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The zero divisor question for supersolvable groups

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Using a theorem of Jacques Lewin, the zero divisor question is solved for group rings of supersolvable groups.

The purpose of this paper is to show that the zero divisor question for supersolvable groups can be settled using a recent theorem of Lewin [3]. This result has been announced by G.O. Michler (unpublished).

THEOREM 1. *Let* F *be a field and* G *a torsion-free supersolvable group. Then the group ring* $F[G]$ *has no zero divisors.*

THEOREM 2 (Lewin [3]). *Let* F *be a field and* $G = H *_N K$ *a free product with amalgamation, where*

(1) N *is normal in both* H *and* K ;

(2) $F[H]$ *and* $F[K]$ *have no zero divisors;*

(3) $F[N]$ *satisfies the Ore condition.*

Then $F[G]$ *has no zero divisors.*

Lewin's Theorem is proved by appealing to a theorem of Cohn on free products of rings [1]. It should be noted that Lewin's Theorem is the first solution of the zero divisor question for groups which are not ordered or close relatives of ordered groups.

It is somewhat surprising that Theorem 2 applies to supersolvable groups since free products with amalgamation always produce groups which have free nonabelian subgroups - unless the amalgamated subgroup has index

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two in both factors. This turns out to be enough to handle supersolvable groups.

LEMMA 3. *Let* G *be an infinite supersolvable group. Then either* $\mathbb{Z}$ *or* $\mathbb{Z}_2 * \mathbb{Z}_2$ *is a quotient of* G .

Proof. Take a normal series for G with cyclic factors. Since G is infinite, one of these is an infinite cycle. Hence there is a group N in the normal series such that G/N is an extension of $\mathbb{Z}$ by a finite group. Let $H = G/N$ and K be the infinite cyclic subgroup.

Consider $C_H(K)$, the centralizer of K in H . It is central-by-finite and hence is an FC-group, so it has a torsion subgroup T (which is normal in H) and $C_H(K)/T$ is torsion-free abelian, all by a theorem of Neumann [4, Theorem 5.1]. Since the torsion-free rank of $C_H(K)/T$ is one, it must be infinite cyclic.

Since $\mathbb{Z}$ has only two automorphisms, $C_H(K)$ has index 1 or 2 in G . In the first case $H/T \cong \mathbb{Z}$ and in the second case $H/T \cong \mathbb{Z}_2 * \mathbb{Z}_2$, the infinite dihedral group. Thus G has a quotient of the type claimed.

Proof of Theorem 1. The proof is by induction on the torsion-free rank of G .

Case I. $\mathbb{Z}$ is a quotient of G . Let $G/N \cong \mathbb{Z}$, with t a generator for $\mathbb{Z}$. By the inductive hypothesis $F[N]$ has no zero divisors. $F[G] = F[N][t^{\pm 1}]$, a twisted polynomial ring, and the usual degree argument shows that $F[G]$ has no zero divisors.

Case II. $\mathbb{Z}_2 * \mathbb{Z}_2$ is a quotient of G . Let $p : G \to A * B$ be a surjection with kernel N , where A and B are cyclic of order 2 . Then $G \cong p^{-1}(A) *_N p^{-1}(B)$. $p^{-1}(A)$ and $p^{-1}(B)$ have lower torsion free rank than G and hence their group rings have no zero divisors by the inductive hypothesis. Further $F[N]$ is a (left and right) noetherian domain and hence satisfies the (left and right) Ore condition [2, Theorem 4.2 or see 3, p. 358]. It follows from Lewin's Theorem that $F[G]$ has no zero divisors.

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