THE YAMABE THEOREM AND GENERAL RELATIVITY

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1. INTRODUCTION

In 1960 H. Yamabe [1] claimed to have proven that every compact Riemannian manifold without boundary could be conformally deformed to one with constant scalar curvature. In 1968 Neil Trudinger [2] showed that the Yamabe proof was incorrect, and gave a partial (correct) proof. This result was improved on by T. Aubin [3] and finally completed by Richard Schoen [4] in 1984. This article consists of a rederivation and slight extension of the Schoen result and shows that the Schoen technique casts light on several problems in General Relativity.

I will consistently assume (because of my interest in the Einstein equations) that we are dealing with three-dimensional Riemannian manifolds. Much of what I do can be repeated in higher dimensions, but I will not deal with these questions.

The key thread that runs through all the analyses of the Yamabe problem is the so called Yamabe invariant of a compact manifold (M,g):

$$Y(g) = \inf_{\theta} \frac{\int\{(\nabla\theta)^2 + \frac{1}{8}R\theta^2\}dv}{[\int\theta^6 dv]^{1/3}} \tag{1.1}$$

The Yamabe constant is a conformal invariant. Consider a conformal transformation of M by some positive function φ

$$\bar{g}_{ab} = \varphi^4 g_{ab} \tag{1.2}$$

Given that the scalar curvature transforms as [5]

$$\bar{R} = \varphi^{-4}R - 8\varphi^{-5}\nabla^2\varphi \tag{1.3}$$

it is easy to show (with $\bar{\theta} = \frac{\theta}{\varphi}$)

(1.4) $$\int_M \{(\bar\nabla\bar\theta)^2 + \tfrac{1}{8}\bar R\,\bar\theta^2\}\,d\bar v = \int\{(\nabla\theta)^2 + \tfrac{1}{8}R\,\theta^2\}dv$$

and

(1.5) $$\int_M \bar\theta^6 d\bar v = \int \theta^6 d\bar v\ .$$

Thus it immediately follows

(1.6) $$Y(M,\bar g) = Y(M,g)\ .$$

The function which minimizes the Yamabe functional satisfies

(1.7) $$-\nabla^2\mu + \tfrac{1}{8}R\mu = \lambda\mu^5$$

with λ a constant. The relationship between λ and Y is

(1.8) $$Y = \lambda[\int\mu^6 dv]^{2/3}\ .$$

Further, the metric $\bar g = \mu^4 g$ satisfies

(1.9) $$\bar R = 8\lambda$$

which is obviously constant. Finally, if we are given a manifold with $R = R_0$, a constant, then the minimizing equation (1.7) is clearly satisfied by μ = constant. In turn, we get

(1.10) $$Y = \tfrac{1}{8}R_0\left[\int_M dv\right]^{2/3} = \tfrac{1}{8}R_0 V^{2/3}$$

Therefore the sign of the Yamabe invariant determines, and is determined by, the sign of the constant scalar curvature one can conformally transform to.

One place where the Yamabe invariant can be easily evaluated is for the three-sphere S^3 with constant scalar curvature. For this case we get

(1.11) $$Y = 3(\pi^2/4)^{2/3}\ .$$

This is a very special number. It is the Sobolev constant for flat space, defined as

$$(1.12) \qquad S = \inf_{\xi \in C_0^\infty} \frac{\int (\nabla\xi)^2 dv}{[\int \xi^6 dv]^{1/3}}$$

evaluated over all functions of compact support.

Trudinger [2] and Aubin [3] showed that (i) $Y \leq 3(\pi^2/4)^{2/3}$ for every compact manifold and that (ii) if $Y < 3(\pi^2/4)^{2/3}$, then that manifold could be conformally transformed to a constant scalar curvature one. Richard Schoen's [4] completion of the Yamabe theorem consisted in showing that the Yamabe invariant for every compact three-manifold without boundary was strictly less than $3(\pi^2/4)^{2/3}$ (except, of course, S^3 with constant scalar curvature).

There are a couple of obvious things one can say about the sign of the Yamabe invariant. If we have a manifold with non-positive scalar curvature, we can use $\theta \equiv 1$ as a test function in (1.1) to give

$$(1.13) \qquad Y \leq \tfrac{1}{8} V^{-1/3} \int R dv < 0$$

(except, of course, $R \equiv 0$, which gives $Y \leq 0$).

If, on the other hand, we have a manifold with non-negative scalar curvature, $R \geq 0$, we have for every test function θ

$$(1.14) \qquad \int \{(\nabla\theta)^2 + \tfrac{1}{8} R\theta^2\} dv > 0 \Rightarrow Y \geq 0 .$$

Thus, on a compact manifold, the global sign of the scalar curvature is a conformal invariant. This can also be seen directly by multiplying (1.3) by φ^5 and integrating.

Trudinger [2] had proved that one could always make a conformal transformation so as to fix the sign of the scalar curvature. Thus Schoen [4] had only to consider those manifolds where the sign of the scalar curvature was everywhere positive.

Flat space, where we found the Sobolev constant $S = 3(\pi^2/4)^{2/3}$, and S^3 with constant curvature, where we found the Yamabe invariant $Y = 3(\pi^2/4)^{2/3}$ are intimately related. We can conformally compactify flat space to give us a sphere of constant curvature. The conformal transformation

$$g_{ab} = u^4 \delta_{ab}, \ u = \frac{\alpha^{1/2}}{(\alpha^2 + r^2)^{1/2}} \quad (\alpha = \text{constant}) \tag{1.15}$$

does exactly that. The function u satisfies

$$\nabla^2 u + 3u^5 = 0 \tag{1.16}$$

and is the minimizing function for the Sobolev constant (1.12). This shows us that the Yamabe invariant is not only an invariant under regular conformal transformations but is also invariant under conformal compactification or decompactification, i.e., we can use in (1.1) functions which blow up like $1/r$ at a point. Thus we can also evaluate the Yamabe invariant of an asymptotically flat manifold (M,g)

$$Y(M,g) = \inf_{\xi \in C_0^\infty} \frac{\int \{(\nabla\xi)^2 + \frac{1}{8} R \xi^2\} dv}{[\int \xi^6 dv]^{1/3}} . \tag{1.17}$$

Richard Schoen [4] considers a compact manifold without boundary with positive scalar curvature and looks at the Green's function of the operator $8\nabla^2 - R$, i.e., a solution ζ to

$$8\nabla^2 \zeta - R\zeta = \delta(x - x_0) . \tag{1.18}$$

It is easy to show that ζ is positive everywhere. Let us assume the opposite, that $\zeta < 0$ on some subset $\bar{M}$ of M with $x_0 \notin \bar{M}$. We will have $\zeta = 0$ on $\partial\bar{M}$. Now multiply (1.18) by ζ and integrate

(1.19)
$$\int_{\bar{M}} \{8\zeta\nabla^2\zeta - R\zeta^2\}dv = 0$$
$$\Rightarrow \; 8\oint_{\partial\bar{M}} \zeta\nabla\zeta\cdot d\vec{s} = \int_{\bar{M}} \{8(\nabla\zeta)^2 + R\zeta^2\}dv \,.$$

The surface integral vanishes and we therefore have the volume integral of a positive definite quantity vanishing. This cannot happen and therefore we can assume $\zeta > 0$. This means we can use ζ as a conformal factor and $(M,\bar{g}) = (M,\zeta^4 g)$ can be regarded as an asymptotically flat manifold. Further since $8\nabla^2\zeta - R\zeta = 0$ everywhere except at x_0, the 'point at infinity', $(M,\bar{g})$ is an asymptotically flat manifold with zero scalar curvature. Thus the Yamabe theorem can be reduced to showing that the Yamabe invariant of asymptotically flat manifolds with zero scalar curvature is less than the flat space value $3(\pi^2/4)^{2/3}$.

