The Zero Surface Tension Limit of Two-Dimensional Water Waves

DAVID M. AMBROSE
Courant Institute

AND

NADER MASMOUDI
Courant Institute

Abstract

We consider two-dimensional water waves of infinite depth, periodic in the horizontal direction. It has been proven by Wu (in the slightly different nonperiodic setting) that solutions to this initial value problem exist in the absence of surface tension. Recently Ambrose has proven that solutions exist when surface tension is taken into account. In this paper, we provide a shorter, more elementary proof of existence of solutions to the water wave initial value problem both with and without surface tension. Our proof requires estimating the growth of geometric quantities using a renormalized arc length parametrization of the free surface and using physical quantities related to the tangential velocity of the free surface. Using this formulation, we find that as surface tension goes to 0, the water wave without surface tension is the limit of the water wave with surface tension. Far from being a simple adaptation of previous works, our method requires a very original choice of variables; these variables turn out to be physical and well adapted to both cases. © 2005 Wiley Periodicals, Inc.

1 Introduction

We study the well-posedness, locally in time, of the initial value problem for water waves in two space dimensions. We consider infinitely deep water with periodic geometry. The fluid is required to be irrotational in the bulk of the fluid, inviscid, and incompressible. The initial data is assumed to be in a class of finite smoothness and the surface of the wave is allowed to be of multiple heights. We only make the assumption of infinite depth for simplicity. Indeed, our method can easily be adapted to the case of finite depth or the case of an incompressible fluid in vacuum. However, one of the major differences between the infinite-depth case and other cases is the fact that the Taylor condition, which is recalled in (1.1) and which is crucial for the well-posedness, can be proven for any initial data in the infinite-depth case (see [17]). In the case of finite depth, that condition should be added to the initial data.

Communications on Pure and Applied Mathematics, Vol. LVIII, 1287–1315 (2005)

The well-posedness of water waves has been studied for many years. Early results include those of [12] and [19]. These works showed that the water wave problem (in the case of infinite and finite depth, respectively) is well-posed for a short time if the initial data are taken sufficiently small. Later, Craig showed that water waves with small, properly scaled initial data exist for fairly long times and demonstrated relationships between the full, nonlinear water wave problem and the Boussinesq and Korteweg–de Vries scaling limits [7]. The work of Craig was extended in [15], in which a longer interval of existence was found and the dynamics of the solution over that time interval was discussed. There have been fewer papers to deal with the water wave without an assumption as to the smallness of the initial data. The most important works of this kind are [17, 18], which establish the well-posedness in Sobolev spaces, locally in time, of the initial value problem in two and three space dimensions, respectively. The proof uses a Cartesian parametrization. It is based on rewriting the system as a quasi-linear hyperbolic system and applying an iteration method. One of the key points is the use of the Riemann mapping theorem and expressing the vertical velocity y_t as a function of the horizontal velocity x_t, namely, $y_t = Kx_t$ for some operator K that has properties similar to the Hilbert transform. The works discussed so far have all neglected surface tension.

Other works have studied the subject of water waves with surface tension. Their results are typically limited to the case in which the height of the wave is a single-valued function of horizontal position. In this case, Iguchi established well-posedness of water waves with surface tension in [10]. Also, in the case of single-valued height, the limit of water waves as surface tension goes to 0 was treated by Yosihara [20]. That work was also restricted to the case in which the initial data are small. Recently Ogawa and Tani have generalized Yosihara's result to the case in which the bulk of the fluid is not necessarily irrotational; they do, however, keep the assumptions of small data and single-valued height [13]. The only work we are aware of which establishes well-posedness of water waves with surface tension that is not limited to the case of single-valued height is [2].

In this paper, we provide a new existence proof for the two-dimensional water wave. We first reformulate the problem by describing the free surface not by its Cartesian coordinates but instead by its arc length and tangent angle. In doing this, we choose an arc length parametrization for the free surface. The estimates we obtain lead to uniform bounds in time that are independent of the amount of surface tension (for small enough surface tension); this is an important, novel feature of this work.

To establish existence of water waves in the absence of surface tension, an additional physical condition should be checked. If p is the pressure and $\hat{\mathbf{n}}$ is the outward normal vector to the free surface, then we must know that

$$-\nabla p \cdot \hat{\mathbf{n}} > c_0 > 0, \tag{1.1}$$

for some positive constant c_0. This is the same condition used by Wu in [17]. It was introduced in [4] in the study of the linearization of the equations of motion of the water wave, and it is a generalization of a condition of G. I. Taylor [16]. It was proven by Wu [17] that condition (1.1) holds for the case of infinitely deep water as long as the surface of the water wave does not intersect itself. A different proof was given in [18]. We do not prove this important lemma here; instead, we give an intuitive argument from [18]. The intuitive argument is simply the maximum principle: without surface tension, the pressure on the free surface is identically 0. The pressure is superharmonic, so the maximum principle applies and the pressure is positive in the bulk of the fluid. Since $\hat{\mathbf{n}}$ points outward, we get $\nabla p \cdot \hat{\mathbf{n}} < 0$. For full details, we refer the interested reader to [18]. We do remark that an easy generalization of the argument shows that (1.1) also holds in the case of small surface tension; in this case, the pressure on the free surface is equal to the surface tension times the curvature of the free surface.

To ensure that the surface does not initially intersect itself, we impose the condition

$$0 < a < \left| \frac{z(\alpha, 0) - z(\alpha', 0)}{\alpha - \alpha'} \right| \tag{1.2}$$

for some constant a. Here, $z(\alpha, t)$ is the location in the complex plane of the free surface, parametrized in space by α. For our existence and convergence to be valid, (1.2) must hold at later times (with a replaced by εa for any fixed, small positive ε). Since we are dealing with regular solutions, this condition will hold at least for a short time.

The method we use in this paper is related to the method of [2], which was strongly influenced by the numerical work of Hou, Lowengrub, and Shelley (HLS) [8, 9]. In those works, HLS efficiently compute vortex sheets in the presence of surface tension. (A vortex sheet is the interface between two irrotational fluids flowing past each other; the two fluids may have different densities, and the irrotational water wave is a special case of the vortex sheet problem in which the upper fluid has density 0.) To do these computations, HLS first reformulate the problem in two important ways: they use the curve's tangent angle and arc length to describe it rather than Cartesian variables, and they always keep the curve parametrized by renormalized arc length. To keep the curve parametrized by its renormalized arc length, a special tangential velocity is introduced. This same formulation of the problem was used in [2] to prove well-posedness of vortex sheets with surface tension. The dependent variables analyzed there are θ, the tangent angle the curve forms with the horizontal; γ, the vortex sheet strength; and L, the length of one period of the curve. (The fluid's vorticity is equal to γ times a Dirac mass centered on the free surface; thus, if it is desired to know the velocity at any point of the fluid, it is sufficient to know the values of these three dependent variables.) The independent variables are α, the space variable, and time, t. In the present work, we also analyze θ and L, but we replace γ with a related variable, δ. This new variable has physical meaning and will be defined in Section 2.

In this paper, we establish two main results. The first is the following:

THEOREM 1.1 *Solutions to the water wave initial value problem* (*with or without surface tension*) *exist if the given data is regular enough and satisifies the non-self-intersection condition* (1.2). *These solutions are unique and depend continuously on the initial data.*

As we have mentioned, this theorem has been proven by Wu in [17] for the case without surface tension; we feel that the proof we present here is more elementary. Also, the theorem was proven in [2] for the case with surface tension. That proof heavily relied upon the regularizing effect of surface tension. In the current proof, this is not the case. Our second result is the following:

THEOREM 1.2 *As the surface tension parameter tends to* 0*, solutions to the water wave initial value problem with surface tension tend to the solution of the water wave initial value problem without surface tension in a suitable norm.*

The precise mathematical statement will be given later (see Theorem 2.2).

Here we make a comment on notation. In what follows, differentiation will sometimes be denoted by application of an operator and sometimes by a subscript. That is, $D_\alpha f$ and f_α both indicate the derivative of f with respect to α. Also, the operator H, which will appear frequently, is the Hilbert transform, which can be written

$$Hf(\alpha) = \frac{1}{\pi}\,\mathrm{PV}\int_{-\infty}^{\infty} \frac{f(\alpha')}{\alpha - \alpha'}\,d\alpha' = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha')\cot\frac{1}{2}(\alpha - \alpha')d\alpha'.$$

Here, f is 2π-periodic.

2 The Choice of Variables

Since the water waves we consider are irrotational in the bulk of the fluid, we will be able to study quantities defined on the free surface only. The quantities we will analyze are the position of the free surface and the vorticity of the fluid. The irrotational water wave has a singular distribution of vorticity: while the vorticity is always identically 0 in the interior of the fluid region, there is vorticity concentrated on the surface. The vorticity is equal to an amplitude multiplied by a Dirac mass on the surface. This amplitude, γ, is called the *vortex sheet strength*. The independent variables we use are the spatial variable α and the time t. Since we are considering two-dimensional fluids, the interface is one-dimensional. Thus, α is in $\mathbb{R}$; in fact, since we consider spatially periodic solutions, we restrict α to be in the interval $[0, 2\pi]$ at all times. We then have $\gamma = \gamma(\alpha, t)$. The position of the free surface is given by $(x(\alpha, t), y(\alpha, t))$, or equivalently, viewing the fluid region as being in the complex plane, $z(\alpha, t) = x(\alpha, t) + iy(\alpha, t)$. The periodicity assumption translates into $z(\alpha + 2\pi, t) = 2\pi + z(\alpha, t)$.

2.1 The Renormalized Arc Length Parametrization

As in [2], we calculate the evolution of the curve's tangent angle θ and arc length s, instead of its Cartesian coordinates x and y. The equations satisfied by θ and s are $s_\alpha^2 = x_\alpha^2 + y_\alpha^2$ and $\theta = \arctan(y_\alpha/x_\alpha)$. Given θ and s_α, we can reconstruct x and y by integrating

$$(x_\alpha, y_\alpha) = (s_\alpha \cos(\theta), s_\alpha \sin(\theta)).$$

As we will see below, the constant of integration is irrelevant to the evolution of θ and γ. Given the evolution of the free surface as

$$(x, y)_t = U\hat{\mathbf{n}} + T\hat{\mathbf{t}},$$

where $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ are the unit normal and tangent vectors to the curve, we can make a straightforward calculation to get

$$\theta_t = \frac{1}{s_\alpha} U_\alpha + \frac{T}{s_\alpha}\theta_\alpha, \tag{2.1}$$

$$s_{\alpha t} = T_\alpha - \theta_\alpha U. \tag{2.2}$$

For the evolution of the water wave, we have no choice regarding the normal velocity U; that is, U is determined from the Euler equations for the motion of an inviscid fluid. The exact form of U is discussed below. However, we may choose the tangential velocity T to be anything we wish. We use this freedom to require that $s_{\alpha t}$ be independent of α. In particular, we set

$$s_{\alpha t} = T_\alpha - \theta_\alpha U = \frac{L_t}{2\pi}. \tag{2.3}$$

Recall that L is the length of one period of the curve; thus, L_t is the time derivative of this length. Integrating this equation yields our choice of T; we set the constant of integration equal to 0. With this choice, the curve will always be parametrized by arc length, normalized so that the parameter α always lies in the interval $[0, 2\pi]$ (as long as it is initially parametrized this way). This is the same choice of T used in [2, 8, 9]. In the rest of this paper, we will frequently use the formula implied by our choice of T,

$$s_\alpha = \frac{L}{2\pi}.$$

The curve's normal velocity, U, is determined from the position of the curve (x, y) and the vortex sheet strength, γ, by the Birkhoff-Rott integral. (A discussion of the Birkhoff-Rott integral can be found, for example, in [3] or [14]. It can be derived by first using the Biot-Savart law to recover the velocity of the fluid from the vorticity together with the fact that our vorticity is a measure concentrated on the free surface. This results in a formula for the velocity at interior points of the fluid. A limit can be taken to find the velocity of material points on the surface; it is important to remember in our case that there is fluid on only one side of the free surface.)

2.2 The Birkhoff-Rott Integral

The Birkhoff-Rott integral $\mathbf{W}$ is the singular integral

$$\mathbf{W}(\alpha,t) = \frac{1}{2\pi}\,\mathrm{PV}\int_{-\infty}^{\infty}\gamma(\alpha',t)\frac{(-(y(\alpha,t)-y(\alpha',t)),x(\alpha,t)-x(\alpha',t))}{(x(\alpha,t)-x(\alpha',t))^2+(y(\alpha,t)-y(\alpha',t))^2}\,d\alpha'; \tag{2.4}$$

we determine U by $U = \mathbf{W}\cdot\hat{\mathbf{n}}$.

