

THE ZEROES OF NONNEGATIVE CURVATURE OPERATORS

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The Riemannian sectional curvature of a Riemannian manifold is a real-valued function σ on the Grassmann bundle of tangent 2-planes of M . Although there exists a large body of theorems relating the curvature of M to various topological and geometric properties of M , relatively little is known of a general nature about the behavior of the function σ itself. In particular, the critical point behavior of σ has been analyzed only in very special cases [3], [4]. In this paper we consider the pointwise behavior of σ ; that is, we consider the restriction of σ to the Grassmann manifold of tangent 2-planes at a point $m \in M$. We are then able to describe completely the structure of the sets of points in this manifold where σ assumes its minimum and maximum. In particular, for spaces of nonnegative curvature we describe the set of points where σ assumes the value zero.

To be more specific, let G denote the Grassmann manifold of oriented tangent 2-planes at m . G is in a natural way a submanifold of the vector space Λ^2 of 2-vectors at m . Since G is a 2-fold covering space of the manifold of (unoriented) 2-planes at m , we may regard σ as a function on G . We shall show that the minimum and maximum sets of σ are intersections with G of linear subspaces of Λ^2 . Moreover every such intersection can occur, for example as the minimum set of some curvature function σ on G .

The case of nonnegative curvature $\sigma \geq 0$ will occupy most of our attention here. One reason for this is that the general result on the minimum set of σ is an elementary consequence of the result for $\sigma \geq 0$, and another is that this case is the one most likely to yield applications. For example, it follows from our description of the minimum set that if $\sigma \geq 0$ and relative to some coordinate system the "diagonal" curvature components R_{ijij} are all zero at m , then in fact the curvature tensor R is zero at m .

Given a space M of nonnegative curvature and given $m \in M$, the linear subspace of Λ^2 whose intersection with G is the zero set of σ is obtained as follows. The curvature tensor R of M at m can be regarded as a self-adjoint linear operator on Λ^2 . Letting \mathcal{R} denote the vector space of all self-adjoint linear operators ("curvature operators") on Λ^2 , the subset \mathcal{B} consisting of those

which come from Riemannian structures (i.e., those satisfying the first Bianchi identity) is a linear subspace of \mathcal{R} . The orthogonal complement \mathcal{S} of \mathcal{B} in \mathcal{R} is the set of all curvature operators whose associated Riemannian curvature function is identically zero. Our theorem asserts that there exists an operator $S \in \mathcal{S}$ such that the zero set of σ (also called the zero set of R) is precisely $G \cap \text{Ker}(R - S)$.

The idea of the proof is first to show that for each P in the zero set there exists an $S \in \mathcal{S}$ such that $P \in \text{Ker}(R - S)$, second to observe that there is a unique such S orthogonal to the subspace of \mathcal{S} annihilating P , and finally to piece these unique operators together to build one which works simultaneously for all P in the zero set.

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1. \mathcal{S} and the Grassmann quadratic 2-relations

We begin by analyzing the space \mathcal{S} complementary in \mathcal{R} to the subspace $\{R \in \mathcal{R} \mid R \text{ satisfies the Bianchi identity}\}$. We shall exhibit a natural isomorphism between \mathcal{S} and Λ^2 and shall establish the relationship between \mathcal{S} and the Grassmann quadratic 2-relations which are necessary and sufficient conditions for decomposability of elements in Λ^2 .

Let V be an n -dimensional real vector space with inner product $\langle \cdot, \cdot \rangle$ (e.g., $V =$ the tangent space at a point of a Riemannian manifold). For k an integer ≥ 0 , let $\Lambda^k = \Lambda^k(V)$ denote the space of k -vectors of V , equipped with inner product given by

$$\langle u_1 \wedge \cdots \wedge u_k, v_1 \wedge \cdots \wedge v_k \rangle = \det [\langle u_i, v_j \rangle], \quad u_i, v_i \in V.$$

Let G denote the Grassmann manifold of oriented 2-dimensional subspaces of V ; we identify G with the submanifold of Λ^2 consisting of decomposable 2-vectors of length 1 by $P \leftrightarrow u \wedge v$ where $\{u, v\}$ is any oriented orthonormal basis for P . Let \mathcal{R} denote the space of self-adjoint linear operators on Λ^2 , equipped with inner product given by $\langle R, S \rangle = \text{trace } R \circ S$, $R, S \in \mathcal{R}$. Elements of \mathcal{R} will be called curvature operators on V . Given $R \in \mathcal{R}$, its sectional curvature is the function $\sigma_R: G \rightarrow \mathbf{R}$ defined by $\sigma_R(P) = \langle RP, P \rangle$, $P \in G$. Each $R \in \mathcal{R}$ can be naturally identified with a 2-form on V with values in the vector space of skew-symmetric endomorphisms of V by

$$\langle R(u, v)(w), x \rangle = R(u \wedge v, w \wedge x), \quad u, v, w, x \in V.$$

We can then consider the subspace \mathcal{B} of \mathcal{R} consisting of those $R \in \mathcal{R}$ which satisfy the first Bianchi identity: $R \in \mathcal{B}$ if and only if

$$R(u, v)w + R(v, w)u + R(w, u)v = 0$$

for all $u, v, w \in V$. Set $\mathcal{S} = \mathcal{B}^\perp$, the orthogonal complement of \mathcal{B} in \mathcal{R} .