2. THE YAMABE INVARIANT OF ASYMPTOTICALLY FLAT MANIFOLDS.

Let us be given an asymptotically flat manifold with zero scalar curvature. Let us further assume (this will be relaxed later) that the metric is conformally flat outside a region of compact support.

Now conformally transform the metric so as to make it flat outside a region of compact support. This means that we are given a manifold (M,g), g flat outside some coordinate radius r_0, and we know that there exists a positive function v which satisfies

(2.1)
$$8\nabla^2 v - Rv = 0$$

(this is the conformal factor which transforms us back to the vanishing scalar curvature manifold). The positive energy theorem [6] tells us that asymptotically v must look like

(2.2)
$$v \to 1 + \frac{M}{2r}, \quad M > 0$$

because we know that the energy of $\bar{g} = v^4 g$ (with $\bar{g} = v^4\delta$ near infinity) is positive, [6] and that v satisfies $\nabla^2 v = 0$ outside the region of compact support. In this calculation we neither know nor care about the interior, the positive energy theorem tells us that R, in general, is negative, but we demand no specific information about it.

built up want to do is get an upper limit for the Yamabe invariant (1.17). We use the following test function

(2.3)
$$\xi = \begin{cases} \beta v \ , & 0 \le r < \bar{r} \\ \dfrac{\alpha^{1/2}}{(a^2+r^2)^{1/2}} , & \bar{r} \le r < \infty \end{cases}$$

with $\bar{r} > r_0$.

Therefore we use the conformal factor in the interior, and the flat-space Sobolev function in the exterior. We have to match the functions and their first derivatives at $\bar{r}$. When we match $1 + M/2r$ with $\dfrac{\alpha^{1/2}}{(\alpha^2+r^2)^{1/2}}$ we get

(2.4)
$$\alpha^2 = \frac{2\,\bar{r}^{\,3}}{M}$$
$$\beta = \alpha^{-1/2}\left[1 + \frac{M}{2\,\bar{r}}\right]^{-3/2} .$$

It is clear that this is nonsense if we had $M \le 0$.

Let us break the integral above the line into two parts

(2.5)
$$\int_0^\infty = \int_0^{\bar{r}} + \int_{\bar{r}}^\infty$$

and we have

(2.6)
$$\int_0^{\bar{r}} \{(\nabla\xi)^2 + \tfrac{1}{8}R\xi^2\}dv = \beta^2 \int_0^{\bar{r}} \{(\nabla v)^2 + \tfrac{1}{8}Rv^2\}dv$$

$$= \beta^2 \oint_{\bar{r}} v\nabla v \cdot d\vec{s} + \beta^2 \int v(\tfrac{1}{8}Rv - \nabla^2 v)dv$$

using (2.1) this reduces to

(2.7)
$$\int_0^{\bar{r}} = \beta^2 \oint_{\bar{r}} v\nabla v \cdot d\vec{s} = \oint_{\bar{r}} \xi\nabla\xi \cdot d\vec{s} \ .$$

The other integral is

(2.8)
$$\int_{\bar{r}}^{\infty} \{(\nabla\xi)^2 + \tfrac{1}{8}R\xi^2\}dv = \int_{\bar{r}}^{\infty} (\nabla u)^2 d^3x$$

$$= \int_{\bar{r}}^{\infty} \{\nabla \cdot (u\nabla u) - u\nabla^2 u\} d^3x$$

$$= -\oint_{\bar{r}} u\nabla u \cdot d\vec{s} + 3\int_{\bar{r}}^{\infty} u^6 dv$$

(on using (1.16)).

When we add the two integrals together the surface integrals at $r = \bar{r}$ cancel because we have matched the functions and first derivatives at $r = \bar{r}$. Therefore we get

(2.9)
$$\int_0^{\infty} \{(\nabla\xi)^2 + \tfrac{1}{8}R\xi^2\}dv = 3\int_{\bar{r}}^{\infty} u^6 dv \ .$$

I can explicitly perform this integration to get

(2.10)
$$\int_{\bar{r}}^{\infty} u^6 dv \simeq \pi^2/4 - \frac{4\pi}{3}(\bar{r}/\alpha)^3$$

$$= \pi^2/4 - \frac{4\pi}{3}\left[\frac{M}{2\,\bar{r}}\right]^{3/2} .$$

(This is in the limit where we choose $\bar{r} >> M$, i.e., $\alpha >> \bar{r}$. In this case we have $u(r \leq \bar{r}) \simeq \alpha^{-1/2}$, a constant. Therefore we have

$$\int_0^{\bar{r}} u^6 dv \simeq \alpha^{-3}\left[\frac{4\pi}{3}\bar{r}^3\right],$$

exactly the correction above).

The denominator in (1.17) we simply estimate by

$$\int_0^{\infty} \xi^6 dv > \int_{\bar{r}}^{\infty} \xi^6 dv = \int_{\bar{r}}^{\infty} u^6 dv . \tag{2.11}$$

Thus we get

$$Y < 3\left[\int_{\bar{r}}^{\infty} u^6 dv\right]^{2/3} = 3\left[\pi^2/4 - \frac{4\pi}{3}\left[\frac{M}{2\bar{r}}\right]^{3/2}\right]^{2/3} . \tag{2.12}$$

This is the desired result and we can immediately see that

$$Y < 3(\pi^2/4)^{2/3} \tag{2.13}$$

so long as $M > 0$, and the positive energy theorem [6] gives us $M = 0$ only if the three-space is flat.

Unfortunately, the argument detailed between (2.3) and (2.13) is flawed. Fortunately, the error made is one that can be easily set right. The error made is in the matching of v and u. While asymptotically v satisfies (2.2), at any finite radius we expect that will have dipole and higher multipole terms (because v satisfies $\nabla^2 v = 0$ outside r_0) and so cannot be smoothly matched to u.