Notice that the integral in (2.4) is taken over the whole real line, even though we are dealing with functions that are periodic in space. This integral is still well defined if the principal value is also taken at infinity; alternatively, it can be summed over the periodic images to yield an integral over one period with a cotangent kernel instead.

Our goal for the rest of this section is to rewrite the evolution equations for the water wave problem. We have already given an evolution equation for θ in (2.1). Now that we have given formulae for U and T, we will rewrite this equation in a convenient way; in particular, we will rewrite the evolution equations as a quasilinear system for θ and δ_α. (We will define δ soon.) In the presence of surface tension, we actually get a semilinear system for θ and δ_α. To be able to find the quasi-linear system, we must make use of a representation of $\mathbf{W}_\alpha$ that was given in [2]. Later in this section, we will give the evolution equation for γ. We are able to infer an evolution equation for δ from the evolution equations for γ and θ. We then make a significant effort to rewrite the evolution equation for δ_α so that it takes a convenient form that is suitable for energy estimates.

We now begin rewriting the evolution equations. To help with future calculations involving $\mathbf{W}$, we introduce some complex notation. We define the map $\Phi : \mathbb{R}^2 \to \mathbb{C}$ by $\Phi(a,b) = a + ib$. We define $z = x + iy$ to be the image under Φ of the free surface of our water wave. Thus, we have the representations for $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$

$$\Phi(\hat{\mathbf{t}}) = \frac{z_\alpha}{s_\alpha}, \quad \Phi(\hat{\mathbf{n}}) = \frac{iz_\alpha}{s_\alpha}.$$

If we let $*$ denote complex conjugation, then the formula for a dot product is

$$\mathbf{a}\cdot\mathbf{b} = \mathrm{Re}\left\{\Phi(\mathbf{a})\Phi(\mathbf{b})^*\right\}.$$

Using Φ, we have the representation for $\mathbf{W}$

$$\Phi(\mathbf{W})^* = \frac{1}{2\pi i}\,\mathrm{PV}\int\frac{\gamma(\alpha')}{z(\alpha)-z(\alpha')}\,d\alpha'. \tag{2.5}$$

To make sense of $\mathbf{W}$ and its derivatives, we will at times approximate the singular integral by a Hilbert transform; this is the reason for the following definition of the integral operator $\mathcal{K}[z]$:

$$\mathcal{K}[z]f(\alpha) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} f(\alpha')\left[\frac{1}{z(\alpha)-z(\alpha')} - \frac{1}{z_\alpha(\alpha')(\alpha-\alpha')}\right]d\alpha'. \tag{2.6}$$

We will also use certain commutators in understanding $\mathbf{W}$ and its derivatives. In particular, the operator $[H, f]$ is given by

$$[H, f]g(\alpha) = \frac{1}{\pi}\int g(\alpha')\frac{f(\alpha') - f(\alpha)}{\alpha - \alpha'}d\alpha'. \tag{2.7}$$

Both of these operators are smoothing operators, and lemmas reflecting this are located in the appendix.

Using the integral remainder operator $\mathcal{K}$ and the commutator above, we get the following representation for $\mathbf{W}_\alpha$:

$$\mathbf{W}_\alpha \cdot \hat{\mathbf{n}} = \frac{\pi}{L}H(\gamma_\alpha) + \mathbf{m}\cdot\hat{\mathbf{n}}, \quad \mathbf{W}_\alpha \cdot \hat{\mathbf{t}} = -\frac{\pi}{L}H(\gamma\theta_\alpha) + \mathbf{m}\cdot\hat{\mathbf{t}}. \tag{2.8}$$

Here, $\mathbf{m}$ is given by

$$\Phi(\mathbf{m})^* = z_\alpha\mathcal{K}[z]\left(\frac{\gamma_\alpha}{z_\alpha} - \frac{\gamma z_{\alpha\alpha}}{z_\alpha^2}\right) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(\gamma_\alpha - \frac{\gamma z_{\alpha\alpha}}{z_\alpha}\right). \tag{2.9}$$

This definition of $\mathbf{m}$ as the sum of various smooth remainders is natural; the full calculation leading to (2.8) can be found in [1, 2]. We also refer the reader to Section 2.6; there a similar argument is used to compute $\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}}$.

We will now give an explanation of the idea behind the calculation leading to (2.8). In (2.5), if we approximate $z(\alpha) - z(\alpha')$ by $z_\alpha(\alpha')(\alpha - \alpha')$, we have

$$\Phi(\mathbf{W})^* \approx \frac{1}{2\pi i}\,\mathrm{PV}\int \frac{\gamma(\alpha')}{z_\alpha(\alpha')(\alpha - \alpha')}\,d\alpha'.$$

Using $s_\alpha = |z_\alpha| = L/2\pi$, we can rewrite this as

$$\Phi(\mathbf{W})^* \approx \frac{1}{L}\,\mathrm{PV}\int \frac{\gamma(\alpha')}{\alpha - \alpha'}\left(\frac{2\pi i z_\alpha(\alpha')}{L}\right)^* d\alpha'.$$

Notice that this is the same as

$$\mathbf{W} \approx \frac{\pi}{L}H(\gamma\hat{\mathbf{n}}).$$

Differentiating, this yields

$$\mathbf{W}_\alpha \approx \frac{\pi}{L}H(\gamma_\alpha\hat{\mathbf{n}}) - \frac{\pi}{L}H(\gamma\theta_\alpha\hat{\mathbf{t}}).$$

Here we have used the identity $\hat{\mathbf{n}}_\alpha = -\theta_\alpha\hat{\mathbf{t}}$. Pulling $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ through the Hilbert transform in this last expression (which costs only a smooth commutator), we have

$$\mathbf{W}_\alpha \approx \frac{\pi}{L}H(\gamma_\alpha)\hat{\mathbf{n}} - \frac{\pi}{L}H(\gamma\theta_\alpha)\hat{\mathbf{t}};$$

this is essentially the same as (2.8).

2.3 The Modified Tangential Velocity

As we have mentioned above, we need to define a new quantity δ in order to perform energy estimates. We make the definition

$$\delta = \frac{\pi}{L}\gamma - (T - \mathbf{W}\cdot\hat{\mathbf{t}}). \tag{2.10}$$

Since the Lagrangian velocity of a fluid particle on the surface is $\mathbf{W} + \frac{\gamma}{2s_\alpha}\hat{\mathbf{t}}$, another way to state the definition is that δ is the difference between the Lagrangian tangential velocity and our tangential velocity T. We will use δ as one of our system's dependent variables. This is because, as we will see later, δ is more regular than γ. We also point out that, for given $z(\alpha)$, the mapping $\gamma \to \delta$ is one-to-one; namely, one can recover γ from z and δ. We refer the reader to [3] and the appendix of [1].

2.4 Evolution Equations

We will frequently use identities involving derivatives of the tangent and normal vectors, such as

$$\begin{gathered}\hat{\mathbf{t}}_\alpha \cdot \hat{\mathbf{n}} = -\hat{\mathbf{n}}_\alpha \cdot \hat{\mathbf{t}} = \theta_\alpha, \quad \hat{\mathbf{t}}_t \cdot \hat{\mathbf{n}} = -\hat{\mathbf{n}}_t \cdot \hat{\mathbf{t}} = \theta_t, \\ \hat{\mathbf{t}}_\alpha \cdot \hat{\mathbf{t}} = \hat{\mathbf{n}}_\alpha \cdot \hat{\mathbf{n}} = \hat{\mathbf{t}}_t \cdot \hat{\mathbf{t}} = \hat{\mathbf{n}}_t \cdot \hat{\mathbf{n}} = 0.\end{gathered} \tag{2.11}$$

We use these identities together with (2.3) and the formula $U = \mathbf{W}\cdot\hat{\mathbf{n}}$ to make the following calculation, which will be useful many times:

$$\begin{aligned} D_\alpha(T - \mathbf{W}\cdot\hat{\mathbf{t}}) &= T_\alpha - \mathbf{W}_\alpha \cdot \hat{\mathbf{t}} - \mathbf{W} \cdot \hat{\mathbf{t}}_\alpha \\ &= \theta_\alpha U + \frac{L_t}{2\pi} - \mathbf{W}_\alpha \cdot \hat{\mathbf{t}} - \theta_\alpha U = \frac{L_t}{2\pi} - \mathbf{W}_\alpha \cdot \hat{\mathbf{t}}. \end{aligned} \tag{2.12}$$

We can now rewrite the evolution equation (2.1) as

$$\begin{aligned} \theta_t &= \frac{2\pi}{L}\big(\mathbf{W}_\alpha \cdot \hat{\mathbf{n}} + (T - \mathbf{W}\cdot\hat{\mathbf{t}})\theta_\alpha\big) \\ &= \frac{2\pi^2}{L^2}H(\gamma_\alpha) + \frac{2\pi}{L}(T - \mathbf{W}\cdot\hat{\mathbf{t}})\theta_\alpha + \frac{2\pi}{L}\mathbf{m}\cdot\hat{\mathbf{n}}. \end{aligned} \tag{2.13}$$

We also need to discuss the evolution of L. Since $L = \int s_\alpha \, d\alpha$, we use (2.2) and the fact that T is periodic to find

$$L_t(t) = -\int_0^{2\pi} \theta_\alpha(\alpha, t) U(\alpha, t) d\alpha. \tag{2.14}$$

The evolution equation for γ is (see Appendix B)

$$\begin{aligned} \gamma_t = &\frac{1}{\mathrm{We}}\frac{2\pi\theta_{\alpha\alpha}}{L} + D_\alpha\left(\frac{2\pi(T - \mathbf{W}\cdot\hat{\mathbf{t}})\gamma}{L}\right) \\ &- 2\left(\frac{L}{2\pi}\mathbf{W}_t \cdot \hat{\mathbf{t}} + \frac{\pi^2}{L^2}\gamma\gamma_\alpha - (T - \mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha \cdot \hat{\mathbf{t}} + g y_\alpha\right). \end{aligned} \tag{2.15}$$

The Weber number We is a dimensionless parameter inversely proportional to surface tension. Thus, We $= \infty$ is the case without surface tension. This form of the equation can be found in [6], for example. We have included a derivation of the equation in the appendix. A similar derivation is included in [3], and another version can be found in [1]. For now, we restrict ourselves to the case We $= \infty$; we will consider We $< \infty$ in Section 4. This allows us to write the evolution equation for γ as

$$\begin{aligned}\gamma_t = {} & \frac{2\pi}{L} D_\alpha((T - \mathbf{W}\cdot\hat{\mathbf{t}})\gamma) \\ & - 2\left(\frac{L}{2\pi}\mathbf{W}_t \cdot \hat{\mathbf{t}} + \frac{\pi^2}{L^2}\gamma\gamma_\alpha - (T - \mathbf{W}\cdot\hat{\mathbf{t}})\mathbf{W}_\alpha \cdot \hat{\mathbf{t}} + g y_\alpha\right).\end{aligned} \tag{2.16}$$

The system formed by equations (2.13) and (2.15) is the water wave system (without surface tension). We rewrite this system in a more convenient form, using the modified tangential velocity δ. In what follows, s is taken large enough ($s \geq 6$ will certainly be sufficient).