We construct, for each $\xi \in A^4$, an operator $S_\xi \in \mathcal{S}$ as follows. Given ξ , define $S_\xi: A^2 \rightarrow A^2$ by

$$\langle S_\xi(\alpha), \beta \rangle = \langle \alpha \wedge \beta, \xi \rangle, \quad \alpha, \beta \in A^2.$$

Clearly $S_\xi \in \mathcal{R}$. To see that $S_\xi \in \mathcal{S}$ we need the following

Lemma 1.1. *Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for V . For $1 \leq i, j, k, l \leq n$, set $S_{ijkl} = S_{e_i \wedge e_j \wedge e_k \wedge e_l}$. Then, for $R \in \mathcal{R}$,*

$$\begin{aligned} \langle R, S_{ijkl} \rangle &= 2[\langle R(e_i \wedge e_j), e_k \wedge e_l \rangle + \langle R(e_j \wedge e_k), e_i \wedge e_l \rangle \\ &\quad + \langle R(e_k \wedge e_l), e_j \wedge e_i \rangle]. \end{aligned}$$

Proof.

$$\begin{aligned} \langle R, S_{ijkl} \rangle &= \text{tr } R \circ S_{ijkl} = \sum_{\alpha < \beta} \langle R \circ S_{ijkl}(e_\alpha \wedge e_\beta), e_\alpha \wedge e_\beta \rangle \\ &= \sum_{\alpha < \beta} \langle S_{ijkl}(e_\alpha \wedge e_\beta), R(e_\alpha \wedge e_\beta) \rangle \\ &= \sum_{\alpha < \beta} \langle S_{ijkl}(e_\alpha \wedge e_\beta), \sum_{\gamma < \delta} \langle R(e_\alpha \wedge e_\beta), e_\gamma \wedge e_\delta \rangle e_\gamma \wedge e_\delta \rangle \\ &= \sum_{\alpha < \beta} \sum_{\gamma < \delta} \langle R(e_\alpha \wedge e_\beta), e_\gamma \wedge e_\delta \rangle \\ &\quad \times \langle e_\alpha \wedge e_\beta \wedge e_\gamma \wedge e_\delta, e_i \wedge e_j \wedge e_k \wedge e_l \rangle. \end{aligned}$$

Collecting terms completes the proof.

Proposition 1.2. $\xi \mapsto S_\xi$ maps A^4 isomorphically onto \mathcal{S} . Moreover $\xi \mapsto (1/\sqrt{6})S_\xi$ is an isometry.

Proof. Clearly $\xi \mapsto S_\xi$ is a linear map from A^4 into \mathcal{R} . Since $\{e_i \wedge e_j \wedge e_k \wedge e_l \mid i < j < k < l\}$ is an (orthonormal) basis for A^4 , and the images S_{ijkl} of the basis vectors are all in \mathcal{S} ($\langle R, S_{ijkl} \rangle = 0$ for all $R \in \mathcal{B}$ by Lemma 1.1), it follows that $\xi \mapsto S_\xi$ maps A^4 into \mathcal{S} . In fact, Lemma 1.1 implies that, given $R \in \mathcal{R}$, $R \in \mathcal{B}$ if and only if $\langle R, S_{ijkl} \rangle = 0$ for all i, j, k, l ; i.e., the S_{ijkl} span \mathcal{S} and $\xi \mapsto S_\xi$ maps onto \mathcal{S} . Injectivity and the fact that $\xi \mapsto (1/\sqrt{6})S_\xi$ is an isometry follow from taking $R = S_{\alpha\beta\gamma\delta}$ in Lemma 1.1 to conclude that $\{S_{ijkl} \mid i < j < k < l\}$ is an orthogonal set and that $\|S_{ijkl}\|^2 = 6$.

Remark. Using the natural isomorphism between A^4 and its dual, the space of alternating 4-forms on V , given by the inner product we can also identify \mathcal{S} with this space of 4-forms. Explicitly, one identifies a 4-form ω on V with the operator $S_\omega \in \mathcal{S}$ given by

$$\langle S_\omega(v_1 \wedge v_2), v_3 \wedge v_4 \rangle = \omega(v_1, v_2, v_3, v_4).$$

Proposition 1.3. *Let $\alpha \in A^2$. Then α is decomposable if and only if $\langle S\alpha, \alpha \rangle = 0$ for all $S \in \mathcal{S}$.*

Proof. The necessity of the condition is clear since each $S \in \mathcal{S}$ is of the form S_ξ for some $\xi \in A^4$ and $\langle S_\xi \alpha, \alpha \rangle = \langle \alpha \wedge \alpha, \xi \rangle = 0$ for α decomposable.

Conversely, given an orthonormal basis $\{e_1, \dots, e_n\}$ for V , it is well-known [2, p. 309 ff] (see also [1]) that the conditions $\langle S_{ijkl}\alpha, \alpha \rangle = 0$ for all $i < j < k < l$ are necessary and sufficient conditions for decomposability.

Remark. The conditions $\langle S_{ijkl}\alpha, \alpha \rangle = 0$ are known as the Grassmann quadratic 2-relations.

Remark. It is clear from Proposition 1.3 that each curvature tensor $S \in \mathcal{S}$ has sectional curvature σ_S identically zero. Conversely, it is easily checked that this property characterizes \mathcal{S} .

2. The uniqueness theorem

In this section we establish the basic uniqueness result which is at the heart of our building process. But first we need some additional notation.

For a subset Z of G , let

$$\mathcal{A}(Z) = \{S \in \mathcal{S} \mid S(P) = 0 \text{ for all } P \in Z\}.$$

Thus $\mathcal{A}(Z)$ is the subspace of \mathcal{S} consisting of all elements of \mathcal{S} which annihilate Z . For a finite subset $Z = \{P_1, \dots, P_k\}$ of G , we shall denote $\mathcal{A}(\{P_1, \dots, P_k\})$ simply by $\mathcal{A}(P_1, \dots, P_k)$. By $\mathcal{A}(Z)^\perp$ with $Z \subset G$ we shall mean the orthogonal complement of $\mathcal{A}(Z)$ in \mathcal{S} .