The test function we need to use has to be more complicated than the one previously defined ((2.3)). We choose a radius $\bar{r}$ (now with $\bar{r} > 2r_0$) and find a smoothing function $\psi(r)$ satisfying

(2.14)
$$\psi(r) = \begin{cases} 0, & r < \bar{r}/2 \\ 1, & r > \bar{r} \end{cases}$$

$$\frac{\partial\psi}{\partial r} \sim 2/\bar{r} \ , \ \frac{\partial^2\psi}{\partial r^2} \sim 1/\bar{r}^2 \ .$$

The test function we use is

$$\xi = \begin{cases} \beta v, & r < \bar{r}/2 \\ \beta\{v + \psi(1 + \frac{M}{2r} - v)\}, & \bar{r}/2 = r < \bar{r} \\ u, & \bar{r} \le r < \infty \end{cases}$$

We still match $\beta\left[1 + \frac{M}{2r}\right]$ and $\frac{\alpha^{1/2}}{(\alpha^2+r^2)^{1/2}}$ at $r = \bar{r}$ and that means that (2.4) remains valid, giving α and β in terms of M and $\bar{r}$. Let us define

(2.16)
$$w = 1 + \frac{M}{2r} - v \ .$$

We know $w \sim O\left[\frac{1}{r^2}\right]$, $\nabla w \sim O\left[\frac{1}{r^2}\right]$ and $\nabla^2 w = 0$ (so long as $r > r_0$).

We now re-evaluate the Yamabe functional with the improved test function (2.15), and especially track the extra terms. We still have

(2.17)
$$\int_0^{\bar{r}} \{(\nabla\xi)^2 + \tfrac{1}{8}R\xi^2\}dv = \oint_{\bar{r}} \xi\nabla\xi\cdot d\vec{s} + \int_0^{\bar{r}} \xi(\tfrac{1}{8}R\xi - \nabla^2\xi)dv \ .$$

The surface term is as in (2.7), and will cancel with the surface integral that arises from the outside integral as in (2.8). Therefore the only terms we need worry about are those that arise in the volume term in (2.17), due to the difference between ξ and βv. The integrand in (2.17) vanishes identically inside $\bar{r}/2$, thus we can write

(2.18)
$$\int_0^{\bar{r}} \xi(\tfrac{1}{8}R\xi - \nabla^2\xi)dv = \int_{\bar{r}/2}^{\bar{r}} \xi(-\beta\nabla^2[\psi w])dv$$

$$= \int_{\bar{r}/2}^{\bar{r}} \{-\beta\xi w\nabla^2\psi - 2\beta\xi\nabla\psi\cdot\nabla w\}dv$$

(because $\nabla^2 w = 0$) .

We now estimate (2.18) in terms of $\bar{r}$, using

$$\beta\xi \sim \beta^2 \sim O(r^{-3/2}) \ , \ w \sim O(r^{-2}) \ , \ \nabla w \sim O(r^{-3})$$
$$\nabla^2\psi \sim O(r^{-2}) \ , \ \nabla\psi \sim O(r^{-1})$$

to give that the integrand is bounded by a term of order $(\bar{r}^{-1/2})$ and the integral by a term of order $(\bar{r}^{-5/2})$. This term, if we choose $\bar{r}$ large enough, will be overwhelmed by the negative-definite term of order $\bar{r}^{-3/2}$ in (2.10). Thus by choosing $\bar{r}$ large enough we can rederive (2.12) and show that the Yamabe invariant is strictly less than $3[\pi^2/4]^{2/3}$.

A key assumption so far is that the given manifold is conformally flat outside some radius. This is not necessary. Let us assume instead that the manifold is conformally flat to leading, i.e., $1/r$ order. In other words, the deviations from conformal flatness fall off like $r^{-(1+\epsilon)}$. This means that we assume that we are given an asymptotically flat metric $\bar{g}$ (with zero scalar curvature) which can be written as

$$\bar{g} = v^4 g \tag{2.19}$$

where v is a conformal factor, going to one at infinity. I wish to assume that the base metric g can be written (outside some finite radius) as

$$g_{ab} = \delta_{ab} + h^{TT}_{ab} + O[r^{-(2+\epsilon)}] \tag{2.20}$$

where h^{TT}_{ab} satisfies

$$\delta^{ab}h^{TT}_{ac,b} = \delta^{ab}h^{TT}_{ab} = O$$

(2.21)

$$h^{TT}_{ab} \sim O[r^{-(1+\epsilon)}] \ .$$

The positive energy theorem gives us (as before) $v \to 1 + \frac{M}{2r}$, $M > 0$. I will show that the Yamabe invariant of the manifold given by (2.20) is less than $3\left[\frac{\pi^2}{4}\right]^{2/3}$. I will do this using the same test function (2.15) as before. The problems now will arise in the integral

$$\int_{\bar{r}}^{\infty}\{(\nabla u)^2 + \tfrac{1}{8}Ru^2\}dv$$

and we have to show that all the extra terms fall off faster than $r^{-3/2}$.

The first thing we realise is that

$$\sqrt{g} \sim 1 + O\left[r^{-(2+\epsilon)}\right]$$

thus corrections which arise from replacing dv by d^3x all fall off faster than $\bar{r}^{-2}$.

Let us next deal with the $\frac{1}{8}Ru^2$ integral. The scalar curvature is dominated by

$$R \sim -\tfrac{1}{4}(h^{TT}_{ab,c})^2 \sim O\left[r^{-(4+\epsilon)}\right] .$$

Therefore we can write

$$\int_{\bar{r}}^{\infty}\tfrac{1}{8}Ru^2dv < \frac{c}{\bar{r}^{2+\epsilon}}\int_{\bar{r}}^{\infty}\frac{\alpha}{\alpha^2+r^2}\frac{4\pi r^2}{r^2}\,dr . \tag{2.22}$$

The integral is finite (approximately $2\pi^2$) and so hence the quantity is bounded by $\bar{r}^{-(2+\epsilon)}$ and can be ignored.

The final quantity we need to consider is

$$\int(\nabla u)^2dv \simeq \int\delta^{ab}\partial_a u\partial_b u\, d^3x + \int h^{ab}_{TT}\partial_a u\partial_b u\, d^3x . \tag{2.23}$$

The first part of (2.23) gives us the desired expression (2.10); the second part can be bounded by

$$\int_{\bar{r}}^{\infty}h^{ab}_{TT}\partial_a u\partial_b u\, d^3x < \frac{c}{\bar{r}^{\epsilon}}\int_{\bar{r}}^{\infty}\frac{1}{r}(\partial_r u)_2 d^3x$$
$$= \frac{c}{\bar{r}^{\epsilon}}\int_{\bar{r}}^{\infty}\frac{1}{r}\frac{\alpha r^2}{(\alpha^2+r^2)^3}4\pi r^2dr = \frac{c}{\bar{r}^{\epsilon}}\int_{\bar{r}}^{\infty}4\pi\frac{\alpha r^3}{(\alpha^2+r^2)^3}\,dr .$$

Substituting $r = \alpha \tan\theta$, this changes to

(2.24) $$\frac{c}{\bar{r}^{\epsilon}}\,\alpha^{-1}\int_{\bar{\theta}}^{\pi/2} 4\pi \sin^3\theta \cos\theta\, d\theta < \frac{\pi c}{\bar{r}^{\epsilon}}\,\alpha^{-1} = \pi c\left[\frac{M}{2}\right]^{1/2} \bar{r}^{-(3/2+\epsilon)}$$

on using (2.4). The corrections to the denominator can be handled in a similar fashion. Thus the Yamabe invariant is less than $3(\pi^2/4)^{2/3}$ for this class of manifolds.