PROPOSITION 2.1 *The water wave system can be rewritten as*

$$\begin{cases}\delta_{\alpha t} = -\frac{2\pi}{L}\delta\delta_{\alpha\alpha} - c\theta_\alpha + \psi, \\ \theta_t = -\frac{2\pi}{L}\delta\theta_\alpha + \frac{2\pi}{L}H(\delta_\alpha) + \phi,\end{cases} \tag{2.17}$$

where $c = -\nabla p \cdot \hat{\mathbf{n}}$ *is positive and* ψ *and* ϕ *are lower-order terms that satisfy*

$$\|\psi\|_{s-1/2} \leq C(\|\theta\|_s, \|\delta\|_{s+1/2}), \quad \|\phi\|_s \leq C(\|\theta\|_{s-1}, \|\delta\|_{s-1/2})\|\theta\|_s.$$

PROOF: For the convenience of the reader, we will give the detailed computation that yields (2.17). Thus, we need to use (2.16) to calculate δ_t. First, however, we restate (2.13) in terms of δ. To do this, we make use of (2.12); we get

$$\theta_t = \frac{2\pi}{L}\big(H(\delta_\alpha) - H(\mathbf{W}_\alpha \cdot \hat{\mathbf{t}}) + (T - \mathbf{W}\cdot\hat{\mathbf{t}})\theta_\alpha + \mathbf{m}\cdot\hat{\mathbf{n}}\big). \tag{2.18}$$

We restate this by using the representation (2.8) for $\mathbf{W}_\alpha \cdot \hat{\mathbf{t}}$ and the definition of δ. We get

$$\theta_t = \frac{2\pi}{L}\left(H(\delta_\alpha) - \delta\theta_\alpha - H(\mathbf{m}\cdot\hat{\mathbf{t}}) + \mathbf{m}\cdot\hat{\mathbf{n}} + \frac{\pi}{L}P_0(\gamma\theta_\alpha)\right). \tag{2.19}$$

Here we have used the fact that $H^2(f) = -f$ whenever f has zero mean. If f does not have zero mean, then $H^2(f) = -f + P_0(f)$, where

$$P_0(f) = \frac{1}{2\pi}\int_0^{2\pi} f(\alpha)\,d\alpha.$$

We now proceed to find an evolution equation for δ. Notice that we can rewrite (2.16) to read

$$\begin{aligned}\gamma_t = &-D_\alpha\left(\left(\frac{\gamma\pi}{L}\right)^2 - 2(T - \mathbf{W}\cdot\hat{\mathbf{t}})\left(\frac{\gamma\pi}{L}\right) + (T - \mathbf{W}\cdot\hat{\mathbf{t}})^2\right)\\ &+ \frac{L_t}{\pi}(T - \mathbf{W}\cdot\hat{\mathbf{t}}) - \frac{L}{\pi}\mathbf{W}_t\cdot\hat{\mathbf{t}} - 2gy_\alpha.\end{aligned} \tag{2.20}$$

This can be rephrased as

$$\gamma_t = -D_\alpha(\delta^2) + \frac{L_t}{\pi}(T - \mathbf{W}\cdot\hat{\mathbf{t}}) - \frac{L}{\pi}\mathbf{W}_t\cdot\hat{\mathbf{t}} - 2gy_\alpha. \tag{2.21}$$

We use (2.10) to calculate δ_t:

$$\delta_t = \frac{\pi}{L}\gamma_t - \frac{\pi L_t}{L^2}\gamma - T_t + \mathbf{W}_t\cdot\hat{\mathbf{t}} + \theta_t(\mathbf{W}\cdot\hat{\mathbf{n}}). \tag{2.22}$$

Substituting from (2.21), this becomes

$$\begin{aligned}\delta_t &= \frac{\pi}{L}\left(-D_\alpha(\delta^2) + \frac{L_t}{L}(T - \mathbf{W}\cdot\hat{\mathbf{t}}) - \frac{L}{\pi}\mathbf{W}_t\cdot\hat{\mathbf{t}} - 2gy_\alpha\right)\\ &\quad - \frac{\pi L_t}{L^2}\gamma - T_t + \mathbf{W}_t\cdot\hat{\mathbf{t}} + \theta_t(\mathbf{W}\cdot\hat{\mathbf{n}})\\ &= -\frac{\pi}{L}D_\alpha(\delta^2) - \frac{L_t}{L}\delta - \frac{2\pi}{L}gy_\alpha - T_t + \theta_t(\mathbf{W}\cdot\hat{\mathbf{n}}).\end{aligned} \tag{2.23}$$

In later sections of this paper, we will perform estimates that require taking α-derivatives of δ. Thus, we now compute $\delta_{\alpha t}$:

$$\begin{aligned}\delta_{\alpha t} = &-\frac{\pi}{L}D_\alpha^2(\delta^2) - \frac{L_t}{L}\delta_\alpha - \frac{2\pi}{L}gy_{\alpha\alpha} - T_{\alpha t}\\ &- \theta_t\theta_\alpha(\mathbf{W}\cdot\hat{\mathbf{t}}) + \theta_{\alpha t}(\mathbf{W}\cdot\hat{\mathbf{n}}) + \theta_t(\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}).\end{aligned} \tag{2.24}$$

Since we know from (2.3) that $T_\alpha = \theta_\alpha(\mathbf{W}\cdot\hat{\mathbf{n}}) + L_t/(2\pi)$, we have

$$T_{\alpha t} = \theta_{\alpha t}(\mathbf{W}\cdot\hat{\mathbf{n}}) + \theta_\alpha\mathbf{W}_t\cdot\hat{\mathbf{n}} - \theta_\alpha\theta_t(\mathbf{W}\cdot\hat{\mathbf{t}}) + \frac{L_{tt}}{2\pi}.$$

We substitute this into (2.24) to get

$$\delta_{\alpha t} = -\frac{\pi}{L}D_\alpha^2(\delta^2) - \frac{L_t}{L}\delta_\alpha - \frac{2\pi}{L}gy_{\alpha\alpha} - \theta_\alpha(\mathbf{W}_t\cdot\hat{\mathbf{n}}) - \frac{L_{tt}}{2\pi} + \theta_t(\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}). \tag{2.25}$$

We want to continue to simplify (2.25), but first we must put some effort into understanding $\mathbf{W}_t\cdot\hat{\mathbf{n}}$. We have thus far been considering the surface of the water wave to be a curve parametrized by the spatial variable α. We now want to think of it as also being parametrized by a Lagrangian spatial variable β. That is, the same curve is given by the set of points $(x(\alpha,t), y(\alpha,t))$ or $(\tilde{x}(\beta,t), \tilde{y}(\beta,t))$. It is natural to use the equation

$$(x(\alpha,t), y(\alpha,t)) = (\tilde{x}(\beta,t), \tilde{y}(\beta,t)) \tag{2.26}$$

to define α as a function of β and t. We define $\mathbf{V}$ by

$$(\tilde{x},\tilde{y})_t(\beta,t)=\mathbf{V}(\beta,t)=\mathbf{W}(\alpha(\beta,t),t)+\frac{\gamma(\alpha(\beta,t),t)}{2s_\alpha(\alpha(\beta,t),t)}\hat{\mathbf{t}}.$$

Given any function $f(\alpha(\beta,t),t)$, we can compute its time derivative by

$$\frac{df}{dt}=\frac{\partial f}{\partial t}+\dot{\alpha}\frac{\partial f}{\partial \alpha}.$$

To calculate $\dot{\alpha}$, we differentiate both sides of (2.26) with respect to t. We find that

$$(x,y)_t+\dot{\alpha}(x,y)_\alpha=(\tilde{x},\tilde{y})_t.$$

Taking the tangential component of this, we see that $\dot{\alpha}$ is the difference in tangential velocities of (x,y) and $(\tilde{x},\tilde{y})$ divided by s_α. That is, $\dot{\alpha}=\delta/s_\alpha$.

Since our goal at the moment is to calculate $\mathbf{W}_t\cdot\hat{\mathbf{n}}$, we now compute $\mathbf{V}_t\cdot\hat{\mathbf{n}}$. We get

$$\mathbf{V}_t\cdot\hat{\mathbf{n}}=\mathbf{W}_t\cdot\hat{\mathbf{n}}+\frac{\delta}{s_\alpha}\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}+\frac{\gamma}{2s_\alpha}\theta_t+\frac{\gamma}{2s_\alpha^2}\delta\theta_\alpha. \tag{2.27}$$

We combine (2.27) with (2.25) to get

$$\begin{aligned}\delta_{\alpha t}=&\left[-\frac{2\pi}{L}gy_{\alpha\alpha}-\theta_\alpha\mathbf{V}_t\cdot\hat{\mathbf{n}}\right]-\frac{\pi}{L}D_\alpha^2(\delta^2)-\frac{L_t}{L}\delta_\alpha-\frac{L_{tt}}{2\pi}\\&+\theta_t\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}+\frac{\pi}{L}\theta_\alpha\gamma\theta_t+\frac{2\pi}{L}\theta_\alpha\delta\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}+\frac{2\pi^2}{L^2}\theta_\alpha^2\delta\gamma.\end{aligned} \tag{2.28}$$

We now look in detail at the first term on the right-hand side of this equation, keeping in mind (1.1). We first notice that since $y_\alpha=\frac{L}{2\pi}\sin(\theta)$, we have that $y_{\alpha\alpha}=\frac{L}{2\pi}\theta_\alpha\cos(\theta)$. We also make the definition $\hat{\mathbf{g}}=(0,-1)$. We can now see that

$$-\frac{2\pi}{L}gy_{\alpha\alpha}=g\theta_\alpha\hat{\mathbf{g}}\cdot\hat{\mathbf{n}}.$$

Following [4], we notice now that the Euler equations (in Lagrangian coordinates) can be stated as

$$V_t-g\hat{\mathbf{g}}=-\nabla p.$$

We make the definition

$$c(\alpha,t)=-\nabla p\cdot\hat{\mathbf{n}}; \tag{2.29}$$

recall that (1.1) implies that c is positive and bounded away from 0. We can now write the term in brackets on the right-hand side of (2.28) as simply $-c\theta_\alpha$. We can then write the equation for $\delta_{\alpha t}$:

$$\begin{aligned}\delta_{\alpha t}=&-c\theta_\alpha-\frac{\pi}{L}D_\alpha^2(\delta^2)-\frac{L_t}{L}\delta_\alpha-\frac{L_{tt}}{2\pi}\\&+\theta_t\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}+\frac{\pi}{L}\theta_\alpha\gamma\theta_t+\frac{2\pi}{L}\theta_\alpha\delta\mathbf{W}_\alpha\cdot\hat{\mathbf{n}}+\frac{2\pi^2}{L^2}\theta_\alpha^2\delta\gamma.\end{aligned} \tag{2.30}$$

We want to understand better the last four terms on the right-hand side above, so we rewrite them as follows:

$$\begin{aligned}\delta_{\alpha t} &= -c\theta_\alpha - \frac{\pi}{L}D_\alpha^2(\delta^2) - \frac{L_t}{L}\delta_\alpha - \frac{L_{tt}}{2\pi} \\ &\quad + \left(\mathbf{W}_\alpha \cdot \hat{\mathbf{n}} + \frac{\pi}{L}\theta_\alpha\gamma\right)\left(\theta_t + \frac{2\pi}{L}\theta_\alpha\delta\right).\end{aligned} \tag{2.31}$$

We will simplify this by using (2.8).

We first simplify the term $\mathbf{W}_\alpha \cdot \hat{\mathbf{n}} + \frac{\pi}{L}\theta_\alpha\gamma$ by recognizing that it is very similar to $H(\delta_\alpha)$. That is, (2.10) implies

$$\begin{aligned}H(\delta_\alpha) &= H\left(\frac{\pi}{L}\gamma_\alpha + \mathbf{W}_\alpha \cdot \hat{\mathbf{t}} - \frac{L_t}{2\pi}\right) \\ &= \frac{\pi}{L}\left(H(\gamma_\alpha) + \gamma\theta_\alpha - P_0(\gamma\theta_\alpha)\right) + H(\mathbf{m}\cdot\hat{\mathbf{t}}).\end{aligned}$$

Now when we use (2.8), we get

$$\begin{aligned}\mathbf{W}_\alpha \cdot \hat{\mathbf{n}} + \frac{\pi}{L}\theta_\alpha\gamma &= \frac{\pi}{L}H(\gamma_\alpha) + \mathbf{m}\cdot\hat{\mathbf{n}} + \frac{\pi}{L}\theta_\alpha\gamma \\ &= H(\delta_\alpha) - H(\mathbf{m}\cdot\hat{\mathbf{t}}) + \mathbf{m}\cdot\hat{\mathbf{n}} + \frac{\pi}{L}P_0(\gamma\theta_\alpha).\end{aligned} \tag{2.32}$$

We next try to simplify $\theta_t + \frac{2\pi}{L}\theta_\alpha\delta$. We first use (2.13) to replace the θ_t term and (2.10) to replace the δ. We get

$$\theta_t + \frac{2\pi}{L}\theta_\alpha\delta = \frac{2\pi}{L}\left(\mathbf{W}_\alpha \cdot \hat{\mathbf{n}} + \frac{\pi}{L}\theta_\alpha\gamma\right). \tag{2.33}$$

Notice that we calculated the quantity on the right-hand side in (2.32). So, our final form of $\delta_{\alpha t}$ is

$$\begin{aligned}\delta_{\alpha t} &= -c\theta_\alpha - \frac{\pi}{L}D_\alpha^2(\delta^2) - \frac{L_t}{L}\delta_\alpha - \frac{L_{tt}}{2\pi} \\ &\quad + \frac{2\pi}{L}\left(H(\delta_\alpha) - H(\mathbf{m}\cdot\hat{\mathbf{t}}) + \mathbf{m}\cdot\hat{\mathbf{n}} + \frac{\pi}{L}P_0(\gamma\theta_\alpha)\right)^2.\end{aligned} \tag{2.34}$$

