Vol. 99, No. 3 DUKE MATHEMATICAL JOURNAL © 1999

THE YAMABE PROBLEM ON MANIFOLDS WITH BOUNDARY: EXISTENCE AND COMPACTNESS RESULTS

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Contents

§0. Introduction. Let (M, g) be an n-dimensional, compact, smooth, Riemannian manifold without boundary. For $n = 2$, we know from the uniformization theorem of Poincaré that there exist metrics that are pointwise conformal to g and have constant Gauss curvature. For $n \geq 3$, the well-known Yamabe conjecture states that there exist metrics that are pointwise conformal to g and have constant scalar curvature. The answer to the Yamabe conjecture is proved to be affirmative through the work of Yamabe [39], Trudinger [38], Aubin [1], and Schoen [31]. See Lee and Parker [23] for a survey. See also Bahri and Brezis [3] and Bahri [2] for works on the Yamabe problem and related ones. For $n \geq 3$, let $\tilde{g} = u^{4/(n-2)} g$ for some positive function $u > 0$ on M; the scalar curvature $R_{\tilde{g}}$ of $\tilde{g}$ can be calculated as

$$R_{\tilde{g}} = u^{-((n+2)/(n-2))} \left(R_g u - \frac{4(n-1)}{n-2} \Delta_g u \right),$$

where R_g denotes the scalar curvature of g. Therefore, the Yamabe conjecture is equivalent to the existence of a solution to

$$-L_g u = \overline{R} u^{(n+2)/(n-2)}, \quad u > 0, \text{in } M, \tag{0.1}$$

Received 7 August 1998.
1991 *Mathematics Subject Classification*. Primary 58, 53; Secondary 35.
Han's work partially supported by a Rutgers University Research Council grant, a Minority Faculty Development grant, and National Science Foundation grant number DMS-9704488.
Li's work partially supported by the Alfred P. Sloan Foundation Research Fellowship, National Science Foundation grant DMS-9401815, and a Rutgers University Research Council grant.

where $L_g = \Delta_g - c(n)R_g$, $c(n) = (n-2)/(4(n-1))$, and $\overline{R} = 0$ or $\pm n(n-2)$.

Consider

$$Q(\varphi) = \frac{\int_M \left(|\nabla_g \varphi|^2 + c(n) R_g \varphi^2\right)}{\left(\int_M |\varphi|^{2n/(n-2)}\right)^{(n-2)/n}},$$

for $\varphi \in H^1(M) \setminus \{0\}$. It is clear that up to some harmless positive constant, a positive critical point of the functional Q is a solution of (0.1).

The Sobolev quotient is given by

$$Q(M, g) = \inf \left\{ Q(\varphi) \mid \varphi \in H^1(M) \setminus \{0\} \right\}.$$

It is clear that $Q(M, g)$ is positive if the first eigenvalue of $-L_g$ is positive, negative if the first eigenvalue of $-L_g$ is negative, and zero if the first eigenvalue of $-L_g$ is zero.

Yamabe attempted in [39] to prove that $Q(M, g)$ is always achieved. However, in [38] Trudinger pointed out that Yamabe's proof was seriously flawed and also corrected Yamabe's proof in the case $Q(M, g) \le 0$.

It is not difficult to see that, for all (M, g), we have

$$Q(M, g) \le Q(\mathbf{S}^n, g_0),$$

where $(\mathbf{S}^n, g_0)$ denotes the standard n sphere. It was proved by Aubin [1] that $Q(M, g)$ is attained if

$$Q(M, g) < Q\big(\mathbf{S}^n, g_0\big). \tag{0.2}$$

In the same paper, Aubin also verified (0.2) for $n \ge 6$ and M not locally conformally flat by choosing test functions supported near a point where the Weyl tensor does not vanish. This confirms the Yamabe conjecture when $n \ge 6$ and M is not locally conformally flat. The remaining cases are more difficult since the local geometry does not contain sufficient information to conclude (0.2). In [31], Schoen established (0.2) by constructing global test functions in the remaining cases based on the positive mass theorems of Schoen and Yau (see [35], [36]). The answer to the Yamabe conjecture was then proved to be affirmative.

More recently, Schoen has obtained compactness results for the Yamabe problem. He proved in [34] that when (M, g) is locally conformally flat but not conformally equivalent to the standard sphere, then all solutions to (0.1) stay in a compact set of $C^2(M)$ and the total degree of all solutions is equal to -1. In the same paper, he also announced, with indication of the proof, the same result for general manifolds (see [32] for more details). This is much stronger than the existence results.

Analogues of the Yamabe problem for compact Riemannian manifolds with boundary have been studied by Cherrier, Escobar, and others. In particular, Escobar proved in [14] that a large class of compact Riemannian manifolds with boundary are conformally equivalent to one with constant scalar curvature and zero mean curvature on the boundary.

From now on in this paper, (M, g) denotes some smooth compact n-dimensional Riemannian manifold with boundary unless we specify otherwise. We use L_g to denote $\Delta_g - c(n)R_g$, $c(n)$ to denote $(n-2)/(4(n-1))$, B_g to denote $(\partial/\partial\nu) + ((n-2)/2)h_g$, ν to denote the outward unit normal on ∂M with respect to g, and h_g to denote the mean curvature of ∂M with respect to the inner normal (balls in $\mathbf{R}^n$ have positive mean curvatures). Let H denote the second fundamental form of ∂M in (M, g) with respect to the inner normal; we denote the traceless part of the second fundamental form by U:

$$U(X, Y) = H(X, Y) - h_g g(X, Y).$$

Definition 0.1. A point $q \in \partial M$ is called an umbilic point if $U = 0$ at q. ∂M is called umbilic if every point of ∂M is an umbilic point.

Let $u > 0$ be some positive function on M, let $\tilde{g} = u^{4/(n-2)}g$, and calculate the mean curvature $h_{\tilde{g}}$ as

$$h_{\tilde{g}} = \frac{2}{n-2} u^{-(n/(n-2))} B_g u.$$

Remark 0.1. The notion of an umbilic point is conformally invariant; namely, if $q \in \partial M$ is an umbilic point with respect to g, it is also an umbilic point with respect to $\varphi^{4/(n-2)}g$ for any positive smooth function φ on M.

Consider the following eigenvalue problem on (M, g):

$$\begin{cases} -L_g\varphi = \lambda\varphi, & \text{in } M^\circ, \\ B_g\varphi = 0, & \text{on } \partial M, \end{cases} \tag{0.3}$$

where $M^\circ = M \setminus \partial M$ denotes the interior of M. Let $\lambda_1(M)$ denote the first eigenvalue. It is well known that

$$\lambda_1(M) = \min_{\varphi \in H^1(M)\setminus\{0\}} \frac{\int_M \left(|\nabla\varphi|^2 + c(n)R_g\varphi^2\right) + \left((n-2)/2\right)\int_{\partial M} h_g\varphi^2}{\int_M \varphi^2}.$$

We say that a manifold M is of positive (negative, zero) type if $\lambda_1(M) > 0$ ($< 0, = 0$). This notion is conformally invariant. As is well known, the existence problems are more difficult for manifolds of positive type. In this paper, we mainly treat this case, though we also include some results for other cases.

Letting (M, g) be a manifold of positive type, we consider for $c \in \mathbf{R}$,

$$\begin{cases} -L_g u = n(n-2)u^{(n+2)/(n-2)}, & u > 0, \text{ in } M^\circ, \\ B_g u = cu^{n/(n-2)}, & \text{on } \partial M. \end{cases} \tag{0.4}$$

Let $\mathcal{M}_c$ denote the set of solutions of (0.4) in $C^2(M)$.

The geometric meaning of (0.4) is that u is a solution of (0.4) if and only if, up to some harmless positive constant, the scalar curvature of $\tilde{g} = u^{4/(n-2)}g$ is 1 and the mean curvature of $\tilde{g}$ on ∂M is c.

Consider

$$Q(\varphi) = \frac{\int_M \left(|\nabla_g \varphi|^2 + c(n) R_g \varphi^2\right) + \left((n-2)/2\right) \int_{\partial M} h_g \varphi^2}{\left(\int_M |\varphi|^{2n/(n-2)}\right)^{(n-2)/n}},$$

for $\varphi \in H^1(M) \setminus \{0\}$. It is clear that, up to some harmless positive constant, $\varphi \in \mathcal{M}_0$ for any positive critical point of the functional Q.

The Sobolev quotient is given by

$$Q(M,g) = \inf \left\{ Q(\varphi) \mid \varphi \in H^1(M) \setminus \{0\} \right\}.$$

It is clear that $Q(M,g)$ is positive if the first eigenvalue of $-L_g$ is positive, negative if the first eigenvalue of $-L_g$ is negative, and zero if the first eigenvalue of $-L_g$ is zero.

Cherrier proved in [10] that, similar to the Yamabe problem, $Q(M,g)$ is achieved if

$$Q(M,g) < Q\left(\mathbf{S}^n_+, g_0\right), \tag{0.5}$$

where $(\mathbf{S}^n_+, g_0)$ denotes the standard half sphere. In the same paper, he also showed the regularity of solutions to such problems. For a large class of manifolds, Escobar established (0.5) in [14], thus showing $\mathcal{M}_0 \neq \emptyset$. More recently, Escobar showed in [15] that, under the same hypotheses as in [14], there exist $c^+ > 0$ and $c^- < 0$ such that $\mathcal{M}_{c^+} \neq \emptyset$ and $\mathcal{M}_{c^-} \neq \emptyset$. Naturally, one wonders whether $\mathcal{M}_c \neq \emptyset$ for all $c \in \mathbf{R}^n$. Our next theorem suggests that it is probably the case.

THEOREM 0.1. *For $n \geq 3$, let (M,g) be a smooth, compact, n-dimensional, locally conformally flat Riemannian manifold of positive type with umbilic boundary. Then, for all $c \in \mathbf{R}, \mathcal{M}_c \neq \emptyset$. Furthermore, if (M,g) is not conformally equivalent to the standard half sphere, then, for all $\bar{c} > 0$, there exists $C = C(M,g,\bar{c})$ such that for all $u \in \cup_{|c| \leq \bar{c}} \mathcal{M}_c$ we have*

$$\frac{1}{C} \leq u(x) \leq C, \quad \forall x \in M; \qquad \|u\|_{C^2(M)} \leq C.$$

In fact, we establish a slightly stronger compactness result.

Consider for $c \in \mathbf{R}$, $1 < p \leq (n+2)/(n-2)$,

$$\begin{cases} -L_g u = n(n-2)u^p, & u > 0, \text{ in } M^\circ, \\ B_g u = cu^{(p+1)/2}, & \text{on } \partial M. \end{cases} \tag*{$(*)_{p,c}$}$$

Let $\mathcal{M}_{p,c}$ denote the set of solutions of $(*)_{p,c}$ in $C^2(M)$.

THEOREM 0.2. *For $n \geq 3$, let (M, g) be a smooth, compact, n-dimensional, locally conformally flat Riemannian manifold of positive type with umbilic boundary. We assume that (M, g) is not conformally equivalent to the standard half sphere. Then, for any $\overline{c} > 0$, there exist constants $\delta_0 = \delta_0(M, g, \overline{c}) > 0$ and $C = C(M, g, \overline{c}) > 0$ such that, for all*

$$u \in \Big(\cup_{((n+2)/(n-2))-\delta_0 \leq p \leq (n+2)/(n-2)} \cup_{|c| \leq \overline{c}} \mathcal{M}_{p,c}\Big) \cup \Big(\cup_{1+\delta_0 \leq p \leq (n+2)/(n-2)} \mathcal{M}_{p,0}\Big),$$

we have

$$\frac{1}{C} \leq u(x) \leq C, \quad \forall x \in M; \qquad \|u\|_{C^2(M)} \leq C. \tag{0.6}$$

In view of Theorems 0.1 and 0.2 and the results in [34], we propose the following two conjectures.

CONJECTURE 1. *Let (M, g) be a smooth, compact n-dimensional Riemannian manifold with boundary of positive type. Then, for all $c \in \mathbf{R}$, $\mathcal{M}_c \neq \emptyset$.*

CONJECTURE 2. *Let (M, g) be a smooth, compact n-dimensional Riemannian manifold with boundary of positive type that is not conformally equivalent to the standard half sphere. Then, for all $\overline{c} > 0$, there exist positive constants $\delta_0 = \delta_0(M, g, \overline{c})$ and $C = C(M, g, \overline{c}) > 0$ such that, for all*

$$u \in \Big(\cup_{((n+2)/(n-2))-\delta_0 \leq p \leq (n+2)/(n-2)} \cup_{|c| \leq \overline{c}} \mathcal{M}_{p,c}\Big),$$

we have (0.6).

Remark 0.2. In another paper [19], we confirm Conjecture 1 for all manifolds of positive type with dimension 5 or higher and boundary not totally umbilic. The remaining cases are discussed in [20].

Remark 0.3. Due to the results in Section 5, we can deduce Conjecture 1 from Conjecture 2. Namely, Conjecture 2 is a stronger one.

To prove Theorems 0.1 and 0.2, we establish compactness results for all solutions of $(*)_{p,c}$ and then show that the total degree of all solutions to (0.4) is equal to -1. The heart of the matter is some fine analysis of possible blow-up behavior of solutions of (0.4) which, together with the positive mass theorem of Schoen and Yau, implies energy-independent estimates of all solutions to $(*)_{p,c}$.

When (M, g) is an n-dimensional ($n \geq 3$) compact locally conformally flat Riemannian manifold without boundary, such fine analysis and energy-independent estimates were obtained by Schoen in [34] for solutions to

$$-L_g u = n(n-2)u^p, \quad u > 0, \text{ in } M,$$

where $1 < 1+\epsilon_0 \leq p \leq (n+2)/(n-2)$. For $n = 3$, Schoen and Zhang established in [37] such fine analysis and energy-independent estimates for solutions to

$$-L_g u = K(x)u^p, \quad u > 0, \text{in } M,$$

where $1 < 1+\epsilon_0 \le p \le 5$, $K > 0$ is some positive function in $C^2(M)$, and M is locally conformally flat.

Along the approach initiated by Schoen, the second author extended in [25] and [26] the above-mentioned results of Schoen and Zhang to dimension $n = 4$, as well as to dimension $n \ge 5$ under a suitable $(n-2)$-flatness hypothesis of K near critical points of K. He also established in [24] such fine analysis and energy-independent estimates in dimension $n = 3$ for solutions to

$$\begin{cases} -L_g u = K u^p, & u > 0, \text{ in } M^\circ, \\ B_g u = 0, & \text{on } \partial M, \end{cases}$$

where $1 < 1+\epsilon_0 \le p \le 5$, $K > 0$ is some positive function in $C^2(M)$, M is locally conformally flat, and ∂M is umbilic. See Brezis, Li, and Shafrir [5], Chang, Gursky, and Yang [7], Chen and Lin [9], and Li and Zhu [28] for related results.

The starting point of most of the above-mentioned results on fine asymptotic analysis and energy-independent estimates is the following Liouville-type theorem in $\mathbf{R}^n$ ($n \ge 3$) of Caffarelli, Gidas, and Spruck [6], which asserts that any solution of

$$-\Delta u = n(n-2)u^{(n+2)/(n-2)}, \quad u > 0, \text{ in } \mathbf{R}^n$$

is of the form

$$u(x) = \left(\frac{\lambda}{1+\lambda^2 |x-\overline{x}|^2} \right)^{(n-2)/2}$$

for some $\lambda > 0$ and $\overline{x} \in \mathbf{R}^n$ (see also Chen and Li [8] for a more direct proof). Under an additional hypothesis that $u(x) = O(|x|^{2-n})$ for large x, the above Liouville-type theorem was obtained by Obata [30] and Gidas, Ni, and Nirenberg [16].

To establish results on fine asymptotic analysis and energy-independent estimates to solutions of $(*)_{p,c}$, one needs to establish some Liouville-type theorem in the half space $\mathbf{R}^n_+$ ($n \ge 3$). This was carried out by the second author and Zhu in [27], where they proved that any solution of

$$\begin{cases} -\Delta u = n(n-2)u^{(n+2)/(n-2)}, & u > 0, \text{ in } \mathbf{R}^n_+, \\ \dfrac{\partial u}{\partial x_n} = c u^{n/(n-2)}, & \text{on } \partial \mathbf{R}^n_+, \end{cases}$$

is of the form

$$u\big(x', x_n\big) = \left(\frac{\lambda}{1+\lambda^2 \left| (x', x_n) - \big(\overline{x}', \overline{x}_n\big) \right|^2} \right)^{(n-2)/2}$$

for some $\lambda > 0$, $\overline{x}' \in \mathbf{R}^{n-1}$, and $\overline{x}_n = (n-2)\lambda^{-1}c$. Under an additional hypothesis that $u(x) = O(|x|^{2-n})$ for large x, the above Liouville-type theorem in $\mathbf{R}^n_+$ was obtained by Escobar in [12].

Our proof of Theorems 0.1 and 0.2 goes along the lines of [25] and [32]. The main difficulties we need to overcome here are caused by the nonlinear boundary term $cu^{n/(n-2)}$ in (0.4). In §1, we make some preliminary reductions based on the Liouville-type theorems of Caffarelli, Gidas, and Spruck in $\mathbf{R}^n$ and of Li and Zhu in $\mathbf{R}^n_+$, and we state various propositions that we prove later on. Proposition 1.2 rules out the possibility of bubble accumulations and therefore establishes that only isolated blow-up points (see Definition 1.1) may occur. Proposition 1.3 asserts that isolated blow-up points are actually isolated simple blow-up points (see Definition 1.2). Geometrically, an isolated simple blow-up point corresponds to one sphere only. To establish Propositions 1.2 and 1.3, we need to have good enough estimates for isolated simple blow-up points. These are stated as in Proposition 1.4. In deriving these estimates, a basic role is played by Harnack inequalities, which include the usual Harnack inequality and one involving boundaries (Lemma A.1). Another important role is played by Pohozaev identities. In §2, we establish Proposition 1.4 where, among other things, we need to construct suitable barrier functions in the proof of Lemma 2.2 to handle boundaries. Proposition 1.3 is proved in §3 and Proposition 1.2 in §4. In §5, we first establish the crucial compactness results, Theorem 0.2, by utilizing Propositions 1.1–1.4 and the positive mass theorem of Schoen and Yau. Then we use Leray-Schauder degree theory to establish the existence results stated in Theorem 0.1. In fact, we show that the total degree of all solutions is equal to -1. The above analysis carried out here is for locally conformally flat manifolds with umbilic boundaries. In the next paragraph, we give some more detailed description of the proof of Theorems 0.1 and 0.2.

We establish Theorem 0.2 by a contradiction argument. Supposing the contrary of Theorem 0.2, we find, in view of Theorem 1.1, sequences $|c_i| \le \bar{c}$, $p_i \le (n+2)/(n-2)$, $p_i \to (n+2)/(n-2)$, and $u_i \in \mathcal{M}_{p_i,c_i}$ such that

$$\max_M u_i \to \infty.$$

It follows from Propositions 1.1–1.4 that, after passing to a subsequence, $\{u_i\}$ has $N(1 \le N < \infty)$ isolated simple blow-up points, denoted as $\{q^{(1)}, \dots, q^{(N)}\}$. Let $\{q_i^{(1)}, \dots, q_i^{(N)}\}$ denote the local maximum points as described in Definition 1.1. It follows from Proposition 1.4 and standard elliptic theories that

$$u_i\big(q_i^{(1)}\big)u_i \to h, \quad \text{in } C^2_{\mathrm{loc}}\big(M \setminus \big\{q^{(1)}, \dots, q^{(N)}\big\}\big).$$

Using the hypothesis $\lambda_1(M) > 0$, we have

$$h = \sum_{l=1}^{N} a_l G\big(\cdot, q^{(l)}\big), \quad \text{on } M,$$

where $a_l > 0, \forall l$, and $G(\cdot, q^{(l)})$ denotes Green's function of $-L_g$ with respect to zero Neumann boundary conditions and centered at $q^{(l)}$. Using the positive mass theorem of Schoen and Yau, we know that in a good coordinate system centered at $q^{(1)}$,

$$h(x) = a_1|x|^{2-n} + A + O(|x|),$$

where $A > 0$ is some positive constant. Applying the Pohozaev identity in $B_\sigma(q^{(1)})$ for $\sigma > 0$ small, we obtain a contradiction for i large by using the estimates we derived for isolated simple blow-up points.

The compactness part of Theorem 0.1 is contained in Theorem 0.2. To establish the existence part of Theorem 0.1, we use Theorem 0.2 and the Leray-Schauder degree theory as follows.

Without loss of generality, we assume $R_g > 0$ and $h_g \equiv 0$. Let $\varphi_1 > 0$ denote the eigenfunction associated with the first eigenvalue $\lambda_1(M)$ satisfying

$$\|\varphi_1\|^2 \equiv \int_M \left(\left|\nabla_g \varphi_1\right|^2 + c(n) R_g \varphi_1^2\right) = 1.$$

For $0 < \alpha < 1$, let $C^{2,\alpha}(M)^+ = \{u \in C^{2,\alpha}(M) : u > 0 \text{ on } M\}$. We define, for $1 \le p \le (n+2)/(n-2)$ and $c \in \mathbf{R}$, a map $T_{p,c} : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ as follows: $u = T_{p,c} v$ if and only if

$$\begin{cases} -L_g u = n(n-2) v^p, & \text{in } M, \\ \dfrac{\partial_g u}{\partial \nu} = c v^{(p+1)/2}, & \text{on } \partial M. \end{cases}$$

It follows from standard elliptic theories and the hypothesis $\lambda_1(M) > 0$ that $T_{p,c}$ is well defined and is a compact operator. It is clear that $\mathcal{M}_c \neq \emptyset$ if and only if $T_{((n+2)/(n-2)),c}$ has a fixed point. Due to our a priori estimates, we can use the Leray-Schauder degree theory to show that $T_{((n+2)/(n-2)),c}$ has a fixed point for all $c \in \mathbf{R}$.