I do not believe that condition (2.20) is necessary. I conjecture that the Yamabe invariant is less than $3(\pi^2/4)^{2/3}$ for all those manifolds with finite ADM mass, i.e., those manifolds which go flat faster than $r^{-1/2}$ [7,8,9].

3. APPLICATIONS.

(a) The Sobolev constant versus the Yamabe constant.

On any asymptotically flat manifold we have two related constants, the Sobolev constant

(3.1) $$S = \inf_{\theta\in C_0^\infty} \frac{\int (\nabla\theta)^2 dv}{[\int \theta^6 dv]^{1/3}}$$

and the Yamabe constant

(3.2) $$Y = \inf_{\theta\in C_0^\infty} \frac{\int \{(\nabla\theta)^2 + \frac{1}{8}R\theta^2\} dv}{[\int \theta^6 dv]^{1/3}}.$$

It is clear that if $R \geq 0$, the term $\int R\theta^2$ must contribute a positive amount to the integral (3.2) over (3.1) and so we must have

THEOREM. *On any asymptotically flat Riemannian manifold with non-negative scalar curvature, we must have*

(3.3) $$S \leq Y \leq 3(\pi^2/4)^{2/3}.$$

Equivalently, on a manifold with non-positive scalar curvature we have

$$Y \leq S .$$

(b) **The average scalar curvature and the size of compact manifolds.**

Let us consider a compact manifold without boundary and define on it the Yamabe constant

$$(3.4) \qquad 3(\pi^2/4)^{2/3} > Y = \inf_{\theta} \frac{\int\{(\nabla\theta)^2 + \frac{1}{8}R\theta^2\}dv}{[\int\theta^6 dv]^{1/3}} .$$

One obvious test function to try is $\theta = 1$.

Then we get

$$(3.5) \qquad Y \leq \frac{\frac{1}{8}\int Rdv}{[\int dv]^{1/3}} = \frac{1}{8}\bar{R}V^{2/3}$$

where $V = \int dv$ is the total volume and $\bar{R} = \int Rdv/V$ is the average scalar curvature. In the special case where the manifold has a constant scalar curvature R_0, the function $\theta = 1$ is the minimizing function and we get

$$(3.6) \qquad Y = \frac{1}{8}R_0 V^{2/3} \leq 3(\pi^2/4)^{2/3} .$$

A more interesting bound on Y can be arrived at in terms of the minimum value of R , call it R_{min}

$$(3.7) \qquad Y = \inf_{\theta} \frac{\int\{(\nabla\theta)^2 + \frac{1}{8}R\theta^2\}dv}{[\int\theta^6 dv]^{1/3}} \geq \inf_{\theta} \frac{\int\{(\nabla\theta)^2 + \frac{1}{8}R_{min}\theta^2\}dv}{[\int\theta^6 dv]^{1/3}} .$$

The minimum value of (3.7) occurs when θ is a constant and equals $\frac{1}{8}R_{min}V^{2/3}$.

Therefore we have

$$(3.8) \qquad 3(\pi^2/4)^{2/3} \geq Y \geq \frac{1}{8}R_{min}V^{2/3}$$

for any compact manifold without boundary.

Thus we can bracket Y by

$$\frac{1}{8}R_{min}V^{2/3} \leq Y \leq \frac{1}{8}\bar{R}V^{2/3} \tag{3.9}$$

and we get equality on both sides if and only if $R = R_0$, a constant.

Both of these bounds have interesting applications. Let us begin with (3.8). Consider a spacelike slice through a pseudo-riemannian manifold satisfying the Einstein equations (with cosmological constant Λ). On the spacelike slice will be defined an intrinsic metric g and extrinsic curvature K (essentially the time derivative of g). An especially interesting slice is a so-called 'maximal' slice, one with $\mathrm{tr}K = g\cdot K = 0$, and, in a compact manifold, it is the slice of largest volume. The Einstein equations imply a relationship between the three scalar curvature R , the square of the extrinsic curvature, the source energy density ρ, and the cosmological constant (the Hamiltonian constraint) [10,11]. On a maximal slice the Hamiltonian constraint gives

$$R = 16\pi\rho + K_{ab}K^{ab} + 2\Lambda . \tag{3.10}$$

Thus, if the sources satisfy the weak energy condition [12] $\rho \geq 0$, then on a maximal slice we have

$$R \geq 2\Lambda$$

and hence

$$R_{min} \geq 2\Lambda . \tag{3.11}$$

This can be substituted into (3.8) to give

$$\frac{1}{4}\Lambda\, V^{2/3} \leq 3(\pi^2/4)^{2/3}$$

or

(3.12) $$V \le \frac{\pi^2}{4}\left[\frac{12}{\Lambda}\right]^{3/2}$$

But, of course, the maximal slice is the one with largest volume, hence (3.12) must be true for any slice through the spacetime. In other words we have shown

THEOREM *Consider any compact solution to the Einstein equations which*

(*i*) *has a maximal slice*

(*ii*) *satisfies the weak energy condition*

(*iii*) *has a positive cosmological constant* Λ .

then any spacelike slice through the manifold satisfies

$$V \le \frac{\pi^2}{4}\left[\frac{12}{\Lambda}\right]^{2/3} .$$

This is an equality only for de Sitter space.

REMARKS 1: The maximal slice condition is satisfied by any solution which goes from a big bang to a big crunch.

2. Some effort has been expended in recent years to find conditions similar to (3.8) which would be valid for a part of a Riemannian manifold, rather than the whole [13,14]. Consider conformally flat space with conformal factor $\varphi = (1+\Lambda^2)^{-1/4}$, i.e., $g_{ab} = \varphi^4\delta_{ab}$. It is possible to find a subset of this manifold for which $R_{min}V^{2/3}$ can be unboundedly large.

The other half of the inequality (3.5) also has applications in Physics. One use is to relate the volume to the total mass content of the universe via the Hamiltonian constraint (3.10) [15]. At a moment-of-time-symmetry (defined by $K^{ab} \equiv 0$) and assuming the cosmological constant vanishes, we have (from (3.10))

(3.11) $$\tilde{M} = \int \rho dv = \frac{1}{16\pi}\int R dv .$$

Thus (3.5) can be written as

(3.12) $$Y \leq 2\pi\tilde{M}V^{-1/3} .$$

Another use: On the compact manifold without boundary we can define the function

(3.13) $$f = \bar{R}V^{2/3} = V^{-1/3}\int R dv .$$

If we consider manifolds which are conformally related to one another, this function achieves a minimum $8Y$ on a manifold with constant scalar curvature. On the other hand, if we change the conformal geometry while holding the scalar curvature constant, f achieves its maximum and only extremum, at the conformally flat metric (this is discussed in Section 4). Therefore, the only extremum of f is at the conformally flat S^3 with constant scalar curvature. Further, this extremum is a saddle point. This may well have relevance to the quantum cosmology programme [16].