Finally, we see that (2.19) and (2.34) can be rewritten as

$$\begin{cases}\delta_{\alpha t} = -\frac{2\pi}{L}\delta\delta_{\alpha\alpha} - c\theta_\alpha + \psi, \\ \theta_t = -\frac{2\pi}{L}\delta\theta_\alpha + \frac{2\pi}{L}H(\delta_\alpha) + \phi,\end{cases} \tag{2.35}$$

where ψ and ϕ are lower-order terms and are given by

$$\begin{aligned}\psi &= -\frac{2\pi}{L}(\delta_\alpha)^2 - \frac{L_t}{L}\delta_\alpha - \frac{L_{tt}}{2\pi} \\ &\quad + \frac{2\pi}{L}\left(H(\delta_\alpha) - H(\mathbf{m}\cdot\hat{\mathbf{t}}) + \mathbf{m}\cdot\hat{\mathbf{n}} + \frac{\pi}{L}P_0(\gamma\theta_\alpha)\right)^2\end{aligned} \tag{2.36}$$

$$\phi = \frac{2\pi}{L}\left(-H(\mathbf{m}\cdot\hat{\mathbf{t}}) + \mathbf{m}\cdot\hat{\mathbf{n}} + \frac{\pi}{L}P_0(\gamma\theta_\alpha)\right). \tag{2.37}$$

To prove the estimates for ϕ and ϕ, we apply Lemma A.8 from Appendix A. We only point out that the term involving L_t and L_{tt} can easily be treated by using the expression of L_t (2.14). □

When surface tension is taken into account we have just to add

$$\frac{1}{\mathrm{We}}\frac{2\pi^2}{L^2}\theta_{\alpha\alpha\alpha}$$

to the equation for $\delta_{\alpha t}$, and system (2.35) should be replaced by

$$\begin{cases}\delta_{\alpha t} = \frac{1}{\mathrm{We}}\frac{2\pi^2}{L^2}\theta_{\alpha\alpha\alpha} - \frac{2\pi}{L}\delta\delta_{\alpha\alpha} - c\theta_\alpha + \psi,\\ \theta_t = -\frac{2\pi}{L}\delta\theta_\alpha + \frac{2\pi}{L}H(\delta_\alpha) + \phi.\end{cases} \tag{2.38}$$

The formulae for ϕ, ψ, and c are exactly the same.

2.5 Statement of the Results

Now we are able to restate Theorems 1.1 and 1.2. For any We, we take an initial data for the water wave problem, namely,

$$\begin{cases}z^{\mathrm{We}}(\alpha, t=0) = z_0^{\mathrm{We}}(\alpha),\\ \gamma^{\mathrm{We}}(\alpha, t=0) = \gamma_0^{\mathrm{We}}(\alpha),\end{cases} \tag{2.39}$$

where $z_0^{\mathrm{We}}(\alpha) - \alpha$ and $\gamma_0^{\mathrm{We}}(\alpha)$ are 2π-periodic and $z_0^{\mathrm{We}}(\alpha)$ satisfies the non-self-intersection condition (1.2) uniformly in We. Moreover, α is an arc length parametrization, namely, $|D_\alpha z_0^{\mathrm{We}}(\alpha)|$ is constant and is equal to $L_0^{\mathrm{We}}/2\pi$. Any initial data for the water wave problem can easily be restated in terms of an initial data for (2.38). Hence, we can define $(\delta_0^{\mathrm{We}}(\alpha), \theta_0^{\mathrm{We}}(\alpha))$ by solving the following equations:

$$D_\alpha z_0^{\mathrm{We}}(\alpha) = \frac{L_0^{\mathrm{We}}}{2\pi}\big(\cos\theta_0^{\mathrm{We}}(\alpha), \sin\theta_0^{\mathrm{We}}(\alpha)\big).$$

Next, we can compute $\mathbf{W}_0^{\mathrm{We}}$ by using (2.4); then, U_0^{We} is determined by $U_0^{\mathrm{We}} = \mathbf{W}_0^{\mathrm{We}}\cdot\hat{\mathbf{n}}$. We compute $L_t^{\mathrm{We}}(t=0) = 2\pi s_{\alpha t}(t=0)$ by using (2.14). And finally, we can compute T by (2.2). This allows us to define δ_0^{We}.

We take s big enough ($s \geq 6$ is enough for our calculation). But we do not intend to give the best regularity here. We assume that for all We, $(\delta_0^{\mathrm{We}}(\alpha), \theta_0^{\mathrm{We}}(\alpha))$ is in $H^{s+1/2}\times H^{s+1}$, that $(\delta_0^{\mathrm{We}}(\alpha), \theta_0^{\mathrm{We}}(\alpha))$ converges to $(\delta_0(\alpha), \theta_0(\alpha))$ in $H^{s+1/2}\times H^s$ when We goes to ∞, and that $\|\theta_0^{\mathrm{We}}(\alpha))\|_{H^{s+1}} \leq C\sqrt{\mathrm{We}}$. Then we have the following:

THEOREM 2.2 *There exist a* We_0 *and a* $T^* > 0$ *such that for all* $\mathrm{We} > \mathrm{We}_0$, *there exists a unique solution* $(\delta^{\mathrm{We}}, \theta^{\mathrm{We}})$ *of* (2.38) *in* $C([0, T^*); H^{s+1/2}\times H^{s+1})$. *Moreover, as* We *goes to* ∞, $(\delta^{\mathrm{We}}, \theta^{\mathrm{We}})$ *converges in* $C_{\mathrm{loc}}([0, T^*); H^{s'+1/2}\times H^{s'})$ *to the unique solution* (δ, θ) *of* (2.35) *for all* $s' < s$.

Remark 2.3. Theorem 2.2 gives a new existence result about the water wave equation (2.35) without surface tension.

The proof of this theorem is based on proving some estimates for (2.38) that are uniform in We and then passing to the limit in We. It is only for this reason that we take We $>$ We$_0$, since the energy we use to solve (2.38) requires that We be big enough to ensure that $c > c_0/2$.

2.6 Calculation of c_α

In order to perform the estimates of the following sections, it will be necessary to understand the smoothness of c_α. To this end, we perform a rather long calculation here which demonstrates that c_α is smoother than might first be expected. In particular, at the end of this section, we will see an important cancellation between two terms with a high number of derivatives.

To begin, we explicitly write what c is. We restate the definition of the previous section:

$$c(\alpha,t) = \mathbf{W}_t \cdot \hat{\mathbf{n}} + \frac{\delta}{s_\alpha}\mathbf{W}_\alpha \cdot \hat{\mathbf{n}} + \frac{\gamma}{2s_\alpha}\theta_t + \frac{\gamma}{2s_\alpha^2}\delta\theta_\alpha - g\hat{\mathbf{g}} \cdot \hat{\mathbf{n}}. \tag{2.40}$$

From this formula it seems that c_α has the same regularity as $\gamma_{\alpha\alpha}$ or $\theta_{\alpha\alpha}$. One of the crucial steps in what follows is to prove that the combination of the derivatives of the terms on the right-hand side of the definition of c is smoother than the individual terms.

PROPOSITION 2.4 *c_α can be written as*

$$c_\alpha = H(\delta_{\alpha t}) + f, \tag{2.41}$$

where, in the case without surface tension,

$$\|f\|_{s-3/2} \le C(\|\delta\|_{s+1/2}, \|\theta\|_s, |L|)$$

and in the case with surface tension

$$\|f\|_{s-3/2} \le C\left(\|\delta\|_{s+1/2}, \|\theta\|_s, \frac{1}{\sqrt{\mathrm{We}}}\|\theta\|_{s+1}, |L|\right)$$

PROOF: To begin, we observe that in order to calculate c_α, we need to calculate $\mathbf{W}_{\alpha t} \cdot \hat{\mathbf{n}}$. We will also need later a convenient form of $\mathbf{W}_{\alpha t} \cdot \hat{\mathbf{t}}$. In the next section, we find the relevant formulae. By the same kind of argument we presented at the end of Section 2.2, we can see that our conclusion will be that

$$\mathbf{W}_{\alpha t} \approx \frac{\pi}{L}H(\gamma_{\alpha t})\hat{\mathbf{n}} - \frac{\pi}{L}H(\gamma\theta_{\alpha t})\hat{\mathbf{t}}.$$

Calculation of $\mathbf{W}_{\alpha t}$

To calculate $\mathbf{W}_{\alpha t} \cdot \hat{\mathbf{n}}$, we use (2.5). We first have

$$\Phi(\mathbf{W}_{\alpha t})^* = \left(\frac{1}{2\pi i}\,\mathrm{PV}\int \frac{\gamma(\alpha')}{z(\alpha) - z(\alpha')}\,d\alpha'\right)_{\alpha t}. \tag{2.42}$$

We use the fact that

$$\mathrm{PV}\int \frac{z_\alpha(\alpha')}{z(\alpha) - z(\alpha')}\,d\alpha' = 0$$

to rewrite (2.42) as

$$
\begin{aligned}
\Phi(\mathbf{W}_{\alpha t})^* &= \left(\frac{1}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}-\frac{\gamma(\alpha)}{z_\alpha(\alpha)}\right)\frac{z_\alpha(\alpha')}{z(\alpha)-z(\alpha')}\,d\alpha'\right)_{\alpha t} \\
&= -\left(\frac{1}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}-\frac{\gamma(\alpha)}{z_\alpha(\alpha)}\right)\frac{z_\alpha(\alpha')z_\alpha(\alpha)}{(z(\alpha)-z(\alpha'))^2}\,d\alpha'\right)_t .
\end{aligned} \tag{2.43}
$$

In the final integral in (2.43), we recognize that there is an α'-derivative, and we integrate by parts:

$$
\begin{aligned}
\Phi(\mathbf{W}_{\alpha t})^* &= -\left(\frac{z_\alpha(\alpha)}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}-\frac{\gamma(\alpha)}{z_\alpha(\alpha)}\right)D_{\alpha'}\frac{1}{z(\alpha)-z(\alpha')}\,d\alpha'\right)_t \\
&= \left(\frac{z_\alpha(\alpha)}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\right)_{\alpha'}\frac{1}{z(\alpha)-z(\alpha')}\,d\alpha'\right)_t .
\end{aligned} \tag{2.44}
$$

Applying the time derivative, we have

$$
\begin{aligned}
\Phi(\mathbf{W}_{\alpha t})^* = {}& \frac{z_{\alpha t}(\alpha)}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\right)_{\alpha'}\frac{1}{z(\alpha)-z(\alpha')}\,d\alpha' \\
&+\frac{z_\alpha(\alpha)}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\right)_{\alpha' t}\frac{1}{z(\alpha)-z(\alpha')}\,d\alpha' \\
&-\frac{z_\alpha(\alpha)}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma(\alpha')}{z_\alpha(\alpha')}\right)_{\alpha'}\frac{z_t(\alpha)-z_t(\alpha')}{(z(\alpha)-z(\alpha'))^2}\,d\alpha' \\
= {}& \Phi(\mathbf{Y}_1)^*+\Phi(\mathbf{Y}_2)^*+\Phi(\mathbf{Y}_3)^*.
\end{aligned} \tag{2.45}
$$

To calculate $\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}}$, we now calculate $\mathbf{Y}_1\cdot\hat{\mathbf{n}}$, $\mathbf{Y}_2\cdot\hat{\mathbf{n}}$, and $\mathbf{Y}_3\cdot\hat{\mathbf{n}}$.

