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THE ZERO-NORMAL-STRESS CONDITION IN PLANE-STRESS AND SHELL ELASTOPLASTICITY

RENÉ DE BORST

Delft University of Technology, Department of Civil Engineering/TNO Institute for Building Materials and Structures, P.O. Box 5048, 2600 GA Delft, The Netherlands

SUMMARY

The use of elastoplastic models in calculations of shell and plate structures, or more generally in stress situations in which one normal stress component is forced to be zero, usually leads to complicated algorithms. This leads to cumbersome programming, which in turn may easily entail errors in the code. Moreover, the fraction model for cyclic plasticity cannot be used within most algorithms since the normal stress perpendicular to the plane of the element is not necessarily zero for each of the fractions. In this contribution an algorithm will be presented that enforces a zero-stress condition at integration point level, while obviating the need to develop special subroutines for plane-stress or shell plasticity. The algorithm is based upon an implicit integration of the stress–strain law and can be combined with the newly developed technique of consistent tangent operators.

INTRODUCTION

A proper implementation of elastoplastic models in constrained stress situations as occur in shells, plates and membranes entails some complications, in particular if implicit integration is considered. In plane-stress elements and in plate/shell elements the normal stress perpendicular to the plane of the element must vanish. We can accomplish this by a priori omitting this stress component, say σ_{zz}, from the stress vector, so that the stress vector only comprises three components for plane-stress situations and five components in plate/shell analyses. This approach, although widely employed in the literature,[1–3] has two disadvantages. First, we need separate subroutines for stress computation in plane-stress elements and in plate/shell structures. This is not so much of a problem in special-purpose finite-element programs, but the task of maintaining general-purpose finite-element programs is greatly alleviated if for instance the stress computation in shell elements can use the subroutines that are also used for the three-dimensional solid elements. The second disadvantage is that the fraction model cannot be used.[4] This is because in this model the individual fraction stresses in the constrained direction need not be zero even though the weighted sum of the fraction stresses at global level (σ_{zz}) must vanish. Consequently, the stress vector of each fraction must explicitly include σ_{zz}, so that we have a three-dimensional stress situation at fraction level.

It is the aim of this communication to outline a simple algorithm that enforces that σ_{zz} vanishes at an integration point level in shell and membrane plasticity. The approach embodies an expansion of the strain vector to generate a non-zero normal strain in the constrained direction (ε_{zz}) in the beginning of each iteration and a compression of the stress vector such that the $\sigma_{zz} = 0$ condition is enforced rigorously. It is of utmost importance that the expansion

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of the strain vector and compression of the stress vector are carried out with the appropriate elastoplastic moduli. If the proper moduli are not used, the quadratic convergence of Newton's method, which is used to solve the set of non-linear equations at a structural level, is lost in a similar fashion as when the conventional tangent operator is used instead of the so-called consistent tangent moduli.[1,3,5]

A similar problem arises when an element possesses internal degrees of freedom, e.g. the Wilson–Taylor element,[6] or with a mixed interpolation of the displacement and the pressure field as is done in incompressible elasticity. Indeed, the algorithm presented here is very similar to the approach needed there.[7]

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For the development of the algorithm we start from the following representation of total equilibrium:

$$\int_V \mathbf{B}^{\mathrm{T}} \sigma_n \, \mathrm{d}V = \mathbf{f}_{\mathrm{e}} \tag{1}$$

with $\mathbf{f}_{\mathrm{e}}$ the vector that assembles the external loads and $\mathbf{B}$ the strain-nodal displacement matrix. In a linearized sense (necessary for the resolution of the correction $\mathrm{d}\mathbf{u}$ to the incremental displacements at a structural level) the updated stresses at the end of iteration n of the current loading step are found as

$$\sigma_n = \sigma_{n-1} + \mathbf{D}_n \, \mathrm{d}\boldsymbol{\varepsilon}_n \tag{2}$$

The matrix $\mathbf{D}_n$ contains the proper or 'consistent' linearization moduli of the elastoplastic solid that are computed at the beginning of iteration n, and $\mathrm{d}\boldsymbol{\varepsilon}_n$ is the correction to strain increment in iteration n. We now partition the stress vector σ, the strain vector $\boldsymbol{\varepsilon}$ and the elastoplastic modular matrix $\mathbf{D}$ as

$$\sigma = [\hat{\sigma}, \sigma_{zz}]^{\mathrm{T}} \tag{3a}$$

$$\boldsymbol{\varepsilon} = [\hat{\boldsymbol{\varepsilon}}, \varepsilon_{zz}]^{\mathrm{T}} \tag{3b}$$

and

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{D}_{21} & D_{22} \end{bmatrix} \tag{3c}$$

with, for plane-stress conditions, $\hat{\sigma} = [\sigma_{xx}, \sigma_{yy}, \sigma_{xy}]^{\mathrm{T}}$ and $\hat{\boldsymbol{\varepsilon}} = [\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{xy}]^{\mathrm{T}}$ and $\mathbf{D}_{11}$ a 3×3 matrix, $\mathbf{D}_{12}$ a 3×1 matrix, $\mathbf{D}_{21}$ a 1×3 matrix and D_{22} a scalar. Since we wish to enforce that $(\sigma_{zz})_n = 0$, equation (1) can be rewritten as

$$\int_V \hat{\mathbf{B}}^{\mathrm{T}} \hat{\sigma}_n \, \mathrm{d}V = \hat{\mathbf{f}}_{\mathrm{e}} \tag{4}$$