For $\Lambda > 1$, let D_Λ denote the the following bounded and open subset of $C^{2,\alpha}(M)^+$:

$$D_\Lambda = \left\{ v \in C^{2,\alpha}(M) : \|v\|_{C^{2,\alpha}(M)} < \Lambda, \min_M v > \frac{1}{\Lambda} \right\}.$$

Since $T_{p,c}$ is compact, we can define the Leray-Schauder degree of $\mathrm{Id} - T_{p,c}$ in D_Λ with respect to $0 \in C^{2,\alpha}(M)$, denoted by $\deg(\mathrm{Id} - T_{p,c}, D_\Lambda, 0)$, provided that 0 does not belong to $(\mathrm{Id} - T_{p,c})(\partial D_\Lambda)$. It follows from Theorem 0.2 that, for Λ large enough, 0 does not belong to $(\mathrm{Id} - T_{((n+2)/(n-2)),tc})(\partial D_\Lambda)$ for all $0 \le t \le 1$. It follows then from the homotopy invariance of the Leray-Schauder degree that

$$\deg\left(\mathrm{Id} - T_{((n+2)/(n-2)),c}, D_\Lambda, 0\right) = \deg\left(\mathrm{Id} - T_{((n+2)/(n-2)),0}, D_\Lambda, 0\right).$$

To evaluate $\deg(\mathrm{Id} - T_{((n+2)/(n-2)),0}, D_\Lambda, 0)$, we introduce, for $0 < \alpha < 1$ and $1 \le p \le (n+2)/(n-2)$, another map $F_p : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ by

$$F_p(v) = v - \left(-L_g\right)^{-1}\left(E(v) v^p\right),$$

where $(-L_g)^{-1}$ denotes the inverse operator of $-L_g$ with respect to the zero Neumann boundary condition and $E(v) = \int_M (|\nabla_g v|^2 + c(n) R_g v^2)$. It is easy to see from standard elliptic theories that F_p is of the form Id + compact, and therefore $\deg(F_p, D_\Lambda, 0)$ is well defined provided 0 does not belong to $F_p(\partial D_\Lambda)$. It follows from the a priori estimates we derived that 0 does not belong to $F_p(\partial D_\Lambda)$ for all $1 \le p \le (n+2)/(n-2)$. Consequently, by the homotopy invariance of the Leray-Schauder degree,

$$\deg(F_p, D_\Lambda, 0) = \deg(F_1, D_\Lambda, 0), \quad \forall 1 \le p \le \frac{n+2}{n-2}.$$

A direct calculation shows that

$$\deg(F_1, D_\Lambda, 0) = -1.$$

Making another homotopy, we also obtain

$$\deg\left(\mathrm{Id} - T_{((n+2)/(n-2)),0}, D_\Lambda, 0\right) = \deg\left(F_{(n+2)/(n-2)}, D_\Lambda, 0\right).$$

Combining the above, we have

$$\deg\left(\mathrm{Id} - T_{((n+2)/(n-2)),c}, D_\Lambda, 0\right) = -1,$$

from which we conclude that $\mathcal{M}_c \neq q\emptyset$ for all $c \in \mathbf{R}$.

We also present some existence and compactness results for manifolds of negative type. Let (M, g) be a compact n-dimensional Riemannian manifold of negative type. Consider for $c \in \mathbf{R}$, $1 < p \le (n+2)/(n-2)$,

$$\begin{cases} -L_g u = -n(n-2)u^p, & u > 0, \text{ in } M^\circ, \\ B_g u = c u^{(p+1)/2}, & \text{on } \partial M. \end{cases} \tag{0.7}$$

Let $\widetilde{\mathcal{M}}_{p,c}$ denote the set of solutions of (0.7) in $C^2(M)$ and let $\widetilde{\mathcal{M}}_c = \widetilde{\mathcal{M}}_{((n+2)/(n-2)),c}$.

THEOREM 0.3. *For $n \ge 3$, let (M, g) be a smooth, compact n-dimensional Riemannian manifold of negative type. Then $\widetilde{\mathcal{M}}_c \neq \emptyset$ for all $c < n-2$. Furthermore, for all $\overline{\epsilon} > 0$, there exist constants $\delta_0 = \delta_0(M, g, \overline{\epsilon}) > 0$ and $C = C(M, g, \overline{\epsilon}) > 0$ such that, for all $u \in (\cup_{((n+2)/(n-2))-\delta_0 \le p \le (n+2)/(n-2)} \cup_{-(\overline{\epsilon})^{-1} \le c \le n-2-\overline{\epsilon}} \widetilde{\mathcal{M}}_{p,c}) \cup (\cup_{1+\delta_0 \le p \le (n+2)/(n-2)} \widetilde{\mathcal{M}}_{p,0})$, we have*

$$\frac{1}{C} \le u(x) \le C, \quad \forall x \in M; \qquad \|u\|_{C^2(M)} \le C. \tag{0.8}$$

The above theorem is established in §6.

§1. Reductions. We point out that if p stays strictly below the critical exponent $(n+2)/(n-2)$ and strictly above 1, the compactness of solutions to $(*)_{p,c}$ is a much easier matter, since it follows directly from the nonexistence of positive solutions to the global equation which one arrives at after a rather standard blow-up argument. The following theorem, not stated in its general form to include $c \neq 0$, is enough for us in establishing Theorems 0.1 and 0.2.

THEOREM 1.1. *Let (M,g) be a smooth, compact n-dimensional Riemannian manifold with boundary. Then, for any $\delta_1 > 0$, there exists $C = C(M,g,\delta_1)$ such that, for all $u \in \cup_{1+\delta_1 \le p \le ((n+2)/(n-2))-\delta_1} \mathcal{M}_{p,0}$, we have*

$$\frac{1}{C} \le u(x) \le C, \quad \forall x \in M; \qquad \|u\|_{C^2(M)} \le C.$$

Proof. Suppose that the theorem were false. Then, in view of the Harnack inequality, Lemma A.1, and standard elliptic estimates, we could find sequences $\{p_i\}$ and $\{u_i\} \in \mathcal{M}_{p_i,0}$ satisfying

$$\lim_{i\to\infty} p_i = p \in \left(1, \frac{n+2}{n-2}\right)$$

and

$$\lim_{i\to\infty} \max_M u_i = \infty.$$

Let q_i be a maximum point of u_i and let x be a geodesic normal coordinate system in a neighborhood of q_i given by $\exp_{q_i}^{-1}$. We write $u_i(x)$ for $u_i(\exp_{q_i}(x))$. We rescale x by $y = \lambda_i x$ with $\lambda_i = u_i^{(p_i-1)/2}(q_i) \to \infty$, and define

$$\hat{v}_i(y) = \lambda_i^{-(2/(p_i-1))} u_i\big(\lambda_i^{-1} y\big).$$

Clearly, $\hat{v}_i(0) = 1, 0 \le \hat{v}_i \le 1$. Let $\delta > 0$ be some small positive number independent of i. We write $g(x) = g_{ab}(x)dx^a dx^b$ in $\exp_{q_i}^{-1}(B_\delta(0))$. Define $g^{(i)}(y) = g_{ab}(\lambda_i^{-1}y)dy^a dy^b$. Then $\hat{v}_i$ satisfies $-L_{g^{(i)}}\hat{v}_i = n(n-2)\hat{v}_i^{p_i}$. If $\partial M \cap \exp_{q_i}^{-1}(B_\delta(0)) \neq \emptyset$, the boundary condition of $\hat{v}_i$ is $B_{g^{(i)}}\hat{v}_i = 0$. Applying L^p-estimates and Schauder estimates, we know that, after passing to a subsequence and a possible rotation of coordinates, $T = \lim_{i\to\infty} d(q_i, \partial M)$ for some $0 \le T \le \infty$, and $\hat{v}_i$ converges to a limit $\hat{v}$ in C^2-norm on any compact subset of $\{y \in \mathbf{R}^n : y^n \ge -T\}$, where $\hat{v} > 0$ satisfies

$$\begin{cases} -\Delta\hat{v} = n(n-2)\hat{v}^p, & \text{in } y^n > -T, \\ -\dfrac{\partial\hat{v}}{\partial y^n} = 0, & \text{on } y^n = -T, \text{in the case of } T < \infty, \\ \hat{v}(0) = 1, & \hat{v} \ge 0. \end{cases} \tag{1.1}$$

It follows from the Liouville-type theorem of Gidas and Spruck [17] (see also [6] and [8]) that (1.1) has no solution. This is a contradiction. Thus, we have established Theorem 1.1. □

The compactness of solutions to $(*)_{p,c}$ is much more difficult to prove when allowing $p \le (n+2)/(n-2)$, since the corresponding global equation does have solutions. On the other hand, due to the Liouville-type theorems of Caffarelli-Gidas-Spruck [6] in $\mathbf{R}^n$ and Li-Zhu [27] in the half space $\mathbf{R}^n_+$, we have the following proposition, similar to Lemma 3.1 of Schoen-Zhang [37]. This proposition gives a preliminary description of large solutions u of $(*)_{p,c}$. Roughly speaking, for such u, one can find a finite collection of disjoint balls $B_{\bar{r}_1}(q_1), \dots, B_{\bar{r}_N}(q_N)$ (N may depend on u) inside which u is very well approximated in strong norms by standard bubbles. Furthermore, u satisfies $u(q) \le C_1[d(q,\{q_1,\dots,q_N\})]^{-(2/(p-1))}$ for all q in M, where C_1 is some positive constant independent of u.

PROPOSITION 1.1. *Let (M,g) be a smooth, compact n-dimensional Riemannian manifold with boundary. For any $\bar{c} > 0$ and any given $R \ge 1$, $0 < \epsilon < 1$, there exist positive constants $\delta_0 = \delta_0(M,g,\bar{c},R,\epsilon)$, $C_0 = C_0(M,g,\bar{c},R,\epsilon)$, and $C_1 = C_1(M,g,\bar{c},R,\epsilon)$ such that, for all $u \in \cup_{((n+2)/(n-2))-\delta_0 \le p \le (n+2)/(n-2)} \cup_{|c| \le \bar{c}} \mathcal{M}_{p,c}$ with $\max_M u \ge C_0$, there exist $\{q_1,\dots,q_N\} \subset M$, with $N \ge 1$ such that the following statements are true.*

(i) *Each q_i is a local maximum point of u in M and*

$$\overline{B_{\bar{r}_i}(q_i)} \cap \overline{B_{\bar{r}_j}(q_j)} = \emptyset, \quad \text{for } i \ne j,$$

where $\bar{r}_i = Ru^{-((p-1)/2)}(q_i)$, and $\overline{B_{\bar{r}_i}(q_i)}$ denotes the geodesic ball in (M,g) of radius $\bar{r}_i$ and centered at q_i.

(ii) *Either $q_i \in M^\circ$,*

$$\left\| u^{-1}(q_i)u\left(\exp_{q_i}\left(\frac{y}{u^{(p-1)/2}(q_i)}\right)\right) - \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2} \right\|_{C^2\left(B^M_{2R}(0)\right)} < \epsilon,$$

or $q_i \in \partial M$, and

$$\left\| u^{-1}(q_i)u\left(\exp_{q_i}\left(\frac{y}{u^{(p-1)/2}(q_i)}\right)\right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2+|y^n+t_c|^2\right)}\right)^{(n-2)/2} \right\|_{C^2\left(B^M_{2R}(0)\right)} < \epsilon,$$

where $B^M_{2R}(0) = \{y \in T_{q_i}M : |y| \le 2R$, and $u^{-((p-1)/2)}(q_i)y \in \exp^{-1}_{q_i}(B_\delta(q_i))\}$, $y = (y', y^n)$, $\lambda_c = 1 + (c/(n-2))^2$, and $t_c = (c/(n-2)\lambda_c)$.

(iii) *$d^{2/(p-1)}(q_j,q_i)u(q_j) \ge C_0$, for $j > i$, while $u(q) \le C_1[d(q,\{q_1,\dots, q_N\})]^{-(2/(p-1))}$, for all $q \in M$, where $d(\cdot,\cdot)$ denotes the distance function in metric g.*

The proof of Proposition 1.1 follows from the next lemma.

LEMMA 1.1. *Let (M,g) be a smooth, compact n-dimensional Riemannian manifold. Given any $R \ge 1$, $0 < \epsilon < 1$, $\bar{c} > 0$. Then there exist positive constants*

$\delta_0 = \delta_0(M, g, R, \epsilon, \overline{c})$ and $C_0 = C_0(M, g, R, \epsilon, \overline{c})$ such that, for any compact $K \subset M$ and any $u \in \cup_{((n+2)/(n-2))-\delta_0 \le p \le (n+2)/(n-2)} \cup_{|c| \le \overline{c}} \mathcal{M}_{p,c}$ with

$$\max_{q \in \overline{M \setminus K}} d^{2/(p-1)}(q, K) u(q) \ge C_0,$$

the following holds.

There exists $q_0 \in M \setminus K$, which is a local maximum point of u in M, and either $q_0 \in M^\circ$,

$$\text{(a)} \qquad \left\| u^{-1}(q_0) u\left(\exp_{q_0}\left(\frac{y}{u^{(p-1)/2}(q_0)}\right)\right) - \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2} \right\|_{C^2\left(B^M_{2R}(0)\right)} < \epsilon,$$

or $q_0 \in \partial M$ and

(b)

$$\left\| u^{-1}(q_0) u\left(\exp_{q_0}\left(\frac{y}{u^{(p-1)/2}(q_0)}\right)\right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2 + |y^n + t_c|^2\right)}\right)^{(n-2)/2} \right\|_{C^2\left(B^M_{2R}(0)\right)} < \epsilon$$

where $B^M_{2R}(0)$, λ_c, and t_c are as in Proposition 1.1, $d(q, K)$ denotes the distance of q to K, and $d(q, K) = 1$ if $K = \emptyset$.

Proof of Lemma 1.1. Suppose no such δ_0 and C_0 exist for some R, ϵ, and $\overline{c}$. Then there exist compact $K_i \subset M$, $((n+2)/(n-2)) - 1/i \le p_i \le (n+2)/(n-2)$, $|c_i| \le \overline{c}$, and solutions u_i of $(*)_{p_i, c_i}$, such that $\max_{q \in \overline{M \setminus K_i}} d^{2/(p_i-1)}(q, K_i) u_i(q) \ge i$, and no q_0 as in Lemma 1.1 exists. It is easy to deduce from the Hopf lemma that $u_i > 0$ in M. Let $\hat{q}_i \in M \setminus K_i$ be such that $d^{2/(p_i-1)}(\hat{q}_i, K_i) u_i(\hat{q}_i) = \max_{q \in M \setminus K_i} d^{2/(p_i-1)}(q, K_i) u_i(q)$. Let x be a geodesic normal coordinate system in a neighborhood of $\hat{q}_i$ given by $\exp^{-1}_{\hat{q}_i}$. We write $u_i(x)$ for $u_i(\exp_{\hat{q}_i}(x))$ and denote $\lambda_i = u_i^{(p_i-1)/2}(\hat{q}_i)$. We rescale x by $y = \lambda_i x$, and define

$$\hat{v}_i(y) = \lambda_i^{-(2/(p_i-1))} u_i\left(\lambda_i^{-1} y\right).$$

In $\exp^{-1}_{\hat{q}_i}(B_\delta(\hat{q}_i))$, write $g(x) = g_{ab}(x) dx^a dx^b$. Define $g^{(i)}(y) = g_{ab}(\lambda_i^{-1} y) dy^a dy^b$. Then $-L_{g^{(i)}} \hat{v}_i = n(n-2) \hat{v}_i^{p_i}$. Fix some small positive constant $\delta > 0$ independent of i. If $\partial M \cap B_\delta(\hat{q}_i) \ne \emptyset$, we may assume, by taking δ smaller, that $\exp^{-1}_{\hat{q}_i}(\partial M) \cap B_\delta(0)$ has only one connected component and may be arranged to let the closest point on $\exp^{-1}_{\hat{q}_i}(\partial M) \cap B_\delta(0)$ to zero be at $(0, \ldots, 0, -t_i)$ and $\exp^{-1}_{\hat{q}_i}(\partial M) \cap B_\delta(0) := \partial' B^M_\delta(0)$ be represented as a graph over $(x^1, \ldots, x^{n-1})$ with horizontal tangent plane at $(0, \ldots, 0, -t_i)$ and uniformly bounded second derivatives. The boundary condition for u_i translates into $B_{g^{(i)}} \hat{v}_i = c_i \hat{v}_i^{(p_i+1)/2}$. Note that $\lambda_i d(\hat{q}_i, K_i) \to \infty$, and for $|y| \le (1/4)\lambda_i d(\hat{q}_i, K_i)$ with $x = \lambda_i^{-1} y \in \exp_{\hat{q}_i}^{-1}(B_\delta(\hat{q}_i))$, we have $d(x, K_i) \ge (1/2) d(\hat{q}_i, K_i)$; therefore,

$$\left(\frac{1}{2} d\big(\hat{q}_i, K_i\big)\right)^{2/(p_i-1)} u_i(x) \le d\big(x, K_i\big)^{2/(p_i-1)} u_i(x) \le d\big(\hat{q}_i, K_i\big)^{2/(p_i-1)} u_i\big(\hat{q}_i\big),$$

which implies, for all $|y| \le (1/4)\lambda_i d(\hat{q}_i, K_i)$ with $\lambda_i^{-1} y \in \exp_{\hat{q}_i}^{-1}(B_\delta(\hat{q}_i))$, that

$$\hat{v}_i(y) \le 2^{2/(p_i-1)} \le 2^{2/(p_0-1)}. \tag{1.2}$$

Standard elliptic theories imply that there exists a subsequence, still denoted as $\hat{v}_i$, such that $T = \lim \lambda_i d(\hat{q}_i, \partial M)$ for some $0 \le T \le \infty$, $c = \lim c_i$ for some $c \in \mathbf{R}$, and $\hat{v}_i$ converges to a limit $\hat{v}$ in C^2 norm on any compact subset of $\{y = (y^1, \dots, y^n) \in \mathbf{R}^n : y^n \ge -T\}$, where $\hat{v} > 0$ satisfies

$$\begin{cases} -\Delta \hat{v} = n(n-2)\hat{v}^{(n+2)/(n-2)}, & \text{in } y^n > -T, \\ -\dfrac{\partial \hat{v}}{\partial y^n} = c\hat{v}^{n/(n-2)}, & \text{on } y^n = -T, \text{in the case of } T < \infty. \end{cases} \tag{1.3}$$

By the Liouville-type theorems of Caffarelli, Gidas, and Spruck [6] in $\mathbf{R}^n$, and of Li and Zhu [27] in $\mathbf{R}^n_+$, we have $\hat{v}(y) = (\lambda/(1+\lambda^2|y-\hat{y}|^2))^{(n-2)/2}$ for some $\hat{y} \in \mathbf{R}^n$, $\lambda > 0$, and $\hat{y}^n = -T - (n-2)^{-1}\lambda^{-1}c$ in the case of $T < \infty$. Since $1 = \hat{v}(0) = (\lambda/(1+\lambda^2|\hat{y}|^2))^{(n-2)/2}$, we have $\lambda \ge 1$ and $|\hat{y}| \le 1$. When $T = \infty$, we obtain from (1.2) that $\hat{v}(y) \le 2^{(n-2)/2}$ for all $y \in \mathbf{R}^n$, and thus $\lambda \le 2$. When $T < \infty$, we take $y = (\hat{y}', -T)$ in (1.2) and obtain, together with $\lambda \ge 1$, that $\lambda \le 2 + 2\bar{c}^2/(n-2)$.

In the following we only consider $T < \infty$ and divide the rest of the proof into three cases. The case $T = \infty$ can be handled similarly to case 1.

Case 1. $c < 0$. In this case, we see from the explicit form of $\hat{v}$ and the boundary condition in (1.3) that $\hat{y}^n > -T$. It follows that there exist $y_i \to \hat{y}$ which are local maximum points of $\hat{v}_i(y)$ such that $\hat{v}_i(y_i) \to \lambda^{(n-2)/2} = \max \hat{v}$. Define $q_i = \exp_{\hat{q}_i}(\lambda_i^{-1} y_i)$; then $q_i \in M^\circ \setminus K_i$ is a local maximum point of u_i, and, if we repeat the scaling with q_i replacing $\hat{q}_i$, we will obtain a new limit v. Because of the new normalization, $v(y) = (1/(1+|y|^2))^{(n-2)/2}$. We redefine $T = \lim d(q_i, \partial M)\lambda_i$. Using the boundary condition for v, we get $T = -(c/(n-2))$. Thus, for sufficiently large i,

$$\left\| u_i^{-1}(q_i) u_i \left(\exp_{q_i} \left(\frac{y}{u_i^{(p_i-1)/2}(q_i)} \right) \right) - \left(\frac{1}{1+|y|^2} \right)^{(n-2)/2} \right\|_{C^2\left(B_{2R}^M(0)\right)} < \epsilon.$$

This shows that, for large i, u_i satisfies (a). This contradicts the contradiction hypothesis we start with.

Case 2. $c = 0$. In this case, $\hat{y}^n = -T$. It follows that there exist $y_i \to \hat{y}$ which are local maximum points of $\hat{v}_i$. We can argue, as in case 1, to reach a contradiction.