Theorem 2.1. *Let $R \in \mathcal{R}$ and $Z \subset G$, and suppose there exists $S \in \mathcal{S}$ such that $Z \subset \text{Ker}(R - S)$. Then there exists a unique $S_0 \in \mathcal{A}(Z)^\perp$ such that $Z \subset \text{Ker}(R - S_0)$. Moreover, given any $S \in \mathcal{S}$, $Z \subset \text{Ker}(R - S)$ if and only if the orthogonal projection of S onto $\mathcal{A}(Z)^\perp$ is S_0 .*

Proof. Existence: Let $S \in \mathcal{S}$ be such that $Z \subset \text{Ker}(R - S)$, and let S_0 denote the orthogonal projection of S onto $\mathcal{A}(Z)^\perp$. Then $S = S_0 + S'$ for some $S' \in \mathcal{A}(Z)$ and

$$Z \subset \text{Ker}(R - S) \cap \text{Ker } S' \subset \text{Ker}(R - S + S') = \text{Ker}(R - S_0).$$

Uniqueness: Suppose $Z \subset \text{Ker}(R - S_0) \cap \text{Ker}(R - S'_0)$ for $S_0, S'_0 \in \mathcal{A}(Z)^\perp$. Then

$$Z \subset \text{Ker}[(R - S_0) - (R - S'_0)] = \text{Ker}(S'_0 - S_0).$$

Thus $S'_0 - S_0 \in \mathcal{A}(Z)$. But S'_0 and $S_0 \in \mathcal{A}(Z)^\perp$, so $S'_0 - S_0$ must be zero.

Finally, it is immediate from the above existence and uniqueness arguments that $Z \subset \text{Ker}(R - S)$ implies S_0 is the orthogonal projection of S onto $\mathcal{A}(Z)^\perp$. Conversely, if $S \in \mathcal{S}$ is such that its orthogonal projection onto $\mathcal{A}(Z)^\perp$ is S_0 , then $S = S_0 + S'$ for some $S' \in \mathcal{A}(Z)$ and

$$Z \subset \text{Ker}(R - S_0) \cap \text{Ker } S' \subset \text{Ker}(R - S_0 - S') = \text{Ker}(R - S).$$

Remark. Note that if $R \in \mathcal{R}$, $S \in \mathcal{S}$ and $P \in G \cap \text{Ker}(R - S)$, then

$$\sigma_R(P) = \langle RP, P \rangle = \langle SP, P \rangle = \sigma_S(P) = 0 .$$

In particular, setting

$$Z(R) = \{P \in G \mid \sigma_R(P) = 0\} ,$$

we see that if, for some $S \in \mathcal{S}$, the subspace $\text{Ker}(R - S)$ has non-null intersection with G then the set $Z(R)$ of zeroes of σ_R is at least big enough to contain this intersection.

Theorem 2.2. *Let $R \in \mathcal{R}$, and suppose there exists $S \in \mathcal{S}$ such that $Z(R) = G \cap \text{Ker}(R - S)$. Then there exists a unique $S_0 \in \mathcal{A}(Z(R))^\perp$ such that $Z(R) = G \cap \text{Ker}(R - S_0)$.*

Proof. By Theorem 2.1, there exists a unique $S_0 \in \mathcal{A}(Z(R))^\perp$ such that $Z(R) \subset G \cap \text{Ker}(R - S_0)$. But, by the remark above, $G \cap \text{Ker}(R - S_0) \subset Z(R)$. Hence we have the equality.

3. Critical zeroes

In studying the critical points of curvature functions, it suffices to consider critical points with critical value zero. For if λ is a critical value of σ_R , $R \in \mathcal{R}$, then the set of critical points of σ_R with critical value λ is the same as the set of critical points of $\sigma_{R-\lambda I}$ with critical value zero, I being the identity operator on A^2 . In this section we show that if P is a critical zero of σ_R , then $P \in \text{Ker}(R - S)$ for some $S \in \mathcal{S}$.

Lemma 3.1. *Let $P \in G$, and let $\{e_1, \dots, e_n\}$ be an orthonormal basis for V . Then*

$$\{P\} \cup \{S_{ijkl}(P) \mid i < j < k < l\}$$

spans the normal space to $G \subset A^2$ at P . If the basis is chosen so that $P = e_1 \wedge e_2$, then

$$\{P\} \cup \{S_{12kl}(P) \mid 2 < k < l\}$$

is an orthonormal basis for this normal space.

Proof. By Proposition 1.3,

$$G = \{\alpha \in A^2 \mid \langle \alpha, \alpha \rangle = 1 \text{ and } \langle S_{ijkl}(\alpha), \alpha \rangle = 0 \text{ for all } i < j < k < l\} .$$

Since the real valued functions $\alpha \mapsto \langle \alpha, \alpha \rangle$ and $\alpha \mapsto \langle S_{ijkl}\alpha, \alpha \rangle$ are constant on G , their gradients $2P$ and $2S_{ijkl}(P)$ at $P \in G$ must be normal to G at P . To see that they span the normal space N_P of G at P , consider first the case where $P = e_1 \wedge e_2$. Then, for $i < j < k < l$,

$$S_{ijkl}(P) = \begin{cases} e_k \wedge e_l , & \text{for } (i, j) = (1, 2) , \\ 0 , & \text{for } (i, j) \neq (1, 2) . \end{cases}$$

It follows that, in this case, $\{P\} \cup \{S_{12kl}(P) \mid 2 < k < l\}$ is an orthonormal set

in N_P . Now the number $[(n - 2)(n - 3)/2] + 1$ of elements in this set is equal to the codimension $[n(n - 1)/2] - 2(n - 2)$ of G in A^2 which in turn is equal to the dimension of N_P . Hence $\{P\} \cup \{S_{12kl}(P) \mid 2 < k < l\}$ is an orthonormal basis for N_P .