Before doing this, we recall the following formulae involving z_α and its derivatives; these stem from the relationship $z_\alpha/s_\alpha = e^{i\theta}$:

$$\mathrm{Re}\left(\frac{z_{\alpha t}}{i z_\alpha}\right)=\mathrm{Re}\left(\frac{(s_\alpha e^{i\theta})_t}{i s_\alpha e^{i\theta}}\right)=\theta_t, \tag{2.46}$$

$$\mathrm{Re}\left(\frac{z_{\alpha t}}{z_\alpha}\right)=\mathrm{Re}\left(\frac{(s_\alpha e^{i\theta})_t}{s_\alpha e^{i\theta}}\right)=\frac{s_{\alpha t}}{s_\alpha}, \tag{2.47}$$

$$\mathrm{Re}\left(\frac{z_{\alpha\alpha}}{i z_\alpha}\right)=\mathrm{Re}\left(\frac{(s_\alpha e^{i\theta})_\alpha}{i s_\alpha e^{i\theta}}\right)=\theta_\alpha, \tag{2.48}$$

$$\mathrm{Re}\left(\frac{z_{\alpha\alpha}}{z_\alpha}\right)=\mathrm{Re}\left(\frac{(s_\alpha e^{i\theta})_\alpha}{s_\alpha e^{i\theta}}\right)=0. \tag{2.49}$$

We see immediately that

$$\Phi(\mathbf{Y}_1)^*=\frac{z_{\alpha t}}{2i z_\alpha^2}H\left(\gamma_\alpha-\frac{\gamma z_{\alpha\alpha}}{z_\alpha}\right)+\frac{z_{\alpha t}}{z_\alpha}\Phi(\mathbf{m})^*.$$

Taking the dot product $\mathbf{Y}_1 \cdot \hat{\mathbf{n}}$, we get

$$\begin{aligned}\mathbf{Y}_1 \cdot \hat{\mathbf{n}} &= \mathrm{Re}\left\{\frac{z_{\alpha t}}{2iz_\alpha^2}\frac{iz_\alpha}{s_\alpha}H(\gamma_\alpha)\right\} + \mathrm{Re}\left\{\frac{z_{\alpha t}}{2iz_\alpha^2}\frac{iz_\alpha}{s_\alpha}H\left(\frac{\gamma z_{\alpha\alpha}}{z_\alpha}\right)\right\} \\ &\quad + \mathrm{Re}\left\{\frac{z_{\alpha t}}{z_\alpha}\Phi(\mathbf{m})^*\frac{iz_\alpha}{s_\alpha}\right\}.\end{aligned} \tag{2.50}$$

The first term on the right-hand side is equal to $\pi L_t H(\gamma_\alpha)/L^2$ by (2.47). To understand the second and third terms on the right-hand side, we use the formula for two complex numbers u and v

$$\mathrm{Re}\{uv\} = \mathrm{Re}\{u\}\,\mathrm{Re}\{v\} - \mathrm{Re}\{iu\}\,\mathrm{Re}\{iv\}.$$

We get that the second term is

$$\mathrm{Re}\left\{\frac{z_{\alpha t}z_\alpha^*}{2s_\alpha^3}\right\}\mathrm{Re}\left\{H\left(\frac{\gamma z_{\alpha\alpha}z_\alpha^*}{s_\alpha^2}\right)\right\} - \mathrm{Re}\left\{\frac{iz_{\alpha t}z_\alpha^*}{2s_\alpha^3}\right\}\mathrm{Re}\left\{H\left(\frac{i\gamma z_{\alpha\alpha}z_\alpha^*}{s_\alpha^2}\right)\right\}.$$

The first of these two is identically 0 by (2.49). The second of these is equal to $\pi\theta_t H(\gamma\theta_\alpha)/L$. Also, we see that the third term on the right-hand side of (2.50) is $L_t\mathbf{m}\cdot\hat{\mathbf{n}}/L - \theta_t\mathbf{m}\cdot\hat{\mathbf{t}}$.

From the definitions of $\mathbf{Y}_2$ and the integral remainder operator $\mathcal{K}$, we see first that we can write

$$\Phi(\mathbf{Y}_2)^* = \frac{z_\alpha}{2i}H\left(\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\frac{1}{z_\alpha}\right) + z_\alpha\mathcal{K}[z]\left(\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right).$$

We apply the derivatives in the first term on the right-hand side to get

$$\begin{aligned}&\frac{z_\alpha}{2i}H\left(\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\frac{1}{z_\alpha}\right) \\ &\quad = \frac{z_\alpha}{2i}H\left(\frac{\gamma_{\alpha t}}{z_\alpha^2} - \frac{\gamma_\alpha z_{\alpha t}}{z_\alpha^3} - \frac{\gamma z_{\alpha\alpha t}}{z_\alpha^3} + 2\frac{\gamma z_{\alpha\alpha}z_{\alpha t}}{z_\alpha^4} - \frac{\gamma_t z_{\alpha\alpha}}{z_\alpha^3}\right) \\ &\quad = \frac{1}{2iz_\alpha}\left(H(\gamma_{\alpha t}) - H\left(\frac{\gamma_\alpha z_{\alpha t}}{z_\alpha}\right) - H\left(\frac{\gamma z_{\alpha\alpha t}}{z_\alpha}\right) + 2H\left(\frac{\gamma z_{\alpha\alpha}z_{\alpha t}}{z_\alpha^2}\right)\right) \\ &\qquad - H\left(\frac{\gamma_t z_{\alpha\alpha}}{z_\alpha}\right) + \frac{z_\alpha}{2i}\left[H, \frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right).\end{aligned} \tag{2.51}$$

When we take the dot product of $\mathbf{Y}_2$ with the normal vector, we get the following contribution from the right-hand side of the above line:

$$\begin{aligned}
&\frac{1}{2s_\alpha}H(\gamma_{\alpha t}) - \frac{1}{2s_\alpha}H\left(\gamma_\alpha \operatorname{Re}\left\{\frac{z_{\alpha t}z_\alpha^*}{s_\alpha^2}\right\}\right) - \frac{1}{2s_\alpha}H\left(\gamma \operatorname{Re}\left\{\frac{z_{\alpha\alpha t}z_\alpha^*}{s_\alpha^2}\right\}\right)\\
&\quad + \frac{1}{s_\alpha}H\left(\gamma \operatorname{Re}\left\{\frac{z_{\alpha\alpha}z_\alpha^*}{s_\alpha^2}\right\}\operatorname{Re}\left\{\frac{z_{\alpha t}z_\alpha^*}{s_\alpha^2}\right\}\right)\\
&\quad - \frac{1}{s_\alpha}H\left(\gamma \operatorname{Re}\left\{\frac{z_{\alpha\alpha}(iz_\alpha)^*}{s_\alpha^2}\right\}\operatorname{Re}\left\{\frac{z_{\alpha t}(iz_\alpha)^*}{s_\alpha^2}\right\}\right)\\
&\quad - \frac{1}{2s_\alpha}H\left(\gamma_t \operatorname{Re}\left\{\frac{z_{\alpha\alpha}z_\alpha^*}{s_\alpha^2}\right\}\right) + \operatorname{Re}\left\{\frac{z_\alpha^2}{2s_\alpha}\left[H,\frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}.
\end{aligned} \tag{2.52}$$

We can make the calculation (similarly to (2.46))

$$\operatorname{Re}\left\{\frac{z_{\alpha\alpha t}z_\alpha^*}{s_\alpha^2}\right\} = -\theta_\alpha\theta_t. \tag{2.53}$$

Finally, we make the conclusion that

$$\begin{aligned}
\mathbf{Y}_2\cdot\hat{\mathbf{n}} &= \frac{\pi}{L}H(\gamma_{\alpha t}) - \frac{\pi L_t}{L^2}H(\gamma_\alpha) - \frac{\pi}{L}H(\gamma\theta_\alpha\theta_t)\\
&\quad + \operatorname{Re}\left\{\frac{iz_\alpha^2}{s_\alpha}\mathcal{K}[z]\left(\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}\\
&\quad + \operatorname{Re}\left\{\frac{z_\alpha^2}{2s_\alpha}\left[H,\frac{1}{z_\alpha^2}\right]\left(z_\alpha\left(\frac{\gamma}{z_\alpha}\right)_{\alpha t}\right)\right\}.
\end{aligned} \tag{2.54}$$

We now rewrite $\mathbf{Y}_3$ as

$$\begin{aligned}
\Phi(\mathbf{Y}_3)^* &= -\frac{z_\alpha}{2\pi i}\,\mathrm{PV}\int\left(\frac{\gamma}{z_\alpha}\right)_{\alpha'}\frac{z_t(\alpha)-z_t(\alpha')}{(z(\alpha)-z(\alpha'))^2}\,d\alpha'\\
&= -\frac{z_\alpha}{2\pi i}\,\mathrm{PV}\int\left(\left(\frac{\gamma}{z_\alpha}\right)_{\alpha'}\frac{1}{z_\alpha}\right)(z_t(\alpha)\\
&\quad - z_t(\alpha'))D_{\alpha'}\left(\frac{1}{z(\alpha)-z(\alpha')}\right)d\alpha'.
\end{aligned} \tag{2.55}$$

We integrate this by parts and get two kinds of terms. The α'-derivative can fall on $z_t(\alpha)-z_t(\alpha')$. This will be the only significant source of terms from $\mathbf{Y}_3$. The other kind of term occurs when the α'-derivative does not fall on $z_t(\alpha)-z_t(\alpha')$; these terms are smooth by Lemmas A.6 and A.7. When the derivative falls on the $z_t(\alpha)-z_t(\alpha')$, we get

$$-\frac{z_\alpha}{2i}H\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\frac{z_{\alpha t}}{z_\alpha^2}\right) - z_\alpha\mathcal{K}[z]\left(\left(\frac{\gamma}{z_\alpha}\right)_\alpha\frac{z_{\alpha t}}{z_\alpha}\right).$$

When we take the dot product with $\hat{\mathbf{n}}$, we get a contribution of

$$-\frac{\pi}{L}H\left(\gamma_\alpha \operatorname{Re}\left\{\frac{z_{\alpha t}}{z_\alpha}\right\}\right) - \frac{\pi}{L}H\left(\gamma \operatorname{Re}\left\{\frac{z_{\alpha\alpha}}{iz_\alpha}\right\}\operatorname{Re}\left\{\frac{z_{\alpha t}}{iz_\alpha}\right\}\right).$$

Using our identities, we conclude that

$$\mathbf{Y}_3 \cdot \hat{\mathbf{n}} = -\frac{\pi L_t}{L^2}H(\gamma_\alpha) - \frac{\pi}{L}H(\gamma\theta_\alpha\theta_t) + \Upsilon_1,$$

where Υ_1 is a large collection of smooth terms.

We are finally able to write

$$W_{\alpha t} \cdot \hat{\mathbf{n}} = \frac{\pi}{L}H(\gamma_{\alpha t}) + f_1 + \Upsilon_2, \tag{2.56}$$

where f_1 is a collection of lower-order terms and Υ_2 is a collection of very smooth terms. We define f_1 as

$$f_1 = -\frac{\pi}{L}\theta_t H(\gamma\theta_\alpha) - \theta_t \mathbf{m} \cdot \hat{\mathbf{t}} - \frac{\pi L_t}{L^2}H(\gamma_\alpha) - \frac{2\pi}{L}H(\gamma\theta_\alpha\theta_t).$$

In both the case without surface tension and the case with surface tension, we will estimate δ in $H^{s+1/2}$. Since $\delta_{\alpha t}$ contains terms that involve $c(\alpha, t)$, we hope to be able to estimate c_α in $H^{s-3/2}$. To simplify the equations, we introduce a new notation. The notation f will appear many times below, and it has a different meaning from line to line. In the case without surface tension, it represents any term whose $H^{s-3/2}$ norm can be estimated in terms of the $H^{s+1/2}$ norm of δ, the H^s norm of θ, and $|L|$, namely,

$$\|f\|_{s-3/2} \le C(\|\delta\|_{s+1/2}, \|\theta\|_s, |L|).$$

In the case with surface tension, f can also depend (uniformly) on $\frac{1}{\sqrt{\mathrm{We}}}\|\theta\|_{s+1}$:

$$\|f\|_{s-3/2} \le C\left(\|\delta\|_{s+1/2}, \|\theta\|_s, \frac{1}{\sqrt{\mathrm{We}}}\|\theta\|_{s+1}, |L|\right).$$

Note that this includes terms that can be estimated in terms of the H^s norm of γ (see the appendix for Lemma A.3). Thus, we can summarize the above calculation as

$$W_{\alpha t} \cdot \hat{\mathbf{n}} = \frac{\pi}{L}H(\gamma_{\alpha t}) + f. \tag{2.57}$$

A similar lengthy calculation yields

$$W_{\alpha t} \cdot \hat{\mathbf{t}} = -\frac{\pi}{L}H(\gamma\theta_{\alpha t}) + f. \tag{2.58}$$

Conclusion of the Calculation of c_α

To begin, we differentiate (2.40) with respect to α to get

$$
\begin{aligned}
c_\alpha = &\boxed{\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}} + \frac{\delta}{s_\alpha}\mathbf{W}_{\alpha\alpha}\cdot\hat{\mathbf{n}} + \frac{\gamma}{2s_\alpha}\,\theta_{\alpha t} + \frac{\gamma}{2s_\alpha^2}\,\delta\theta_{\alpha\alpha}} \\
&-\theta_\alpha \mathbf{W}_t\cdot\hat{\mathbf{t}} + \frac{\delta_\alpha}{s_\alpha}\mathbf{W}_\alpha\cdot\hat{\mathbf{n}} - \frac{\delta}{s_\alpha}\theta_\alpha\mathbf{W}_\alpha\cdot\hat{\mathbf{t}} \\
&+\frac{\gamma_\alpha}{2s_\alpha}\theta_t + \frac{\gamma_\alpha}{2s_\alpha^2}\delta\theta_\alpha + \frac{\gamma}{2s_\alpha^2}\delta_\alpha\theta_\alpha + g\theta_\alpha\hat{\mathbf{g}}\cdot\hat{\mathbf{t}}.
\end{aligned}
\tag{2.59}
$$

The most singular terms are those that appear in the box. The rest of the terms are obviously bounded in $H^{s-3/2}$ by $\|\gamma\|_s$, $\|\theta\|_s$, $\|\delta\|_{s-1/2}$ (and by $\frac{1}{\sqrt{\mathrm{We}}}\|\theta\|_{s+1}$ in the case with surface tension), and $|L|$; see Appendix A for relevant lemmas.

