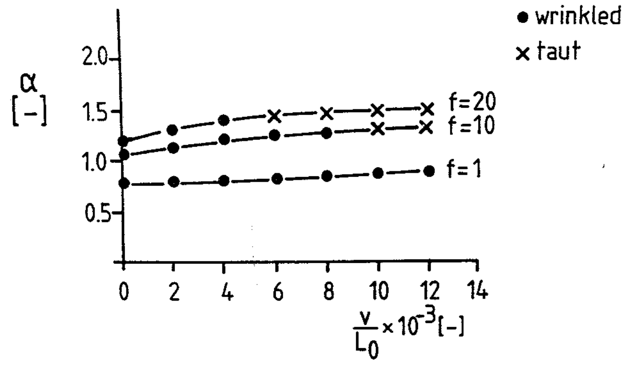

Fig. 6 The angle α as a function of the stretching distance v for different values of the anisotropy parameter f (see equation (16))

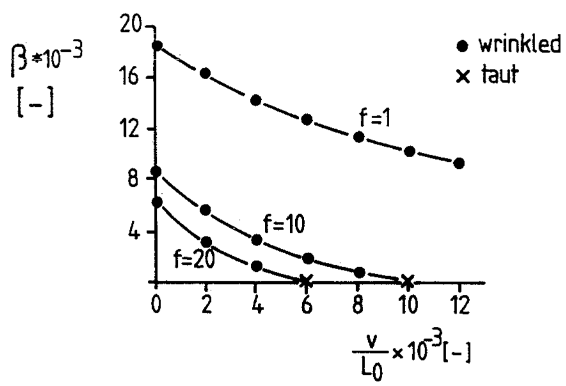

Fig. 7 The parameter β as a function of the stretching distance v for different values of the anisotropy parameter f

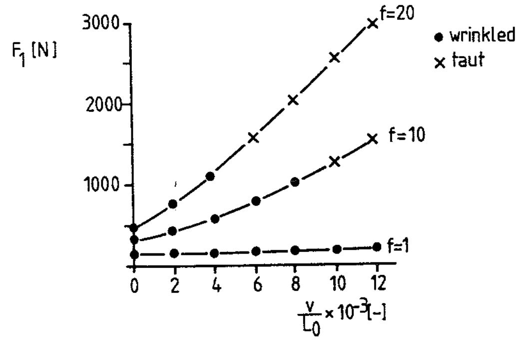

Fig. 8 The force F_1 as a function of the stretching distance v for different values of the anisotropy parameter f

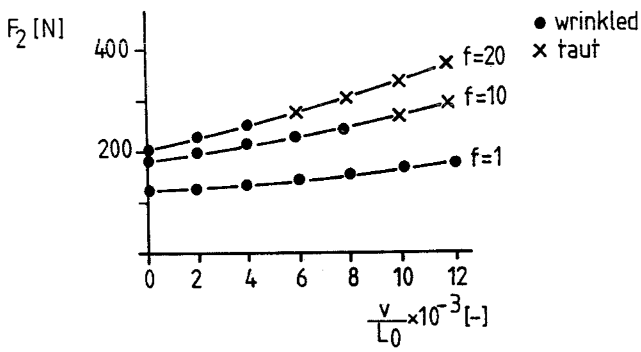

Fig. 9 The force F_2 as a function of the stretching distance v for different values of the anisotropy parameter f

β, and w:

$$h_1(\alpha,\beta,w)=0 \tag{26}$$

$$h_2(\alpha,\beta,w)=0 \tag{27}$$

$$h_3(\alpha,\beta,w)=0 \tag{28}$$

The unknowns α, β, and w can be solved numerically by using equations (26) to (28), for example by means of a Newton-Raphson procedure.

We have chosen the following numerical values for the model parameters:

$$L_0=100[\text{mm}];\ B_0=100[\text{mm}];\ D_0=1[\text{mm}];$$

$$E=100[\text{N/mm}^2];\quad \upsilon=0.3$$

The membrane is sheared with $u=5$ [mm] and then stretched until $v=1.2$ [mm]. Results for the geometrically nonlinear analysis are given in Figs. 6 to 9.

In Fig. 6 it can be seen that if the membrane stiffens in $\vec{e}_1$ direction the lines of tension tend to this direction. Notice that if the stretching is strong enough the wrinkles are pulled out of the membrane. This is the point at which the parameter β becomes zero (Fig. 7). If the membrane stiffens, the forces necessary to the deform the membrane increase (Fig. 8 and Fig. 9).

For isotropic materials ($f=1$) the formulation of Wu (1981) would generate the same results because, in this situation, equation (10) degenerates to the same modification of the deformation tensor as Wu proposes in order to describe the wrinkling of isotropic membranes. The results for anisotropic materials, however, can only be found by making use of the theoretical model given above.

Similar figures can be found in a geometrical linear analysis. These results turn out to be the same as results found by geometrical linear formulations, for example the formulation of Mansfield (1977).

Conclusions

The model given above is easy to understand and pretends to describe the general situation of the wrinkling of anisotropic membranes in geometrically nonlinear analysis. However, the preceding analysis of wrinkling membranes is only theoretical. A confrontation between this theory and experiments still has to be done.

In situations in which no analytical solutions to problems are known we want to use numerical approximation methods. We choose to use the Finite Element Method. Based on the theoretical method, a membrane element which is able to wrinkle has been developed. We plan to discuss this element in a paper to follow.

Acknowledgments

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