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THE ZERO-COMPLETION OF A MEDIAN ALGEBRA

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A distributive lattice $(L, \wedge, \vee)$ gives rise to a self-dual symmetric ternary operation, viz.,

$$(*) \qquad x, y, z \to (xyz) := (x \wedge y) \vee (x \wedge z) \vee (y \wedge z),$$

named the median operation of L. This operation satisfies the identities

$$(xxy) = x$$
$$(vw(xyz)) = ((vwx)(vwy)z).$$

A *median algebra* M is a symmetric ternary algebra satisfying these two identities. Such an algebra is close to a distributive lattice: for any element a of M one obtains a *median semilattice* $(M, \wedge)$ with partial join $\vee$ (distributing over $\wedge$) and least element a via

$$x \wedge y := (xay)$$

such that the median of any x, y, z is recovered by the expression $(*)$. In general, $(M, \wedge, \vee)$ is not a lattice, but still admits a representation as a lower set of some distributive lattice.

Typically, a property of a distributive lattice L that is invariant under interchanging meet and join often is expressible merely in terms of the median operation. Most concepts, though, are not self-dual. For instance, the *translational hull* ΩL of $(L, \wedge, \vee)$ usually refers to the meet $\wedge$. It consists of all $\wedge$-*translations* of L, i.e., mappings $\tau\colon L \to L$ satisfying

$$\tau(x \wedge y) = x \wedge \tau y \quad \text{for all} \quad x, y \in L.$$

ΩL is a distributive lattice (with the identity map as its largest element) with respect to the pointwise order. L embeds in ΩL via $a \to \omega_a$, where

$$\omega_a\colon x \to x \wedge a \quad (x \in L)$$

is the ("inner") translation associated with $a \in L$. The translational hull is a standard construction in semigroup theory; cf. Petrich (1970). Now, in order to obtain a self-dual extension concept, one may first form the translational hull ΩL and then take the dual translational hull (that is, the $\vee$-translational hull) Ω^d of ΩL. But what is a convenient description of this "double" translational hull $\Omega^d \Omega L$ or $\Omega\Omega^d L$? Certainly its members can be regarded as *retractions* of the corresponding median algebra, i.e., mappings φ satisfying

$$\varphi(xyz) = (x\varphi y\varphi z) \quad \text{for all} \quad x, y, z;$$

cf. Bandelt & Hedlíková (1983).

Not all retractions can qualify simultaneously as members of $\Omega^d\Omega L$ since there are too many of them. Actually, for each pair u, v of elements, the mapping $x \to (xuv)$ is a retraction. In any case, median algebra is the appropriate framework for studying the double translational hull of distributive lattices. It turns out that for an arbitrary median algebra M one can define this sort of extension. This can be accomplished without reference to a particular orientation of the median algebra as a median semilattice.

Some further terminology is needed here. A *convex* subalgebra N of a median algebra M is a subset satisfying $(vwz) \in N$ for all $v, w \in N$ and $z \in M$. The smallest convex subalgebra containing a given subset A is called the *convex hull* of A in M. A *split* of M is a congruence relation with exactly two blocks P and Q; necessarily, P and Q are ("prime") convex subalgebras. An n-ary operation f on the set M is said to *preserve* a binary relation ϱ if and only if $x_i \varrho y_i$ for $i = 1, \ldots, n$ implies

$$f(x_1, \ldots, x_n) \varrho f(y_1, \ldots, y_n) \quad (x_i, y_i \in M).$$

Particular interest attaches to the semilattice orders which are preserved under the median operation: these orders and their meet operations are then called *compatible* (with the algebra M). A semilattice operation $\wedge$ on M is known to be compatible if and only if the median operation of the algebra M can be written in terms of $\wedge$ and the associated partial join $\vee$ as in (*) above. For this and further information on median algebras, see Bandelt & Hedlíková (1983).

Lemma. *Let $\wedge$ be an idempotent, commutative operation on a median algebra M. Then $\wedge$ is a compatible semilattice operation on M if and only if*

$$(w \wedge xyz) = (wyz) \wedge (xyz) \quad \textit{for all} \quad w, x, y, z \in M,$$

or equivalently, if $\wedge$ preserves all splits of M.