Case 3. $c > 0$. In this case, $\hat{y}^n < -T$. It follows that there exists $y_i \in \partial' B^M_{\lambda_i\delta}(0)$ that is a local maximum point of $\hat{v}_i$ such that $y_i \to (\hat{y}', -T)$. If we redo the rescaling with respect to $q_i = \exp_{\hat{q}_i}(\lambda_i^{-1} y_i) \in \partial M$, we obtain a limit

$$v(y) = \left(\frac{\lambda}{1+\lambda^2\big(|y'|^2+|y^n+t_c|^2\big)}\right)^{(n-2)/2},$$

where $(n-2)\lambda t_c = c$ and $\lambda = 1+\lambda^2|t_c|^2$. Thus, $\lambda = \lambda_c = 1+(c/(n-2))^2$, $t_c = \lambda_c^{-1}(c/(n-2))$, and the conclusion of (b) holds. This again contradicts the initial contradiction hypothesis. We have thus established Lemma 1.1. $\square$

Proof of Proposition 1.1. First, we apply Lemma 1.1 by taking $K = \emptyset$ and $d(q,K) \equiv 1$ to obtain q_1, which is a maximum point of u, and the conclusion of Lemma 1.1 holds. Next, we take $K_1 = \overline{B_{\overline{r}_1}(q_1)}$, $\overline{r}_1 = Ru^{-((p-1)/2)}(q_1)$. If $\max_{q\in M\setminus K_1} d^{2/(p-1)}(q,K_1)u(q) \le C_0$, we stop. Otherwise, we obtain q_2 given by Lemma 1.1. It is clear from the lemma that $\overline{B_{\overline{r}_1}(q_1)} \cap \overline{B_{\overline{r}_2}(q_2)} = \emptyset$, since $\epsilon > 0$ can be made very small from the beginning. We continue this process. The process will stop after a finite number of steps, since there exists some dimensional constant $a(n) > 0$ such that $\int_{B_{\overline{r}_i}(q_i)} |\nabla u|^2 \ge a(n)$. Thus, we obtain $\{q_1,\dots,q_N\} \subset M$ as in (i) and (ii), and $d^{2/(p-1)}(q, \cup_{i=1}^N B_{\overline{r}_i}(q_i))u(q) \le C_0$, for any $q \in M \setminus \cup_{i=1}^N B_{\overline{r}_i}(q_i)$. Now, for any $q \in M$, either $q \in B_{2\overline{r}_i}(q_i)$ for some i, or $d(q,q_i) > 2\overline{r}_i$, for all $1 \le i \le N$. In the first case, $d(q,\{q_1,\dots,q_N\}) \le d(q,q_i) < 2\overline{r}_i$, so (ii) implies $u(q) \le 2u(q_i) = 2R^{2/(p-1)}\overline{r}_i^{-(2/(p-1))}$, and so $d^{2/(p-1)}(q,\{q_1,\dots,q_N\})u(q) \le 2(2R)^{2/(p-1)} \le 2(2R)^{2/(p_0-1)}$. In the second case, $d(q,\{q_1,\dots,q_N\}) \le 2d(q,\cup_{i=1}^N B_{\overline{r}_i}(q_i))$, so $d^{2/(p-1)}(q,\{q_1,\dots,q_N\})u(q) \le 2^{2/(p-1)}C_0 \le 2^{2/(p_0-1)}C_0$. Taking $C_1 = \max(2^{2/(p_0-1)}C_0, 2(2R)^{2/(p_0-1)})$, we obtain the conclusion of Proposition 1.1. $\square$

Though we know from Proposition 1.1 that u is very well approximated in strong norms by standard bubbles in disjoint balls $B_{\overline{r}_1}(q_1),\dots,B_{\overline{r}_N}(q_N)$, it is far from the compactness results we wish to establish. Interactions between all these standard bubbles have to be analyzed in order to rule out the possibility of blowing-ups. The next proposition rules out possible accumulations of these standard bubbles, which implies that only isolated blow-up points (see Definition 1.1) may occur to a blowing-up sequence of solutions. This proposition plays a crucial role in the proof of Theorem 0.2, and its proof is much more involved than that of Proposition 1.1.

PROPOSITION 1.2. *Let (M,g) be a smooth, compact n-dimensional locally conformally flat Rimannian manifold with umbilic boundary. Given any $\overline{c} > 0$, then for suitably large R and small $\epsilon > 0$, there exist positive constants $\delta_1 = \delta_1(M,g,\overline{c},R,\epsilon)$ and $d = d(M,g,\overline{c},R,\epsilon)$ such that, for all $u \in \cup_{((n+2)/(n-2))-\delta_1\le p\le (n+2)/(n-2)} \cup_{|c|\le\overline{c}} \mathcal{M}_{p,c}$ with* $\max_M \ge C_0$, *we have*

$$\min\big\{d\big(q_i,q_j\big) \mid 1\le i,j\le N, i\ne j\big\} \ge d,$$

where C_0 is the constant in Proposition 1.1 and $q_1, \ldots, q_N$ are the points in M given by Proposition 1.1.

This proposition is proved in §4. We now begin the process of the reductions. If Theorem 0.2 were false, then we could find, in view of the L^p-estimates, Schauder estimates, the Harnack inequality, Lemma A.1, and Theorem 1.1, sequences $\{p_i\}$, $\{c_i\}$, and $\{u_i\} \in \mathcal{M}_{p_i,c_i}$ satisfying

$$\frac{n+2}{n-2} - \frac{1}{i} \le p_i \le \frac{n+2}{n-2}, \qquad \lim_{i\to\infty} c_i = c \in \mathbf{R},$$

and

$$\lim_{i\to\infty} \max_M u_i = \infty.$$

Fix a large R and a small $\epsilon > 0$. Let $\mathcal{S}(u_i) = \{q_{i,1}, \ldots, q_{i,N(i)}\}$ be the set of points selected according to Proposition 1.1. It follows from Proposition 1.2 that $\min_{a\neq b} d(q_{i,a}, q_{i,b}) \ge d > 0$. This shows in particular that $N(i)$ stays bounded. After passing to a subsequence, there exist $N = N(M,g) \ge 1$, $q_1, \ldots, q_N \in M$, such that, for some $1 \le a \le N$, $q_{i,a} \to q_a$ and $u_i(q_{i,a}) \to \infty$, as $i \to \infty$. For each such a, $q_{i,a} \to q_a$ is a so-called isolated blow-up point for $\{u_i\}$ defined as follows.

Definition 1.1. Let (M,g) be a smooth, compact n-dimensional Riemannian manifold with boundary, and let $\bar r > 0, \bar c > 0$, $\bar x \in M$, $f \in C^0(\overline{B_{\bar r}(\bar x)})$ be some positive function where $B_{\bar r}(\bar x)$ denotes the geodesic ball in (M,g) of radius $\bar r$ centered at $\bar x$. Suppose for sequences $|c_i| \le \bar c$, $p_i \le (n+2)/(n-2)$, $p_i \to (n+2)/(n-2)$, $f_i \to f$ in $C^0(\overline{B_{\bar r}(\bar x)})$, $\{u_i\}$ satisfies, for $\tau_i = p_i - ((n+2)/(n-2))$, that

$$\begin{cases} -L_g u_i = n(n-2) f_i^{\tau_i} u_i^{p_i}, & u_i > 0 \text{ in } B_{\bar r}(\bar x), \\ B_g u_i = c_i f_i^{\tau_i/2} u_i^{(p_i+1)/2}, & \text{on } \partial M \cap B_{\bar r}(\bar x), \end{cases} \tag{1.4}$$

and there exists a sequence of local maximum points $\{x_i\}$ of u_i such that $x_i \to \bar x$, and, for some $\overline{C} > 0$,

$$u_i(x) \le \overline{C}\big[d(x,x_i)\big]^{-(2/(p_i-1))}, \quad \forall x \in B_{\bar r}(x_i), \forall i,$$

$$\lim_{i\to\infty} u_i(x_i) \to \infty.$$

Then we say that $x_i \to \bar x$ is an isolated blow-up point of $\{u_i\}$.

To describe the behavior of blowing-up solutions near an isolated blow-up point, we define spherical averages of u_i centered at x_i as follows:

$$\bar u_i(r) = \frac{1}{\mathrm{vol}_g\left(M \cap \partial B_r(x_i)\right)} \int_{M\cap\partial B_r(x_i)} u_i(z).$$

Remark 1.1. Let $\{x_i\}$ and $\{\tilde{x}_i\}$ be two sequences tending to $\overline{x}$ satisfying the properties in Definition 1.1; then $x_i = \tilde{x}_i$ for large i.

In the following, we present a Harnack inequality that is used often in treating isolated blow-up points. We use $\mathbf{R}^n_+$ to denote the upper half space of $\mathbf{R}^n$ and, for $\sigma > 0$ and $\overline{x} \in \mathbf{R}^n$, $B^+_\sigma(\overline{x}) = \{x = (x^1, \ldots, x^n) \in \mathbf{R}^n_+ | |x - \overline{x}| < \sigma\}$, $B^+_\sigma = B^+_\sigma(0)$, $\partial' B^+_\sigma(\overline{x}) = \partial B^+_\sigma(\overline{x}) \cap \partial \mathbf{R}^n_+$, $\partial'' B^+_\sigma(\overline{x}) = \partial B_\sigma(\overline{x}) \cap \overline{\mathbf{R}^n_+}$.

Let $f_i, h_i \in C^0(\overline{B^+_2})$ satisfy

$$\|f_i\|_{L^\infty(B^+_2)} + \|h_i\|_{L^\infty(B^+_2)} \le \overline{C}, \tag{1.5}$$

and, for some $p_0 > 1$ and $\overline{c} > 0$,

$$p_0 \le p_i \le \frac{n+2}{n-2}, \qquad |c_i| \le \overline{c}. \tag{1.6}$$

Suppose that $u_i \in C^2(\overline{B^+_2})$ is a sequence of solutions to

$$\begin{cases} -\Delta u_i = f_i u_i^{p_i}, & u_i > 0, \text{ in } B^+_2, \\ \dfrac{\partial u_i}{\partial x^n} = h_i u_i^{(p_i+1)/2}, & \text{on } \partial' B^+_2. \end{cases} \tag{1.7}$$

LEMMA 1.2. *Assume (1.5), (1.6), and that $\{u_i\}$ satisfies (1.7). Let $0 < \overline{r} < 1/8$, $\overline{x} \in \overline{B^+_{1/8}}$, and suppose that $x_i \to \overline{x}$ is an isolated blow-up point of $\{u_i\}$. Then, for all $0 < r < \overline{r}$,*

$$\sup_{B^+_{2r}(x_i)\setminus B^+_{r/2}(x_i)} u_i \le C \inf_{B^+_{2r}(x_i)\setminus B^+_{r/2}(x_i)} u_i,$$

where $C > 0$ is some constant independent of i and r.

Proof. For $0 < r < \overline{r}$, we consider $\widetilde{u}_i(y) = r^{2/(p_i-1)} u_i(ry + x_i)$. Then $\widetilde{u}_i$ satisfies

$$\begin{cases} -\Delta \widetilde{u}_i = f_i(ry+x_i)\widetilde{u}_i^{p_i}, & \widetilde{u}_i > 0, \text{ in } A_i, \\ \dfrac{\partial \widetilde{u}_i}{\partial y^n} = h_i(ry+x_i)\widetilde{u}_i^{(p_i+1)/2}, & \text{on } \partial' A_i, \end{cases}$$

where $A_i = \{y \in \mathbf{R}^n : 1/3 < |y| < 3, ry + x_i \in \mathbf{R}^n_+\}, \partial' A_i = \partial A_i \cap \partial \mathbf{R}^n_+$.

We know from the first property in Definition 1.1 that $\widetilde{u}_i \le C$ in A_i. In view of (1.5) and (1.6), it follows from the Harnack inequality and Lemma A.1 in the appendix that

$$\max_{\widetilde{A}_i} \widetilde{u}_i \le C \min_{\widetilde{A}_i} \widetilde{u}_i,$$

where $\widetilde{A}_i = \{y \in \mathbf{R}^n : 1/2 < |y| < 2, ry + x_i \in \mathbf{R}^n_+\}$. Lemma 1.2 is thus established. $\square$

In the following, we use the Harnack inequality and Lemma A.1 to derive some properties of solutions of such equations. Let $c \in \mathbf{R}$, $f \in C^1(\overline{B_3^+})$ be some positive function, $1 < p_i \le ((n+2)/(n-2))$, $p_i \to ((n+2)/(n-2))$, $c_i \to c$, $f_i \to f$ in $C^1(\overline{B_3^+})$. Consider, for $\tau_i = ((n+2)/(n-2)) - p_i$,

$$\begin{cases} -\Delta v_i = n(n-2) f_i^{\tau_i} v_i^{p_i}, & v_i > 0, \text{ in } B_3^+, \\ \dfrac{\partial v_i}{\partial x^n} = -c_i f_i^{\tau_i/2} v_i^{(p_i+1)/2}, & \text{on } \partial' B_3^+. \end{cases} \tag{1.8}$$

LEMMA 1.3. *Suppose that* $\{v_i\}$ *satisfies (1.8) and* $\{x_i\} \subset B_1^+ \cup \partial' B_1^+$ *is a sequence of local maximum points of* $\{v_i\}$ *in* $\overline{B_3^+}$ *satisfying*

$$\{v_i(x_i)\} \text{ is bounded},$$

and, for some constant C_1,

$$|x - x_i|^{2/(p_i-1)} v_i(x) \le C_1, \quad \forall x \in B_3^+. \tag{1.9}$$

Then

$$\limsup_{i\to\infty} \max_{B_{1/4}^+(x_i)} v_i < \infty. \tag{1.10}$$

Proof. Suppose that (1.10) did not hold under the hypotheses of Lemma 1.3. Then, along a subsequence, we would have, for some $\tilde{x}_i \in \overline{B_{1/4}^+(x_i)}$, that

$$v_i(\tilde{x}_i) = \max_{\overline{B}_{1/4}^+(x_i)} v_i \to \infty.$$

It follows from (1.9) that $|\tilde{x}_i - x_i| \to 0$. Consider, for $\tilde{T}_i = \tilde{x}_i^n v_i(\tilde{x}_i)^{(p_i-1)/2}$,

$$\xi_i(z) = v_i(\tilde{x}_i)^{-1} v_i\big(\tilde{x}_i + v_i(\tilde{x}_i)^{-(p_i-1)/2} z\big), \quad z \in B^{-\tilde{T}_i}_{v_i(\tilde{x}_i)^{(p_i-1)/2}/8}(0),$$

where $B^{-\tilde{T}_i}_{v_i(\tilde{x}_i)^{(p_i-1)/2}/8}(0) = \{z \in \mathbf{R}^n : |z| < v_i(\tilde{x}_i)^{(p_i-1)/2}/8, z^n > -\tilde{T}_i\}$. We derive from (1.8) that

$$\begin{cases} -\Delta \xi_i = n(n-2) f^{\tau_i} \xi_i^{p_i}, & \xi_i > 0, z \in B^{-\tilde{T}_i}_{v_i(\tilde{x}_i)^{(p_i-1)/2}/8}(0), \\ \dfrac{\partial \xi_i}{\partial z^n} = -c_i f^{\tau_i/2} \xi_i^{(p_i+1)/2}, & z \in \partial' B^{-\tilde{T}_i}_{v_i(\tilde{x}_i)^{(p_i-1)/2}/8}(0), \end{cases}$$

and

$$\xi_i(z) \le \xi_i(0) = 1, \quad \forall z \in B^{-\tilde{T}_i}_{v_i(\tilde{x}_i)^{(p_i-1)/2}/8}(0),$$

where $\partial' B^{-\tilde{T}_i}_{v_i(\tilde{x}_i)^{(p_i-1)/2}/8}(0) = \{z \in \mathbf{R}^n : |z| < v_i(\tilde{x}_i)^{(p_i-1)/2}/8, z^n = -\tilde{T}_i\}$.

Applying L^p-estimates, Schauder estimates, the Harnack inequality, and Lemma A.1, we have, after passing to a subsequence, that

$$\lim_{i\to\infty} \|\xi_i - \xi\|_{C^2(\mathbf{R}^n_{-\tilde{T}_i} \cap \overline{B}_R)} = 0, \quad \forall R > 1,$$

where $\mathbf{R}^n_{-\tilde{T}_i} = \{z \in \mathbf{R}^n : z^n > -\tilde{T}_i\}$, ξ satisfies, for $\tilde{T} = \lim_{i\to\infty} \tilde{T}_i \in [0,\infty]$ and $c = \lim_{i\to\infty} c_i \in \mathbf{R}$, that

$$\begin{cases} -\Delta \xi(z) = n(n-2)\xi(z)^{(n+2)/(n-2)}, & \xi(z) > 0, \text{in } \mathbf{R}^n_{-\tilde{T}}, \\ \dfrac{\partial \xi}{\partial z^n} = -c\xi^{n/(n-2)}, & \text{on } \partial\mathbf{R}^n_{-\tilde{T}}, \text{in the case } \tilde{T} < \infty. \end{cases}$$

It follows that, for all $R > 1$,

$$\min_{\overline{B}^{-\tilde{T}_i}_{Rv_i(\tilde{x}_i)^{-(p_i-1)/2}}(\tilde{x}_i)} v_i = v_i(\tilde{x}_i) \min_{\overline{B}^{-\tilde{T}_i}_R(0)} \xi_i \to \infty.$$

Since $\{v_i(x_i)\}$ stays bounded, we have, in view of the above, that, for all $R > 1$, x_i does not belong to $\overline{B}^{-\tilde{T}_i}_{Rv_i(\tilde{x}_i)^{-(p_i-1)/2}}(\tilde{x}_i)$ for large i, namely, $|\tilde{x}_i - x_i| > Rv_i(\tilde{x}_i)^{-(p_i-1)/2}$ for large i, which violates (1.9). Lemma 1.3 is established. □

Definition 1.2. Let $x_i \to \bar{x}$ be an isolated blow-up point of $\{u_i\}$ as in Definition 1.1. We say that $x_i \to \bar{x}$ is an isolated simple blow-up point if for some positive constants $\tilde{r} \in (0, \bar{r})$ and $\tilde{C} > 1$, the function $\bar{w}_i(r) := r^{2/(p_i-1)}\bar{u}_i(r)$ satisfies, for large i,

$$\bar{w}_i'(r) < 0 \quad \text{for } r \text{ satisfying } \tilde{C}u_i(x_i)^{-(p_i-1)/2} \le r \le \tilde{r}. \tag{1.11}$$

Remark 1.2. It is not difficult to see that, for $\bar{x} \in M^\circ$, the notion of isolated simple blow-up point we introduced here is equivalent to the one introduced by Schoen (see [25, p. 322]).

The following proposition is established in §3.

PROPOSITION 1.3. *Let (M, g) be a smooth, compact n-dimensional locally conformally flat Riemannian manifold with umbilic boundary, and let $x_i \to \bar{x}$ be an isolated blow-up point of $\{u_i\}$. Then, it is necessarily an isolated simple blow-up point.*

Strong estimates can be obtained for isolated simple blow-up points as shown in the next proposition, which is established in §2.

PROPOSITION 1.4. *Let (M, g) be a smooth, compact n-dimensional locally conformally flat Riemannian manifold with umbilic boundary, and let $x_i \to \overline{x}$ be an isolated simple blow-up point of $\{u_i\}$. Then, for any sequences of positive numbers $R_i \to \infty$, $\epsilon_i \to 0$, there exists a subsequence $\{u_{j_i}\}$ (still denoted as $\{u_i\}$) such that*

$$r_i := R_i u_i^{-((p_i-1)/2)}(x_i) \to 0,$$

and either

$$\left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p_i-1)/2}(x_i)} \right) \right) - \left(\frac{1}{1+|y|^2} \right)^{(n-2)/2} \right\|_{C^2\left(B^M_{3R_i}(0)\right)}$$
$$+ \left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p-1)/2}(x_i)} \right) \right) - \left(\frac{1}{1+|y|^2} \right)^{(n-2)/2} \right\|_{H^1\left(B^M_{3R_i}(0)\right)} < \epsilon_i,$$

or, $x_i \in \partial M$,

$$\left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p_i-1)/2}(x_i)} \right) \right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2+|y^n+t_c|^2\right)} \right)^{(n-2)/2} \right\|_{C^2\left(B^M_{3R_i}(0)\right)}$$
$$+ \left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p_i-1)/2}(x_i)} \right) \right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2+|y^n+t_c|^2\right)} \right)^{(n-2)/2} \right\|_{H^1\left(B^M_{3R_i}(0)\right)} < \epsilon_i.$$

Moreover, for all $2r_i \le d(x, x_i) \le \tilde{r}/2$,

$$u_i(x) \le C u_i(x_i)^{-1} d\left(x, x_i\right)^{2-n} \tag{1.12}$$

where C *is some positive constant independent of* i, *and*

$$u_i(x_i) u_i \to h, \quad \text{in } C^2_{\text{loc}}\left(\overline{B_{\tilde{r}}(\overline{x})} \setminus \{\overline{x}\}\right),$$

for some $h \in C^2(\overline{B_{\tilde{r}}(\overline{x})} \setminus \{0\})$ *satisfying*

$$\begin{cases} L_g h = 0, & \text{in } B_{\tilde{r}}(\overline{x}) \setminus \{\overline{x}\}, \\ h(x) \to \infty, & \text{as } x \to \overline{x}, \\ B_g h = 0, & \text{in } B_{\tilde{r}}(\overline{x}) \cap \partial M \text{ if } B_{\tilde{r}}(\overline{x}) \cap \partial M \neq \emptyset. \end{cases}$$

The following Pohozaev identity (for a proof, see Theorem 1.1 in [25]) is used often.