Returning to the general case, let $\{e_1, \dots, e_n\}$ be an arbitrary orthonormal basis for V , and let $\{e'_1, \dots, e'_n\}$ be one such that $P = e'_1 \wedge e'_2$. Let $\{S_{ijkl} \mid i < j < k < l\}$ and $\{S'_{ijkl} \mid i < j < k < l\}$ be the corresponding bases for \mathcal{S} . Then, from above, $\{P\} \cup \{S'_{12kl}(P) \mid 2 < k < l\}$ spans N_P . But each S'_{12kl} is a linear combination of the S_{ijkl} and hence each $S'_{12kl}(P)$ is a linear combination of the $S_{ijkl}(P)$. Thus $\{P\} \cup \{S_{ijkl}(P) \mid i < j < k < l\}$ spans N_P .

Theorem 3.2. *Let $R \in \mathcal{R}$ and suppose $P \in G$ is a critical zero of σ_R . Then there exists $S \in \mathcal{S}$ such that $P \in \text{Ker}(R - S)$.*

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis for V such that $P = e_1 \wedge e_2$. Since P is a critical point of σ_R , and σ_R is the restriction to G of the function $\alpha \mapsto \langle R(\alpha), \alpha \rangle$, the gradient $2R(P)$ of this function at P must be normal to G at P . By Lemma 3.1, this implies that

$$RP = \lambda P + \sum_{2 < k < l} \mu_{kl} S_{12kl}(P)$$

for some $\lambda, \mu_{kl} \in \mathbf{R}$. But $\lambda = \langle RP, P \rangle = \sigma_R(P) = 0$, so $P \in \text{Ker}(R - S)$ where $S = \sum_{2 < k < l} \mu_{kl} S_{12kl}$.

Corollary 3.3. *Let $R \in \mathcal{R}$ and suppose $P \in G$ is a critical zero of σ_R . Then there exists a unique $S \in \mathcal{A}(P)^\perp$ such that $P \in \text{Ker}(R - S)$.*

Proof. Immediate from Theorems 3.2 and 2.1.

Remark. The operator S constructed in the proof of Theorem 3.2 is in fact the unique $S \in \mathcal{A}(P)^\perp$ such that $P \in \text{Ker}(R - S)$. Indeed, by Lemma 1.1 together with the fact that each $S' \in \mathcal{S}$ is an S_ω for some alternating 4-form ω on V , we have

$$\langle S', S_{12kl} \rangle = 6 \langle S'(e_1 \wedge e_2), e_k \wedge e_l \rangle,$$

and this is zero for all $S' \in \mathcal{A}(P)$; thus $S_{12kl} \in \mathcal{A}(P)^\perp$ for $2 < k < l$.

Note also that, since $\{S_{12kl} \mid 2 < k < l\}$ is linearly independent, the numbers μ_{kl} above are uniquely determined. In fact, they are curvature components of R relative to the basis $\{e_i\}$:

$$\begin{aligned} \mu_{kl} &= \left\langle \sum_{2 < \alpha < \beta} \mu_{\alpha\beta} e_\alpha \wedge e_\beta, e_k \wedge e_l \right\rangle = \left\langle \sum_{2 < \alpha < \beta} \mu_{\alpha\beta} S_{12\alpha\beta}(e_1 \wedge e_2), e_k \wedge e_l \right\rangle \\ &= \langle R(e_1 \wedge e_2), e_k \wedge e_l \rangle. \end{aligned}$$

4. The case $n = 4$

We consider now the case when V has dimension 4, and establish our main theorem in this case. The validity of the result in dimension 4 will play a crucial role in establishing the theorem in general.

Theorem 4.1. *Let $\dim V = 4$, and suppose $R \in \mathcal{R}$ is such that $\sigma_R \geq 0$ and $Z(R) \neq \emptyset$. Then there exists a unique $S \in \mathcal{S}$ such that $Z(R) = G \cap \text{Ker}(R - S)$.*

Proof. Since $\dim V = 4$, \mathcal{S} is 1-dimensional. Given $\{e_1, \dots, e_4\}$ an orthonormal basis for V , the operator S_{1234} is just the Hodge star operator $*$ and so $\{*\} = \{S_{1234}\}$ is a basis for \mathcal{S} . Given $P \in Z(R)$, P is a minimum, hence a critical point, of σ_R so by Theorem 3.2 there exists $\mu \in \mathbf{R}$ such that $P \in \text{Ker}(R - \mu*)$; i.e., such that

$$RP = \mu * P .$$

If P_1 and P_2 are two zeroes of σ_R , then $RP_i = \mu_i * P_i$ for some $\mu_i \in \mathbf{R} (i = 1, 2)$. We shall show that $\mu_1 = \mu_2$. This is clear if $\{P_1, P_2\}$ is linearly dependent in Λ^2 , so we may assume linear independence. We have

$$\mu_1 \langle *P_1, P_2 \rangle = \langle RP_1, P_2 \rangle = \langle P_1, RP_2 \rangle = \mu_2 \langle P_1, *P_2 \rangle = \mu_2 \langle *P_1, P_2 \rangle .$$

Hence, if $\langle *P_1, P_2 \rangle \neq 0$ we must have $\mu_1 = \mu_2$. On the other hand, if $\langle *P_1, P_2 \rangle = 0$, then $\langle P_1 + P_2, *(P_1 + P_2) \rangle = 0$, so $P_1 + P_2$ is decomposable. Let $Q = (P_1 + P_2)/l$ where $l = \|P_1 + P_2\|$. Then $Q \in G$ and

$$RQ = (\mu_1 * P_1 + \mu_2 * P_2)/l ,$$

so $\sigma_R(Q) = \langle RQ, Q \rangle = 0$. Thus Q is also a zero of σ_R ; hence $RQ = \mu * Q$ for some $\mu \in \mathbf{R}$, and

$$\mu_1 * P_1 + \mu_2 * P_2 = lRQ = l\mu * Q = \mu(*P_1 + *P_2) .$$

This implies that

$$(\mu_1 - \mu)P_1 + (\mu_2 - \mu)P_2 = 0 ,$$

and hence $\mu_1 = \mu_2 = \mu$ since $\{P_1, P_2\}$ is linearly independent in Λ^2 .