For the boxed terms, we first calculate the sum of the first two:

$$\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}} + \frac{\delta}{s_\alpha}\mathbf{W}_{\alpha\alpha}\cdot\hat{\mathbf{n}} = \frac{\pi}{L}H(\gamma_{\alpha t}) + \frac{2\pi^2}{L^2}\delta H(\gamma_{\alpha\alpha}) + f. \tag{2.60}$$

We now rewrite (2.60) by using the equation

$$\delta_{\alpha t} = \frac{\pi}{L}\gamma_{\alpha t} + W_{\alpha t}\cdot\hat{\mathbf{t}} + 2U\theta_{\alpha t} + f, \tag{2.61}$$

which can be found by differentiating (2.22) with respect to α. We find

$$
\begin{aligned}
\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}} + \frac{\delta}{s_\alpha}\mathbf{W}_{\alpha\alpha}\cdot\hat{\mathbf{n}} &= H(\delta_{\alpha t}) - H(W_{\alpha t}\cdot\hat{\mathbf{t}}) \\
&\quad - 2H(U\theta_{\alpha t}) + \frac{2\pi^2}{L^2}\,\delta H(\gamma_{\alpha\alpha}) + f.
\end{aligned}
\tag{2.62}
$$

Using (2.58), we rewrite this as

$$\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}} + \frac{\delta}{s_\alpha}\mathbf{W}_{\alpha\alpha}\cdot\hat{\mathbf{n}} = H(\delta_{\alpha t}) - \frac{\pi}{L}\,\gamma\theta_{\alpha t} + \frac{2\pi^2}{L^2}\,\delta H(\gamma_{\alpha\alpha}) + f. \tag{2.63}$$

Substituting in $\theta_t = \frac{2\pi^2}{L^2}H(\gamma_\alpha) + \frac{2\pi}{L}(T - \mathbf{W}\cdot\hat{\mathbf{t}})\theta_\alpha + f$ and $\delta = \frac{\pi}{L}\gamma - (T - \mathbf{W}\cdot\hat{\mathbf{t}})$, we get

$$
\begin{aligned}
\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}} + \frac{\delta}{s_\alpha}\mathbf{W}_{\alpha\alpha}\cdot\hat{\mathbf{n}} &= H(\delta_{\alpha t}) - \frac{2\pi^3}{L^3}\gamma H(\gamma_{\alpha\alpha}) - \frac{2\pi^3}{L^3}\gamma(T - \mathbf{W}\cdot\hat{\mathbf{t}})\theta_{\alpha\alpha} \\
&\quad + \frac{2\pi^2}{L^2}\left(\frac{\pi}{L}\gamma - (T - \mathbf{W}\cdot\hat{\mathbf{t}})\right)H(\gamma_{\alpha\alpha}).
\end{aligned}
\tag{2.64}
$$

There is an important cancellation here between the two terms with $\gamma H(\gamma_{\alpha\alpha})$; we are left with

$$\mathbf{W}_{\alpha t}\cdot\hat{\mathbf{n}} + \frac{\delta}{s_\alpha}\mathbf{W}_{\alpha\alpha}\cdot\hat{\mathbf{n}} = H(\delta_{\alpha t}) - \frac{2\pi}{L}(T - \mathbf{W}\cdot\hat{\mathbf{t}})H(\delta_{\alpha\alpha}) + f. \tag{2.65}$$

This concludes the calculation of the first two of the boxed terms. For the third and the fourth of the boxed terms, we get

$$\frac{\pi\gamma}{L}\left(\theta_{\alpha t}+\frac{2\pi}{L}\delta\theta_{\alpha\alpha}\right)=\frac{2\pi^2\gamma}{L^2}H(\delta_{\alpha\alpha})+f.$$

In summary, we have found the following formula for c_α:

$$c_\alpha=H(\delta_{\alpha t})+\frac{2\pi}{L}\delta H(\delta_{\alpha\alpha})+f. \tag{2.66}$$

and the proof of Proposition 2.4 is complete. □

3 Estimates Without Surface Tension

In this section, we perform energy estimates for triples (θ,δ,L). The energy function is related to the H^s norm of θ, the $H^{s+1/2}$ norm of δ, and the absolute value of L. First, notice that we only include the absolute value of L for technical reasons—since it is also an unknown, we need to bound its growth in order to achieve an existence theorem. However, if θ and δ are sufficiently regular, we can see immediately from (2.14) that the growth of L is bounded by their norms. We define the energy as

$$E(t)=\|\theta\|_0^2(t)+\|\delta\|_0^2(t)+\|\gamma\|_{s-1}^2(t)+\sum_{k=1}^{s}E_k(t)+L^2(t), \tag{3.1}$$

where E_k is like the square of the H^k norm of θ and the square of the $H^{k+1/2}$ norm of δ. In particular,

$$E_k(t)=\frac{1}{2}\int_0^{2\pi}c\frac{L}{2\pi}\left(D_\alpha^k\theta\right)^2+\left(D_\alpha^k\delta\right)\Lambda\left(D_\alpha^k\delta\right)d\alpha. \tag{3.2}$$

The operator Λ is equal to HD_α. This is a positive operator since the symbol of the Fourier transform of Λ is $|\xi|$. Thus, for any function f, the integral $\int f\Lambda f$ is related to Sobolev norms of f in half-integer spaces. We write the integral in (3.2) as $E_{k,1}+E_{k,2}$. We remark here that although only $s-1$ derivatives of γ appear explicitly in the definition of the energy, we are actually able to estimate s derivatives of γ. This is explained in Appendix A, and it is because of the close relationship between δ and γ and θ.

PROPOSITION 3.1 *If* (δ,θ) *solves* (2.35)*, then the time derivative of the energy,* E*, is bounded in terms of* E *as long as* (1.2) *holds with* a *replaced by* εa*. This leads to an a priori estimate for* E *until a certain time.*

PROOF: We first address the least important parts of the energy; that is, we will first discuss how to bound the time derivative of all of the components of the energy except E_k. As we have mentioned above, we can bound L_t by the energy as long as s is large enough. Clearly, we can also bound θ_t and δ_t in H^0 as long as s is large enough. By recalling equation (2.21), we can see that the H^{s-1} norm of γ_t

can be bounded by $\|\delta\|_s$, $|L_t|$, $|L|$, $\|T - \mathbf{W}\cdot\hat{\mathbf{t}}\|_{s-1}$, $\|\mathbf{W}_t \cdot \hat{\mathbf{t}}\|_{s-1}$, and $\|z\|_s$. The first three of these are clearly bounded by the energy. By Lemma A.1 (in the appendix) $\|z\|_s$ can be bounded by $\|\theta\|_{s-1}$ and $|L|$. Lemma A.5 demonstrates that $\mathbf{W}_t \cdot \hat{\mathbf{t}}$ can be bounded by the energy. Instead of considering $\|T - \mathbf{W}\cdot\hat{\mathbf{t}}\|_{s-1}$, we consider $\|D_\alpha(T - \mathbf{W}\cdot\hat{\mathbf{t}})\|_{s-2}$; using (2.12) and (2.8), we see that this can be bounded by $\|\gamma\|_{s-2}$, $\|\theta\|_{s-1}$, $\|m\|_{s-2}$, and $\|\hat{\mathbf{t}}\|_{s-2}$. Of these, the first two are clearly bounded by the energy. The second two are also, by Lemmas A.2 and A.8.

We begin to investigate the time derivative of E_k by calculating

$$\frac{dE_{k,1}}{dt} = \frac{L}{2\pi}\int \left(cD_\alpha^k\theta\right)\left(D_\alpha^k\theta_t\right) + \frac{c_t}{2}\left(D_\alpha^k\theta\right)^2 \, d\alpha + \frac{L_t}{L}E_{k,1}. \tag{3.3}$$

For θ and δ sufficiently smooth, we can bound $|c_t|$ and $|L_t|$ by the energy; so we concern ourselves with the first term. Using (2.35), it is

$$\int c\left(D_\alpha^k\theta\right)\left(D_\alpha^k H(\delta_\alpha) - D_\alpha^k(\delta\theta_\alpha) + \frac{L}{2\pi}D_\alpha^k\phi\right)d\alpha. \tag{3.4}$$

Of these terms, only the one that includes $H(\delta_\alpha)$ is significant. The second term is a transport term, and we only need to treat the case where all the k derivatives hit on θ_α. Then, by integration by parts we get

$$\int c\left(D_\alpha^k\theta\right)\left(\delta D_\alpha^k\theta_\alpha\right) = -\frac{1}{2}\int (c\delta)_\alpha\left(D_\alpha^k\theta\right)^2, \tag{3.5}$$

and we can use the energy to bound it. Recalling the definition of ϕ in (2.37), we see that we can bound the term in (3.4), which includes ϕ by the energy using Lemmas A.2 and A.8.

Now we turn our attention to $E_{k,2}$. We clearly have

$$\frac{dE_{k,2}}{dt} = \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^{k+1}\delta_t\right)d\alpha. \tag{3.6}$$

Using the equation $\delta_{\alpha t} = -\frac{2\pi}{L}\delta\delta_{\alpha\alpha} - c\theta_\alpha + \psi$ from (2.35), we have

$$\begin{aligned}\frac{dE_{k,2}}{dt} &= \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k\delta_{\alpha t}\right)d\alpha\\ &= -\int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k(c\theta_\alpha)\right)d\alpha\\ &\quad - \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k\left(\frac{2\pi}{L}\delta\delta_{\alpha\alpha}\right)\right)d\alpha + \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k\psi\right)d\alpha.\end{aligned} \tag{3.7}$$

The second term on the right-hand side is a transport term and can be treated in a slightly more complicated way than (3.5). Using the fact that $\Lambda = HD_\alpha$ is self-adjoint, we have

$$\int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k(\delta\delta_{\alpha\alpha})\right)d\alpha = \int \left(HD_\alpha^{k+1}\delta\right)\left(D_\alpha^{k-1}(\delta\delta_{\alpha\alpha})\right)d\alpha.$$

Distributing the $k-1$ derivatives in the second factor on the right-hand side, we have

$$\int \big(D_\alpha^k\delta\big)H\big(D_\alpha^k(\delta\delta_{\alpha\alpha})\big)d\alpha = \int \big(HD_\alpha^{k+1}\delta\big)\big(\delta D_\alpha^{k+1}\delta\big)d\alpha + \text{ l.o.t.}$$

We rewrite the integral on the right-hand side using the fact that the adjoint of H is $-H$,

$$\int \big(HD_\alpha^{k+1}\delta\big)\big(\delta D_\alpha^{k+1}\delta\big)d\alpha = -\int \big(D_\alpha^{k+1}\delta\big)H\big(\delta D_\alpha^{k+1}\delta\big)d\alpha.$$

We pull δ through the Hilbert transform to find

$$\begin{aligned}\int \big(HD_\alpha^{k+1}\delta\big)\big(\delta D_\alpha^{k+1}\delta\big)d\alpha = &-\int \delta\big(D_\alpha^{k+1}\delta\big)H\big(D_\alpha^{k+1}\delta\big)d\alpha\\ &-\int \big([H,\delta]D_\alpha^{k+1}\delta\big)\,\big(D_\alpha^{k+1}\delta\big)d\alpha.\end{aligned}$$

Rearranging this slightly, we have

$$\int \delta\big(D_\alpha^{k+1}\delta\big)H\big(D_\alpha^{k+1}\delta\big)d\alpha = -\frac{1}{2}\int \big([H,\delta]D_\alpha^{k+1}\delta\big)\,\big(D_\alpha^{k+1}\delta\big)d\alpha.$$

We can use Lemma A.7 (in the appendix) to control the commutator term. Thus, the second term in (3.7) can be bounded in terms of the energy.