LEMMA 1.4. *Let* Ω *be a piecewise smooth bounded domain in* $\mathbf{R}^n (n \ge 3)$, *let* $K \in C^1(\overline{\Omega})$, *and let* $u > 0$ *be a* $C^2(\overline{\Omega})$ *solution of*

$$-\Delta u = n(n-2) K u^p,$$

in Ω. *Then,*

$$n(n-2)\left(\frac{n}{p+1} - \frac{n-2}{2} \right) \int_\Omega K u^{p+1} + \frac{n(n-2)}{p+1} \int_\Omega x \cdot \nabla K(x) u^{p+1}$$
$$= \int_{\partial\Omega} \left\{ x \cdot \nu \left(\frac{n(n-2)}{p+1} K u^{p+1} - \frac{|\nabla u|^2}{2} \right) + \frac{\partial u}{\partial \nu} x \cdot \nabla u + \frac{n-2}{2} u \frac{\partial u}{\partial \nu} \right\},$$

where ν *denotes the unit outer normal of* $\partial\Omega$.

§2. The proof of Proposition 1.4. The following lemma, essentially established in the proof of Proposition 2.4 in [24], gives some properties of boundary-isolated blow-up points.

LEMMA 2.1. *Let (M, g) be a smooth, compact, n-dimensional, locally conformally flat Riemannian manifold with umbilic boundary, $\bar{r} > 0, \bar{c} > 0$, $\bar{x} \in \partial M$, $f \in C^1(\overline{B_{\bar{r}}(\bar{x})})$ be some positive function. Suppose, for sequences $|c_i| \le \bar{c}$, $p_i \le ((n+2)/(n-2))$, $p_i \to ((n+2)/(n-2))$, $f_i \to f$ in $C^1(\overline{B_{\bar{r}}(\bar{x})})$, $\{u_i\}$ satisfies (1.4) with $\tau_i = p_i - ((n+2)/(n-2))$, and $x_i \to \bar{x}$ is an isolated blow-up point. Then $\{d(x_i, \partial M) u_i(x_i)^{(p_i-1)/2}\}$ stays bounded.*

Proof. The proof is similar to that of Proposition 2.4 in [24]. For the reader's convenience, we include the proof. Since we are assuming that M is locally conformally flat and ∂M is umbilic, and since the form of the equations together with the boundary conditions are invariant under conformal transformations, we may work locally and assume that $M = B_2^+$ is equipped with the Euclidean metric, and $x_i = (0, \dots, 0, x_{in}) \to 0$ is an isolated blow-up point where $x_{in} \ge 0$. Set $T_i = x_{in} u_i(x_i)^{(p_i-1)/2}$. We need to show that

$$\{T_i\} \text{ stays bounded.} \tag{2.1}$$

We establish the above by a contradiction argument. Suppose the contrary of (2.1). We consider, for a subsequence along which $T_i \to \infty$,

$$\xi_i(z) = x_{in}^{2/(p_i-1)} u_i(x_i + x_{in} z), \quad z \in B_{1/x_{in}}^{-1},$$

where $B_{1/x_{in}}^{-1} = B_{1/x_{in}} \cap \{z \mid z^n > -1\}$.

Clearly ξ_i satisfies

$$\begin{cases} -\Delta \xi_i(z) = n(n-2) \tilde{f}_i^{\tau_i} \xi_i(z)^{p_i}, & z \in B_{1/x_{in}}^{-1}, \\ \dfrac{\partial \xi_i}{\partial z^n} = -c_i \tilde{f}_i^{\tau_i/2} \xi_i^{(p_i+1)/2}, & z \in \partial' B_{1/x_{in}}^{-1}, \\ |z|^{2/(p_i-1)} \xi_i(z) \le \overline{C}, & z \in B_{1/x_{in}}^{-1}, \\ \lim\limits_{i\to\infty} \xi_i(0) = \lim\limits_{i\to\infty} T_i^{2/(p_i-1)} = \infty, \end{cases}$$

where $\tilde{f}_i(z) = f_i(x_i + x_{in} z)$ and $\partial' B_{1/x_{in}}^{-1} = \{(z^1, \dots, z^{n-1}, -1) : |(z^1, \dots, z^{n-1}, -1)| < 1/x_{in}\}$.

It follows that $\{0\}$ is an interior isolated blow-up point of $\{\xi_i\}$. It follows from Proposition 3.1 in [25] that it is an interior isolated simple blow-up point of $\{\xi_i\}$. In turn, it follows from Proposition 2.3 in [25], the Harnack inequality, Lemma A.1, and standard elliptic estimates that, after passing to a subsequence, we have

$$\xi_i(0)\xi_i(z) \to h(z), \quad \text{in } C^2_{\text{loc}}\big(\overline{\mathbf{R}^n_{-1}} \setminus \{0\}\big),$$

where $\mathbf{R}^n_{-1} = \{(z^1, \ldots, z^n) \in \mathbf{R}^n : z^n > -1\}$, and h is singular near $\{0\}$ and satisfies

$$\begin{cases} -\Delta h(z) = 0, & h(z) > 0, \mathbf{R}^n_{-1} \setminus \{0\}, \\ \dfrac{\partial h}{\partial z^n}(z) = 0, & \partial \mathbf{R}^n_{-1}. \end{cases}$$

Extending h evenly across the hyperplace $z^n = -1$, we derive from Böcher's theorem (see e.g., [22]) and the maximum principle that

$$h(z) = a\big(|z|^{2-n} + |z - (0, 0, -2)|^{2-n}\big) + b,$$

for some constants $a > 0$ and $b \geq 0$. Consequently, for $A = a|(0, 0, -2)|^{2-n} + b > 0$, we have

$$h(z) = a|z|^{2-n} + A + O(|z|). \tag{2.2}$$

In the following, we derive a contradiction as in the proof of Proposition 3.1 in [25].

For $0 < \sigma < 1$, we apply Lemma 1.4 to ξ_i on B_σ to obtain

$$\begin{aligned} n(n-2)\left(\frac{n}{p_i+1} - \frac{n-2}{2}\right)\int_{B_\sigma} \tilde{f}_i^{\tau_i}\xi_i^{p_i+1} + \frac{n(n-2)}{p_i+1}\int_{B_\sigma} \tau_i \tilde{f}_i^{\tau_i-1}\left(z\cdot\nabla \tilde{f}_i\right)\xi_i^{p_i+1} \\ = \int_{\partial B_\sigma} B\left(\sigma, z, \xi_i, \nabla\xi_i\right) + \frac{n(n-2)}{p+1}\sigma\int_{\partial B_\sigma} \tilde{f}_i\xi_i^{p_i+1}, \end{aligned}$$

where $B(\sigma, z, u, \nabla u) = ((n-2)/2)u(\partial u/\partial\nu) - (\sigma/2)|\nabla u|^2 + \sigma(\partial u/\partial \nu)^2$. Multiplying the above identity by $\xi_i(0)^2$ and sending i to ∞, we obtain, by using Proposition 2.3, Lemma 2.4, and Lemma 2.7 in [25], that

$$\xi_i(0)^2\tau_i\int_{B_\sigma} \tilde{f}_i^{\tau_i-1}\big(z\cdot\nabla\tilde{f}_i\big)\xi_i^{p_i+1} \to 0$$

and

$$\xi_i(0)^2\int_{\partial B_\sigma} \tilde{f}_i\xi_i^{p_i+1} \to 0,$$

which implies

$$\int_{\partial B_\sigma} B\big(\sigma, z, h, \nabla h\big) \geq 0.$$

On the other hand, in view of (2.2), a direct computation shows that

$$\lim_{\sigma\to 0}\int_{\partial B_\sigma} B\big(\sigma, z, h, \nabla h\big) = -\frac{(n-2)^2}{2}A\left|\mathbf{S}^{n-1}\right|,$$

which contradicts $A > 0$. Thus, we have established (2.1). The proof of Lemma 2.1 is completed. $\square$

The following lemma is an important step toward establishing Proposition 1.4. The main difference between this lemma and Proposition 1.4 is that we do not know yet whether we can take δ_i in (2.3) to be equal to zero, which would be the same as (1.12).

LEMMA 2.2. *Let (M, g) be a smooth, compact n-dimensional locally conformally flat Riemannian manifold with umbilic boundary, and let $x_i \to \overline{x}$ be an isolated simple blow-up point of $\{u_i\}$. Then, for any sequences of positive numbers $R_i \to \infty$, $\epsilon_i = \circ(R_i^{2-n})$, there exists a subsequence $\{u_{j_i}\}$ (still denoted as $\{u_i\}$) such that*

$$r_i := R_i u_i^{-((p_i-1)/2)}(x_i) \to 0,$$

and either

$$\left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p_i-1)/2}(x_i)} \right) \right) - \left(\frac{1}{1+|y|^2} \right)^{(n-2)/2} \right\|_{C^2\left(B^M_{3R_i}(0)\right)}$$
$$+ \left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p-1)/2}(x_i)} \right) \right) - \left(\frac{1}{1+|y|^2} \right)^{(n-2)/2} \right\|_{H^1\left(B^M_{3R_i}(0)\right)} < \epsilon_i,$$

or, $x_i \in \partial M$,

$$\left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p_i-1)/2}(x_i)} \right) \right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2+|y^n+t_c|^2\right)} \right)^{(n-2)/2} \right\|_{C^2\left(B^M_{3R_i}(0)\right)}$$
$$+ \left\| u_i^{-1}(x_i) u_i \left(\exp_{x_i} \left(\frac{y}{u_i^{(p_i-1)/2}(x_i)} \right) \right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2+|y^n+t_c|^2\right)} \right)^{(n-2)/2} \right\|_{H^1\left(B^M_{3R_i}(0)\right)} < \epsilon_i.$$

Moreover, for all $r_i \le d(x, x_i) \le \tilde{r}/2$,

$$u_i(x) \le C u_i^{-\lambda_i}(x_i) d(x, x_i)^{2-n+\delta_i}, \tag{2.3}$$

where $\lambda_i = (n-2-\delta_i)(p_i-1)/2 - 1$ for some $0 < \delta_i = O(R_i^{-1+\circ(1)})$.

Proof. In the case where the blow-up point is an interior point, see Lemma 2.2 in [25]. Thus, we concentrate on the case of a boundary blow-up point. Since we are assuming that M is locally conformally flat and ∂M is umbilic, we may work locally and assume that $M = B_2^+$ is equipped with the Euclidean metric, and $y_i = (0, \dots, 0, y_{in}) \to 0$ is an isolated simple blow-up point for a sequence of solutions u_i. The equation takes the form

$$\begin{cases} -\Delta u_i = n(n-2) f_i^{\tau_i} u_i^{p_i}, & u_i > 0, \text{in } B_2^+, \\ \dfrac{\partial u_i}{\partial y^n} = -c_i f_i^{\tau_i/2} u_i^{(p_i+1)/2}, & \text{on } \partial' B_2^+, \end{cases} \tag{2.4}$$

where, as usual, $\partial' B_2^+ = \{(y^1, \dots, y^{n-1}, 0) : |(y^1, \dots, y^{n-1}, 0)| < 2\}$.

The proof goes along the lines of the proof of Lemma 2.2 in [25]. The difference is that we need to find different comparison functions when using the maximum principle. Arguing as in the proof of Proposition 1.1, it is easy to see that, for any sequences of positive numbers $R_i \to \infty$, $\epsilon_i \to 0$, we have, after passing to a subsequence $\{u_{j_i}\}$ (still denoted as $\{u_i\}$), that

$$\left\| u_i^{-1}(y_i) u_i\left(y_i + u_i^{-((p_i-1)/2)}(y_i) y\right) - \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2} \right\|_{C^2\left(B^M_{3R_i}(0)\right)} + \left\| u_i^{-1}(y_i) u_i\left(y_i + u_i^{-((p_i-1)/2)}(y_i) y\right) - \left(\frac{1}{1+|y|^2}\right)^{(n-2)/2} \right\|_{H^1\left(B^M_{3R_i}(0)\right)} < \epsilon_i,$$

or $y_i \in \partial' M = \{(x', 0) : |x'| < 2\}$, and

$$\left\| u_i^{-1}(y_i) u_i\left(y_i + u_i^{-((p_i-1)/2)}(y_i) y\right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2 + |y^n + t_c|^2\right)}\right)^{(n-2)/2} \right\|_{C^2\left(B^M_{3R_i}(0)\right)} + \left\| u_i^{-1}(y_i) u_i\left(y_i + u_i^{-((p_i-1)/2)}(y_i) y\right) - \left(\frac{\lambda_c}{1+\lambda_c^2\left(|y'|^2 + |y^n + t_c|^2\right)}\right)^{(n-2)/2} \right\|_{H^1\left(B^M_{3R_i}(0)\right)} < \epsilon_i,$$

and $r_i = R_i u_i^{-((p_i-1)/2)}(y_i) \to 0$. As a consequence, if we choose $\epsilon_i = \circ(R_i^{2-n})$,

$$u_i(y) \le C u_i(y_i) R_i^{2-n}, \quad \text{for all } \frac{r_i}{2} \le |y - y_i| \le 3r_i, \tag{2.5}$$

and we only need to provide the bound for $r_i \le |y - y_i| \le 1$. (We take $\tilde{r} = 2$ in the definition of an isolated simple blow-up point for simplicity.) It follows from the estimate above, Lemma 1.2, and (1.11), that, for $r_i < |y - y_i| \le 7/4$, we have

$$\begin{aligned} |y - y_i|^{2/(p_i-1)} u_i(y) &\le C |y - y_i|^{2/(p_i-1)} \bar{u}_i(|y - y_i|) \\ &\le C r_i^{2/(p_i-1)} \bar{u}_i(r_i) \\ &\le C R_i^{(2-n)/2 + \circ(1)}. \end{aligned}$$

Therefore,

$$u_i^{p_i-1}(y) \le O\left(R_i^{-2+\circ(1)}\right) |y - y_i|^{-2}, \quad \text{for all } r_i \le |y - y_i| \le \frac{7}{4}. \tag{2.6}$$

Set $T_i = y_{in} u_i(y_i)^{(p_i-1)/2}$. It follows from Lemma 2.1 that $\{T_i\}$ stays bounded. Consequently,

$$|y_i| = \circ(r_i) \quad \text{and} \quad B_1^+(0) \setminus B_{2r_i}^+(0) \subset \left\{ y : \frac{3}{2} r_i \le |y - y_i| \le \frac{7}{4} \right\} \quad \text{for large } i. \tag{2.7}$$

We construct comparison functions and apply Lemma A.2 to u_i. Consider the second-order elliptic operator $L_i = \Delta + n(n-2) f_i^{\tau_i} u_i^{p_i-1}$ in B_2^+, and the boundary operator $B_i = -(\partial/\partial y^n) - c_i f_i^{\tau_i/2} u_i^{(p_i-1)/2}$ on $\partial' B_2^+$. Then $L_i u_i = 0$, $u_i > 0$ in B_2^+, and $B_i u_i = 0$ on $\partial' B_2^+$. It follows from Lemma A.2 that the maximum principle holds for (L_i, B_i). Our construction of comparison functions is carried out according to two different cases.

Case 1. $c_i \le 0$. It is clear, in view of (2.7), that

$$\frac{1}{10}|y|^{-2} \le |y-y_i|^{-2} \le 10|y|^{-2}, \quad \text{in } B_1^+(0) \setminus B_{2r_i}^+(0). \tag{2.8}$$

We can easily derive from (2.6), (2.7), and (2.8) that for $0 \le \mu \le n-2$,

$$L_i\big(|y|^{-\mu}\big) = \left\{-\mu(n-2-\mu) + O\left(R_i^{-2+\circ(1)}\right)\right\}|y|^{-2-\mu}$$

in $B_1^+(0) \setminus B_{2r_i}^+(0)$.

On $\partial' B_1^+(0) \setminus \partial' B_{2r_i}^+(0) = \{(y^1, \dots, y^{n-1}, 0) : 2r_i < |(y^1, \dots, y^{n-1}, 0)| < 1\}$,

$$B_i\big(|y|^{-\mu}\big) = -c_i f_i^{\tau_i/2} u_i^{(p_i-1)/2} |y|^{-\mu} \ge 0.$$

Set $M_i = \max_{\partial'' B_1^+(0)} u_i$ and $\lambda_i = (n-2-\delta_i)(p_i-1)/2 - 1$, where $\partial'' B_1^+(0) = \{(y^1, \dots, y^n) : |(y^1, \dots, y^n)| = 1, y^n \ge 0\}$, and $0 < \delta_i = O(R_i^{-2+\circ(1)})$ is chosen so that $-\delta_i(n-2-\delta_i) + O(R_i^{-2+\circ(1)}) \le 0$. Let

$$\varphi_i(y) = M_i|y|^{-\delta_i} + A u_i^{-\lambda_i}(y_i)|y|^{2-n+\delta_i} - u_i(y), \quad \text{in } B_1^+(0) \setminus B_{2r_i}^+(0),$$

where A will be chosen in a moment. It follows from our computation that φ_i satisfies

$$\begin{cases} L_i(\varphi_i) \le 0, & B_1^+(0) \setminus B_{2r_i}^+(0), \\ B_i(\varphi_i) \ge 0, & \partial' B_1^+(0) \setminus \partial' B_{2r_i}^+(0), \\ \varphi_i \ge 0, & \partial'' B_1^+(0) \cup \partial'' B_{2r_i}^+(0), \end{cases}$$

provided we choose A large enough so that, on $|y| = 2r_i$, $\varphi_i \ge 0$. Such an A can be chosen with an absolute bound independent of i because of (2.5) and the choice of λ_i. Then we derive from Lemma A.2 that $\varphi_i(y) \ge 0$ on $B_1^+(0) \setminus B_{2r_i}^+(0)$. Thus, we use the above estimate, (1.11), and Lemma 1.2 to obtain, for $r_i < \theta < 1$, that

$$\begin{aligned} M_i &\le C\bar{u}_i(1) \le C\theta^{2/(p_i-1)}\bar{u}_i(\theta) \\ &\le C\theta^{2/(p_i-1)}\big\{M_i\theta^{-\delta_i} + A u_i^{-\lambda_i}(y_i)\theta^{2-n+\delta_i}\big\}. \end{aligned}$$

Choosing θ sufficiently small, but independent of i, so that $C\theta^{2/(p_i-1)-\delta_i} < 1/2$, we obtain

$$M_i \le C u_i^{-\lambda_i}(y_i). \tag{2.9}$$

Combining (2.9) and the estimate $\varphi_i \geq 0$, we have, for some constant C independent of i, that

$$u_i(y) \leq C u_i^{-\lambda_i}(y_i)|y|^{2-n+\delta_i}, \quad \text{for } 2r_i \leq |y| \leq 1.$$

Estimate (2.3) follows from the above and (2.8).

Case 2. $c_i > 0$. In this case, we modify $|y|^{-\mu}$ by $|y|^{-\mu} - \epsilon|y|^{-\mu-1}y^n$ and obtain, for all $0 \leq \mu \leq n-2$ and $\epsilon > 0$, that

$$B_i\big(|y|^{-\mu} - \epsilon|y|^{-\mu-1}y^n\big) = |y|^{-\mu-1}\left\{\epsilon + O\left(R_i^{-1+\circ(1)}\right)\right\}, \quad \forall y \in \partial' B_1^+(0) \setminus \partial' B_{2r_i}^+(0),$$

and, for all $y \in B_1^+(0) \setminus B_{2r_i}^+(0)$, that

$$L_i\big(|y|^{-\mu} - \epsilon|y|^{-\mu-1}y^n\big) = |y|^{-\mu-2}\left\{-\mu(n-2-\mu) + \epsilon(\mu+1)(n-1-\mu)y^n/|y| + O\left(R_i^{-2+\circ(1)}\right)\right\}.$$

Apparently we can find $0 < \delta_i = O(R_i^{-1+\circ(1)})$ and $0 < \epsilon_i = O(R_i^{-1+\circ(1)})$ so that, for all $y \in \partial' B_1^+(0) \setminus \partial' B_{2r_i}^+(0)$,

$$B_i\big(|y|^{-\delta_i} - \epsilon_i|y|^{-\delta_i-1}y^n\big) \geq 0, \qquad B_i\big(|y|^{2-n+\delta_i} - \epsilon_i|y|^{1-n+\delta_i}y^n\big) \geq 0,$$

and, for all $y \in B_1^+(0) \setminus B_{2r_i}^+(0)$,

$$L_i\big(|y|^{-\delta_i} - \epsilon_i|y|^{-\delta_i-1}y^n\big) \leq 0, \qquad L_i\big(|y|^{2-n+\delta_i} - \epsilon_i|y|^{1-n+\delta_i}y^n\big) \leq 0.$$

We define in $B_1^+(0) \setminus B_{2r_i}^+(0)$,

$$\varphi_i(y) = M_i\big(|y|^{-\delta_i} - \epsilon_i|y|^{-\delta_i-1}y^n\big) + A u_i^{-\lambda_i}(y_i)\big(|y|^{2-n+\delta_i} - \epsilon_i|y|^{1-n+\delta_i}y^n\big) - \frac{1}{2}u_i(y),$$

where M_i and λ_i are as in case 1. Arguing as in case 1, we obtain the desired upper bound of u_i. We have thus established Lemma 2.2. □

LEMMA 2.3. *Suppose that $\{u_i\}$ satisfies (2.4) and $y_i \to \overline{y} \in B_1^+ \cup \partial' B_1^+$ is an isolated simple blow-up point. Then, for any $0 < \sigma < \overline{r}$, there exists $C > 1$ independent of i such that*

$$n(n-2)\left(\frac{n}{p_i+1} - \frac{n-2}{2}\right)\int_{B_\sigma^+} f_i^{\tau_i} u_i^{p_i+1} + c_i\left(\frac{2(n-1)}{p_i+3} - \frac{n-2}{2}\right)\int_{\partial' B_\sigma^+} f_i^{\tau_i/2} u_i^{(p_i+3)/2} \geq \frac{\tau_i u_i(y_i)^{(n-2)\tau_i/2}}{C}.$$

Proof. If $\overline{y} \in B_1^+$, the proof follows easily from Lemma 2.2. We assume without loss of generality that $\overline{y} = 0$ and $y_i = (0, \ldots, 0, y_{in})$. Set $T_i = y_{in} u_i(y_i)^{(p_i-1)/2}$, and, after passing to a subsequence, $T = \lim_{i\to\infty} T_i$. It follows from Lemma 2.1 that $0 \le T < \infty$. Consider

$$\eta_i(z) = u_i(y_i)^{-1} u_i\left(y_i + u_i(y_i)^{-(p_i-1)/2} z\right), \quad z \in B^{-T_i}_{u_i(y_i)^{(p_i-1)/2}/8},$$

where $B^{-T_i}_{u_i(y_i)^{(p_i-1)/2}/8} = \{z \in \mathbf{R}^n : |z| < u_i(y_i)^{(p_i-1)/2}/8, z^n > -T_i\}$.