Proof. Assume that $(M, \wedge)$ is a compatible semilattice (with partial join operation $\vee$). Then

$$[(w\wedge y)\vee(w\wedge z)\vee(y\wedge z)]\wedge[(x\wedge y)\vee(x\wedge z)\vee(y\wedge z)] = (w\wedge x\wedge y)\vee(w\wedge x\wedge z)\vee(y\wedge z)$$

since the meet is distributive over the partial join. This proves necessity.

Next we show that this identity implies that $\wedge$ preserves every split $\sim$ of M. For $v, w, x \in M$ with $v \sim w$, we get

$$\begin{aligned} v \wedge x &= (vvx) \wedge (xvx) \\ &= (v \wedge xvx) \\ &\sim (v \wedge xwx) \\ &= (vwx) \wedge (xwx) \\ &= (vwx) \wedge x. \end{aligned}$$

Similarly we obtain that

$$w \wedge x \sim (vwx) \wedge x.$$

Therefore $v \wedge x \sim w \wedge x$, as required.

Conversely, assume that $\wedge$ preserves all splits. Suppose $(x \wedge y) \wedge z$ and $x \wedge (y \wedge z)$ were incongruent modulo some split $\sim$. If $x \sim z$, then

$$x \wedge (y \wedge z) \sim x \wedge (y \wedge x) = (x \wedge y) \wedge x \sim (x \wedge y) \wedge z,$$

yielding a contradiction. So, without loss of generality assume that $x \sim y$. Then $(x \wedge y) \wedge z \sim y \wedge z$. If $y \wedge z \sim y$, then

$$x \wedge (y \wedge z) \sim y \wedge (y \wedge z) \sim y \wedge z,$$

again a contradiction. Hence $y \wedge z \sim y$. But this yields

$$x \wedge (y \wedge z) \sim (y \wedge z) \wedge (y \wedge z) = y \wedge z,$$

a final contradiction. We conclude that $\wedge$ is a semilattice operation.

Finally assume that $\wedge$ is a semilattice operation preserving all splits. We claim that for w, x, y, z in M the identity in the Lemma holds. Let $\sim$ be any split of M. If $y \sim z$, then $(w \wedge xyz), (wyz), (xyz)$, and hence their meets would be congruent to y, z. So assume that $x \sim y$ but z is incongruent with x, y. If $w \sim x$, then again either side of the asserted equality would be congruent to w, x, y. Otherwise $w \sim z$, and then

$$(w \wedge xyz) \sim (z \wedge yyz) = z \wedge y = (zyz) \wedge (yyz) \sim (wyz) \wedge (xyz).$$

This proves the claim. In particular, the partial order associated with $\wedge$ is a compatible relation of M. Then, by Lemma 3.3 of Bandelt & Hedlíková (1983), $(M\wedge)$ is a compatible semilattice. □

Theorem 1. *Let M be a median algebra. Then the set ξM of all compatible semilattice operations on M is a median algebra with respect to the operation $\alpha, \beta, \gamma \to (\alpha\beta\gamma)$ defined by*

$$x(\alpha\beta\gamma)y := (x\alpha y\ x\beta y\ x\gamma y) \quad \text{for} \quad x, y \in M.$$

Then ξM is up to isomorphism the unique median algebra N such that (i) *M is a convex subalgebra of N,* (ii) *every compatible semilattice operation on M uniquely extends to one on N,* (iii) *every compatible semilattice operation on N has a zero.*

In particular, M embeds in ξM via

$$u \to \hat{u}, \quad \text{where} \quad x\hat{u}y := (xuy) \quad \text{for} \quad x, y \in M.$$

Proof. For $\alpha, \beta, \gamma \in \xi M$ the binary operation $(\alpha\beta\gamma)$ on M is evidently idempotent and commutative since α, β, γ are such. From the Lemma we infer that $(\alpha\beta\gamma)$ belongs to ξM.