$\hat{\mathbf{B}}$ contains the first three rows of $\mathbf{B}$, and since we only consider in-plane loads, $\mathbf{f}_{\mathrm{e}} = \hat{\mathbf{f}}_{\mathrm{e}}$. Using equations (3), equation (2) can also be partitioned:

$$\begin{bmatrix} \hat{\sigma}_n \\ (\sigma_{zz})_n \end{bmatrix} = \begin{bmatrix} (\mathbf{D}_{11})_n & (\mathbf{D}_{12})_n \\ (\mathbf{D}_{21})_n & (D_{22})_n \end{bmatrix} \begin{bmatrix} \mathrm{d}\hat{\boldsymbol{\varepsilon}}_n \\ (\mathrm{d}\varepsilon_{zz})_n \end{bmatrix} + \begin{bmatrix} \hat{\sigma}_{n-1} \\ (\sigma_{zz})_{n-1} \end{bmatrix} \tag{5}$$

Because of the condition $(\sigma_{zz})_n = 0$ the normal strain in the z-direction can be computed locally as

$$(\mathrm{d}\varepsilon_{zz})_n = -(D_{22})_n^{-1} [(\mathbf{D}_{21})_n \, \mathrm{d}\hat{\boldsymbol{\varepsilon}}_n + (\sigma_{zz})_{n-1}] \tag{6}$$

It is emphasized that, although σ_{zz} converges to zero during the iterative process, $(\sigma_{xzz})_{n-1}$ itself is not necessarily zero and has to be included in equation (6). Convergence of $\boldsymbol{\sigma}_{zz}$ to zero is of the same order as convergence at structural level and is even quadratic if proper linearization moduli are employed.

To obtain the proper elastoplastic moduli and the consistent internal force vector for a plane-stress situation from their plane-strain counterparts we substitute equation (6) into equation (5). This results in

$$\hat{\sigma}_n = \left[(\mathbf{D}_{11})_n - \frac{(\mathbf{D}_{12})_n(\mathbf{D}_{21})_n}{(D_{22})_n}\right] \mathrm{d}\hat{\boldsymbol{\varepsilon}}_n + \hat{\boldsymbol{\sigma}}_{n-1} - \frac{(\mathbf{D}_{12})_n}{(D_{22})_n}(\sigma_{zz})_{n-1} \tag{7}$$

Inserting equation (7) in equation (4) and invoking the strain-nodal displacement relation $\hat{\varepsilon} = \hat{\mathbf{B}}\mathbf{u}$ for plane-stress conditions yields

$$\int_V \hat{\mathbf{B}}^{\mathrm{T}}\left[(\mathbf{D}_{11})_n - \frac{(\mathbf{D}_{12})_n(\mathbf{D}_{21})_n}{(D_{22})_n}\right]\hat{\mathbf{B}}\ \mathrm{d}\mathbf{u}_n\ \mathrm{d}V = \hat{\mathbf{f}}_e - \int_V \hat{\mathbf{B}}^{\mathrm{T}}\left[\hat{\sigma}_{n-1} - \frac{(\mathbf{D}_{12})_n}{(D_{22})_n}(\sigma_{zz})_{n-1}\right]\mathrm{d}V \tag{8}$$

The crucial point is now that the operations on the left- and the right-hand side must be carried out with the same $\mathbf{D}_{12}$ and D_{22} in order to preserve the quadratic convergence if Newton's method is used. Hence, $(\mathbf{D}_{12})_n$ and $(D_{22})_n$ must be set up before the internal force vector

$$(\hat{\mathbf{f}}_i)_{n-1} = \int_V \hat{\mathbf{B}}^{\mathrm{T}}\left[\hat{\sigma}_{n-1} - \frac{(\mathbf{D}_{12})_n}{(D_{22})_n}(\sigma_{zz})_{n-1}\right]\mathrm{d}V \tag{9}$$

is computed. This disturbs the usual flow of an iterative procedure, in which $(\hat{\mathbf{f}}_i)_{n-1}$ is calculated at the end of iteration $n-1$ before the new tangential moduli $\mathbf{D}_n$ are constructed at the beginning of iteration n. In a modular program that is extremely inconvenient. A better way to implement the proposed procedure is to calculate the internal force vector $(\hat{\mathbf{f}}_i)_{n-1}$ using the 'old' tangential matrix $\mathbf{D}_{n-1}$ and then to apply a correction

$$\hat{\mathbf{f}}_c = \int_V \hat{\mathbf{B}}^{\mathrm{T}}\left[\frac{(\mathbf{D}_{12})_n}{(D_{22})_n} - \frac{(\mathbf{D}_{12})_{n-1}}{(D_{22})_{n-1}}\right](\sigma_{zz})_{n-1}\ \mathrm{d}V \tag{10}$$

after having calculated the new tangential moduli. A flow chart of this procedure is presented in Figure 1.