It follows that $Z(R) \subset \text{Ker}(R - \mu*)$ for some unique $\mu \in \mathbf{R}$. By the Remark in § 2, $G \cap \text{Ker}(R - \mu*) \subset Z(R)$. Hence, setting $S = \mu*$ we have $Z(R) = G \cap \text{Ker}(R - S)$.

Corollary 4.2. *Let $\dim V = 4$ and $R \in \mathcal{R}$, and let λ denote the minimum (or maximum) value of σ_R . Then there exists a unique $S \in \mathcal{S}$ such that*

$$\{P \in G \mid \sigma_R(P) = \lambda\} = G \cap \text{Ker}(R - \lambda I - S) .$$

Proof. Follows immediately from Theorem 4.1 upon replacing R in that theorem by $R - \lambda I$ (or, in the case where λ is the maximum value of σ_R , by $\lambda I - R$).

Remark. The hypotheses of Corollary 4.2 cannot be weakened to include the case where λ is an arbitrary critical value of σ_R . Indeed, if we define $R \in \mathcal{R}$ by

$$\begin{aligned} R(e_1 \wedge e_2) &= e_3 \wedge e_4, & R(e_3 \wedge e_4) &= e_1 \wedge e_2, \\ R(e_1 \wedge e_3) &= 0, & R(e_2 \wedge e_4) &= 0, \\ R(e_2 \wedge e_3) &= -e_1 \wedge e_4, & R(e_1 \wedge e_4) &= -e_2 \wedge e_3, \end{aligned}$$

then each of the basis planes $e_i \wedge e_j$ is a critical zero of σ_R (critical because $(\text{grad } \sigma_R)(e_i \wedge e_j) = 2R(e_i \wedge e_j) = \pm 2 * e_i \wedge e_j$ which is normal to G at $e_i \wedge e_j$). Hence, if either $\sigma_R^{-1}(0)$ or the critical set of σ_R with critical value zero were the intersection of G with a linear subspace of \mathcal{A}^2 , it would have to be all of G . But this is not the case: setting

$$Q = \frac{1}{2}(e_1 \wedge e_2 + e_3 \wedge e_4 + e_2 \wedge e_3 - e_1 \wedge e_4)$$

we have $Q \in G$ and $\sigma_R(Q) = 1$.

Note that the R of this example satisfies the first Bianchi identity, and also observe that this example illustrates the necessity of the assumption $\sigma_R \geq 0$ (or $\sigma_R \leq 0$) in Theorem 4.1.

Remark. Perhaps a word about the 3-dimensional case is in order at this point, even though it is included in the general case to be considered in the next section. For $n = 3$, every 2-vector is decomposable and hence G is the entire unit sphere in \mathcal{A}^2 . Hence the critical values of σ_R are just the eigenvalues of R , and the set of critical points of σ_R with critical value λ is just the intersection with G of the λ -eigenspace of R . Note that this description (in dimension 3) is valid for each critical value λ , not just for the minimum and maximum values.

5. The main theorem

We now proceed to our main result by way of a sequence of rather technical lemmas.

Lemma 5.1. *Let $R \in \mathcal{R}$ be such that $\sigma_R \geq 0$, and suppose $P, Q \in Z(R)$. Then there exists $S \in \mathcal{S}$ such that $\{P, Q\} \subset \text{Ker}(R - S)$.*

Proof. Choose an orthonormal basis $\{e_1, \dots, e_n\}$ for V such that $P = e_1 \wedge e_2$ and Q is contained in the span of $\{e_1, \dots, e_4\}$, so that $Q = \sum_{i < j \leq 4} q_{ij} e_i \wedge e_j$ for some $q_{ij} \in \mathbf{R}$. Since Q is a critical point (a minimum) of σ_R , $RQ = \frac{1}{2}(\text{grad } \sigma_R)(Q)$ is normal to G at Q so, by Lemma 3.1,

$$(1) \quad RQ = \sum_{i < j < k < l} \nu_{ijkl} S_{ijkl}(Q)$$

for some $\nu_{ijkl} \in \mathbf{R}$ (the component of RQ in the direction of Q is zero since $\langle RQ, Q \rangle = \sigma_R(Q) = 0$). Note that the ν_{ijkl} are not uniquely determined since the $S_{ijkl}(Q)$ are not linearly independent.

Similarly (see the proof of Theorem 3.2),

$$(2) \quad RP = \sum_{2 < k < l} \mu_{12kl} S_{12kl}(P),$$

where now the μ_{12kl} are uniquely determined since the $S_{12kl}(P)$ are orthonormal. Moreover, by the Remark following Corollary 3.3, $S_1 = \sum \mu_{12kl} S_{12kl}$ is the unique operator in $\mathcal{A}(P)^\perp$ such that $P \in \text{Ker}(R - S_1)$. Thus, by Theorem 2.1, it suffices to construct an $S_2 \in \mathcal{S}$ such that $Q \in \text{Ker}(R - S_2)$ and such that the orthogonal projection of S_2 onto $\mathcal{A}(P)^\perp$ is just S_1 . But $\{S_{ijkl} \mid i < j < k < l\}$ is an orthogonal set in \mathcal{S} , $S_{12kl} \in \mathcal{A}(P)^\perp$ for $2 < k < l$, and $S_{ijkl} \in \mathcal{A}(P)$ for $(i, j) \neq (1, 2)$, and so the orthogonal projection into $\mathcal{A}(P)^\perp$ of $\sum_{i < j < k < l} \nu_{ijkl} S_{ijkl}$ is just $\sum_{2 < k < l} \nu_{12kl} S_{12kl}$. Thus we must show that we can choose $\tilde{\nu}_{ijkl} \in \mathbf{R}$ such that

$$RQ = \sum_{i < j < k < l} \tilde{\nu}_{ijkl} S_{ijkl}(Q) \text{ and } \tilde{\nu}_{12kl} = \mu_{12kl} \quad \text{for } 2 < k < l.$$