(c) Counting solutions to the Einstein Equations

One way of constructing solutions to the Einstein equations is to choose initial data for the gravitational field. As already mentioned, these data consist of a Riemannian three-metric and a symmetric tensor K^{ab} (the extrinsic curvature). In addition, we also need the energy-density and current density (ρ, J^a) of the sources. These cannot be freely specified, they must satisfy the constraints [10,11]

(3.14) $$R - K^{ab}K_{ab} + (trK)^2 = 16\pi\rho$$

(3.15) $$\nabla_a\{K^{ab} - (trK)g^{ab}\} = 8\pi J^b .$$

If the initial data is asymptotically flat, we can define a number of conserved quantities (total energy, total linear momentum, total electric charge) which can all be expressed as surface integrals at infinity. These objects can take essentially any value (of course, they must satisfy the requirements of the positive energy theorem).

The situation is very different in the compact, without boundary, case. All these conserved quantities must be zero. This means that the positive energy of the gravitational waves must exactly balance the negative binding energy; if the sources are charged, we must have exactly the same number of protons and electrons. In the asymptotically flat case (or any solution with a boundary) the flux lines can be allowed to leak out at infinity, whereas they all have to be neatly tied off in the compact case.

This means that it is much harder to construct compact solutions that non-compact solutions to the Einstein equations and there should be many more asymptotically flat than closed solutions. The Schoen technique [4], as described in Section 2, of going from a compact to a non-compact manifold using the Green function as a conformal factor can be used to give a more precise version of this counting argument.

The key idea is that the Einstein constraints have a natural conformal invariance, especially when $\mathrm{tr}K = 0$ [12]. Therefore, let us consider a compact solution to the vacuum Einstein equations with a maximal slice. On this slice we have metric and extrinsic curvature satisfying

$$R = K^{ab}K_{ab} \tag{3.16a}$$

$$\nabla_a K^{ab} = 0 \ , \ g_{ab}K^{ab} = 0 \ . \tag{3.16b}$$

Choosing a point x_0, find a solution to

$$8\nabla^2\varphi - R\varphi + R\varphi^{-7} = \delta(x-x_0) \ . \tag{3.17}$$

It is easy to show (using $R \geq 0$) that a unique positive solution exists to this equation. Further, it can be shown that [17]

$$\bar{g}_{ab} = \varphi^4 g_{ab} \ , \ \bar{K}^{ab} = \varphi^{-10}K^{ab} \tag{3.18}$$

form an asymptotically flat solution to

$$\bar{R} = \bar{K}_{ab}\bar{K}^{ab} \quad , \quad \bar{\nabla}_a\bar{K}^{ab} = 0 \quad , \quad \bar{g}_{ab}\bar{K}^{ab} = 0 \tag{3.19}$$

and so are maximal, asymptotically flat data for the Einstein equations. Therefore we have a natural mapping from compact maximal initial data to asymptotically flat maximal initial data.

However, this mapping is not one-one, it is many-one. For each choice of x_0, the support of the delta function, which becomes the 'point at infinity', we get a different asymptotically flat three-space. Therefore, there are enormously more (of the order of the number of points in three-space) asymptotically flat maximal solutions to the constraints than there are compact maximal data.

One cannot go directly from this argument to a claim that there are many more asymptotically flat solutions that compact solutions. The problem is that a standard cosmological solution (going from a big bang to a big crunch) has only a single maximal slice, whereas any asymptotically flat solution with a maximal slice has a three-fold infinity of such slices. A further piece of information is required to eliminate the possibility that many of the different asymptotically flat maximal data sets generate the same space-time.

The key point is that the maximal date sets generated by (3.18) are of a very special kind, so that a general asymptotically flat spacetime which can be maximally sliced should contain at most one, and so we do have a direct link between the number of these special maximal data and the number of different spacetimes. Near x_0, φ is of the form

$$\varphi = 1/r + A + O(r^2) , \quad A \text{ a positive constant .} \tag{3.20}$$

Near infinity we get

$$\bar{g}_{ij} = \delta_{ij} + \frac{4A}{\bar{r}}\delta_{ij} + O(r^{-2}) \tag{3.21}$$

i.e., $\bar{g}$ is Schwarzschildian near infinity. The transformed extrinsic curvature $\bar{K}$ is of the form [18]

$$\bar{K}_m^{\ell} = \frac{K^i_j(x_0)}{\bar{r}^6}\{\delta^{\ell}_i\delta^j_m - 2\delta^{\ell}_i u^j u_m - 2u^{\ell}u_i\delta^j_m + 4u^{\ell}u^j u_i u_m\} \tag{3.22}$$

with $u^{\ell} = x^{\ell}/\bar{r}$, the unit radial vector, $K^i_j(x_0)$ is a constant tracefree tensor, the value of K^{ij} at x_0.

These asymptotic conditions are not preserved by the evolution. In particular, the essentially arbitrary $1/\bar{r}^2$ part of $\bar{g}$ will generate a $1/r^4$ part in $\bar{K}$. Further, if we boost the slice $\bar{K}$ must pick up a $1/r^2$ part to carry the linear momentum. Therefore, a maximal slice satisfying (3.21), (3.22) will be essentially unique in a spacetime, and most spacetimes, even those which can be maximally sliced, will have no slice satisfying (3.21), (3.22). For example, not only is the linear momentum zero, but the angular momentum is also zero. Therefore, at least within the class of maximally slicable solutions to the Einstein equations, there are immensely more asymptotically flat solutions than compact solutions. This surely will have relevance to any 'statistical mechanics' approach to quantum gravity.

(d) A Poincaré inequality for compact manifolds

Consider the Yamabe functional on a compact manifold of constant scalar curvature R_0:

$$Y(u) = \frac{\int\{(\nabla u)^2 + \frac{1}{8}R_0 u^2\}dv}{[\int u^6 dv]^{1/3}}. \tag{3.23}$$

Choose as test-function $u = 1 + f$, and expand $Y(u)$ in terms of powers of f. To quadratic order we get

$$\frac{\int(\nabla u)^2 dv}{[\int u^6 dv]^{1/3}} = V^{-1/3}\int(\nabla f)^2 dv$$

(3.24)

$$\frac{1}{8}R_0 \frac{\int u^2 dv}{[\int u^6 dv]^{1/3}} = \frac{1}{8}R_0 V^{2/3}\{1 - 4\bar{f}^2 + u(\bar{f})^2\}$$

where

$$\bar{f}^2 = V^{-1}\int f^2 dv, \quad \bar{f} = V^{-1}\int f dv .$$

In other words

(3.25) $$Y(1+f) = Y(1) + V^{-1/3}\{\int(\nabla f)^2 - \tfrac{1}{2}R_0 V[\bar{f}^2 - (\bar{f})^2]\} + \ldots$$

We know that the Yamabe functional is minimized on a manifold with constant scalar curvature by the constant functions, so we should not be surprised to find no term linear in f in (3.25). The interesting property is that since $Y(1)$ is a minimum, the quadratic term, the second variation of Y, must be non-negative for any function f. Hence, we must have

(3.26) $$\tfrac{1}{2}R_0 V[\bar{f}^2 - (\bar{f})^2] \leq \int(\nabla f)^2 dv$$

for any function f, on a compact manifold with constant scalar curvature.