The following identity will be used in the sequel:

$$x(\alpha\beta\gamma)y = (x\alpha y)\beta(x\gamma y) \quad \text{for all} \quad x, y \in M.$$

To prove this, suppose by way of contradiction that there exists a split $\sim$ of M such that $v := x(\alpha\beta\gamma)y$ and $w := (x\alpha y)\beta(x\gamma y)$ are not congruent modulo $\sim$. If $x\alpha y \sim x\gamma y$, then

$$v \sim (x\alpha y\ x\beta y\ x\alpha y) = x\alpha y = (x\alpha y)\beta(x\alpha y) \sim w,$$

contradicting the hypothesis. Hence $x\alpha y$ and $x\gamma y$ are incongruent. Then, without loss of generality, $x\beta y \sim x\gamma y$ and consequently

$$v \sim (x\alpha y\ x\beta y\ x\beta y) = x\beta y.$$

Now, x and y are not congruent, for otherwise, we would get $x\alpha y \sim x \sim x\gamma y$. Say, $x\alpha y \sim x$ and $x\gamma y \sim y$. This, however, yields $w \sim x\beta y \sim v$, a final contradiction.

Next we show that ξM is a median algebra. The identity $(\alpha\alpha\beta) = \alpha$ and symmetry are clear from the definition of the ternary operation on ξM. The third axiom required for a median algebra is readily checked as well:

$$
\begin{aligned}
x((\alpha\beta\gamma)\delta\varepsilon)y &= ((x\alpha y\ x\beta y\ x\gamma y)x\delta y\ x\varepsilon y)\\
&= (x\alpha y\ (x\beta y\ x\delta y\ x\varepsilon y)(x\gamma y\ x\delta y\ x\varepsilon y))\\
&= x(\alpha(\beta\delta\varepsilon)(\gamma\delta\varepsilon))y
\end{aligned}
$$

for all $x, y \in M$ and $\alpha, \beta, \gamma, \delta, \varepsilon \in \xi M$.

Every element $u \in M$ is the zero of its associated semilattice operation $\hat{u}$. Therefore $\hat{u} = \hat{v}$ implies $u = v$ for $u, v \in M$. Further,

$$x(\hat{u}\hat{v}\hat{w})y = ((xuy)(xvy)(xwy)) = (x(uvw)y)$$

for $u, v, w, x, y \in M$, whence

$$(\widehat{uvw}) = (\hat{u}\hat{v}\hat{w}).$$

We conclude that $u \to \hat{u}$ constitutes an embedding of M into ξM. Denote the image of M under this embedding by $\hat{M}$.

Note that every retraction φ of M is a homomorphism with respect to any member $\wedge$ of ξM. Indeed, let $x, y \in M$, and put $w = x \wedge y \wedge \varphi x \wedge \varphi y$. Then

$$\varphi(x \wedge y) = \varphi(wxy) = (w\ \varphi x\ \varphi y) = \varphi x \wedge \varphi y.$$

In particular, for $u, v \in M$ and $\alpha \in \xi M$,

$$x(\hat{u}\alpha\hat{v})y = (x\hat{u}y)\alpha(x\hat{v}y) = (xuy)\alpha(xvy) = (x(u\alpha v)y)$$

since $w \to (xwy)$ is a retraction. Therefore $\hat{u}\alpha\hat{v} \in \hat{M}$, and thus $\hat{M}$ is a convex subalgebra of ξM.

Every compatible semilattice operation $*$ on $\hat{M}$ extends to ξM by the rule

$$\widehat{x(\alpha * \beta)y} := (\hat{x}\alpha\hat{y}) * (\hat{x}\beta\hat{y}) \quad \text{for} \quad x, y \in M, \alpha, \beta \in \xi M.$$

This operation is certainly idempotent and commutative and belongs to $\xi\xi M$ because

$$
\begin{aligned}
\widehat{x((\alpha\gamma\delta) * (\beta\gamma\delta))y} &= (\hat{x}(\alpha\gamma\delta)\hat{y}) * (\hat{x}(\beta\gamma\delta)\hat{y})\\
&= ((\hat{x}\alpha\hat{y})(\hat{x}\gamma\hat{y})(\hat{x}\delta\hat{y})) * (\hat{x}\beta\hat{y})(\hat{x}\gamma\hat{y})(\hat{x}\delta\hat{y}))\\
&= ((\hat{x}\alpha\hat{y}) * (\hat{x}\beta\hat{y})(\hat{x}\gamma\hat{y})(\hat{x}\delta\hat{y}))\\
&= (\widehat{x(\alpha * \beta)y}\widehat{x\gamma y}\widehat{x\delta y})\\
&= \widehat{x(\alpha * \beta\gamma\delta)y},
\end{aligned}
$$

by virtue of the Lemma. The operation $*$ on ξM actually restricts to the given operation $*$ on $\hat{M}$ since

$$\widehat{x(\hat{u} * \hat{v})y} = (\hat{x}(\hat{u} * \hat{v})\hat{y}) = (\hat{x}\hat{u}\hat{y}) * (\hat{x}\hat{v}\hat{y}) \quad \text{for} \quad u, v \in M.$$