Step I. Given any $\nu_{ijkl}(i < j < k < l)$ such that (1) is satisfied, we shall show that $\nu_{1234} = \mu_{1234}$. Let $W = e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \Lambda^4$. Identifying W with the oriented 4-dimensional subspace of V spanned by $\{e_1, \dots, e_4\}$ we have $P \subset W$ and $Q \subset W$, i.e., $P, Q \in \Lambda^2(W) \subset \Lambda^2(V)$. Letting $\pi_W: \Lambda^2(V) \rightarrow \Lambda^2(W)$ denote orthogonal projection, we have

$$\begin{aligned} \nu_{1234} &= \langle \nu_{1234} S_{1234}(Q), S_{1234}(Q) \rangle = \langle \pi_W \sum \nu_{ijkl} S_{ijkl}(Q), S_{1234}(Q) \rangle \\ &= \langle \pi_W \circ R(Q), *_{\mathcal{W}} Q \rangle, \end{aligned}$$

where $*_{\mathcal{W}}$ is the star operator of W . Similarly,

$$\mu_{1234} = \langle \pi_W \circ R(P), *_{\mathcal{W}} P \rangle.$$

But the restriction of $\pi_W \circ R$ to $\Lambda^2(W)$ is a curvature operator (with sectional curvature ≥ 0) on the 4-dimensional space W , and $\{P, Q\}$ is contained in the zero set of this curvature operator. Hence, by Theorem 4.1, there exists a unique $\mu \in \mathbf{R}$ such that $P, Q \in \text{Ker}(\pi_W \circ R - S')$ where $S' = \mu *_{\mathcal{W}}$. Thus

$$\nu_{1234} = \langle \pi_W \circ R(Q), *_{\mathcal{W}} Q \rangle = \langle \mu *_{\mathcal{W}} Q, *_{\mathcal{W}} Q \rangle = \mu,$$

and similarly $\mu_{1234} = \mu$, so $\nu_{1234} = \mu_{1234}$.

Step II. We shall take advantage of the non-uniqueness of the remaining ν_{ijkl} in (1) to make essential alterations. In terms of the basis $\{e_i \wedge e_j \mid i < j\}$ for Λ^2 , (1) becomes

$$\begin{aligned} (3) \quad RQ &= \nu_{1234} S_{1234}(Q) + \sum_{5 \leq k} [(\nu_{123k} q_{23} + \nu_{124k} q_{24} + \nu_{134k} q_{34}) e_1 \wedge e_k \\ &\quad + (-\nu_{123k} q_{13} - \nu_{124k} q_{14} + \nu_{234k} q_{34}) e_2 \wedge e_k \\ &\quad + (\nu_{123k} q_{12} - \nu_{134k} q_{14} - \nu_{234k} q_{24}) e_3 \wedge e_k \\ &\quad + (\nu_{124k} q_{12} + \nu_{134k} q_{13} + \nu_{234k} q_{23}) e_4 \wedge e_k] \\ &\quad + \sum_{5 \leq k < l} [\nu_{12kl} q_{12} + \nu_{13kl} q_{13} + \nu_{14kl} q_{14} \\ &\quad + \nu_{23kl} q_{23} + \nu_{24kl} q_{24} + \nu_{34kl} q_{34}] e_k \wedge e_l. \end{aligned}$$

Case I. Assume $q_{34} \neq 0$. Then, given ν_{ijkl} satisfying (1), we can choose, for each $k \geq 5$, $\tilde{\nu}_{134k}$ and $\tilde{\nu}_{234k} \in \mathbf{R}$ so that

$$(4) \quad \mu_{123k}q_{23} + \mu_{124k}q_{24} + \tilde{\nu}_{134k}q_{34} = \nu_{123k}q_{23} + \nu_{124k}q_{24} + \nu_{134k}q_{34},$$

$$(5) \quad -\mu_{123k}q_{13} - \mu_{124k}q_{14} + \tilde{\nu}_{234k}q_{34} = -\nu_{123k}q_{13} - \nu_{124k}q_{14} + \nu_{234k}q_{34}.$$

(Compare (4) and (5) with the coefficients of $e_1 \wedge e_k$ and $e_2 \wedge e_k$ in (3).) Having chosen $\tilde{\nu}_{134k}$ and $\tilde{\nu}_{234k}$ to satisfy (4) and (5), note that

$$\begin{aligned} \mu_{123k}q_{12} - \tilde{\nu}_{134k}q_{14} - \tilde{\nu}_{234k}q_{24} &= \nu_{123k}(q_{13}q_{24} - q_{14}q_{23})/q_{34} \\ &\quad - \nu_{134k}q_{14} - \nu_{234k}q_{24} + \mu_{123k}[q_{12} + (q_{14}q_{23} - q_{13}q_{24})/q_{34}]. \end{aligned}$$

But

$$q_{12}q_{34} + q_{14}q_{23} - q_{13}q_{24} = \frac{1}{2} \langle Q, *WQ \rangle = 0,$$

so the above equation reduces to

$$(6) \quad \mu_{123k}q_{12} - \tilde{\nu}_{134k}q_{14} - \tilde{\nu}_{234k}q_{24} = \nu_{123k}q_{12} - \nu_{134k}q_{14} - \nu_{234k}q_{24}.$$

(Compare (6) with the coefficient of $e_3 \wedge e_k$ in (3).)

Similarly we can check that

$$(7) \quad \mu_{124k}q_{12} + \tilde{\nu}_{134k}q_{13} + \tilde{\nu}_{234k}q_{23} = \nu_{124k}q_{12} + \nu_{134k}q_{13} + \nu_{234k}q_{23}.$$

(Compare (7) with the coefficient of $e_4 \wedge e_k$ in (3).)