It is important that the correction of equation (10) is indeed applied. If it is omitted the quadratic convergence of Newton's method is disturbed and in extreme cases divergence of the solution may result. A simple example of the impact of inclusion of $\hat{\mathbf{f}}_c$ has been carried out for a single plane-stress element in uniaxial tension (Figure 2). The number of iterations with and without including $\hat{\mathbf{f}}_c$ is listed in Table I for the loading step in which the element first plastifies (energy norm $\varepsilon = 10^{-9}$). It is interesting to observe that for the present case in which we have no stress rotations the continuum and so-called consistent tangent operators[1,3,5] both achieve convergence within three iterations, but that omission of $\hat{\mathbf{f}}_c$ more severely deteriorates the convergence rate when a classical, continuum tangent operator is used than for a consistent tangent matrix.

It is finally noted that the idea which has been worked out here in a proper fashion was first suggested in conjunction with an implementation of the fraction model.[4,8] Owing to the differences in lateral contraction normal stresses in the z-direction develop in the subelements of this model, thus necessitating the inclusion of σ_{zz} anyway.

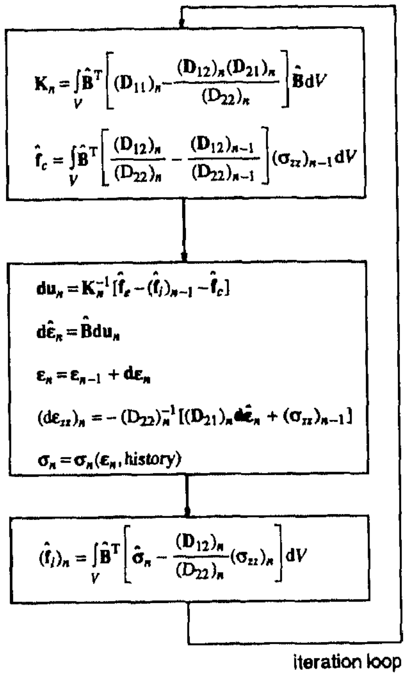

Figure 1. Flow chart of proposed procedure for enforcing that $\sigma_{zz} = 0$

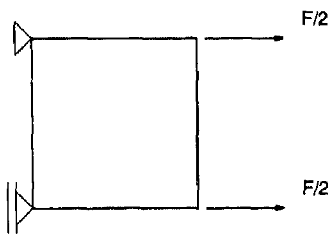

Figure 2. Test problem: uniaxial stressing of a plane-stress element

Table I. Effect of proper correction for σ_{zz} on convergence

Proper correction	No correction, consistent tangent	No correction, continuum tangent
3	5	12

CONCLUDING REMARKS

In stress configurations in which one of the normal stresses, say σ_{zz}, is assumed to vanish, two possible approaches exist for enforcing $\sigma_{zz} = 0$. The first is that σ_{zz} is completely left out of consideration, i.e. σ_{zz} is omitted from the stress vector. Since the normal strain ε_{zz} in that direction is also not taken into account, the relation between stress-rate vector and strain-rate vector reduces to a 5×5 and a 3×3 matrix, respectively, for shell and plane-stress configurations. This approach has two disadvantages. First, one needs separate subroutines for stress computation in shells and in membrane elements. For special-purpose finite-element codes this is not a major issue, but for general-purpose computer codes it is important to keep the cost of maintaining the software as low as possible. It is therefore desirable that, for instance, the stress computation in shells can make use of the subroutines developed for solid elements. A second disadvantage is the fact that the fraction model which is used to describe cyclic plasticity of metals cannot be used, since the normal stesses in the z-direction of the fractions need not be zero, even though the (weighted) sum of the fraction stresses must vanish.

In this contribution an algorithm has been proposed and fully elaborated that includes the normal stress and the normal strain in the z-direction and therefore does not suffer from the drawbacks mentioned above. It is based on an expansion of the strain vector in the beginning of each iteration and on a compression of the stress vector at the end of the iteration. The importance of using the appropriate tangential moduli and right-hand-side vectors has been emphasized. It has been shown by an example that failure to do so results in a loss of the quadratic convergence of Newton's method. Then, the method also becomes numerically less efficient than the alternative approach in which the stress space is reduced to three or five dimensions (plane-stress and shell elastoplasticity, respectively). When appropriate tangential moduli and right-hand-side vectors are employed the convergence characteristics of both approaches are identical, so that the additional computations that have to be carried out are limited to expansion of the strain vector (equation (6)), compression of the stress vector (equation (7)) and the computation of the correction vector $\hat{\mathbf{f}}_c$.

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