It follows that η_i satisfies

$$\begin{cases} -\Delta \eta_i(z) = n(n-2) \tilde{f}_i^{\tau_i} \eta_i(z)^{p_i}, \quad \eta_i(z) > 0,\ z \in B^{-T_i}_{u_i(y_i)^{(p_i-1)/2}/8}, \\ \dfrac{\partial \eta_i}{\partial z^n} = -c_i \tilde{f}_i^{\tau_i/2} \eta_i^{(p_i+1)/2}, \quad z \in \partial' B^{-T_i}_{u_i(y_i)^{(p_i-1)/2}/8}, \\ |z|^{2/(p_i-1)} \eta_i(z) \le \overline{C}, \quad z \in B^{-T_i}_{u_i(y_i)^{(p_i-1)/2}/8}, \\ \eta_i(0) = 1 \text{ and } 0 \text{ are local maximum points of } \eta_i, \end{cases} \tag{2.10}$$

where $\tilde{f}_i(z) = f_i(y_i + u_i(y_i)^{-(p_i-1)/2} z)$.

It follows from Lemma 1.3, the Harnack inequality, and Lemma A.1 that $\{\eta_i\}$ is locally bounded. Therefore, for any $R_i \to \infty$, we deduce from standard elliptic estimates that, after passing to a subsequence $\{u_{j_i}\}$ (still denoted as $\{u_i\}$), $R_i = \circ(u_i(y_i)^{(p_i-1)/2}/8)$ and

$$\|\eta_i - \eta\|_{C^2\left(\overline{B}^{-T_i}_{3R_i}\right)} + \|\eta_i - \eta\|_{H^1\left(\overline{B}^{-T_i}_{3R_i}\right)} < e^{-R_i}, \tag{2.11}$$

for some η satisfying

$$\begin{cases} -\Delta \eta(z) = n(n-2)\eta(z)^{(n+2)/(n-2)}, \quad \eta(z) > 0,\ \mathbf{R}^n_{-T}, \\ \dfrac{\partial \eta}{\partial z^n} = -c\eta^{n/(n-2)}, \qquad \partial \mathbf{R}^n_{-T}, \\ \eta(0) = 1 \text{ and } 0 \text{ are local maximum points}, \end{cases} \tag{2.12}$$

where $c = \lim_{i\to\infty} c_i$, $\mathbf{R}^n_{-T} = \{(z^1, \ldots, z^n) : z^n > -T\}$. It follows from the Liouville-type theorem in [27] that when $c < 0$, $T = -(c/(n-2)) > 0$, and

$$\eta(z) = \frac{1}{\left(1 + |z'|^2 + |z^n + (c/n-2)|^2\right)^{(n-2)/2}}; \tag{2.13}$$

and when $c \ge 0$, $T = 0$, and

$$\eta(z) = \left(\frac{\lambda_c}{1 + \lambda_c^2\left(|z'|^2 + |z^n + t_c|^2\right)}\right)^{(n-2)/2}, \tag{2.14}$$

where λ_c and t_c are defined as in Proposition 1.1.

It follows from (2.11) and Lemma 2.2 that

$$\begin{aligned}\int_{B_\sigma^+} f^{\tau_i} u_i^{p_i+1} &= \big(1+\circ(1)\big)u_i(y_i)^{(n-2)\tau_i/2}\int_{B_{R_i}^{-T_i}} \eta^{p_i+1}\\ &= \big(1+\circ(1)\big)u_i(y_i)^{(n-2)\tau_i/2}\int_{\mathbf{R}^n_{-T}} \eta^{2n/(n-2)},\\ \int_{\partial' B_\sigma^+} f^{\tau_i/2}u_i^{(p_i+3)/2} &= \big(1+\circ(1)\big)u_i(y_i)^{(n-2)\tau_i/2}\int_{\partial' B_{R_i}^{-T_i}} \eta^{(p_i+3)/2}\\ &= \big(1+\circ(1)\big)u_i(y_i)^{(n-2)\tau_i/2}\int_{\partial\mathbf{R}^n_{-T}} \eta^{2(n-1)/(n-2)}.\end{aligned}$$

It follows that

(2.15)
$$\begin{aligned} n(n-2)&\left(\frac{n}{p_i+1}-\frac{n-2}{2}\right)\int_{B_\sigma^+} f^{\tau_i}u_i^{p_i+1} + c_i\left(\frac{2(n-1)}{p_i+3}-\frac{n-2}{2}\right)\int_{\partial' B_\sigma^+} f^{\tau_i/2}u_i^{(p_i+3)/2}\\ &= \big(1+\circ(1)\big)\frac{8(n-1)}{(n-2)^2}u_i(y_i)^{(n-2)\tau_i/2}\tau_i\left\{2(n-1)(n-2)\int_{\mathbf{R}^n_{-T}} \eta^{2n/(n-2)}\right.\\ &\qquad\left. + c\int_{\partial\mathbf{R}^n_{-T}} \eta^{2(n-1)/(n-2)}\right\}.\end{aligned}$$

Multiplying the first line of (2.12) by η and integrating by parts on $\mathbf{R}^n_{-T}$, we have, in view of the boundary condition in (2.12), that

$$n(n-2)\int_{\mathbf{R}^n_{-T}} \eta^{2n/(n-2)} + c\int_{\partial\mathbf{R}^n_{-T}} \eta^{2(n-1)/(n-2)} = \int_{\mathbf{R}^n_{-T}} |\nabla\eta|^2 > 0. \tag{2.16}$$

Lemma 2.3 follows from (2.15) and (2.16). □

LEMMA 2.4. *Suppose that* $\{u_i\}$ *satisfies (2.4) and* $y_i \to \overline{y} \in B_1^+ \cup \partial' B_1^+$ *is an isolated simple blow-up point. Then*

$$\tau_i = O\big(u_i(y_i)^{-2}\big), \tag{2.17}$$

and therefore,

$$u_i(y_i)^{\tau_i} = 1+\circ(1).$$

Proof. For $\overline{y} \in \partial' B_1^+$, we assume, without loss of generality, that $\overline{y} = 0$ and $y_i = (0,\dots,0,y_{in})$. Applying Lemma 1.4 to u_i on B_σ^+ for some $\sigma > 0$, we have, in view of Lemma 2.2 and the boundary condition of u_i in (2.4), that

$$
\begin{aligned}
&n(n-2)\left(\frac{n}{p_i+1}-\frac{n-2}{2}\right)\int_{B_\sigma^+} f^{\tau_i}u_i^{p_i+1}\\
&\quad=\int_{\partial'' B_\sigma^+} B(\sigma,y,u_i,\nabla u_i)+c_i\int_{\partial' B_\sigma^+}\left(\sum_{j=1}^{n-1} y^j\frac{\partial u_i}{\partial y^j}+\frac{n-2}{2}u_i\right) f^{\tau_i/2}u_i^{(p_i+1)/2}\\
&\qquad+\circ\big(u_i(y_i)^{-2}\big)+\circ(\tau_i)\\
&\quad=-c_i\left(\frac{2(n-1)}{p_i+3}-\frac{n-2}{2}\right)\int_{\partial' B_\sigma^+} f^{\tau_i/2}u_i^{(p_i+3)/2}+O\big(u_i(y_i)^{-2}\big)+\circ(\tau_i),
\end{aligned}
$$

where $B(\sigma,y,u_i,\nabla u_i)$ is defined as before, and we have integrated by parts on the term $\int_{\partial' B_\sigma^+}\sum_{j=1}^{n-1} y^j(\partial u_i/\partial y^j)f^{\tau_i/2}u_i^{(p_i+1)/2}$.

It follows from Lemma 2.3 that

$$
\begin{aligned}
n(n-2)\left(\frac{n}{p_i+1}-\frac{n-2}{2}\right)&\int_{B_\sigma^+} f^{\tau_i}u_i^{p_i+1}\\
&+c_i\left(\frac{2(n-1)}{p_i+3}-\frac{n-2}{2}\right)\int_{\partial' B_\sigma^+} f^{\tau_i/2}u_i^{(p_i+3)/2}\geq\frac{\tau_i}{C}.
\end{aligned}
$$

Estimate (2.17) follows from the above two inequalities. For $\overline{y}\in B_1^+$, we apply the Pohozaev identity in $B_\sigma(y_i)$ and conclude the same way. □

Proof of Proposition 1.4. We follow pages 338–341 of [25] with rather obvious modifications. □

§3. The proof of Proposition 1.3. Due to the conformal invariance of L_g and B_g, Proposition 1.3 follows easily from the next proposition.

PROPOSITION 3.1. *Let $y_i\to\overline{y}\in B_1^+\cup\partial' B_1^+$ be an isolated blow-up point of $\{u_i\}$, solutions of (2.4). Then $y_i\to\overline{y}$ is an isolated simple blow-up point.*

If $\overline{y}\in B_1^+$, the conclusion is known (see, for example, Proposition 3.1 in [25]). We only consider $\overline{y}\in\partial' B_1^+$. Without loss of generality, we assume $\overline{y}=0$, $y_i=(0,\dots,0,y_{in})$, and $\overline{r}=1/4$.

Recall that

$$\overline{u}_i(r)=\frac{1}{\left|\partial'' B_r^+(y_i)\right|}\int_{\partial'' B_r^+(y_i)} u_i,$$

where, as usual, $B_r^+(y_i)=\{y\in\mathbf{R}_+^n:|y-y_i|<r\}$, $\partial'' B_r^+(y_i)=\partial B_r(y_i)\cap\overline{\mathbf{R}_+^n}$. Also,

$$\overline{w}_i(r)=r^{2/(p_i-1)}\overline{u}_i(r).$$

Proof of Proposition 3.1. Suppose the contrary; then there exist some sequences of positive numbers $\tilde{r}_i\to 0$ and $\tilde{C}_i\to\infty$ satisfying $\tilde{C}_i u_i(y_i)^{-(p_i-1)/2}\leq\tilde{r}_i$ such that, after passing to a subsequence,

$$\overline{w}_i'(\tilde{r}_i) \geq 0. \tag{3.1}$$

Set $T_i = y_{in} u_i(y_i)^{(p_i-1)/2}$ and, after passing to a subsequence, $T = \lim_{i\to\infty} T_i$. It follows from Lemma 2.1 that $0 \leq T < \infty$. Consider

$$\eta_i(z) = u_i(y_i)^{-1} u_i\big(y_i + u_i(y_i)^{-(p_i-1)/2} z\big), \quad z \in B^{-T_i}_{u_i(y_i)^{(p_i-1)/2}/8},$$

where $B^{-T_i}_{u_i(y_i)^{(p_i-1)/2}/8} = \{z \in \mathbf{R}^n : |z| < u_i({}_i)^{(p_i-1)/2}/8, z^n > -T_i\}$. It follows that η_i satisfies (2.10).

It follows from Lemma 1.3, the Harnack inequality, and Lemma A.1 that $\{\eta_i\}$ is locally bounded. Let $R_i \gg \tilde{C}_i$; after passing to a subsequence, as before, we have (2.11) for some η satisfying (2.12). It follows from the Liouville-type theorem in [27] that when $c < 0$, we have $T = -(c/(n-2)) > 0$, and η is given by (2.13); when $c \geq 0$ we have $T = 0$, and η is given by (2.14) where λ_c and t_c are defined as in Proposition 1.1.

Let $s = u_i(x_i)^{(p_i-1)/2} r$, and set

$$\overline{\eta}_i(s) = \frac{1}{\big|\partial'' B_s^{-T_i}(0)\big|} \int_{\partial'' B_s^{-T_i}(0)} \eta_i, \qquad \overline{\eta}(s) = \frac{1}{\big|\partial'' B_s^{-T_i}(0)\big|} \int_{\partial'' B_s^{-T_i}(0)} \eta.$$

It follows from (2.11) that, after passing to a subsequence,

$$\Big\| s^{2/(p_i-1)} \overline{\eta}_i(s) - s^{(n-2)/2} \overline{\eta}(s) \Big\|_{C^2([0,3R_i])} \leq e^{-\sqrt{R_i}}. \tag{3.2}$$

It is easy to see from the explicit form of $s^{(n-2)/2}\overline{\eta}(s)$ that, for some positive constant C independent of i,

$$\frac{d}{ds}\big(s^{(n-2)/2}\overline{\eta}(s)\big) \leq -\frac{s^{-(n/2)}}{C}, \quad \forall C \leq s \leq 3R_i,$$

which, together with (3.2), yields

$$\frac{d}{ds}\big(s^{2/(p_i-1)}\overline{\eta}_i(s)\big) \leq -\frac{s^{-(n/2)}}{C}, \quad \forall C \leq s \leq R_i.$$

Making a change of variables, the above implies

$$\bar{w}_i'(r) < 0, \quad \forall C u_i(y_i)^{-(p_i-1)/2} \leq r \leq r_i,$$

where $r_i = R_i u_i(y_i)^{-(p_i-1)/2}$.

We derive from the above and (3.1) that $\tilde{r}_i \geq r_i$ and $\bar{w}_i$ has at least one critical point in the interval $[r_i, \tilde{r}_i]$. Let μ_i be the smallest critical point of $\bar{w}_i$ in this interval. It is clear that

$$\tilde{r}_i \geq \mu_i \geq r_i, \qquad \lim_{i\to\infty} \mu_i = 0.$$

Consider $\xi_i(x) = \mu_i^{2/(p_i-1)} u_i(\mu_i x + y_i)$. Set $T_i = y_{in}/\mu_i$, $T = \lim_{i\to\infty} T_i$. We have

$$
\tag{3.3}
\begin{cases}
-\Delta \xi_i(x) = n(n-2)\tilde{f}_i^{\tau_i} \xi_i(x)^{p_i}, \quad |x| < \dfrac{\bar{r}}{\mu_i}, x^n > -T_i,\\
\dfrac{\partial \xi_i}{\partial x^n} = -c_i \tilde{f}_i^{\tau_i/2} \xi_i(x)^{(p_i+1)/2}, \quad |x| < \dfrac{\bar{r}}{\mu_i}, x^n = -T_i,\\
|x|^{2/(p_i+1)} \xi_i(x) \le C, \quad |x| < \dfrac{\bar{r}}{\mu_i}, x^n > -T_i,\\
\lim\limits_{i\to\infty} \xi_i(0) = \infty, \text{and } 0 \text{ is a local maximum point of } \xi_i,\\
r^{2/(p_i+1)} \overline{\xi}_i(r) \text{ has negative derivative in } C\xi_i(0)^{-(p_i-1)/2} < r < 1,\\
\dfrac{d}{dr}\left\{ r^{2/(p_i+1)} \overline{\xi}_i(r) \right\}\Big|_{r=1} = 0,
\end{cases}
$$

where $\overline{\xi}_i(r) = (1/|\partial'' B_r^{-T_i}(0)|) \int_{\partial'' B_r^{-T_i}(0)} \xi(r)$, $B_r^{-T_i}(0) = \{x \in \mathbf{R}^n : |x| \langle r, x^n \rangle - T_i\}$, $\partial'' B_r^{-T_i}(0) = \partial B_r^{-T_i}(0) \cap \partial B_r^{-T_i}(0)$, and $\tilde{f}_i(x) = f_i(\mu_i x + y_i)$.

If $T = \infty$, we can derive a contradiction exactly the same way as in the proof of Proposition 3.1 in [25]. If $0 < T < \infty$, we know from Proposition 1.4 that

$$
\xi_i(0)\xi_i(x) \to h(x), \quad \text{in } C^2_{\text{loc}}\big(\overline{\mathbf{R}}^n_{-T_i} \setminus \{0\}\big),
$$

where h satisfies

$$
\begin{cases}
-\Delta h = 0, \quad h > 0, \text{in } \mathbf{R}^n_{-T} \setminus \{0\},\\
\dfrac{\partial h}{\partial x^n} = 0, \quad \text{on } x^n = -T.
\end{cases}
$$

Clearly, for some constant $A > 0$,

$$
h(x) = A\big(|x|^{2-n} + |x - (0, \dots, 0, -2T)|^{2-n}\big) + b(x),
$$

where $b(x)$ is some harmonic function on $\mathbf{R}^n$. Due to the positivity of h, $\liminf_{|x|\to\infty} b(x) \ge 0$. Consequently, $b(x) \equiv b$ for some constant $b \ge 0$. For $\sigma > 0$ small, applying the Pohozaev identity as usual, we reach a contradiction.

Now we only need to rule out the possibility of $T = 0$. Indeed, we know from Proposition 1.4 that

$$
\tag{3.4}
\xi_i(0)\xi_i(x) \to h(x), \quad \text{in } C^2_{\text{loc}}\big(\overline{\mathbf{R}^n_{-T_i}} \setminus \{0\}\big),
$$

where h satisfies

$$
\begin{cases}
-\Delta h = 0, \quad h > 0, \text{in } \mathbf{R}^n_+,\\
\dfrac{\partial h}{\partial x^n} = 0, \quad \text{on } \partial \mathbf{R}^n_+ \setminus \{0\}.
\end{cases}
$$

Arguing as before, we have, for some constants $A > 0$ and $b \geq 0$, that

$$h(x) = A|x|^{2-n} + b.$$

Using the last line in (3.3) and (3.4), we have

$$\begin{aligned}\frac{n-2}{2}(-A+b) &= \frac{d}{dr}\left\{r^{(2-n)/2} + br^{(n-2)/2}\right\}\Big|_{r=1} \\ &= \frac{d}{dr}\left\{\frac{r^{(n-2)/2}}{|\partial'' B_r^+(0)|}\int_{\partial'' B_r^+(0)} h\right\}\Big|_{r=1} \\ &= \lim_{i\to\infty}\frac{d}{dr}\left\{r^{2/(p_i-1)}\xi_i(0)\overline{\xi}_i(r)\right\}\Big|_{r=1} \\ &= 0.\end{aligned}$$

Consequently, $b = A > 0$, and

$$h(x) = A\big(|x|^{2-n} + 1\big).$$

Set $\hat{\xi}_i(x) = \xi_i(x^1, \ldots, x^{n-1}, x^n - T_i)$, $\hat{f}_i(x) = \tilde{f}_i(x^1, \ldots, x^{n-1}, x^n - T_i)$, and apply Lemma 1.4 to $\hat{\xi}_i$ on B_σ^+ for $\sigma > 0$ small. We have

$$\begin{aligned}&n(n-2)\left(\frac{n}{p_i+1} - \frac{n-2}{2}\right)\int_{B_\sigma^+}\hat{f}_i^{\tau_i}\hat{\xi}_i^{p_i+1} \\ &\quad = \int_{\partial'' B_\sigma^+} B\big(\sigma, x, \hat{\xi}_i, \nabla\hat{\xi}_i\big) + \frac{n(n-2)\sigma}{p_i+1}\int_{\partial'' B_\sigma^+}\hat{f}_i^{\tau_i/2}\big|\hat{\xi}_i\big|^{p_i+1} \\ &\qquad + C\tau_i\int_{B_\sigma^+}|x|\big|\hat{\xi}_i\big|^{p_i+1} + c_i\int_{\partial' B_\sigma^+}\left(\sum_{j=1}^{n-1} x^j\frac{\partial\hat{\xi}_i}{\partial x^j} + \frac{n-2}{2}\hat{\xi}_i\right)\hat{f}_i^{\tau_i/2}\hat{\xi}_i^{(p_i+1)/2}.\end{aligned}$$

We calculate the last term:

$$\begin{aligned}&\int_{\partial' B_\sigma^+}\left(\sum_{j=1}^{n-1} x^j\frac{\partial\hat{\xi}_i}{\partial x^j} + \frac{n-2}{2}\hat{\xi}_i\right)\hat{f}_i^{\tau_i/2}\hat{\xi}_i^{(p_i+1)/2} \\ &\quad = \int_{\partial' B_\sigma^+}\left(\frac{2}{p_i+3}\sum_{j=1}^{n-1}\frac{\partial\big(\hat{\xi}_i^{(p_i+3)/2}\big)}{\partial x^j}\hat{f}_i^{\tau_i/2} + \frac{n-2}{2}f^{\tau_i/2}\hat{\xi}_i^{(p_i+3)/2}\right) \\ &\quad = -\left(\frac{2(n-1)}{p_i+3} - \frac{n-2}{2}\right)\int_{\partial' B_\sigma^+}\hat{f}_i^{\tau_i/2}\hat{\xi}_i^{(p_i+3)/2} + O\left(\sigma\int_{x^n=0,|x|=\sigma}\hat{\xi}_i^{(p_i+3)/2}\right) \\ &\qquad + O\left(\tau_i\int_{\partial' B_\sigma^+}|x|\hat{\xi}_i^{(p_i+3)/2}\right).\end{aligned}$$

Multiplying the above by $\xi_i(0)^2$, and sending i to ∞, we have, by using (3.4), Lemma 2.3, and Lemma 2.4, that

$$\int_{\partial'' B_\sigma^+} B(\sigma, x, h, \nabla h) \geq 0.$$

This, as usual, contradicts the fact that

$$\lim_{\sigma\to 0^+} \int_{\partial'' B_\sigma^+} B(\sigma, x, h, \nabla h) = -\frac{(n-2)^2 A}{4} \left|\mathbf{S}^{n-1}\right| < 0.$$

Proposition 3.1 is established. □

§4. The proof of Proposition 1.2. In this section, we establish Proposition 1.2.