Finally, since $q_{34} \neq 0$ we can choose, for each $l > k \geq 5$, $\tilde{\nu}_{34kl}$ such that

$$(8) \quad \mu_{12kl}q_{12} + \tilde{\nu}_{34kl}q_{34} = \nu_{12kl}q_{12} + \nu_{34kl}q_{34}.$$

(Compare (8) with the coefficient of $e_k \wedge e_l$ in (3).)

Then, setting $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$ and $\tilde{\nu}_{ijkl} = \nu_{ijkl}$ for all i, j, k, l for which $\tilde{\nu}_{ijkl}$ has not been previously defined, it follows from (1)–(8), together with step I, that

$$RQ = \sum \nu_{ijkl} S_{ijkl}(Q) = \sum \tilde{\nu}_{ijkl} S_{ijkl}(Q),$$

and $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$. This completes the proof in the case where $q_{34} \neq 0$.

Case II. Suppose $q_{34} = 0$. Then

$$0 = q_{34} = \langle Q, e_3 \wedge e_4 \rangle = \langle Q, *W e_1 \wedge e_2 \rangle = \langle Q, *WP \rangle = \langle P \wedge Q, W \rangle.$$

But $P, Q \in \Lambda^2(W)$ implies $P \wedge Q$ is a multiple of W . Therefore $P \wedge Q = 0$. It follows that the 2-planes P and Q have non-trivial intersection. Hence we can choose our basis $\{e_1, \dots, e_n\}$ for V so that $P = e_1 \wedge e_2$ and

$$Q = e_1 \wedge (q_{12}e_2 + q_{13}e_3) = q_{12}e_1 \wedge e_2 + q_{13}e_1 \wedge e_3$$

for some $q_{12}, q_{13} \in \mathbf{R}$. Since $q_{14} = q_{23} = q_{24} = q_{34} = 0$, (3) becomes

$$\begin{aligned} (3') \quad RQ &= \nu_{1234}S_{1234}(Q) + \sum_{5 \leq k} [\nu_{123k}(-q_{13}e_2 \wedge e_k + q_{12}e_3 \wedge e_k) \\ &\quad + (\nu_{124k}q_{12} + \nu_{134k}q_{13})e_4 \wedge e_k] \\ &\quad + \sum_{5 \leq k < l} (\nu_{12kl}q_{12} + \nu_{13kl}q_{13})e_k \wedge e_l. \end{aligned}$$

Now $\nu_{1234} = \mu_{1234}$ since P and Q both lie in the 4-plane $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ (Step I). Similarly, $\nu_{123k} = \mu_{123k}$ for all $k \geq 4$ since P and Q both lie in the 4-plane $e_1 \wedge e_2 \wedge e_3 \wedge e_k$. Moreover, $q_{13} \neq 0$ since $Q \neq P$, and hence we can choose $\tilde{\nu}_{134k} (k \geq 5)$ and $\tilde{\nu}_{13kl} (l > k \geq 5)$ such that

$$(7') \quad \mu_{124k}q_{12} + \tilde{\nu}_{134k}q_{13} = \nu_{124k}q_{12} + \nu_{134k}q_{13},$$

$$(8') \quad \mu_{12kl}q_{12} + \tilde{\nu}_{13kl}q_{13} = \nu_{12kl}q_{12} + \nu_{13kl}q_{13}.$$

Then, setting $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$ and $\tilde{\nu}_{ijkl} = \nu_{ijkl}$ for all i, j, k, l for which $\tilde{\nu}_{ijkl}$ has not been previously defined, it follows from (1), (3'), (7') and (8') that $RQ = \sum \tilde{\nu}_{ijkl}S_{ijkl}(Q)$ and $\tilde{\nu}_{12kl} = \mu_{12kl}$ for $2 < k < l$, as required.

Lemma 5.2. *Let $Z \subset G$. Then there exists a finite subset $\{P_1, \dots, P_k\}$ of Z such that if $R \in \mathcal{R}$ and $P_i \in \text{Ker}(R)$ for all $i \leq k$, then $Z \subset \text{Ker} R$.*

Proof. Suppose not. Then there exists an infinite sequence $\{P_k\}$ in Z such that, for each $k, P_{k+1} \notin \text{Ker}(R)$ for some $R \in \mathcal{R}$ with $\{P_1, \dots, P_k\} \subset \text{Ker}(R)$. But then

$$\mathcal{R}_k = \{R \in \mathcal{R} \mid \{P_1, \dots, P_k\} \subset \text{Ker}(R)\}$$

is a strictly decreasing infinite sequence of subspaces of \mathcal{R} , contradicting the finite dimensionality of \mathcal{R} .

Lemma 5.3. *Let X be an inner product space, and $X_i (1 \leq i \leq k)$ subspaces of X such that $X = \sum_{i=1}^k X_i$. Let $\pi_i: X \rightarrow X_i$ and $\pi_{ij}: X \rightarrow X_i \cap X_j (1 \leq i, j \leq k)$ denote orthogonal projections, and $x_i \in X_i (1 \leq i \leq k)$ be such that $\pi_{ij}x_i = \pi_{ij}x_j$ for all $i \neq j$. Then there exists a unique $x \in X$ such that $\pi_i x = x_i$ for all i .*

Proof. An easy induction on k .