This is exactly of the same form of the well-known Poincaré inequality [19] for any convex subset Ω of $\mathbb{R}^3$

(3.27) $$\|f - f_\Omega\|_2 \leq \left[\frac{4\pi}{3|\Omega|}\right]^{2/3} d^3 \|\nabla f\|_2$$

where $|\Omega| = \mathrm{vol}(\Omega)$, $d = \mathrm{diam}(\Omega)$, $f_\Omega = 1/|\Omega|\int_\Omega f dv$. The two are identical, except for replacing $\left[\frac{4\pi}{3|\Omega|}\right]^{2/3} d^3$ by $(2/R_0)^{1/3}$.

This inequality (3.26) can be easily generalised to the case where the compact, without boundary, manifold has nonconstant scalar curvature. In this case, all one has to do is replace R_0 by R_{min}, the minimum value of the scalar curvature on the manifold. The inequality now takes the form

$$\frac{1}{2}R_{min}[\bar{f^2} - (\bar{f})^2] \leq \int(\nabla f)^2 dv \tag{3.28}$$

for any function f.

The trick is to take the standard Yamabe functional (3.2) and replace R in it by R_{min}. This functional is now again minimized by the constant functions. Take the second variation and (3.28) immediately emerges. Of course, it is obvious that (3.26) and (3.28) are nontrivial only in the case where the scalar curvature is positive.

4. THE YAMABE CONSTANT AND THE ADM MASS

Consider any asymptotically flat Riemannian manifold. This manifold will have some Yamabe constant Y. If $Y > 0$, then the manifold can be conformally transformed to an asymptotically flat one satisfying $R = 0$ [24]. On the other hand, if $Y \leq 0$, the manifold cannot be conformally transformed to $R = 0$. One convoluted way of proving the first claim is to conformally compactify the manifold (keeping $Y > 0$), and then open it up again at the same point as in Section 2 to a manifold with $R = 0$.

The second claim can be proven by contradiction. Let us assume that one can transform to $R = 0$. On this manifold the Yamabe constant is defined as

$$Y = \inf_{\theta \in C_0^\infty} \frac{\int(\nabla\theta)^2 dv}{[\int \theta^6 dv]^{1/3}} > 0 \tag{4.1}$$

contradicting the assumption $Y \leq 0$.

On any manifold with $R = 0$ we can define the ADM mass [10,11]

$$m = \frac{1}{16\pi}\oint_\infty g^{ab}g^{cd}(g_{ac,d} - g_{cd,a})dS_b \tag{4.2}$$

and we know that $m > 0$. Thus, there is some connection between positive energy and positive Yamabe costant. The linkage goes much deeper, however.

The first point of similarity is that, on the set of metrics satisfying $R = 0$, the only extremum of the energy is flat space [20]. It is a minimum there. The only extremum of the Yamabe constant is also at flat space. However, it is a maximum there.

This has been demonstrated by Lars Andersson [21]. He shows that the first derivative of the Yamabe constant is

$$D_h Y(g) = \int [-\tfrac{1}{8}u^2 \mathrm{Ric}(u^4 g) + \tfrac{1}{3} Y(g) u^6 g] \circ h \, dv \tag{4.3}$$

where u is the function that minimizes the Yamabe functional at g . The stationary points of the Yamabe constant are those metrics which satisfy

$$-3u^2 \mathrm{Ric}(u^4 g) + Y(g) u^6 g = 0 \, . \tag{4.4}$$

If we conformally transform, i.e., write $\bar{g} = u^4 g$, (4.4) can be written as

$$-3\mathrm{Ric}(\bar{g}) + Y(\bar{g})\bar{g} = 0 \, . \tag{4.5}$$

The only solution of (4.5) is that $\bar{g}$ is the constant curvature S^3 and hence g must be conformally flat. This is the point where the Yamabe constant has its largest value $3(\pi^2/4)^{2/3}$ and so must be a maximum.

Another linkage between Y and m is the behavious of m as Y approaches zero. It can be shown that $Y = 0$ corresponds to $m = \infty$. More precisely, consider the set of all asymptotically flat three-metrics with the standard finite-energy asymptotic structure [4,8,9], i.e., $g - \delta \sim O[r^{-(1/2+\epsilon)}]$. Consideı a smooth curve of metrics in this space, g_t, with $Y(g_t) > 0$ for all $t > 0$ and $Y(g_0) = 0$. All the metrics g_t, $t > 0$, can be regularly conformally transformed to ones with zero scalar curvature, call them $\tilde{g}_t(R=0)$, and the positive energy theorem guarantees that the mass of $\tilde{g}_t$ (as defined by (4.2)) is finite and positive. The mass of g_0 cannot be equivalently defined, because we cannot conformally transform it to an asymptotically flat metric with zero scalar curvature. What we will show is that $m(\tilde{g}_t)$ becomes unboundedly large as $t \to 0$.

Sine we are dealing with conformal transformations on metrics, we can use this conformal freedom to place restrictions on the sequence of metrics g_t we consider.

THEOREM [22] *All asymptotically flat metrics satisfying* $g - \delta \sim O[r^{-(1/2+\epsilon)}]$ $g_{ab,c} \sim O[r^{-(3/2+\epsilon)}]$ *can be conformally transformed to metrics of negative scalar curvature of compact support. Further, on such transformed metrics the surface integral* $B = \oint g^{ab}g^{cd}(g_{ca,d}-g_{cd,a})dS_b$ *is finite.*

PROOF Let us first make the scalar curvature negative. Start with an asymptotically flat metric g with scalar curvature R. Solve

$$\nabla^2\theta = \tfrac{1}{8}R \ , \quad \theta \to 0 \ \text{ at } \infty . \tag{4.6}$$

Conformally transform the metric with conformal factor $\varphi = e^{\theta}$. Since θ is finite and goes to zero at infinity, φ is positive and goes to one at infinity. If we define $\bar{g} = \varphi^4 g$, we get

$$\bar{R} = \varphi^{-4}R - 8\varphi^{-5}\nabla^2\varphi . \tag{4.7}$$

Using

$$\begin{aligned} \nabla^2\varphi &= \nabla^2(e^{\theta}) = \varphi\nabla^2\theta + \varphi(\nabla\theta)^2 \\ &= \tfrac{1}{8}R\varphi + \varphi\nabla^2\theta \end{aligned} \tag{4.8}$$

we finally get

$$\begin{aligned} \bar{R} &= \varphi^{-4}R - \varphi^{-4}R - 8\varphi^{-4}(\nabla\theta)^2 \\ &= -8\varphi^{-4}(\nabla\theta)^2 \le 0 . \end{aligned} \tag{4.9}$$