If $\bullet$ is any member of $\xi\xi M$ restricting to $*$ on $\hat{M}$, then

$$\begin{aligned}(\hat{x}(\alpha \bullet \beta)\hat{y}) &= (\hat{x}\alpha\hat{y}) \bullet (\hat{x}\beta\hat{y}) \\ &= (\hat{x}\alpha\hat{y}) * (\hat{x}\beta\hat{y}) = (\hat{x}(\alpha * \beta)\hat{y}),\end{aligned}$$

and therefore every member of $\xi\hat{M}$ extends uniquely to ξM.

Finally, let $*$ be any member of $\xi\xi M$. We wish to show that $*$ has a zero. Since $\hat{M}$ is a convex subalgebra of ξM, it is closed under $*$, that is: $\hat{u} * \hat{v} \in \hat{M}$ for all $u, v \in M$. The restriction of $*$ to $\hat{M}$ thus corresponds to a compatible semilattice operation α on the isomorphic copy M, so that

$$\widehat{u\alpha v} = \hat{u} * \hat{v} \quad \text{for} \quad u, v \in M.$$

We claim that α is the zero of $*$. For $\beta, \gamma \in \xi M$ and $x, y \in M$ we get

$$\begin{aligned}(\hat{x}(\beta * \gamma)\hat{y}) &= (\hat{x}\beta\hat{y}) * (\hat{x}\gamma\hat{y}) \\ &= \widehat{(x\beta y)\alpha(x\gamma y)} \\ &= \widehat{x(\beta\alpha\gamma)y} \\ &= (\hat{x}(\beta\alpha\gamma)\hat{y}),\end{aligned}$$

whence

$$\beta * \gamma = (\beta\alpha\gamma).$$

So, in particular, $\xi\xi M \cong \xi M$.

Now assume that N is a median algebra satisfying (i), (ii), (iii). Then $\xi M \cong \xi N$ by (i) and (ii), and $\xi N \cong N$ by (iii). This completes the proof of Theorem 1. □

For a median algebra M the algebra ξM described in Theorem 1 is referred to as the *zero-completion* of M. If ξM coincides with M, then M is said to be *zero-complete*. Every bounded distributive lattice is zero-complete. From the subdirect representation theorem we infer that every median algebra M embeds in some algebra 2^X such that 2^X is the convex hull of M. Then the zero-completion of M can be described within 2^X as follows.

Theorem 2. *Let M be a subalgebra of the median algebra 2^X of all subsets of some set X. If 2^X is the convex hull of M, then ξM is isomorphic to the largest subalgebra N of 2^X which contains M as a convex subalgebra, viz.:*

$$N = \{z \in 2^X \mid (uvz) \in M \quad \textit{for all} \quad u, v \in M\}.$$

Proof. First observe that N is in fact a subalgebra of 2^X, as

$$(uv(z_1z_2z_3)) = ((uvz_1)(uvz_2)(uvz_3)) \in M$$

for all $u, v \in M$ and $z_1, z_2, z_3 \in N$. Clearly M is a convex subalgebra of N, and every other subalgebra of 2^X in which M is convex is necessarily contained in N.

One may identify X as the set of all splits of M. Since every compatible semilattice operation on M preserves all splits of M it extends uniquely to 2^X. Therefore ξM embeds in 2^X by virtue of Theorem 1. Furthermore, ξM actually embeds in the algebra N.

If $\wedge$ is a compatible semilattice operation on N, then its extension to 2^X has a least element 0. For $u, v \in M$,

$$(uv0) = u \wedge v \in M,$$

whence $0 \in N$. Therefore N is zero-complete and thus meets the three conditions in Theorem 1. We conclude that N is isomorphic to ξM. □

Assume that M is a subalgebra of a median algebra M'. Let us call

$$N = \{z \in M' \mid (uvz) \in M \text{ for all } u, v \in M\}$$

the *convexizer* of M in M'. It is the largest subalgebra of M' containing M as a convex subalgebra. The convexizers play a role similar to that of the idealizers in the framework of distributive lattices. So, Theorem 2 is the analogue of a result concerning the translational hull ΩL of a distributive lattice L; see Figa-Talamanca & Franklin (1968) and Cornish (1974). If L is a distributive sublattice of L', then the convexizer of L in L' is just the dual idealizer of the idealizer of L in L' since for $z \in L'$,

$$(z \wedge s) \vee t \in L \text{ for all } s, t \in L$$

if and only if

$$(uvz) \in L \text{ for all } u, v \in L.$$

So not unexpectedly, we have the following result.