Proof of Proposition 1.2. If the conclusion of Proposition 1.2 does not hold, then there exist sequences $((n+2)/(n-2)) - 1/i \leq p_i \leq ((n+2)/(n-2))$, $|c_i| \leq \bar{c}$, $u_i \in \mathcal{M}_{p_i,c_i}$ such that $\min\{d(q_{i,a}, q_{i,b}) | 1 \leq a, b \leq N_i, a \neq b\} \to 0$ as $i \to \infty$ where $q_{i,1}, \ldots, q_{i,N_i}$ are the points determined by Proposition 1.1 for $u = u_i$. Notice that when we apply Proposition 1.1 to determine these points, we fix some very large constant R, and then fix some very small constant $\epsilon > 0$ (which may very well depend on R), and in all the arguments, i is large (which may very well depend on R and ϵ). Let $d_i = d(q_{i,1}, q_{i,2}) = \min_{a\neq b} d(q_{i,a}, q_{i,b})$, and $q_0 = \lim_{i\to\infty} q_{i,1} = \lim_{i\to\infty} q_{i,2} \in M$. We distinguish two cases.

Case 1. $q_0 \in \partial M$.

Case 2. $q_0 \in M^\circ$.

Case 2 is simpler to handle and can be ruled out as in [34] or [25, Proposition 4.2] (see also the argument below). We only work out the detail to rule out case 1. In case 1, due to the hypothesis that M is locally conformally flat and ∂M is umbilic, we can find a diffeomorphism $\varphi : B_2^+ \to B_\delta^M(q_0)$, $\varphi(0) = q_0$, $B_{\delta/8}^M(q_0) \subset \varphi(B_1^+) \subset B_{\delta/4}^M(q_0)$, and $\varphi^* g = f^{4/(n-2)} g_0$, where $g_0 = \sum_{j=1}^n (dx^j)^2$ is the flat metric on B_2^+ and $f \in C^2(\overline{B_2^+})$ is some positive function. It follows from the conformal invariance of L_g and B_g that

$$\begin{cases} -\Delta v_i = n(n-2) f^{\tau_i} v_i^{p_i}, & \text{in } B_2^+, \\ \dfrac{\partial v_i}{\partial x^n} = -c_i f^{\tau_i/2} v_i^{(p_i+1)/2}, & \text{on } \partial' B_2^+, \end{cases}$$

where $\tau_i = ((n+2)/(n-2)) - p_i$, $v_i = f u_i \circ \varphi$.

It follows from property (i) in Proposition 1.1 that, for all $a \neq b$,

$$R u_i^{-(p_i-1)/2}(q_{i,a}) \leq d(q_{i,a}, q_{i,b}),$$

which in turn implies, in view of $\lim_{i\to\infty} d(q_{i,1}, q_{i,2}) = 0$,

$$\lim_{i\to\infty} u_i(q_{i,a}) = \infty, \quad \text{for } a = 1, 2,$$

and

$$d(q_{i,1},q_{i,2})^{2/(p_i-1)}u_i(q_{i,a}) \geq R^{2/(p_i-1)} \geq R^{(n-2)/2}, \quad \text{for } a=1,2.$$

Due to the factor f, $q_{i,a}$ may not be a local maximum point of v_i anymore, but it is not difficult to see that for each $q_{i,a} \in B^M_{\delta/2}(q_0)$, there exists $x_{i,a} \in \overline{B_2^+}$ such that, for large i,

$$\left|x_{i,a}-\varphi^{-1}(q_{i,a})\right|u_i(q_{i,a})^{(p_i-1)/2} \leq C\epsilon,$$

$$x_{i,a} \text{ is a local maximum point of } v_i, \text{ and } v_i(x_{i,a}) \to \infty, \quad a=1,2, \tag{4.1}$$

$$d\big(x,\cup_a\{x_{i,a}\}\big)^{2/(p_i-1)}v_i(x) \leq C_1, \quad \forall x \in B_1^+, \tag{4.2}$$

$$0<\sigma_i := |x_{i,1}-x_{i,2}| \leq 2\min_{a\neq b}|x_{i,a}-x_{i,b}| \to 0,$$

$$\sigma_i^{2/(p_i-1)}v_i(x_{i,a}) \geq \frac{R^{2/(p_i-1)}}{C} \geq \frac{R^{(n-2)/2}}{C}, \quad \text{for } a=1,2, \tag{4.3}$$

where $C>1$ is some universal constant independent of ϵ, R, and i.

Without loss of generality, we assume that $x_{i,1}=(0,\dots,0,x^n_{i,1})$. Consider

$$w_i(y)=\sigma_i^{2/(p_i-1)}v_i(x_{i,1}+\sigma_i y),$$

and set, for $x_{i,a} \in \overline{B_1^+}$,

$$y_{i,a}=\frac{x_{i,a}-x_{i,1}}{\sigma_i}.$$

Clearly,

(4.4)

$$\begin{cases} -\Delta w_i(y)=n(n-2)f(x_{i,1}+\sigma_i y)^{\tau_i}w_i(y)^{p_i}, & w_i(y)>0,\ |y|<\dfrac{1}{\sigma_i}, y^n>-T_i,\\ \dfrac{\partial w_i(y)}{\partial y^n}=-c_i f(x_{i,1}+\sigma_i y)^{\tau_i/2}w_i(y)^{(p_i+1)/2}, & |y|<\dfrac{1}{\sigma_i}, y^n=-T_i, \end{cases}$$

where $T_i=x^n_{i,1}/\sigma_i$.

It is also clear that

$$|y_{i,a}-y_{i,b}|=\frac{|x_{i,a}-x_{i,b}|}{\sigma_i} \geq 1, \quad \forall a \neq b, \tag{4.5}$$

and

$$y_{i,1}=0, \qquad |y_{i,2}|=1.$$

After passing to a subsequence, we have

$$\bar{y}=\lim_{i\to\infty}y_{i,2}, \qquad |\bar{y}|=1.$$

The following can be derived easily from (4.3), (4.1), and (4.2):

$$\begin{cases} w_i(0), w_i(y_{i,2}) \geq C_0', \\ \text{each } y_{i,a} \text{ is a local maximum point of } w_i, \\ \min\limits_a |y - y_{i,a}|^{2/(p_i-1)} w_i(y) \leq C_1, \quad |y| \leq \dfrac{1}{(2\sigma_i)}, y^n \geq -T_i, \end{cases} \tag{4.6}$$

where $C_0' > 0$ is independent of i. The next lemma follows immediately from Lemma 1.3.

LEMMA 4.1. *If along some subsequence both $\{y_{i,a_i}\}$ and $\{w_i(y_{i,a_i})\}$ remain bounded, then along the same subsequence*

$$\limsup_{i\to\infty} \max_{\overline{B}_{1/4}^{-T_i}(y_{i,a_i})} w_i < \infty,$$

where $B_{1/4}^{-T_i}(y_{i,a_i}) = \{y : |y - y_{i,a_i}| < 1/4, y^n > -T_i\}$.

In the following, we show that

$$w_i(0), w_i(y_{i,2}) \to \infty. \tag{4.7}$$

If one of them tends to infinity along a subsequence, say, $w_i(0) \to \infty$, then $\{0\}$ is an isolated blow-up point. According to Proposition 1.3, it has to be an isolated simple blow-up point. In view of Proposition 1.4, the Harnack inequality, and Lemma A.1, this implies that w_i tends to zero on any compact subset of $(B_{3/4}^{-T_i}(0) \cup B_{3/4}^{-T_i}(y_{i,2})) \setminus \{0, y_{i,2}\}$. This, together with the Harnack inequality and Lemma A.1, implies that either w_i is not bounded in $\overline{B}_{1/4}^{-T_i}(y_{i,2})$ or $w_i(y_{i,2})$ tends to zero, but we know from the first line of (4.6) that $w_i(y_{i,2})$ does not tend to zero. So w_i is not bounded in $\overline{B}_{1/4}^{-T_i}(y_{i,2})$, which in turn implies, in view of Lemma 4.1, that $w_i(y_{i,2}) \to \infty$. This shows that either (4.7) holds or, along some subsequence, both $\{w_i(0)\}$ and $\{w_i(y_{i,2})\}$ stay bounded.

On the other hand, if both $\{w_i(0)\}$ and $\{w_i(y_{i,2})\}$ stay bounded along a subsequence, then $\{w_i\}$ is locally bounded. This can be seen as follows. Suppose the contrary; then, in view of Lemma 4.1, $w_i(y_{i,a_i}) \to \infty$ along some bounded subsequence $\{y_{i,a_i}\}$. So we have an isolated simple blow-up point $\{y_{i,a_i}\}$. Applying Proposition 1.4, the Harnack inequality, and Lemma A.1 as before, we deduce that $w_i(0) \to 0$. This is a contradiction.

Since $\{w_i\}$ is locally bounded, we deduce by applying L^p estimates, Schauder estimates, the Harnack inequality, and Lemma A.1 that

$$\lim_{i\to\infty} \|w_i - w\|_{C^2(\overline{\mathbf{R}}^n_{-T_i} \cap B_R)} = 0, \quad \forall R > 1,$$

where, for $T := \lim_{i\to\infty} T_i \in [0,\infty]$, w satisfies that

$$(4.8) \qquad \begin{cases} -\Delta w = n(n-2)w^{(n+2)/(n-2)}, & w>0, \text{in } \{y\in\mathbf{R}^n : y^n > -T\}, \\ \dfrac{\partial w}{\partial y^n} = -cw^{n/(n-2)}, & \text{on } y^n = -T, \text{in the case } T<\infty. \end{cases}$$

All the solutions of (4.8) are classified by Caffarelli, Gidas, and Spruck [6] for $T=\infty$ and by Li and Zhu [27] for $T<\infty$. It is clear from their work that there is no solution of (4.8) having two distinct local maximum points. However, w apparently has zero and $\bar y$ as its local maximum points. This is a contradiction. We have thus established (4.7). This, together with the third line in (4.6), implies that both $\{0\}$ and $y_{i,2}\to\bar y$ are isolated blow-up points. According to Proposition 1.3, they are isolated simple blow-up points. We proceed by distinguishing two subcases.

Case 1.1. $T=\lim_{i\to\infty} T_i = \infty$. In this case, we argue as in the proof of Proposition 4.1 in [25] as follows. We see from (4.4) that $w_i(0)w_i$ satisfies

(4.9)
$$\begin{cases} -\Delta\big(w_i(0)w_i\big) = n(n-2)f^{\tau_i} w_i(0)^{1-p_i}\big(w_i(0)w_i\big)^{p_i}, & |y|<\dfrac{1}{\sigma_i}, y^n > -T_i, \\ \dfrac{\partial\big(w_i(0)w_i\big)}{\partial y^n} = -c_i f^{\tau_i/2} w_i(0)^{(1-p_i)/2}\big(w_i(0)w_i\big)^{(p_i+1)/2}, & |y|<\dfrac{1}{\sigma_i}, y^n = -T_i. \end{cases}$$

We have shown in the above that if $\{y_{i,a_i}\}$ stays bounded along a subsequence, then after passing to a subsequence either $w_i(y_{i,a_i})\to\infty$ is an isolated simple blow-up point or $\{w_i\}$ is bounded in $B^{-T_i}_{1/4}(y_{i,a_i})$. Since $\{0\}$ and $\{y_{i,2}\}$ are isolated simple blow-up points, we can derive from Proposition 1.4 and the Harnack inequality that $\{w_i(0)w_i\}$ is locally bounded in $\mathbf{R}^n_{-T_i}\setminus\cup_i\{y_{i,a_i}\}$. In view of (4.5), we apply L^p-estimates, Schauder estimates, and the Harnack inequality to (4.9) to obtain, after passing to a subsequence, a set $\mathcal{S}_1\subset\mathbf{R}^n$ such that $\{0,\bar y\}\in\mathcal{S}_1$,

$$\min\big\{|x-y| \mid x,y\in\mathcal{S}_1\big\}\ge 1,$$

$$\begin{cases} \lim\limits_{i\to\infty} w_i(0)w_i(y) = h(y), & \text{in } C^2_{\text{loc}}\big(\mathbf{R}^n\setminus\mathcal{S}_1\big), \\ h(y)>0, & y\in\mathbf{R}^n\setminus\mathcal{S}_1, \\ h(y) \text{ is unbounded near any point in } \mathcal{S}_1, \end{cases}$$

and

$$\Delta h(y) = 0, \quad y\in\mathbf{R}^n\setminus\mathcal{S}_1.$$

We then deduce from Böcher's theorem (see, e.g., [22]) and the maximum principle that there exists some nonnegative function $b(y)$ and some positive constants $a_1, a_2 > 0$, such that

$$\begin{cases} b(y)\ge 0, & y\in\mathbf{R}^n\setminus\big\{\mathcal{S}_1\setminus\{0,\bar y\}\big\}, \\ \Delta b(y) = 0, & y\in\mathbf{R}^n\setminus\big\{\mathcal{S}_1\setminus\{0,\bar y\}\big\}, \end{cases}$$

and

$$h(y) = a_1|y|^{2-n} + a_2|y-\bar{y}|^{2-n} + b(y), \quad y \in \mathbf{R}^n \setminus \{\mathcal{S}_1 \setminus \{0, \bar{y}\}\}.$$

Therefore, for some constant $A > 0$,

$$h(y) = a_1|y|^{2-n} + A + O(|y|), \quad \text{for } y \text{ close to zero.}$$

As usual, we derive a contradiction by using Lemma 1.4 as follows. For $0 < \sigma < 1$, we apply Lemma 1.4 to w_i on B_σ to obtain

$$\begin{aligned} n(n-2)\left(\frac{n}{p_i+1} - \frac{n-2}{2}\right)\int_{B_\sigma} f^{\tau_i} w_i^{p_i+1} + \frac{n(n-2)}{p_i+1}\int_{B_\sigma} \tau_i f^{\tau_i - 1}(y\cdot\nabla f) w_i^{p_i+1} \\ = \int_{\partial B_\sigma} B(\sigma, y, w_i \nabla w_i) + \frac{n(n-2)}{p+1}\sigma\int_{\partial B_\sigma} f w_i^{p_i+1}, \end{aligned}$$

where $B(\sigma, x, u, \nabla u) = ((n-2)/2)u(\partial u/\partial \nu) - (\sigma/2)|\nabla u|^2 + \sigma(\partial u/\partial\nu)^2$. Multiplying the above identity by $w_i(0)^2$ and sending i to ∞, we obtain, by using Proposition 1.4, that

$$\int_{\partial B_\sigma} B(\sigma, y, h, \nabla h) \geq 0,$$

but a direct computation shows that

$$\lim_{\sigma\to 0}\int_{\partial B_\sigma} B(\sigma, y, h, \nabla h) = -\frac{(n-2)^2}{2} A\left|\mathbf{S}^{n-1}\right|,$$

which contradicts $A > 0$.

Case 1.2. $T = \lim_{i\to\infty} T_i < \infty$. In this case, we argue similarly to case 1.1 while using Lemma A.1 and the interior Harnack inequality to obtain a set $\mathcal{S}_1 \subset \overline{\mathbf{R}^n_{-T}}$ ($\mathbf{R}^n_{-T} := \{y \in \mathbf{R}^n : y^n > -T\}$) such that $\{0, \bar{y}\} \subset \mathcal{S}_1$,

$$\min\{|x-y| \mid x, y \in \mathcal{S}_1\} \geq 1,$$

$$\begin{cases} \lim\limits_{i\to\infty} w_i(0) w_i(y) = h(y), & \text{in } C^2_{\text{loc}}\left(\overline{\mathbf{R}^n_{-T}} \setminus \mathcal{S}_1\right), \\ h(y) > 0, & y \in \overline{\mathbf{R}^n_{-T}} \setminus \mathcal{S}_1, \\ h(y) \text{ is unbounded near any point in } \mathcal{S}_1, \end{cases}$$

and

$$\begin{cases} \Delta h(y) = 0, & y \in \mathbf{R}^n_{-T} \setminus \mathcal{S}_1, \\ \dfrac{\partial h}{\partial y^n} = 0, & y \in \partial\mathbf{R}^n_{-T} \setminus \mathcal{S}_1. \end{cases}$$

Making an even extension of $h(y)$ across the hyperplane $y^n = -T$ and arguing as in case 1.1, we find some nonnegative function $b(y)$ and some positive constants

$a_1, a_2 > 0$ such that

$$\begin{cases} b(y) \geq 0, & y \in \mathbf{R}^n_{-T} \setminus \{\mathcal{S}_1 \setminus \{0, \bar{y}\}\}, \\ \Delta b(y) = 0, & y \in \mathbf{R}^n_{-T} \setminus \{\mathcal{S}_1 \setminus \{0, \bar{y}\}\}, \end{cases}$$

$$h(y) = a_1 |y|^{2-n} + a_2 |y - \bar{y}|^{2-n} + b(y), \quad y \in \mathbf{R}^n_{-T} \setminus \{\mathcal{S}_1 \setminus \{0, \bar{y}\}\}.$$

Therefore, for some constant $A > 0$,

$$h(y) = a_1 |y|^{2-n} + A + O(|y|), \quad \text{for } y \text{ close to zero.}$$

If $T > 0$, we derive a contradiction exactly the same way as in case 1.1. If $T = 0$, we set $\widetilde{w}_i(y) = w_i(y^1, \ldots, y^{n-1}, y^n - T_i)$ and derive a contradiction exactly the same way as in the proof of Proposition 3.1. □

§5. The proof of Theorems 0.1 and 0.2. Let φ_1 denote the first eigenfunction of (0.3), and consider $g_1 = \varphi_1^{4/(n-2)} g$. Then $R_{g_1} > 0$ and $h_{g_1} \equiv 0$. We can work with g_1 instead of g. For simplicity, we still denote it as g. We first establish Theorem 0.2.

Proof of Theorem 0.2. In view of the L^p-estimates, Schauder estimates, Harnack inequality, and Lemma A.1, we only need to establish the L^∞-bound of u. We prove it by a contradiction argument. Suppose the contrary; then in view of Theorem 1.1, there exist $|c_i| \leq \bar{c}$, $p_i = ((n+2)/(n-2)) - \tau_i$, $\tau_i \geq 0, \tau_i \to 0$, and $u_i \in \mathcal{M}_{p_i, c_i}$ such that

$$\max_{\overline{M}} u_i \to \infty.$$

It follows from Propositions 1.1–1.4 that, after passing to a subsequence, $\{u_i\}$ has $N (1 \leq N < \infty)$ isolated simple blow-up points, denoted as $\{q^{(1)}, \ldots, q^{(N)}\}$. Let $\{q_i^{(1)}, \ldots, q_i^{(N)}\}$ denote the local maximum points as described in Definition 1.1. It follows from Proposition 1.4 and standard elliptic theories that

$$u_i(q_i^{(1)}) u_i \to h, \quad \text{in } C^2_{\mathrm{loc}}\big(M \setminus \{q^{(1)}, \ldots, q^{(N)}\}\big).$$

It is not difficult to see, using the hypothesis $\lambda_1(M) > 0$, that

$$h = \sum_{l=1}^{N} a_l G(\cdot, q^{(l)}), \quad \text{on } M,$$

where $a_l > 0, \forall l$, and $G(\cdot, q^{(l)})$ denotes Green's function of $-L_g$ with respect to zero Neumann boundary conditions centered at $q^{(l)}$. Since $R_g > 0$, it is clear that $G(\cdot, q^{(l)}) > 0$ on $M \setminus \{q^{(l)}\}$. We assume that $q^{(1)} \in \partial M$, since otherwise it is easier and can be handled similarly. Since M is locally conformally flat, we can find a local conformal diffeomorphism Φ that maps $B^M_\delta(q^{(1)})$ $(\delta > 0)$ into $\mathbf{R}^n$ with $\Phi(q^{(1)}) = 0$. Let g_0 denote the flat metric of $\mathbf{R}^n$; then $\Phi^*(g_0) = \varphi^{4/(n-2)} g$ for some positive function φ. Since ∂M is umbilic, $\Phi(\partial M \cap B_\delta^M(q^{(1)}))$ has to be a piece of sphere or a piece of hyperplane. Since spheres and hyperplanes are locally conformal to each other, we can assume without loss of generality that $\partial' B_2^+(0) \subset \Phi(\partial M \cap B_\delta^M(q^{(1)}))$ and $\Phi(M^\circ \cap B_\delta^M(q^{(1)})) \subset \mathbf{R}_+^n$. Since $\partial' B_2^+(0)$ clearly has zero mean curvature in $\overline{B}_2^+(0)$, we know that $(\partial\varphi/\partial\nu) = 0$ on $\partial M \cap B_\delta^M(q^{(1)})$. Extending φ to be a positive smooth function on M such that $(\partial\varphi/\partial\nu) = 0$ on ∂M, consider the conformal metric $g_2 = \varphi^{4/(n-2)} g$. Then we know that g_2 has the property that $h_{g_2} = 0$ on ∂M, and it is Euclidean in a neighborhood of $q^{(1)}$.

Clearly, Green's function $\hat{G}(x, q^{(1)})$ of g_2 has the following expansion near $q^{(1)}$:

$$\hat{G}\big(x, q^{(1)}\big) = |x|^{2-n} + A + O(|x|).$$

It follows from the positive mass theorem of Schoen and Yau [36] (see also the appendix of [13]) that $A \geq 0$ with equality if and only if (M, g) is conformally equivalent to the half sphere with the standard metric.