The third term is a lower-order term and is easily bounded by the energy. We rewrite the first one to emphasize the fact that the most important contribution is when all derivatives fall on θ or when they all fall on c. We get

$$\begin{aligned}\frac{dE_{k,2}}{dt} = &-\int \big(D_\alpha^k\delta\big)\Lambda\big(cD_\alpha^k\theta\big)d\alpha - \int \big(D_\alpha^k\delta\big)\Lambda\left(\big(D_\alpha^{k-1}c\big)(\theta_\alpha)\right)d\alpha\\ &-\int \big(D_\alpha^k\delta\big)\Lambda\left(\sum_{j=1}^{k-2}\binom{k-1}{j}\big(D_\alpha^j c\big)\big(D_\alpha^{k-j}\theta\big)\right)d\alpha\\ &-\frac{2\pi}{L}\int \big(D_\alpha^k\delta\big)H\big(D_\alpha^k(\delta\delta_\alpha)\big)d\alpha + \int \big(D_\alpha^k\delta\big)H\big(D_\alpha^k\psi\big)d\alpha.\end{aligned} \tag{3.8}$$

Adding (3.4) and (3.8), we see that the first term on the right-hand side of each cancels with the other (because Λ is self-adjoint). To complete the proof of the proposition, we need only prove that we can estimate c_α in the space $H^{s-3/2}$, which allows us to estimate the second term on the right-hand of (3.8). Indeed, using Proposition 2.4, we see that

$$\begin{aligned}\left|\int \big(D_\alpha^k\delta\big)\Lambda\left(\big(D_\alpha^{k-1}c\big)(\theta_\alpha)\right)d\alpha\right| &\le C\|\delta\|_{k+1/2}\|c_\alpha\|_{k-3/2}\|\theta\|_2\\ &\le CE_k^3.\end{aligned}$$

The proof of Proposition 3.1 is complete. □

4 Estimates with Surface Tension and Convergence

In this section, we study the problem with surface tension by adding an extra term to the energy.

4.1 Estimates with Surface Tension

We can prove estimates similar to those proven in the last section as long as c is bounded away from 0. The argument given in [18] can easily be adapted to the case with small surface tension, and we can easily prove that if We is big enough, then c is uniformly bounded away from 0. Hence, we consider the case $\mathrm{We} > \mathrm{We}_0$ for some We_0. Then, we introduce the following modified energy:

$$E^{\mathrm{We}}(t) = \|\theta\|_0^2(t) + \|\delta\|_0^2(t) + \|\gamma\|_{s-1}^2(t) + \sum_{k=1}^{s} E_k^{\mathrm{We}}(t) + L^2(t), \tag{4.1}$$

where $E_k^{\mathrm{We}}(t)$ is given by

$$E_k^{\mathrm{We}}(t) = \frac{1}{2}\int_0^{2\pi} c\frac{L}{2\pi}\left(D_\alpha^k\theta\right)^2 + \frac{1}{\mathrm{We}}\frac{\pi}{L}\left(D_\alpha^{k+1}\theta\right)^2 + \left(D_\alpha^k\delta\right)\Lambda\left(D_\alpha^k\delta\right)d\alpha. \tag{4.2}$$

PROPOSITION 4.1 *If* $(\delta^{\mathrm{We}}, \theta^{\mathrm{We}})$ *solves* (2.38)*, then the time derivative of the energy,* E^{We}*, is uniformly bounded in terms of* E^{We} *as long as* (1.2) *holds with* a *replaced by* εa*. This leads to an a priori estimate for* E^{We} *until a certain time* T^**, which is independent of* We.

PROOF: The proof is similar to the proof of Proposition 3.1. We can decompose as above $E_k^{\mathrm{We}} = E_{k,1}^{\mathrm{We}} + E_{k,2}^{\mathrm{We}}$. Then

$$\begin{aligned}\frac{dE_{k,1}^{\mathrm{We}}}{dt} &= \int \frac{L}{2\pi}\left(cD_\alpha^k\theta\right)\left(D_\alpha^k\theta_t\right) + \frac{1}{\mathrm{We}}\frac{\pi}{L}\left(D_\alpha^{k+1}\theta\right)\left(D_\alpha^{k+1}\theta_t\right)\\ &\quad + \frac{L}{2\pi}c_t\left(D_\alpha^k\theta\right)^2 + L_t\left(\frac{c}{2\pi}\left(D_\alpha^k\theta\right)^2 - \frac{\pi}{L^2\mathrm{We}}\left(D_\alpha^{k+1}\theta\right)^2\right)d\alpha,\end{aligned} \tag{4.3}$$

and the important contributions can be rewritten as

$$\begin{aligned}&\int c\left(D_\alpha^k\theta\right)D_\alpha^k H(\delta_\alpha) + \frac{1}{\mathrm{We}}\frac{2\pi^2}{L^2}\left(D_\alpha^{k+1}\theta\right)D_\alpha^{k+1}H(\delta_\alpha)d\alpha\\ &\qquad + \int -\frac{1}{\mathrm{We}}\frac{2\pi^2}{L^2}\left(D_\alpha^{k+1}\theta\right)\left(D_\alpha^{k+1}\delta\right)\theta_\alpha + \frac{1}{\mathrm{We}}\left(D_\alpha^{k+1}\theta\right)\left(D_\alpha^{k+1}\phi\right).\end{aligned} \tag{4.4}$$

On the other hand, we have

$$\begin{aligned}\frac{dE_{k,2}^{\mathrm{We}}}{dt} &= \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k\delta_{\alpha t}\right)d\alpha\\ &= \int \frac{1}{\mathrm{We}}\frac{2\pi^2}{L^2}\left(D_\alpha^k\delta\right)H\left(D_\alpha^k(\theta_{\alpha\alpha\alpha})\right)d\alpha - \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k(c\theta_\alpha)\right)d\alpha\\ &\quad - \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k\left(\frac{2\pi}{L}\delta\delta_{\alpha\alpha}\right)\right)d\alpha + \int \left(D_\alpha^k\delta\right)H\left(D_\alpha^k\psi\right)d\alpha.\end{aligned} \tag{4.5}$$

By a simple integration by parts, the first term on the right-hand side of (4.5) cancels with the second term of (4.4). The second term of the right-hand side of (4.5) has two important contributions, namely when all the k derivatives hit on c or on θ_α:

$$-\int \big(D_\alpha^k\delta\big)H\big(cD_\alpha^{k+1}\theta\big)d\alpha - \int \big(D_\alpha^k\delta\big)H\big(\big(D_\alpha^k c\big)\theta_\alpha\big)d\alpha \tag{4.6}$$

By a simple integration by parts the first term of (4.6) cancels with the first term of (4.4) modulo some low-order terms.

Using Proposition 2.4, we deduce that

$$D_\alpha^k c = \frac{1}{\mathrm{We}}\frac{2\pi^2}{L^2}H\big(D_\alpha^{k+2}\theta\big) + D_\alpha^{k-1}f.$$

Hence the second term of (4.6) can be written as

$$-\frac{1}{\mathrm{We}}\frac{2\pi^2}{L^2}\int \big(D_\alpha^k\delta\big)H\big(H\big(D_\alpha^{k+2}\theta\big)\theta_\alpha\big)d\alpha - \int \big(D_\alpha^k\delta\big)\Lambda\big(\big(D_\alpha^{k-2}f\big)\theta_\alpha\big)d\alpha. \tag{4.7}$$

By a simple integration by parts, the first term of (4.7) cancels with the third term of (4.4) modulo low-order terms. The second term of (4.7) can easily be controlled by the energy.

Finally, let us explain how we can control the fourth term of (4.4). Using that

$$\|\phi\|_{k+1} \le C(\|\theta\|_k, \|\delta\|_{k+1/2})\|\theta\|_{k+1},$$

we deduce that the fourth term of (4.4) can be controlled by the energy. This ends the proof of the proposition. □

Remark 4.2. The condition (1.2) with a replaced by εa holds on some time interval that is uniform in We since the energy controls $\partial_t z$ (and its spatial derivatives) in the sup norm.

Remark 4.3. If we take $\mathrm{We} \le \mathrm{We}_0$, then the energy introduced in (4.1) can also be used to yield existence for the water wave with surface tension (2.38) by using the fact that θ^{We} is uniformly bounded in H^{s+1}.

4.2 Convergence Proof

From the uniform bounds we have proven in the last section, there exists a family of solutions $(\delta^{\mathrm{We}}, \theta^{\mathrm{We}})$ of (2.38) that is bounded in $C_{\mathrm{loc}}([0, T^*); H^{s+1/2} \times H^s)$. We can then extract a subsequence that converges weakly to some (δ, θ) and by a very standard compactness argument we can prove that (δ, θ) is a solution of the water wave without surface tension (2.35). We only point out that to get compactness in time, we have to use that $(\delta_t^{\mathrm{We}}, \theta_t^{\mathrm{We}})$ are bounded in $C_{\mathrm{loc}}([0, T^*); H^{s-2} \times H^{s-1})$ by using the evolution equations.

Hence, we deduce that for all $s' < s$ as $\mathrm{We} \to \infty$, $(\delta^{\mathrm{We}}, \theta^{\mathrm{We}})$ converges to (δ, θ) in $C_{\mathrm{loc}}([0, T^*); H^{s'+1/2} \times H^{s'})$.

Appendix A: Basic Estimates

In this section, we provide details of several estimates used in earlier sections. Much of what we state here is proven in more detail in [1] and [2]. In the following lemmas, if the range of possible values of s is not specified, it is understood that s is taken to be large enough ($s \geq 6$ is always sufficient).

LEMMA A.1 *If $\theta \in H^s$, then $z \in H^{s+1}$, with the estimate $\|z\|_{s+1} \leq C|L|(1+\|\theta\|_s)$.*

PROOF: This follows from the relationship $(x_\alpha, y_\alpha) = \frac{L}{2\pi}(\cos(\theta), \sin(\theta))$ and a standard composition estimate. Note that C can depend on $|\theta|_\infty$. □

COROLLARY A.2 *If $\theta \in H^s$, the vectors $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are in H^s.*

PROOF: This follows from the relationship $\hat{\mathbf{t}} = (\cos(\theta), \sin(\theta))$ and the fact that $\hat{\mathbf{n}}$ and $\hat{\mathbf{t}}$ have the same regularity. □

We next give a lemma that tells us that γ has the same regularity as θ. Recall that our energy functional in Section 3 only included estimates for $s-1$ derivatives of γ. Since we estimated θ in H^s, this lemma gives a gain of one derivative for γ.

LEMMA A.3 *If $\delta \in H^{s+1/2}$, $\theta \in H^s$, and $\gamma \in H^{s-1}$, then $\gamma \in H^s$.*

PROOF: The definition of δ is $\delta = \frac{\pi}{L}\gamma - (T - \mathbf{W}\cdot\hat{\mathbf{t}})$. Solving this for γ and differentiating, we get

$$\gamma_\alpha = \frac{L}{\pi}\left(\delta_\alpha + \frac{L_t}{2\pi} - \mathbf{W}_\alpha \cdot \hat{\mathbf{t}}\right). \tag{A.1}$$

Using the representation $\mathbf{W}_\alpha \cdot \hat{\mathbf{t}} = -\frac{\pi}{L}H(\gamma\theta_\alpha) + \mathbf{m}\cdot\hat{\mathbf{t}}$, we see that everything on the left side of (A.1) can be estimated in H^{s-1}. (See Lemma A.8 for an estimate for $\mathbf{m}$.) □

In Section 3, we needed to provide an estimate for c_α. This required estimating both $\mathbf{W}_t \cdot \hat{\mathbf{t}}$ and $\mathbf{W}_{\alpha t} \cdot \hat{\mathbf{n}}$. To make these estimates, we first need to provide estimates for γ_t. To this end, first notice that (2.16) is actually an integral equation for γ_t because of the presence of $\mathbf{W}_t \cdot \hat{\mathbf{t}}$ on the right-hand side. It was demonstrated in [3] that this integral equation is solvable; discussion of this can be found in [5] and [1].