Corollary 1. *For every distributive lattice L,*

$$\xi L \cong \Omega^d \Omega L \cong \Omega \Omega^d L.$$

Proof. Assume that L is given by its subdirect representation, that is: L is a sublattice of some power set lattice 2^X so that 2^X is the convex hull of L. Then, up to isomorphism, L is an ideal of ΩL, and ΩL is a dual ideal of $\Omega^d \Omega L$, whence L is a convex sublattice of $\Omega^d \Omega L$, the latter being a sublattice of 2^X. Since $\Omega^d \Omega L$ is bounded, it is zero-complete. Further, every member of ξL uniquely extends to $\Omega^d \Omega L$ (even to 2^X). We conclude from Theorem 1 that ξL is isomorphic to $\Omega^d \Omega L$, and analogously, to $\Omega \Omega^d L$ as well. Alternatively, one may argue that $\Omega^d \Omega L$ is the largest sublattice in which L is a convex sublattice, and thus conclude the proof with Theorem 2. □

In particular, if L is a distributive lattice with zero, then $\xi L \cong \Omega L$. More generally, consider the following subset of the translational hull of a median semilattice $(M, \wedge, \vee)$ with least element 0:

$$\Omega_\ell M = \{\tau \in \Omega M \mid \tau x \ \vee \ \tau y \text{ exists for all } x, y \in M\}.$$

So, a translation τ of $(M, \wedge)$ belongs to $\Omega_\ell M$ if and only if the image im τ is a (distributive) lattice. It is easy to see that $\Omega_\ell M$ is a subsemilattice of ΩM, as well as a subalgebra of the median algebra of all retractions, where the median of three retractions is given by

$$(\varphi_1 \varphi_2 \varphi_3)x = (\varphi_1 x \varphi_2 x \varphi_3 x) \quad \text{for} \quad x \in M.$$

Therefore $\Omega_\ell M$ is a median semilattice extending M. Now, Theorem 5.5 of Bandelt & Hedlíková (1983) establishes a one-to-one correspondence between the sets ξM and $\Omega_\ell M$. In fact, to each member $*$ of ξM one can associate a retraction τ via

$$\tau x := x * 0 \quad \text{for} \quad x \in M.$$

Since

$$x \wedge \tau x = x \wedge (x * 0) = (x \wedge x) * (x \wedge 0) = x * 0 = \tau x,$$

the image im τ is a lower set in $(M, \wedge)$, whence τ belongs to ΩM. Moreover, $\tau(x * y)$ is an upper bound of τx and τy because

$$(x * 0) \wedge (x * y * 0) = x * (0 \wedge (y * 0)) = x * 0,$$

and the analogous identity holds for τy. Therefore τ is a member of $\Omega_\ell M$. On the other hand, given $\tau \in \Omega_\ell M$ one can uniquely extend the join $\vee$ on the distributive lattice $(\mathrm{im}\,\tau, \wedge, \vee)$ to M, thus giving a member $*$ of ξM, by virtue of Theorem 5.4 of Bandelt & Hedlíková (1983). It is then not difficult to check that $* \leftrightarrow \tau$ constitutes an isomorphism between the median algebras ξM and $\Omega_\ell M$, thus proving the concluding corollary.

Corollary 2. *For a median semilattice* $(M, \wedge)$ *with zero,* $\Omega_\ell M$ *is isomorphic to* ξM.

As every median algebra M can be turned into a median semilattice with zero, the preceding corollary provides a convenient method to determine the zero-completion of M. For instance, given a tree algebra T, choose any compatible semilattice order with zero. Then ξT can be regarded as the set of all chains between the zero and the elements of T and all unbounded maximal chains; cf. Corollary 6.6 of Bandelt & Hedlíková (1983).

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