Let $v_i = (\varphi \circ \Phi)^{-1} u_i \circ \Phi^{-1}$; then v_i satisfies

$$\begin{cases} -\Delta v_i = n(n-2)\varphi^{\tau_i} v_i^{p_i}, & \text{in } B_2^+(0), \\ \dfrac{\partial v_i}{\partial x^n} = -c_i \varphi^{\tau_i/2} v_i^{(p_i+1)/2}, & \text{on } \partial' B_2^+(0), \end{cases}$$

where $\tau_i = ((n+2)/(n-2)) - p_i$. We can also deduce that $x_i \to 0$ is an isolated simple blow-up point of $\{v_i\}$ and

$$v_i(x_i) v_i \to \hat{h}, \quad \text{in } C^2_{\text{loc}}\big(\overline{B}_1^+(0) \setminus \{0\}\big),$$

where $h(x) = |x|^{2-n} + \hat{A} + O(|x|)$ for some $\hat{A} > 0$. Applying the Pohozaev identity in B_σ^+ as usual, we reach a contradiction. Theorem 0.2 is established. $\square$

In the following, we use Leray-Schauder degree theory and Theorem 0.2 to establish the existence part of Theorem 0.1. Apparently, we can assume that (M, g) is not conformally equivalent to the half sphere with the standard metric since it is trivial otherwise.

As remarked earlier, we can assume without loss of generality that $R_g > 0$ and $h_g \equiv 0$. Consequently, $B_g = (\partial_g/\partial\nu)$. We still use $\varphi_1 > 0$ to denote the eigenfunction associated with the first eigenvalue $\lambda_1(M)$ satisfying

$$\|\varphi_1\|^2 \equiv \int_M \big(|\nabla_g \varphi_1|^2 + c(n) R_g \varphi_1^2\big) = 1. \tag{5.1}$$

Consider for $1 \leq p \leq (n+2)/(n-2)$,

$$\begin{cases} -L_g v = E(v) v^p, & v > 0, \text{ in } M^\circ, \\ \dfrac{\partial_g v}{\partial \nu} = 0, & \text{on } \partial M, \end{cases} \tag{5.2}$$

where $E(v) = \int_M (|\nabla_g v|^2 + c(n) R_g v^2)$.

LEMMA 5.1. *There exists some constant $C = C(M, g) > 0$ such that for all $1 \le p \le (n+2)/(n-2)$ and v satisfying (5.2), we have*

$$\frac{1}{C} < v < C, \quad \text{on } M. \tag{5.3}$$

Proof. Multiplying (5.2) by v and integrating by parts, we have

$$\int_M v^{p+1} = 1. \tag{5.4}$$

Multiplying (5.2) by the first eigenfunction of (0.3) and integrating by parts over M, we have $E(v) > 0$.

Let δ_0 be the positive constant in Theorem 0.2. For $1 + \delta_0 \le p \le (n+2)/(n-2)$, we deduce from Theorem 0.2 and Theorem 1.1 that

$$\frac{1}{C} \le E(v)^{1/(p-1)} v \le C. \tag{5.5}$$

It follows easily from (5.4) and (5.5) that $1/C \le E(v) \le C$, which, together with (5.5), yields (5.3). For $1 \le p \le 1 + \delta_0$, we apply Lemma A.3 to obtain that $E(v) \le C$, and then applying standard elliptic estimates to (5.2), we obtain the upper bound in (5.3). The lower bound follows from the upper bound by using the Harnack inequality and Lemma A.1.

For $0 < \alpha < 1$, let $C^{2,\alpha}(M)^+ = \{u \in C^{2,\alpha}(M) : u > 0 \text{ on } M\}$. We define, for $1 \le p \le (n+2)/(n-2)$, a map $F_p : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ by

$$F_p(v) = v - \left(-L_g\right)^{-1}\left(E(v) v^p\right),$$

where $(-L_g)^{-1}$ denotes the inverse operator of $-L_g$ with respect to the zero Neumann boundary condition.

For $\Lambda > 1$, let D_Λ denote the following bounded and open subset of $C^{2,\alpha}(M)^+$:

$$D_\Lambda = \left\{ v \in C^{2,\alpha}(M) : \|v\|_{C^{2,\alpha}(M)} < \Lambda, \min_M v > \frac{1}{\Lambda} \right\}. \tag{5.6}$$

It is easy to see from standard elliptic theories that F_p is of the form Id + compact, and therefore, we may define the Leray-Schauder degree of F_p in D_Λ with respect to $0 \in C^{2,\alpha}(M)$, denoted by $\deg(F_p, D_\Lambda, 0)$, provided zero does not belong to $F_p(\partial D_\Lambda)$. It follows from Lemma 5.1 that, for Λ large, zero does not belong to $F_p(\partial D_\Lambda)$ for all $1 \le p \le (n+2)/(n-2)$. Consequently, by the homotopy invariance of the Leray-Schauder degree (see [29]),

$$\deg(F_p, D_\Lambda, 0) = \deg(F_1, D_\Lambda, 0), \quad \forall 1 \le p \le \frac{n+2}{n-2}. \tag{5.7}$$

It is easy to see that $F_1(v) = 0$ if and only if $E(v) = \lambda_1$ and $v = \sqrt{\lambda_1}\varphi_1$. Set $\overline{v} = \sqrt{\lambda_1}\varphi_1$. We calculate the derivative of F_1 at $\overline{v}$ in the following. For $v \in C^{2,\alpha}(M)$, we write

$$v = \langle v, \varphi_1\rangle\varphi_1 + w,$$

where $\langle v, \varphi_1\rangle := \int_M (\nabla_g v \cdot \nabla_g \varphi_1 + c(n) R_g v\varphi_1)$. It follows that

$$\begin{aligned} F_1'(\overline{v})v &= \frac{d}{dt} F_1(\overline{v} + tv)\Big|_{t=0} \\ &= v - (-L_g)^{-1}\big\{2\langle \overline{v}, v\rangle\overline{v} + E(\overline{v})v\big\} \\ &= v - \frac{2}{\lambda_1}\langle \overline{v}, v\rangle\overline{v} - \lambda_1(-L_g)^{-1}v. \end{aligned}$$

Let $\mu \le 0$ be an eigenvalue of $F_1'(\overline{v})$; then for some nonzero v, we have

$$F_1'(\overline{v})v = v - \frac{2}{\lambda_1}\langle \overline{v}, v\rangle\overline{v} - \lambda_1(-L_g)^{-1}v = \mu v. \tag{5.8}$$

It follows that

$$\frac{\partial v}{\partial \nu} = 0. \tag{5.9}$$

Applying $-L_g$ to (5.8), we have

$$(1-\mu)(-L_g v) = \lambda_1 v + 2\langle \overline{v}, v\rangle\overline{v}. \tag{5.10}$$

Multiplying the above by $\overline{v}$ and integrating by parts over M, we have, using (5.9), that

$$\begin{aligned} (1-\mu)\langle \overline{v}, v\rangle &= \frac{2}{\lambda_1}\langle \overline{v}, \overline{v}\rangle\langle \overline{v}, v\rangle + \lambda_1\int_M \overline{v}v \\ &= 2\langle \overline{v}, v\rangle + \int_M (-L_g\overline{v})v \\ &= 3\langle \overline{v}, v\rangle, \end{aligned}$$

namely,

$$(2+\mu)\langle \overline{v}, v\rangle = 0.$$

If $\langle \overline{v}, v\rangle = 0$, we have $-L_g v = ((\lambda_1)/(1-\mu))v$. Since $\mu \le 0$, this implies that $\mu = 0$ and $-L_g v = \lambda_1 v$. Since λ_1 is a simple eigenvalue, v is a nonzero multiple of $\overline{v}$. This violates $\langle \overline{v}, v\rangle = 0$. In the following, we assume that $\langle \overline{v}, v\rangle \ne 0$, and therefore, $\mu = -2$. In turn, it follows from (5.10) that

$$-3L_g v = \lambda_1 v + 2\langle \overline{v}, v\rangle\overline{v}.$$

Multiplying the above by w, integrating by parts over M, and using the fact that $\langle w, \varphi_1 \rangle = 0$, we have

$$3\langle w, w\rangle = \lambda_1 \int_M w^2.$$

On the other hand, since λ_1 is the first eigenvalue, we have

$$\langle w, w\rangle \geq \lambda_1 \int_M w^2.$$

It follows that $w \equiv 0$. Therefore, the eigenspace associated with $\mu = -2$ is the one-dimensional space spanned by φ_1. To sum up, we have shown that $F_1'(\overline{v})$ is invertible with exactly one simple negative eigenvalue -2. Therefore,

$$\deg(F_1, D_\Lambda, 0) = -1. \tag{5.11}$$

For $1 \leq p \leq (n+2)/(n-2)$ and $c \in \mathbf{R}$, we define an operator $T_{p,c} : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ as follows: $u = T_{p,c} v$ if and only if

$$\begin{cases} -L_g u = n(n-2)v^p, & \text{in } M, \\ \dfrac{\partial_g u}{\partial \nu} = c v^{(p+1)/2}, & \text{on } \partial M. \end{cases}$$

It follows from standard elliptic theories and the hypothesis $\lambda_1(M) > 0$ that $T_{p,c}$ is well defined and is a compact operator.

It follows from Theorem 0.2 that, for $\Lambda_c > 2$ large enough (depending only on M, g, and c),

$$\left\{ u \in C^{2,\alpha}(M)^+ : \big(\mathrm{Id} - T_{((n+2)/(n-2)),tc}\big) u = 0 \text{ for some } 0 \leq t \leq 1 \right\} \subset D_{\Lambda_c - 1}.$$

It follows from the homotopy invariance of the Leray-Schauder degree that, for all $\Lambda \geq \Lambda_c$,

$$\deg\big(\mathrm{Id} - T_{((n+2)/(n-2)),c}, D_{\Lambda_c}, 0\big) = \deg\big(\mathrm{Id} - T_{((n+2)/(n-2)),0}, D_{\Lambda_c}, 0\big). \tag{5.12}$$

For $0 \leq s \leq 1$, we define $G_s : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ by

$$G_s(u) = u - \big(-L_g\big)^{-1} \left\{ \big[n(n-2)s + (1-s)E(u)\big] u^{(n+2)/(n-2)} \right\},$$

where $(-L_g)^{-1}$ denotes the inverse operator of $-L_g$ with respect to the zero Neumann boundary. Clearly, $G_1 = \mathrm{Id} - T_{((n+2)/(n-2)),0}$ and $G_0 = F_{(n+2)/(n-2)}$. $\square$

LEMMA 5.2. *There exists $\overline{\Lambda}_c > 2$ depending only on M, g, and c, such that, for all $\Lambda \geq \overline{\Lambda}_c$,*

$$G_s(u) \neq 0, \quad \forall 0 \leq s \leq 1, u \in \partial D_\Lambda.$$

Proof. Let $G_s(u)=0$ for some $u\in C^{2,\alpha}(M)^+$, $0\le s\le 1$. Then u satisfies

$$\begin{cases} -L_g u=\left[n(n-2)s+(1-s)E(u)\right]u^{(n+2)/(n-2)}, & \text{in } M,\\ \dfrac{\partial_g u}{\partial\nu}=0, \quad \text{on } \partial M. \end{cases}$$

Multiplying the above equation by u and integrating by parts, we have

$$E(u)=\left[n(n-2)s+(1-s)E(u)\right]\int_M u^{2n/(n-2)}. \tag{5.13}$$

We deduce from Theorem 0.2 that for some constant $C=C(M,g)>1$,

$$\frac{1}{C}\le\left[n(n-2)s+(1-s)E(u)\right]^{(n-2)/4}u\le C. \tag{5.14}$$

It follows from (5.13) and (5.14) that

$$\frac{1}{C}\le E(u)\left[n(n-2)s+(1-s)E(u)\right]^{(n-2)/2}\le C.$$

Consequently,

$$\frac{1}{C}\le n(n-2)s+(1-s)E(u)\le C,$$

which, in view of (5.14), implies that

$$\frac{1}{C}\le u\le C, \quad \text{in } M.$$

We can then apply standard elliptic theories to the equation of u to conclude that for some $\overline{\Lambda}_c>1$, u does not belong to ∂D_Λ for all $\Lambda\ge\overline{\Lambda}_c$. □

Proof of Theorem 0.1 completed. Using Lemma 5.2 and the homotopy invariance of the Leray-Schauder degree, we have, for all $\Lambda\ge\overline{\Lambda}_c$, that

$$\deg\left(\mathrm{Id}-T_{((n+2)/(n-2)),0},D_\Lambda,0\right)=\deg\left(F_{(n+2)/(n-2)},D_\Lambda,0\right). \tag{5.15}$$

Combining (5.12), (5.15), (5.7), and (5.11), we have, for Λ sufficiently large, that

$$\deg\left(\mathrm{Id}-T_{((n+2)/(n-2)),c},D_\Lambda,0\right)=\deg\left(F_{(n+2)/(n-2)},D_\Lambda,0\right)=\deg(F_1,D_\Lambda,0)=-1,$$

which, in particular, implies that $\mathcal{M}_c\cap D_\Lambda\ne\emptyset$. We have thus completed the existence part of the proof of Theorem 0.1. □

§6. The proof of Theorem 0.3. In this section, we establish Theorem 0.3. Throughout this section, we assume $\lambda_1(M)<0$. We first establish estimate (0.8).

Proof of estimate (0.8). We first show that $u \le C$ in M under the hypothesis. For all $u \in \cup_{1+\delta_0 \le p \le (n+2)/(n-2)} \widetilde{\mathcal{M}}_{p,0}$, this can be obtained by a blow-up argument as in the proof of Theorem 1.1 and the well-known fact that $\Delta v = v^p$ has no positive solution in $\mathbf{R}^n$ for $p > 1$ (see [4]). To obtain the upper bound of u for $u \in \cup_{((n+2)/(n-2))-\delta_0 \le p \le (n+2)/(n-2)} \cup_{-(\overline{\epsilon})^{-1} \le c \le n-2-\overline{\epsilon}} \widetilde{\mathcal{M}}_{p,c}$, we use a contradiction argument as follows. Suppose the contrary; then there exist sequences $\{p_i\}$, $\{c_i\}$, and $\{u_i\} \in \widetilde{\mathcal{M}}_{p_i,c_i}$ satisfying

$$\frac{n+2}{n-2} - \frac{1}{i} \le p_i \to \frac{n+2}{n-2}, \qquad c_i \to c \in \left[-(\overline{\epsilon})^{-1}, n-2-\overline{\epsilon}\right],$$

and

$$\lim_{i\to\infty} \max_M u_i = \infty.$$

By making a conformal transformation using the first eigenfunction associated with $\lambda_1(M)$, we can assume without loss of generality that $R_g < 0$ in M and $h_g \equiv 0$ on ∂M. Let $x_i \in M$ be some maximum point of u_i, namely,

$$u_i(x_i) = \max_M u_i \to \infty.$$

Since, at an interior local maximum point, one has $c(n)R_g u_i \le -n(n-2)u_i^{p_i}$, we have $x_i \in \partial M$ for large i. Let $y^1, \dots, y^n$ be the geodesic normal coordinates given by some exponential map $\exp_{x_i}$, with $(\partial/\partial y^n) = -\nu$ at x_i. Consider $\tilde{u}_i(z) = u_i(x_i)^{-1} u_i(\exp_{x_i}(u_i(x_i)^{-(p_i-1)/2} z))$. It is not difficult to see, after passing to a subsequence, that $\tilde{u}_i$ converges in C^2_{loc}-norm to some $\tilde{u}$ satisfying

$$\begin{cases} -\Delta\tilde{u} = -n(n-2)\tilde{u}^{(n+2)/(n-2)}, & \tilde{u} > 0, \text{in } \mathbf{R}^n_+, \\ \dfrac{\partial\tilde{u}}{\partial z^n} = -c\tilde{u}^{n/(n-2)}, & \text{on } \partial\mathbf{R}^n_+, \end{cases} \tag{6.1}$$

and $\tilde{u}(0) = 1$, $\tilde{u} \le 1$ on $\mathbf{R}^n_+$. Applying the same method in the proof of Theorem 1.1 and Theorem 1.2 in [27] (see also [11]), we see that (6.1) does not have any solution. This is a contradiction. Thus, we have established the upper bound of u in (0.8). The lower bound in (0.8) follows from the upper bound, the Harnack inequality, and Lemma A.1. The rest of the estimates in (0.8) follow from standard elliptic estimates. □

As remarked earlier, we can assume, without loss of generality, that $R_g < 0$ and $h_g \equiv 0$. Consequently, $B_g = (\partial_g/\partial\nu)$. Throughout the rest of this section, we assume, without loss of generality, that the metric g has this property. For convenience, we introduce the following quadratic form: $E(u,v) = \int_M (\nabla_g u \cdot \nabla_g v + c(n)R_g uv)$, and $E(u) = E(u,u)$. We still use $\varphi_1 > 0$ to denote the positive eigenfunction associated with the first eigenvalue $\lambda_1(M)$ satisfying

$$E(\varphi_1) = \int_M \left(|\nabla_g\varphi_1|^2 + c(n)R_g\varphi_1^2\right) = -1.$$

Consider, for $1 \leq p \leq (n+2)/(n-2)$,

$$\begin{cases} -L_g v = E(v) v^p, & v > 0, \text{ in } M^\circ, \\ \dfrac{\partial_g v}{\partial \nu} = 0, & \text{on } \partial M, \end{cases} \tag{6.2}$$

where $E(v) = \int_M (|\nabla_g v|^2 + c(n) R_g v^2)$.

LEMMA 6.1. *There exists some constant $C = C(M, g) > 0$ such that, for all $1 \leq p \leq (n+2)/(n-2)$, and v satisfying (6.2), we have*

$$\frac{1}{C} < v < C. \tag{6.3}$$

Proof. Multiplying (6.2) by v and integrating by parts over M, we have

$$\int_M v^{p+1} = 1. \tag{6.4}$$

Multiplying (6.2) by the first eigenfunction of (0.3) and integrating by parts over M, we have $E(v) < 0$.

Let δ_0 be the positive constant in Theorem 0.2. For $1 + \delta_0 \leq p \leq (n+2)/(n-2)$, we deduce from (0.8) that

$$\frac{1}{C} \leq \big(-E(v)\big)^{1/(p-1)} v \leq C. \tag{6.5}$$

It follows easily from (6.4) and (6.5) that $1/C \leq -E(v) \leq C$, which, together with (6.5), yields (6.3). For $1 \leq p \leq 1 + \delta_0$, we apply Lemma A.4 to obtain $0 < -E(v) \leq C$. The upper bound in (6.3) then follows from the L^p-theory of linear elliptic equations. The lower bound follows from the Harnack inequality and Lemma A.1.

Let $\lambda_1 < \lambda_2 < \cdots$ denote all the eigenvalues of $-L_g$ in (0.3). Pick some constant $A \in (-\lambda_2, -\lambda_1)$. For $0 < \alpha < 1$ and $1 \leq p \leq (n+2)/(n-2)$, we define $F_p : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ by

$$F_p(v) = v - (-L_g + A)^{-1}\big[E(v) v^p + A v\big],$$

where $(-L_g + A)^{-1}$ denotes the inverse operator of $-L_g + A$ with respect to the zero Neumann boundary condition.

For $\Lambda > 1$, let $D_\Lambda \subset C^{2,\alpha}(M)^+$ be given as in (5.6). It follows from Lemma 6.1 that, for Λ large, zero is not contained in $F_p(\partial D_\Lambda)$ for all $1 \leq p \leq (n+2)/(n-2)$. Consequently,

$$\deg(F_p, D_\Lambda, 0) = \deg(F_1, D_\Lambda, 0), \quad \forall 1 \leq p \leq \frac{n+2}{n-2}. \tag{6.6}$$

□

PROPOSITION 6.1. *Suppose that $\lambda_1(M) < 0$, $h_g \equiv 0$. Then*

$$\deg(F_p, D_\Lambda, 0) = -1, \quad \forall 1 \leq p \leq \frac{n+2}{n-2}.$$

Proof. It is easy to see that $F_1(v) = 0$ if and only if $E(v) = \lambda_1$ and $v = \sqrt{-\lambda_1}\varphi_1$. Set $\overline{v} = \sqrt{-\lambda_1}\varphi_1$. We calculate the derivative of F_1 at $\overline{v}$ in the following. For $v \in C^{2,\alpha}(M)$, we write

$$v = -E(v, \varphi_1)\varphi_1 + w.$$

It follows that

$$\begin{aligned} F_1'(\overline{v})v &= \frac{d}{dt} F_1(\overline{v} + tv)\Big|_{t=0} \\ &= v - (-L_g + A)^{-1}\left\{2E(\overline{v}, v)\overline{v} + E(\overline{v})v + Av\right\} \\ &= v - \frac{2}{\lambda_1 + A} E(\overline{v}, v)\overline{v} - (\lambda_1 + A)(-L_g + A)^{-1} v. \end{aligned}$$

Let $\mu \leq 0$ be an eigenvalue of $F_1'(\overline{v})$. Then, for some nonzero v, we have

$$F_1'(\overline{v})v = v - \frac{2}{\lambda_1 + A} E(\overline{v}, v)\overline{v} - (\lambda_1 + A)(-L_g + A)^{-1} v = \mu v. \tag{6.7}$$

It follows that

$$\frac{\partial v}{\partial \nu} = 0. \tag{6.8}$$

Applying $-L_g + A$ to (6.7), we have

$$(1 - \mu)(-L_g + A)v = (\lambda_1 + A)v + 2E(\overline{v}, v)\overline{v}. \tag{6.9}$$

Multiplying the above by $\overline{v}$ and integrating by parts over M, we have, using (6.8), that

$$(1 - \mu)E(\overline{v}, v) + (1 - \mu)\frac{A}{\lambda_1} E(\overline{v}, v) = \left(\frac{\lambda_1 + A}{\lambda_1}\right) E(\overline{v}, v) + 2E(\overline{v}, v).$$

It follows that

$$\left(2 + \left(1 + \frac{A}{\lambda_1}\right)\mu\right) E(\overline{v}, v) = 0.$$

If $E(\overline{v}, v) = 0$, then it follows from (6.9) that

$$-L_g v = \left(\left[-1 + \frac{1}{1 - \mu}\right] A + \frac{\lambda_1}{1 - \mu}\right) v.$$

We see easily from the fact that $\mu \leq 0$ and $-\lambda_2 < A < -\lambda_1$ that $\lambda_1 \leq [-1 + (1/(1 - \mu))]A + (\lambda_1/(1 - \mu)) < \lambda_2$. Since $E(\overline{v}, v) = 0$ and λ_1 is a simple eigenvalue, we have $v = 0$, which is a contradiction.