Theorem 5.4. *Let $R \in \mathcal{R}$ be such that $\sigma_R \geq 0$. Then there exists $S \in \mathcal{S}$ such that $Z(R) = G \cap \text{Ker}(R-S)$.*

Proof. We shall construct the unique (see Theorem 2.2) $S \in \mathcal{S}(Z(R))^\perp$ which will do the job. By Lemma 5.2, there exists a finite subset $\{P_1, \dots, P_k\}$ in $Z(R)$ such that every curvature operator which annihilates $\{P_1, \dots, P_k\}$ annihilates $Z(R)$. In particular,

$$\mathcal{A}(Z(R)) = \mathcal{A}(P_1, \dots, P_k) = \bigcap_{1 \leq i \leq k} \mathcal{A}(P_i),$$

and

$$\mathcal{A}(Z(R))^\perp = \sum_{i=1}^k \mathcal{A}(P_i)^\perp.$$

For $i, j \leq k$, let $\pi_i: \mathcal{S} \rightarrow \mathcal{A}(P_i)^\perp$ and $\pi_{ij}: \mathcal{S} \rightarrow \mathcal{A}(P_i)^\perp \cap \mathcal{A}(P_j)^\perp$ denote orthogonal projections. By Corollary 3.3, for each $i \leq k$ there exists $S_i \in \mathcal{A}(P_i)^\perp$ such that $P_i \in \text{Ker}(R - S_i)$. Moreover, for $i \neq j$, $\pi_{ij}(S_i) = \pi_{ij}(S_j)$. Indeed, by Lemma 5.1, there exists $S_{ij} \in \mathcal{S}$ such that $\{P_i, P_j\} \subset \text{Ker}(R - S_{ij})$ and, by Theorem 2.1, $S_i = \pi_i(S_{ij})$ and $S_j = \pi_j(S_{ij})$ so $\pi_{ij}(S_i) = \pi_{ij}(S_{ij}) = \pi_{ij}(S_j)$. Hence, by Lemma 5.3, there exists $S \in \sum \mathcal{A}(P_i)^\perp = \mathcal{A}(Z(R))^\perp$ such that $\pi_i(S) = S_i$ for all $i \leq k$. By Theorem 2.1 again, this implies that $P_i \in \text{Ker}(R - S)$ for all $i \leq k$, and hence $Z(R) \subset \text{Ker}(R - S)$ by the defining property of the set $\{P_1, \dots, P_k\}$. Finally, $G \cap \text{Ker}(R - S) \subset Z(R)$ by the remark in §2 and so we have the equality.

Corollary 5.5. *Let $R \in \mathcal{R}$ and let λ denote the minimum (or maximum) value of σ_R . Then there exists $S \in \mathcal{S}$ such that*

$$\{P \in G \mid \sigma_R(P) = \lambda\} = G \cap \text{Ker}(R - \lambda I - S).$$

Proof. Immediate from Theorem 5.4 upon replacing R in that theorem by $R - \lambda I$ (or, in the maximum case, by $\lambda I - R$).

Remarks. (i) It is interesting to note that the only use of the assumption that λ be the minimum or maximum of σ_R or, in Theorem 5.4, the assumption that $\sigma_R \geq 0$, occurs in the proof of the 4-dimensional case (Theorem 4.1). Thus, if it were true for 4-dimensional spaces that the set of critical points of σ_R with critical value λ were of the form $G \cap \text{Ker}(R - S)$ for some $S \in \mathcal{S}$, then it would be true in general. Of course, it is not. The counterexample in §4 easily extends to all dimensions ≥ 4 .

(ii) Corollary 5.5 implies that there are linear subspaces L_1 and L_2 of A^2 such that $G \cap L_1$ is the minimum set of σ_R and $G \cap L_2$ is the maximum set of σ_R . These subspaces can have non-trivial intersection. For example, let $\dim V = 4$ and let $R \in \mathcal{R}$ be defined by

$$\begin{aligned} R(e_1 \wedge e_2) &= R(e_3 \wedge e_4) = e_1 \wedge e_2 + e_3 \wedge e_4, \\ R(e_1 \wedge e_3) &= R(e_2 \wedge e_4) = 0, \\ R(e_1 \wedge e_4) &= R(e_2 \wedge e_3) = -e_1 \wedge e_4 - e_2 \wedge e_3. \end{aligned}$$

Then $L_1 = \text{Ker}(R + I + *)$, $L_2 = \text{Ker}(R - I - *)$, and $\dim(L_1 \cap L_2) = 3$.

(iii) Given any linear subspace L of A^2 , there exists $R \in \mathcal{R}$ such that $\sigma_R \geq 0$ and $Z(R) = G \cap L$. Indeed, given L , the curvature operator R which is zero on L and identity on L^\perp will have these properties. Moreover, the curvature

operator obtained by projecting the one just described orthogonally onto $\mathcal{B} = \mathcal{S}^\perp$ will have these properties and will in addition satisfy the first Bianchi identity.

(iv) It is a consequence of Corollary 5.5 that if M is an almost Kaehler manifold with almost complex structure J and $m \in M$, then both the set of holomorphic 2-planes at m (planes invariant under J) and the set of anti-holomorphic 2-planes at m (planes P such that $v \in P$ implies $Jv \perp P$) are intersections with G of linear subspaces of $\Lambda^2(V)$ where $V = M_m$ is the tangent space of M at m . Indeed, the automorphism J of V induces a curvature operator, also denoted by J , on V by $J(u \wedge v) = Ju \wedge Jv$ ($u, v \in V$) and one easily checks that σ_J assumes its maximum value 1 on holomorphic 2-planes and its minimum value 0 on anti-holomorphic 2-planes. A further computation verifies that in fact $P \in G$ is holomorphic if and only if $P \in \text{Ker}(J - I)$, and $P \in G$ is anti-holomorphic if and only if $P \in \text{Ker}(J - S)$ where $S \in \mathcal{S}$ is the operator corresponding under the isomorphisms of §1 to the 4-form $\varphi \wedge \varphi$, φ being the fundamental 2-form given by $\varphi(u, v) = \langle Ju, v \rangle$.

Added in proof. Theorem 5.4 has recently been generalized by A. Stehney to curvature operators on Λ^p for arbitrary p . Using her techniques, it is possible to eliminate the intricate computations in the proof of Lemma 5.1.

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