Now, to make the scalar curvature have compact support we need a cut off function $\psi(r)$ satisfying

$$0 \leq \psi(r) \leq 1, \quad \psi(r) = \begin{cases} 0, & r \leq R_0 \\ 1, & r \geq 2R_0 \end{cases}$$

If we choose R_0 large enough we can ensure that $\psi\bar{R}$ is small in any reasonable norm. This guarantees the existence of a positive solution to

$$8\bar{\nabla}^2\bar{\varphi} - (\psi\bar{R})\bar{\varphi} = 0 \ , \ \bar{\varphi} \to 1 \ \text{at} \ \infty . \tag{4.11}$$

Now conformally transform to $\bar{\bar{g}} = \bar{\varphi}^4\bar{g}$ to give

$$\bar{\bar{R}} = \bar{\varphi}^{-4}\bar{R} = 8\bar{\varphi}^{-5}\bar{\nabla}^2\bar{\varphi} .$$

This gives

$$\bar{\bar{R}} = \bar{\varphi}^{-4}(\bar{R} - \psi\bar{R}) . \tag{4.12}$$

Obviously $\bar{\bar{R}}$ has compact support and is non-positive.

These conformal transformations do not change the asymptotic behaviour of the metric. Since $\bar{\bar{R}}$ has compact support, it is obvious that $\int\bar{\bar{R}}dv$ is finite. The difference between the surface integral $\bar{\bar{B}}$ and $\int\bar{\bar{R}}dv$ is an integral of the form $\int(\bar{\bar{g}}_{ij,k})^2$ which is finite, hence $\bar{\bar{B}}$ is finite. Q.E.D.

REMARK It is clear that one could make yet another conformal transformation so as to set the surface integral to zero while keeping the scalar curvature non-positive and of compact support. This, however, is an unnecessary luxury.

This theorem means that instead of a sequence of metrics along which the Yamabe constant goes to zero, we can consider a sequence of metrics each with non-positive scalar curvature of compact support, along which the Yamabe constant goes to zero. Since $Y(g_t) > 0$ for $t > 0$, this means that g_t can be conformally transformed to an asymptotically flat manifold of zero scalar curvature. Hence there exists a function φ_t which satisfies

(4.13) $$8\nabla_t^2\varphi_t - R_t\varphi_t = 0 \ , \ \varphi_t > 0 \ , \ \varphi_t \to 1 \text{ at } \infty .$$

Since $Y(g_0) = 0$, we can conformally compactify g_0 to a compact manifold with zero scalar curvature. Hence there exists a function μ_0 which satisfies

(4.14) $$8\nabla_0^2\mu_0 - R_0\mu_0 = 0 \ , \ \mu_0 > 0 \ , \ \mu_0 \to 0 \text{ at } \infty .$$

Asymptotically, it is clear

(4.15) $$\varphi_t \sim 1 + \frac{\alpha_t}{r} , \ \alpha_t = -\frac{1}{32\pi}\int R_t\varphi_t dv$$

(4.16) $$\mu_0 \sim \beta r \ , \ \beta = -\frac{1}{32\pi}\int R_0\mu_0 dv .$$

It is clear $\beta \neq 0$ and I will show $\alpha_t \to \infty$ as $t \to 0$. We also have, for any $t \neq 0$,

(4.17) $$\begin{aligned} 4\pi\beta &= \oint(\mu_0\nabla\varphi_t - \varphi_t\nabla\mu_0)\cdot d\vec{s} \\ &= \int^{\infty}\sqrt{g_t}\,(\mu_0\nabla^2\varphi_t - \varphi_t\nabla^2\mu_0)d^3x \\ &= \int\sqrt{g}\,[1/8\mu_0\varphi_t(R_t-R_0) - \varphi_t(\nabla_t^2 - \nabla_0^2)\mu_0]d^3x . \end{aligned}$$

It is clear that $R_t \to R_0$ and $\nabla_t^2\mu_0 \to \nabla_0^2\mu_0$ as $t \to 0$. Therefore the terms multiplying φ_t in (4.17) smoothly approach zero as $t \to 0$. The only way that the volume integral can maintain a fixed value is if $\varphi_t \to \infty$. In other words

(4.18) $$\begin{aligned} 4\pi\beta &= \int\sqrt{g_t}\,[\tfrac{1}{8}\mu_0\varphi_t(R_t-R_0) - \varphi_t(\nabla_t^2-\nabla_0^2)\mu_0]d^3x \\ &\leq (\max\varphi_t)\int\sqrt{g_t}\,[\tfrac{1}{8}|\mu_0(R_t-R_0)| + |(\nabla_t^2-\nabla_0^2)\mu_0|]d^3x . \end{aligned}$$

The integral on the right-hand-side of (4.18) becomes small as $t \to 0$ and so $(\max\varphi_t)$ must become unboundedly large.

Since φ_t satisfies (4.13) we know that it must achieve its maximum on the support of R_t. Further, we know that φ_t becomes large everywhere, not just at an isolated point. This is the content of the Harnack inequality which says that for any function u which is a solution to a linear elliptic equation, on any ball B_R of radius R

$$\sup_{B_R} u \le C \inf_{B_R} u \tag{4.19}$$

which says that if the maximum becomes large the minimum also becomes large.

This means in particular that the minimum value of φ_t on the support of R_t becomes large and

$$\alpha_t = -\frac{1}{32\pi}\int R_t\varphi_t dv \to \infty \text{ as } t \to 0 . \tag{4.20}$$

The ADM mass of the metric with zero scalar curvature is given by (remembering $\bar{g} = \varphi^4 g$)

$$\begin{aligned} m &= \frac{1}{16\pi}\oint_\infty \bar{g}^{ab}\bar{g}^{cd}(\bar{g}_{ac,d} - \bar{g}_{cd,a})dS_b \\ &= \frac{1}{16\pi}\oint_\infty g^{ab}g^{cd}(g_{ac,d} - g_{cd,a})dS_b - \frac{1}{2\pi}\oint_\infty \varphi_{,b}dS^b \\ &= \frac{1}{16\pi}\oint_\infty g^{ab}g^{cd}(g_{ac,d} - g_{cd,a})dS_b - \frac{1}{16\pi}\int\sqrt{g}\, R\varphi d^3x . \end{aligned} \tag{4.21}$$

The metrics g_t are chosen so that the surface integral remains finite whereas $-\frac{1}{16\pi}\int\sqrt{g}\, R\varphi\, d^3x \to \infty$ as $t \to 0$. Therefore the mass becomes unboundedly large as the Yamabe constant approaches zero.