On the other hand, if $E(\overline{v}, v) \neq 0$, then $\mu = -2\lambda_1/(\lambda_1 + A) < 0$. Multiplying (6.9) by w and integrating by parts over M, we have, in view of $E(\varphi_1, w) = 0$, that

$$(1 - \mu)\left(E(w, w) + A \int_M w^2\right) = (\lambda_1 + A) \int_M w^2.$$

Since $E(\varphi_1, w) = 0$, we have

$$E(w,w) \geq \lambda_2 \int_M w^2.$$

Using the fact that $\mu < 0, -\lambda_2 < A < -\lambda_1$, we can easily derive from the above that $\int_M w^2 = 0$, that is, $w = 0$. This shows that $\mu = -2\lambda_1/(\lambda_1 + A)$ is the only negative eigenvalue of $F_1'(\overline{v})$ having $\operatorname{span}\{\varphi_1\}$ as its eigenspace.

The above discussion shows that $F_1'(\overline{v})$ is invertible and the eigenspace of $F_1'(\overline{v})$ associated with negative eigenvalues is $\operatorname{span}\{\varphi_1\}$. Consequently,

$$\deg(F_1, D_\Lambda, 0) = -1.$$

This completes the proof of Proposition 6.1. □

For $1 \leq p \leq ((n+2)/(n-2)), c < n-2$, we define an operator $T_{p,c} : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ by the following: $u = T_{p,c} v$ if and only if

$$\begin{cases} (-L_g + A)u = -n(n-2)v^p + Av, & v > 0, \text{in } M^\circ, \\ \dfrac{\partial_g u}{\partial \nu} = c v^{(p+1)/2}, & \text{on } \partial M. \end{cases}$$

Since zero is not an eigenvalue of $-L_g + A$, $T_{p,c}$ is well defined. It follows from the Schauder theory that $T_{p,c}$ is compact. It follows from (0.8) that, for $c < n-2$ and $\overline{\Lambda}_c > 2$ large enough (depending on M, g, and c),

$$\left\{u \in C^{2,\alpha}(M) : \left(\mathrm{Id} - T_{((n+2)/(n-2)),tc}\right) u = 0 \text{ for some } 0 \leq t \leq 1\right\} \subset D_{\overline{\Lambda}_c - 1}.$$

PROPOSITION 6.2. *Suppose that* $\lambda_1(M) < 0$, $h_g \equiv 0$, $c < n-2$. *Then, for* Λ *large, we have*

$$\deg\left(\mathrm{Id} - T_{((n+2)/(n-2)),c}, D_\Lambda, 0\right) = \deg(F_1, D_\Lambda, 0) = -1.$$

Proof. It follows from the homotopy invariance of the Leray-Schauder degree that

$$\deg\left(\mathrm{Id} - T_{((n+2)/(n-2)),c}, D_\Lambda, 0\right) = \deg\left(\mathrm{Id} - T_{((n+2)/(n-2)),0}, D_\Lambda, 0\right).$$

For $0 \leq s \leq 1$, we define $G_s : C^{2,\alpha}(M)^+ \to C^{2,\alpha}(M)$ by

$$G_s(u) = u - (-L_g + A)^{-1}\left\{\left[-n(n-2)s + (1-s)E(u)\right]u^{(n+2)/(n-2)} + Au\right\},$$

where $(-L_g + A)^{-1}$ denotes the inverse operator of $-L_g + A$ with respect to the zero Neumann boundary condition. Clearly, $G_1 = \mathrm{Id} - T_{((n+2)/(n-2)),0}$ and $G_0 = F_{(n+2)/(n-2)}$. □

LEMMA 6.2. *For large* Λ,

$$G_s(u) \neq 0, \quad \forall 0 \leq s \leq 1, u \in \partial D_\Lambda.$$

Proof. If $G_s(u) = 0$, then u satisfies

$$\begin{cases} -L_g u = \big[-n(n-2)s + (1-s)E(u)\big]u^{(n+2)/(n-2)}, & \text{in } M^\circ, \\ \dfrac{\partial_g u}{\partial \nu} = 0, & \text{on } \partial M. \end{cases} \tag{6.10}$$

Multiplying the above equation by φ_1 and integrating by parts over M, we have, using $\lambda_1 < 0$, that $n(n-2)s - (1-s)E(u) > 0$. Multiplying the equation in (6.10) by u and integrating by parts, we have

$$E(u) = \big[-n(n-2)s + (1-s)E(u)\big] \int_M u^{2n/(n-2)}. \tag{6.11}$$

We deduce from (0.8) that

$$\frac{1}{C} \le \big[n(n-2)s - (1-s)E(u)\big]^{(n-2)/4} u \le C. \tag{6.12}$$

It follows from (6.11) and (6.12) that

$$\frac{1}{C} \le -E(u)\big(n(n-2)s - (1-s)E(u)\big)^{(n-2)/2} \le C.$$

Consequently,

$$\frac{1}{C} \le n(n-2)s - (1-s)E(u) \le C,$$

which, in view of (6.12), implies that

$$\frac{1}{C} \le u \le C.$$

We can then apply standard elliptic theories to the equation of u to conclude that, for some $\overline{\Lambda}_c > 1$, u does not belong to ∂D_Λ for all $\Lambda \ge \overline{\Lambda}_c$.

Using Lemma 6.2 and the homotopy invariance of the Leray-Schauder degree, we have

$$\deg\big(\mathrm{Id} - T_{((n+2)/(n-2)),0}, D_\Lambda, 0\big) = \deg\big(F_{(n+2)/(n-2)}, D_\Lambda, 0\big). \tag{6.13}$$

Combining (6.13), (6.6), and Proposition 6.1, we have

$$\deg\big(\mathrm{Id} - T_{((n+2)/(n-2)),c}, D_\Lambda, 0\big) = \deg\big(F_{(n+2)/(n-2)}, D_\Lambda, 0\big) = \deg(F_1, D_\Lambda, 0) = -1.$$

We have completed the proof of Proposition 6.2. $\square$

Proof of Theorem 0.3. Since we have already established (0.8), we only need to show that, for all $c < n-2$, $\widetilde{\mathcal{M}}_c \ne \emptyset$. This follows from Proposition 6.2. $\square$

APPENDIX

We present three analytical facts used in our arguments. First, we present a Harnack inequality for divergence form second-order elliptic equations with Neumann-type boundary condition. Consider

$$Lu = \partial_i \big(a_{ij}(x)\partial_j u + b_i(x)u\big) + c_i(x)\partial_i u + d(x)u$$

on $B_3^+ \subset \mathbf{R}^n, n \geq 2$ where, as usual, $B_3^+ = \{(x = (x^1, \ldots, x^n) \in \mathbf{R}^n : |x| < 3, x^n > 0\}$. For some constant $\Lambda > 1$, the coefficient functions satisfy that

$$\Lambda^{-1}|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall x \in B_3^+, \xi \in \mathbf{R}^n, \tag{A.1}$$

$$|b_i(x)| + |c_i(x)| + |d(x)| \leq \Lambda, \quad \forall x \in B_3^+. \tag{A.2}$$

LEMMA A.1. *Assume (A.1), (A.2), and* $|h(x)| \leq \Lambda$, $\forall x \in B_3^+$. *Let* $u \in C^2(B_3^+) \cap C^1(\overline{B_3^+})$ *satisfy*

$$\begin{cases} -Lu = 0, & u > 0, \textit{in } B_3^+, \\ a_{nj}(x)\partial_j u = h(x)u, & \textit{on } \partial' B_3^+. \end{cases}$$

Then there exists $C = C(n, \Lambda) > 1$ *such that*

$$\max_{\overline{B_1^+}} u \leq C \min_{\overline{B_1^+}} u.$$

Proof. Without loss of generality, we assume that $u > 0$ in $\overline{B}_3^+$. For $k \geq k_0 > 0$, let η be some C^∞ function with $\eta(x) \equiv 0, 5/2 \leq |x| \leq 3$. Multiplying the equation by $\eta^2 u^k$ and integrating by parts, we have, by using the boundary condition of u, that

$$\begin{aligned} &\int_{B_3^+} a_{ij}\partial_j u \partial_i \left(\eta^2 u^k\right) \\ &\qquad = \int_{B_3^+} c_i \partial_i u \eta^2 u^k + \int_{B_3^+} d\eta^2 u^{k+1} - \int_{B_3^+} b_i u \partial_i \big(\eta^2 u^k\big) - \int_{\partial' B_3^+} (h + b_n)\eta^2 u^{k+1}. \end{aligned}$$

It is easy to see from (A.1) and (A.2) that

$$\begin{aligned} a_{ij}\partial_j u \partial_i \left(\eta^2 u^k\right) &\geq \frac{k}{2\Lambda}\eta^2 u^{k-1}|\nabla u|^2 - \frac{C(\Lambda)}{k} u^{k+1}|\nabla \eta|^2, \\ \left|c_i \partial_i u \eta^2 u^k\right| &\leq \frac{k}{8\Lambda}\eta^2 u^{k-1}|\nabla u|^2 + \frac{C(\Lambda)}{k}\eta^2 u^{k+1}, \\ \left|b_i u \partial_i \left(\eta^2 u^k\right)\right| &\leq \frac{k}{8\Lambda}\eta^2 u^{k-1}|\nabla u|^2 + C(\Lambda)(1+k)\left(\eta^2 + |\nabla \eta|^2\right) u^{k+1}. \end{aligned}$$

It follows that

$$\frac{k}{4\Lambda}\int_{B_3^+}\eta^2u^{k-1}|\nabla u|^2 \le C(\Lambda)\left(1+k+\frac{1}{k}\right)\int_{B_3^+}\big(\eta^2+|\nabla\eta|^2\big)u^{k+1}+2\Lambda\int_{\partial' B_3^+}\eta^2u^{k+1}.$$

Set $w=u^{(k+1)/2}$. It is clear that

$$|\nabla(\eta w)|^2 \le 2\big(\eta^2|\nabla w|^2+w^2|\nabla\eta|^2\big).$$

Combining the above two inequalities, we have

$$\int_{B_3^+}|\nabla(\eta w)|^2 \le C(\Lambda,k_0)k^2\int_{B_3^+}\big(\eta^2+|\nabla\eta|^2\big)w^2+C(\Lambda,k_0)k\int_{\partial' B_3^+}(\eta w)^2.$$

It follows from the Sobolev embedding theorems that

$$C(\Lambda,k_0)k\int_{\partial' B_3^+}(\eta w)^2 \le \frac{1}{2}\int_{B_3^+}|\nabla(\eta w)|^2+C(\Lambda,k_0)k^2\int_{B_3^+}(\eta w)^2.$$

Combining the above two inequalities, we have

$$\int_{B_3^+}|\nabla(\eta w)|^2 \le C(\Lambda,k_0)k^2\int_{B_3^+}\big(\eta^2+|\nabla\eta|^2\big)w^2.$$

Using the above and the Sobolev embedding theorems, we have

$$\|\eta w\|_{L^{2n/(n-2)}(B_3^+)} \le C(\Lambda,k_0)k\big\|(\eta+|\nabla\eta|)w\big\|_{L^2(B_3^+)}, \quad \text{for } n\ge 3,$$

and

$$\|\eta w\|_{L^p(B_3^+)} \le C(\Lambda,k_0,p)k\big\|(\eta+|\nabla\eta|)w\big\|_{L^2(B_3^+)}, \quad \text{for } n=2,\ 0<p<\infty.$$

We iterate as usual (see [18, page 197]) and obtain

$$\max_{\overline{B_1^+}} u \le C(\Lambda,n)\|u\|_{L^2\left(B_{3/2}^+\right)}.$$

It is also standard that we can then obtain

$$\max_{\overline{B_1^+}} u \le C(\Lambda,n,p)\|u\|_{L^p\left(B_{3/2}^+\right)}, \quad \forall p>0.$$

It is easy to see that $\xi=u^{-1}$ satisfies

$$\begin{cases} -\partial_i\big(a_{ij}(x)\partial_j\xi-b_i(x)\xi\big)+\big(-c_i(x)-2b_i(x)\big)\partial_i\xi+d(x)\xi\le 0, & \text{in } B_3^+,\\ a_{nj}(x)\partial_j\xi=-h(x)\xi, & \text{on } \partial' B_3^+. \end{cases}$$

Substituting ξ for u in the previous argument, we obtain that

$$\max_{\overline{B_1^+}} \xi \le C(\Lambda, n, p)\|\xi\|_{L^p\left(B_{3/2}^+\right)}, \quad \forall p > 0,$$

namely,

$$\left(\min_{\overline{B_1^+}} u\right)^{-1} \le C(\Lambda, n, p)\left(\int_{B_2^+} u^{-p}\right)^{1/p}, \quad \forall p > 0.$$

It is clear that Lemma A.1 will follow if we can show that, for some $p > 0$,

$$\text{(A.3)} \qquad \int_{B_2^+} u^p \int_{B_2^+} u^{-p} \le C(\Lambda, n, p).$$

For $x_0 \in \overline{B_2^+}$, $0 < R < 1/10$, let η be some smooth cut-off function satisfying $\eta(x) = 1$ for $|x - x_0| < R$, $\eta(x) = 0$ for $|x - x_0| > 2R$, and $|\nabla\eta(x)| \le C/R$ for all x. Multiplying the equation of u by $v = \eta^2 u^{-1}$ and integrating by parts, we have

$$\begin{aligned}&\int_{B_{2R}^+(x_0)} a_{ij}\partial_j u \partial_i v \\ &\qquad = \int_{B_{2R}^+(x_0)} c_i \partial_i u v + \int_{B_{2R}^+(x_0)} d u v - \int_{B_{2R}^+(x_0)} b_i u \partial_i v - \int_{\partial' B_{2R}^+(x_0)} (h + b_n) u v.\end{aligned}$$

Arguing as before, we have

$$\begin{aligned} a_{ij}\partial_j u \partial_i v &\le -\frac{1}{2\Lambda}\eta^2 |\nabla \log u|^2 + C(\Lambda)|\nabla\eta|^2, \\ |c_i \partial_i u v| &\le \frac{1}{8\Lambda}\eta^2|\nabla \log u|^2 + C(\Lambda)\eta^2, \\ |b_i u \partial_i v| &\le \frac{1}{8\Lambda}\eta^2|\nabla\log u|^2 + C(\Lambda)\left(\eta^2 + |\nabla\eta|^2\right). \end{aligned}$$

It follows that

$$\int_{B_{2R}^+(x_0)} \eta^2|\nabla \log u|^2 \le C\int_{B_{2R}^+(x_0)} \left(\eta^2 + |\nabla\eta|^2\right) + C\int_{\partial' B_{2R}^+(x_0)} \eta^2 \le CR^{n-2}.$$

Using Hölder's inequality, we have

$$\int_{B_R^+(x_0)} |\nabla \log u| \le CR^{n-1}.$$

It follows from the John-Nirenberg estimate (see [18, Theorem 7.21]) that there exists some $p > 0$ such that

$$\int_{B_2^+} e^{p\left|\log u - (\log u)_{B_2^+}\right|} \le C,$$

where $(\log u)_{B_2^+} = (1/|B_2^+|)\int_{B_2^+} \log u$. Estimate (A.3) can be derived easily as follows:

$$\begin{aligned}\int_{B_2^+} u^p \int_{B_2^+} u^{-p} &= \int_{B_2^+} e^{p\left[\log u - (\log u)_{B_2^+}\right]} \int_{B_2^+} e^{-p\left[\log u - (\log u)_{B_2^+}\right]} \\ &\le \left(\int_{B_2^+} e^{p\left|\log u - (\log u)_{B_2^+}\right|}\right)^2 \le C.\end{aligned}$$

We have thus established Lemma A.1. $\square$

LEMMA A.2. *Let Ω be a bounded domain in $\mathbf{R}^n$ with piecewise smooth boundary $\partial\Omega = \Gamma \cup \Sigma$, $V \in L^\infty(\Omega)$, $h \in L^\infty(\Sigma)$. Suppose that $u \in C^2(\Omega)\cap C^1(\overline{\Omega})$, $u > 0$ in $\bar{\Omega}$ satisfies*

$$\begin{cases} \Delta u + Vu \le 0, & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} \ge hu, & \text{on } \Sigma, \end{cases}$$

and $v \in C^2(\Omega)\cap C^1(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta v + Vv \le 0, & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} \ge hv, & \text{on } \Sigma, \\ v \ge 0, & \text{on } \Gamma, \end{cases}$$

where ν denotes the unit outer normal of Σ. Then $v \ge 0$ in $\overline{\Omega}$.

Proof. Let $w = v/u$; then

$$\begin{cases} \Delta w + 2\dfrac{\nabla u}{u}\cdot\nabla w + \dfrac{\Delta u + Vu}{u}\cdot w \le 0, & \text{in } \Omega, \\ \dfrac{\partial w}{\partial \nu} + u^{-1}\left(\dfrac{\partial u}{\partial \nu} - hu\right) w \ge 0, & \text{on } \Sigma, \\ w \ge 0, & \text{on } \Gamma. \end{cases}$$

We conclude from the maximum principle that $w \ge 0$; therefore, $v \ge 0$. $\square$

LEMMA A.3. *Let (M, g) be a smooth compact Riemannian manifold with first eigenvalue $\lambda_1(M) > 0$ and the boundary mean curvature $h_g \equiv 0$. Let $\epsilon_0 > 0$, and $1 \le p \le ((n+2)/(n-2)) - \epsilon_0$. Suppose that u satisfies*

$$\begin{cases} -L_g u = \mu u^p, & u > 0, \text{in } M, \\ \dfrac{\partial_g u}{\partial \nu} = 0, & \text{on } \partial M, \\ \int_M u^{p+1} = 1. \end{cases}$$

Then,

$$0 < \mu = \int_M \left(|\nabla_g u|^2 + c(n) R_g u^2\right) \le C(M, g, \epsilon_0).$$

Proof. For $1 + \epsilon_0 \le p \le ((n+2)/(n-2)) - \epsilon_0$, it follows from Theorem 1.1 that $C^{-1} \le \mu^{1/(p-1)} u \le C$, which, together with $\int_M u^{p+1} = 1$, immediately gives the estimate of μ. Here and in the following, C denotes various positive constants depending only on ϵ_0, M, and g. So we only need to establish the estimate of μ for $1 \le p \le 1+\epsilon_0$ for small ϵ_0. In the following, we give a proof for $1 \le p \le (n/(n-2)) - \epsilon_0$. Multiplying the equation by the first eigenfunction φ_1 with the normalization (5.1), we have

$$\lambda_1 \int_M \varphi_1 u = \mu \int_M \varphi_1 u^p. \tag{A.4}$$

Clearly $\mu > 0$. For $p = 1$, $\mu = \lambda_1$. In the following, we assume $1 < p \le (n/(n-2)) - \epsilon_0$. Since $1/C \le \varphi_1 \le C$, we derive from (A.4) and Hölder's inequality that

$$\mu \|u\|_{L^p}^{p-1} \le C. \tag{A.5}$$

From Hölder's inequality, we have

$$\|u\|_{L^{p+1}} \le \|u\|_{L^p}^{\theta} \|u\|_{L^{2n/(n-2)}}^{1-\theta},$$

where $\theta^{-1} = (1/(p+1)) - ((n-2)/2n)^{-1}((1/p) - ((n-2)/2n))$. It is clear that $0 < \theta < 1, \theta^{-1} \le C$, and $(1-\theta)^{-1} \le C$. Therefore, by using the Sobolev embedding theorems, we have that

$$1 = \|u\|_{L^{p+1}} \le C \|u\|_{L^p}^{\theta} \mu^{(1-\theta)/2} \le C \left(\mu \|u\|_{L^p}^{2\theta/(1-\theta)}\right)^{(1-\theta)/2}.$$

It follows that

$$\mu \|u\|_{L^p}^{2\theta/(1-\theta)} \ge \frac{1}{C}. \tag{A.6}$$

Combining (A.5) and (A.6), we have

$$\mu^{1-((1-\theta)(p-1)/2\theta)} \le C^{1+((1-\theta)(p-1)/2\theta)}.$$

Since, for $1 \le p \le (n/(n-2)) - \epsilon_0$, we have

$$1 - \frac{(1-\theta)(p-1)}{2\theta} \ge \delta(\epsilon_0) > 0.$$

The estimate of μ follows immediately from the above two estimates. □

LEMMA A.4. *Let (M, g) be a smooth compact Riemannian manifold with $\lambda_1(M) < 0$ and $h_g \equiv 0$. Let $\epsilon_0 > 0$ and $1 \le p < \infty$. Suppose that u satisfies*

$$\begin{cases} -L_g u = \mu u^p, \quad u > 0, \textit{in } M, \\ \dfrac{\partial_g u}{\partial \nu} = 0, \quad \textit{on } \partial M, \\ \int_M u^{p+1} = 1. \end{cases}$$

Then,

$$0 < -\mu = -\int_M \left(|\nabla_g u|^2 + c(n) R_g u^2 \right) \le -c(n) \int_M R_g u^2 \le C(M, g).$$

Proof. It is obvious. □

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