LEMMA A.4 *If $\theta \in H^s$, $\delta \in H^{s+1/2}$, and $\gamma \in H^{s-1}$, then $\gamma_t \in H^{s-1}$ in the case without surface tension. In the case with surface tension, $\gamma_t \in H^{s-1}$ if we also have $\theta \in H^{s+1}$.*

PROOF: We first write (2.16) as $\gamma_t = \mathcal{J}\gamma_t + \tau$, where $\mathcal{J}$ is the integral operator given by

$$\mathcal{J}[z]f(\alpha) = -\mathrm{PV}\int f(\alpha')\,\mathrm{Re}\left(iz_\alpha(\alpha)\cot\frac{1}{2}(z(\alpha) - z(\alpha'))\right)d\alpha'.$$

The theorem proven in [3] says that the operator $(I - \mathcal{J})^{-1}$ is bounded from H^0 to H^0. We have taken s large enough to guarantee that $\tau \in H^0$, which implies $\gamma_t \in H^0$. Then, as in the proof of lemma 6.2 in [2], we can examine the integral operator to conclude that $\mathcal{J}\gamma_t$ has the same regularity as z_α. Thus, $\mathcal{J}\gamma_t$ is in H^s. Examining τ, we see that it has the same regularity as γ_α in the case without surface tension. Using the previous lemma, we see that γ_α is in H^{s-1}. In the case with surface tension, τ has the regularity of $\theta_{\alpha\alpha}$, which is also in H^{s-1}. Thus, we conclude that γ_t is in H^{s-1}. □

LEMMA A.5 *If $\theta \in H^s$, $\delta \in H^{s+1/2}$, and $\gamma \in H^{s-1}$, then $\mathbf{W}_t \cdot \hat{\mathbf{t}} \in H^{s-1}$ in the case without surface tension. With surface tension, $\mathbf{W}_t \cdot \hat{\mathbf{t}} \in H^{s-1}$ if we also have that $\theta \in H^{s+1}$.*

Remark. This is immediate from the definition of $\mathbf{W}$ and the previous lemma.

Our final goal in this appendix is to provide estimates for the operator $\mathcal{K}$ and for the commutator of the Hilbert transform and multiplication by a smooth function. These are both integral operators, and they were defined in (2.6) and (2.7). The kernels of both of these operators involve divided differences; for $\mathcal{K}$, the kernel is q_2/q_1, where

$$q_1(\alpha, \alpha') = \frac{z(\alpha) - z(\alpha')}{\alpha - \alpha'} = \int_0^1 z_\alpha(t\alpha + (1-t)\alpha')\, dt,$$

$$q_2(\alpha, \alpha') = \frac{z(\alpha) - z(\alpha') - z_\alpha(\alpha)(\alpha - \alpha')}{(\alpha - \alpha')^2}$$
$$= \int_0^1 (t-1) z_{\alpha\alpha}((1-t)\alpha + t\alpha')\, dt.$$

The proof of Lemma A.6 makes use of this representation of the kernel. Similarly, the kernel in (2.7) is a divided difference for f, and the proof of Lemma A.7 makes use of the corresponding representation. We omit these proofs.

LEMMA A.6 *Let s be an integer such that $s \geq 2$. If $z \in H^s$, then $\mathcal{K}[z] : H^1 \to H^{s-1}$, and, in particular, there is a positive function C_1 such that*

$$\|\mathcal{K}[z] f\|_{s-1} \leq C_1(\|z\|_{s-1}) \|f\|_1 \, \|z\|_s.$$

Similarly, $\mathcal{K}[z] : H^0 \to H^{s-2}$ and $\|\mathcal{K}[z] f\|_{s-2} \leq C_2(\|z\|_{s-1}) \|f\|_0 \, \|z\|_s$.

LEMMA A.7 *For $s \geq 3$ and $g \in H^s$, the operator $[H, g]$ is bounded from H^{s-2} to H^s. For $s \geq 4$ and $g \in H^{s-1/2}$, $[H, g]$ is bounded from H^{s-2} to $H^{s-1/2}$. For $i = 0$ or $i = -\frac{1}{2}$, we have the estimates $\|[H, g] f\|_{s+i} \leq C \|f\|_{s-2} \|g\|_{s+i}$.*

By the definition of $\mathbf{m}$ (see (2.9)) and the above lemmas on the regularity of the associated operators, we have the following lemma:

LEMMA A.8 *For $s \geq 4$, if $\theta \in H^s$ and $\gamma \in H^{s-1}$, then $\mathbf{m} \in H^s$. Moreover,*

$$\|m\|_s \leq C(\|\gamma\|_{s-1}, \|\theta\|_{s-1}) \|\theta\|_s.$$

Appendix B: Details of the Formulation

In this appendix, we derive the evolution equation (2.15) for γ from the Euler equations with the appropriate boundary conditions. The main tools used are Bernoulli's equations of motion for potential flow and the Laplace-Young jump condition for the pressure. Our derivation is a generalization of that found in [3]; a version also appears in the appendix of [1]. We give a derivation for an Atwood number that is not necessarily equal to 1. We will take an upper fluid of density ρ_2 and a lower fluid of density ρ_1.

Since the flow is both irrotational and incompressible, there exist both a velocity potential ϕ and a stream function ψ. That is, the fluid velocity (u, v) is equal to both (ϕ_x, ϕ_y) and $(\psi_y, -\psi_x)$. If we let $\Phi = \phi + i\psi$ be the complex potential, then the complex velocity is given by

$$u - iv = \left(\frac{d\Phi}{dz}\right)^* = \left(\frac{\Phi_\alpha}{z_\alpha}\right)^*.$$

For z away from the interface, we have a double-layer potential representation of Φ. We give the name μ to the dipole strength associated with this double-layer potential. This gives the formula

$$\Phi(z) = \frac{1}{4\pi i}\int \mu(\alpha) z_\alpha(\alpha) \cot\left(\frac{1}{2}\left(z - z(\alpha)\right)\right) d\alpha.$$

The vortex sheet strength is $\gamma = \mu_\alpha$.

For irrotational flow the Euler equation reads $\nabla\phi_t + \nabla\phi \cdot \nabla(\nabla\phi) = -\frac{1}{\rho}\nabla p - (0, g)$. Hence, in Eulerian coordinates, Bernoulli's equation is $\phi_t + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gy = 0$. More generally, if we follow a particle of coordinates (x, y), we have

$$\frac{d\phi_i}{dt} - \nabla\phi_i \cdot (x_t, y_t) + \frac{1}{2}|\nabla\phi_i|^2 + \frac{p_i}{\rho_i} + gy = 0. \tag{B.1}$$

The limiting values of the velocity from below and above the interface can be found by the Plemelj formulae (see [11]):

$$\nabla\phi_1 = \mathbf{W} + \frac{\gamma}{2s_\alpha}\hat{\mathbf{t}}, \quad \nabla\phi_2 = \mathbf{W} - \frac{\gamma}{2s_\alpha}\hat{\mathbf{t}}. \tag{B.2}$$

Also, for (x, y) on the interface, we write $(x_t, y_t) = \mathbf{W} + (T - \mathbf{W}\cdot\hat{\mathbf{t}})\hat{\mathbf{t}}$. The Laplace-Young condition for the pressure jump at the interface is

$$p_2 - p_1 = S\kappa = S\frac{\theta_\alpha}{s_\alpha}, \tag{B.3}$$

where S is the coefficient of surface tension and κ is the curvature.

Subtracting (B.1) for $i = 2$ from (B.1) for $i = 1$, we have (since the stream function is continuous across the interface but the potential is discontinuous)

$$\frac{\partial\mu}{\partial t} - \frac{\gamma}{s_\alpha}(T - \mathbf{W}\cdot\hat{\mathbf{t}}) + \frac{p_1}{\rho_1} - \frac{p_2}{\rho_2} = 0. \tag{B.4}$$

If we instead add the same equations, we get

$$2\,\mathrm{Re}\left(\frac{\partial\Phi}{\partial t}\right) - \mathbf{W}\cdot\mathbf{W} - 2(\mathbf{W}\cdot\hat{\mathbf{t}})(T - \mathbf{W}\cdot\hat{\mathbf{t}}) + \frac{\gamma^2}{4s_\alpha^2} + \frac{p_1}{\rho_1} + \frac{p_2}{\rho_2} + 2gy = 0. \tag{B.5}$$

We solve (B.4) and (B.5) for p_1 and p_2, and subtract. We substitute (B.3) for the pressure difference. The result is

$$\begin{aligned}\frac{\partial\mu}{\partial t} &= \frac{2S\kappa}{\rho_1+\rho_2} + \frac{\gamma}{s_\alpha}(T - \mathbf{W}\cdot\hat{\mathbf{t}}) \\ &\quad - \frac{\rho_1-\rho_2}{\rho_1+\rho_2}\left(2\,\mathrm{Re}\left(\frac{\partial\Phi}{\partial t}\right)\right. \\ &\qquad\left. - \mathbf{W}\cdot\mathbf{W} - 2(\mathbf{W}\cdot\hat{\mathbf{t}})(T - \mathbf{W}\cdot\hat{\mathbf{t}}) + \frac{\gamma^2}{4s_\alpha^2} + 2gy\right).\end{aligned} \tag{B.6}$$

Differentiating (B.6) with respect to α, denoting $\mathrm{At} = \frac{\rho_1-\rho_2}{\rho_1+\rho_2}$ and $\mathrm{We} = \frac{\rho_1+\rho_2}{2S}$ and simplifying yields equation (2.15) (for the water wave, $\mathrm{At} = 1$, and in the formulation used in this paper, $s_\alpha = L/2\pi$).

Acknowledgment. The first author was partially supported by National Science Foundation Grant DMS-9983190. The second author was partially supported by NSF Grant DMS-0100946 and by an Alfred P. Sloan Fellowship.

Bibliography

[1] Ambrose, D. M. Well-posedness of vortex sheets with surface tension. Doctoral dissertation, Duke University, 2002.

[2] Ambrose, D. M. Well-posedness of vortex sheets with surface tension. *SIAM J. Math. Anal.* **35** (2003), no. 1, 211–244.

[3] Baker, G.; Meiron, D.; Orszag, S. Generalized vortex methods for free-surface flow problems. *J. Fluid Mech.* **123** (1982), 477–501.

[4] Beale, J. T.; Hou, T. Y.; Lowengrub, J. Growth rates for the linearized motion of fluid interfaces away from equilibrium. *Comm. Pure Appl. Math* **46** (1993), no. 9, 1296–1301.

[5] Beale, J. T.; Hou, T. Y.; Lowengrub, J. Convergence of a boundary integral method for water waves. *SIAM J. Numer. Anal.* **33** (1996), no. 5, 1797–1843.

[6] Ceniceros, H.; Hou, T. Y. Convergence of a non-stiff boundary integral method for interfacial flows with surface tension. *Math. Comp.* **67** (1998), no. 221, 137–182.

[7] Craig, W. An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits. *Comm. Partial Differential Equations* **10** (1985), no. 8, 787–1003.

[8] Hou, T. Y.; Lowengrub, J.; Shelley, M. Removing the stiffness from interfacial flows with surface tension. *J. Comput. Phys.* **114** (1994), no. 2, 312–338.

[9] Hou, T. Y.; Lowengrub, J.; Shelley, M. The long-time motion of vortex sheets with surface tension. *Phys. Fluids* **9** (1997), no. 7, 1933–1954.

[10] Iguchi, T. Well-posedness of the initial value problem for capillary-gravity waves. *Funkcial. Ekvac.* **44** (2001), no. 2, 219–241.

[11] Muskhelishvili, N. I. *Singular integral equations: boundary problems of function theory and their applications to mathematical physics*. 2nd ed. Dover, New York, 1992.

[12] Nalimov, V. I. The Cauchy-Poisson problem. *Dinamika Splošn. Sredy* **18** (1974), 104–210.

[13] Ogawa, M; Tani, A. Free boundary problem for an incompressible ideal fluid with surface tension. *Math. Models Methods Appl. Sci.* **12** (2002), no. 12, 1725–1740.

[14] Saffman, P. G. *Vortex dynamics*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, New York, 1992.

[15] Schneider, G.; Wayne, C. E. The long-wave limit for the water wave problem. I. The case of zero surface tension. *Comm. Pure Appl. Math.* **53** (2000), no. 12, 1475–1535.

[16] Taylor, G. I. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. *Proc. Roy. Soc. London Ser. A* **201** (1950), 192–196.

[17] Wu, S. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. Math.* **130** (1997), no. 1, 39–72.

[18] Wu, S. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.* **12** (1999), no. 2, 445–495.

[19] Yosihara, H. Gravity waves on the free surface of an incompressible perfect fluid of finite depth. *Publ. Res. Inst. Math. Sci.* **18** (1982), no. 1, 49–96.

[20] Yosihara, H. Capillary-gravity waves for an incompressible ideal fluid. *J. Math. Kyoto Univ.* **23** (1983), no. 4, 649–694.

D. AMBROSE
Department of Mathematics
Courant Institute
251 Mercer Street
New York, NY 10012
E-mail: ambrose@cims.nyu.edu

N. MASMOUDI
Department of Mathematics
Courant Institute
251 Mercer Street
New York, NY 10012
E-mail: masmoudi@cims.nyu.edu

Received April 2004.
Revised January 2005.