The converse of this is easy to prove: if the mass goes to infinity then the Yamabe constant must go to zero. Consider a smooth curve of asymptotically flat metrics, each with negative scalar curvature of compact support, which can be conformally transformed to zero scalar curvature and along which the ADM mass becomes unboundedly large. In other words, we assume a curve of metrics g_t, and a set of solutions to

(4.22) $$8\nabla_t^2\varphi_t - R_t\varphi_t = 0 \ , \ \varphi_t > 1 \ , \ \varphi_t \to 1 \text{ at } \infty$$

with the property that

(4.23) $$-\int R_t\varphi_t dv_t \to \infty \text{ as } t \to 0 .$$

Rewrite (4.22) and (4.23) in terms of $\theta_t = \theta_t - 1$ as

(4.24) $$8\nabla_t^2\theta_t - R_t\theta_t = R_t \ , \ \theta_t > 0 \ , \ \theta_t \to 0 \text{ at } \infty$$

and

(4.25) $$-\int R_t\theta_t dv_t \to \infty \text{ as } t \to 0 .$$

Multiplying (4.24) by θ_t to give

(4.26) $$-8(\nabla\theta_t)^2 - R_t\theta_t^2 + 8\nabla\circ(\theta_t\nabla\theta_t) = R_t\theta_t .$$

Integrating gives

(4.27) $$\int\{(\nabla\theta_t)^2 + \tfrac{1}{8}R_t\theta_t^2\}dv = -1/8\int R_t\theta_t dv_t .$$

Now

(4.28) $$-\int R_t\theta_t dv_t \leq \Big[\int |R_t|^{6/5}dv\Big]^{5/6}\Big[\int \theta_t^6 dv\Big]^{1/6} .$$

Combining (4.27) and (4.28) we get

(4.29) $$\frac{\int\{(\nabla\theta_t)^2 + \frac{1}{8}R\theta_t^2\}dv}{[\int \theta_t^6 dv]^{1/3}} \leq \frac{1}{8}\,\frac{[\int |R_t|^{6/5}dv]^{5/6}}{[\int \theta_t^6 dv]^{1/6}} \to 0 \text{ as } t \to 0 .$$

From (4.21) we have

$$-\frac{1}{16\pi}\int R_t\theta_t dv_t = m_t + \frac{1}{16\pi}\int R_t dv_t = \frac{1}{16\pi}\oint_\infty g^{ab}g^{cd}(g_{ac,d} - g_{cd,a})dS_b .$$

Define C_1 as the minimum value of $\frac{1}{16\pi}\int R_t dv_t - \frac{1}{16\pi}\oint_\infty g^{ab}g^{cd}(g_{ac,d}-g_{cd,a})dS_b$ for some range of t, $0 \le t \le t_0$.

Therefore we have

$$m_t + C_1 \le -\frac{1}{16\pi}\int R_t \theta_t dv_t . \tag{4.30}$$

Now (4.28) gives

$$m_t + C_1 \le \frac{1}{16\pi}[\int |R_t|^{6/5} dv_t]^{5/6}[\int \theta_t^6 dv_t]^{1/6} \tag{4.31}$$

while (4.29) gives

$$Y(g_t) \le \frac{1}{8}[\int |R_t|^{6/5} dv_t]^{5/6}/[\int \theta_t^6 dv_t]^{1/6} . \tag{4.32}$$

Multiplying (4.31) and (4.32) gives

$$Y_t(m_t+C_1) \le \frac{1}{128\pi}[\int |R_t|^{6/5} dv_t]^{5/3} . \tag{4.33}$$

Obviously on the same interval we can define

$$C_2 = \max\left\{\frac{1}{128\pi}[\int |R_t|^{6/5} dv_t]^{5/3}\right\} \tag{4.34}$$

So therefore on the whole interval $0 \le t \le t_0$ we have

$$m_t Y_t \le C_2 - Y_t C_1$$

or

$$m_t Y_t \le C_3 \tag{4.35}$$

where C_3 is finite, but depends on the curve of metrics.

To get a lower bound on mY we use an argument similar in spirit to the one used earlier in this section to prove that the maximum of φ_t went to infinity as $t \to 0$. Since g_t has positive Yamabe constant, we know that there exists a positive function μ_t, $\mu_t \to 0$ at infinity, satisfying

(4.36) $$\nabla_t^2\mu_t - \frac{1}{8}R_t\mu_t + C_t\mu_t^5 = 0 \ , \ \mu_t > 0 \ , \ \mu_t \to 0 \ \text{ at } \infty$$

where C_t is a positive constant, and if we normalize μ_t to satisfy

(4.37) $$\int \mu_t^6 dv = 1$$

we get

(4.38) $$C_t = Y_t \ .$$

Asymptotically, we have $\mu_t \sim \beta_t/4\pi r$ where

(4.39) $$\beta_t = \int [-\frac{1}{8}R_t\mu_t + C_t\mu_t^5]dv_t > 0 \ .$$

Now we have

(4.40) $$\begin{aligned} \beta_t &= \oint_\infty (\mu_t\nabla\varphi_t - \varphi_t\nabla\mu_t)\cdot d\vec{s} \\ &= \int(\mu_t\nabla^2\varphi_t - \varphi_t\nabla^2\mu_t)dv_t \\ &= \int\{\mu_t(1/8R_t\varphi_t) - \varphi_t(1/8R_t\mu_t - C_t\mu_t^5)\}dv_t \\ &= C_t\int\mu_t^5\varphi_t dv_t \end{aligned}$$

(Aside: This can also be used to show that $\max \varphi_t \to \infty$, because as $t \to 0$, $C_t \to 0$, μ_t remains regular and β_t remains bounded away from zero.)

From (4.40) we get

(4.41) $$C_t = Y_t = \frac{\beta_t}{\int\mu_t^5\varphi_t dv_t} \geq \frac{\beta_t}{(\max\ \varphi_t)[\int\mu_t^5 dv_t]} \ .$$

If we define $C_4 = \max[-\frac{1}{16\pi}\oint g^{ab}g^{cd}(g_{ca,d}-g_{cd,a})dS_b]$ over some interval, we get from (4.21)

(4.42) $$m_t + C_4 \geq -\frac{1}{16\pi}\int R_t\varphi_t dv_t \ .$$

Let us define a ball B which encloses the support of all the R_t's. From (4.42) we get

$$m_t + C_4 \geq -\frac{1}{16\pi}(\min \varphi_t)_B \int R_t dv_t$$

(4.43)

$$\geq -\frac{1}{16\pi} \frac{(\max \ \varphi_t)_B}{C} \int R_t dv_t$$

where C is the constant that enters the Harnack inequality (4.19)[23]. Of course we know

(4.44)
$$(\max \varphi_t)_B = \max \varphi_t$$

so multiplying (4.41) and (4.43) we finally get

(4.45)
$$m_t Y_t \geq C_5 > 0 \ .$$

We now can combine (4.35) and (4.45) to give

(4.46)
$$0 < C_1 < m_t Y_t < C_2 < \infty$$

for any curve of metrics along which $Y \to 0$.

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