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# Theorems of Existence and Multiplicity for Nonlinear Elliptic Problems with Noninvertible Linear Part (*).

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*dedicated to Jean Leray*

**Summary.** – *The paper deals with nonlinear elliptic boundary value problems with linear part at resonance. Existence, nonexistence and multiplicity results are given both for bounded and some unbounded nonlinearities, extending a preceding paper* [1] *to the case in which the linear part has general finite dimensional nullspace.*

## 0. – Introduction.

Let $\Omega \subset R^N$ be a bounded domain with boundary $\partial\Omega$, $\mathfrak{L}$ an uniformly elliptic variational operator, $\lambda_k$ any eigenvalue of $\mathfrak{L}u + \lambda u = 0$ with zero Dirichlet data. In a preceding paper [1] we studied the nonlinear Dirichlet problem

$$(0.1)\qquad \begin{aligned} \mathfrak{L}u + \lambda_k u + f(x, u) &= g(x) \qquad \text{on } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

both for bounded and some unbounded nonlinearities, assuming $\lambda_k$ be a simple eigenvalue and proving existence and multiplicity results for (0.1). It is our goal in the present paper to state similar results removing the condition on the semplicity of the eigenvalue. As in [1] the idea of the

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proofs is to use a global Lyapunov-Schmidt method: (0.1) is equivalent to a system of an equation on the Cokernel of $\mathcal{L}+\lambda_k$ (solved by means of a global inversion theorem) and a finite dimensional equation on the Kernel of $\mathcal{L}+\lambda_k$. This latter contains the main contribute of the nonlinearity and can be studied with bit more precision using elementary degree theory or elementary critical point theory. With these techniques we state existence and nonexistence theorems on (0.1) related with several previous papers (see references listed in [1]; see also [8]). But besides to prove such results by means of simple arguments, another purpose of this paper is to show that in some cases (0.1) possesses multiple solutions. As far as we are concerned, this kind of results are not studied before for such equations, with exception of [2], where multiplicity results are given, but for odd nonlinearities, and [1].

The outline of the paper is the following. Section 1 contains the description of the problem and some preliminary lemmas, where the equation is studied in the Cokernel. In § 2 it is stated the main existence result in the case $f$ is bounded. The existence of multiple solutions is investigated in § 3. In § 4 we consider briefly the case of unbounded $f$. In sections 2-4 we assume some bound on $f'$ in order to give the proofs as simple as possible. In § 5 we show as this assumption can be eliminated by a more appropriate use of the finite dimensional topological degree.

For brevity we will assume the reader is somewhat familiar with [1] and we refer often to such paper.

## 1. – Position of the problem and preliminary lemmas.

We first describe briefly our problem. For more details see [1]. Let $\Omega$ be a bounded domain in $R^N$ with boundary $\partial\Omega$ and let $E=W_0^{m,2}(\Omega)$. We will denote by $\|\cdot\|$ the norm in $E$, by $\|\cdot\|_0$ the norm in $L^2(\Omega)$ and by $(\ ,\ )_m$ (resp. $(\ ,\ )$) the scalar product in $E$ (resp. in $L^2$). Let $a_{\alpha\beta}=a_{\beta\alpha}\in L^\infty(\Omega)$ be such that $\exists\gamma>0$ with

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}\xi_\alpha\xi_\beta\geqslant\gamma|\xi|^{2m}\qquad\forall\xi\in R^N$$

and set, for $u, v\in E$

$$((u, v))=\int_\Omega\sum a_{\alpha\beta}D^\alpha u D^\beta v\,.$$

Let us consider the linear operator $L\colon E\to E$ defined by $(Lu, v)_m=-((u, v))$. $L$ is a selfadjoint operator with infinitely many eigenvalues $0<\lambda_1\leqslant\lambda_2\leqslant\ldots$

and a corresponding complete orthonormal system of eigenfunctions $\varphi_i$. Let us denote by $L_k$ the linear operator defined by $(L_k u, v)_m = (Lu, v)_m + \lambda_k(u, v)$; it is known $L_k$ is a Fredholm mapping of index zero. We will suppose $\lambda_k$ is an eigenvalue of multiplicity $p > 1$, namely

$$\lambda_{k-1} = \lambda_k < \lambda_{k+1} = \ldots = \lambda_{k+p-1} < \lambda_{k+p} .$$

We set $V = \operatorname{Ker} L_k$, $V^\perp$ its orthogonal complement in such a way that $E = V \oplus V^\perp$ and $u \in E$ can be put in the form $u = v + w$, where $v \in V$ and $w \in V^\perp$.

Let $f: \Omega \times R \to R$ be a function which throughout in the paper will be assumed to be measurable in $x \in \Omega$ $\forall s \in R$ and $C^1$ in $s$ for a.e. $x \in \Omega$. In sections 1-3 we deal with bounded $f$. Precisely we assume:

HYPOTHESIS I:

i) $\exists M: |f(x, s)| \leqslant M, \ \forall (x, s) \in \Omega \times R$;

ii) $\lambda_{k-1} < \text{const} \leqslant \lambda_k + f'_s(x, s) \leqslant \text{const} < \lambda_{k+p}$ *if* $k > 1$
$\text{const} \leqslant f'_s(x, s) + \lambda_1 \leqslant \text{const} < \lambda_{p+1}$ *if* $k = 1$.

If $f$ satisfies (I), we can define a mapping $F: E \to E$ by $(Fu, \psi)_m = (f(x, u), \psi)$ $\forall \psi \in E$, and $F$ is $C^1$.

It is our purpose to study the following nonlinear Dirichlet problem:

PROBLEM. *Given* $g \in E$, *find* $u \in E$ *such that*

$$L_k u + Fu = g . \tag{1.1}$$

To study (1.1) we use, as in [1], a global Lyapunov-Schmidt method. Let us denote by $P$ the $L^2$-orthogonal projection on $V$ and set $Q = I - P$, where $I$ is the identity on $E$. Applying $P$ and $Q$ to (1.1) we obtain the following system:

$$L_k w + QF(v + w) = Qg \tag{1.2}$$

$$PF(v + w) = Pg . \tag{1.3}$$

It is evident that:

LEMMA 1.1. *Problem* (1.1) *is equivalent to the system* (1.2)-(1.3).

The following lemma studies (1.2). Since the proof is very close to those of lemmas 2.2 and 2.3 of [1], we give here only the outline.

LEMMA 1.2. *Let us assume* (I) *and let* $g \in E$ *be fixed. Then* $\forall v \in V$ *equation* (1.2) *has one solution* $w_{Qg}(v) \in V^{\perp}$. *Such* $w_{Qg}(v)$ *is a* $C^1$ *function of* $v$ *and* $\exists k$ *such that* $\|w_{Qg}(v)\| \leqslant k$.

PROOF. Consider the mapping $w \mapsto L_k w + QF(v+w)$. By I-ii) and using the variational characterization of the eigenvalues $\lambda_i$, it is possible to show such application is locally invertible in every point $w \in V^{\perp}$. Moreover, since $f$ is bounded, the mapping above is proper, and therefore it is a global homeomorphism on $V^{\perp}$. The last statements are easy consequence of the fact that $f$ is $C^1$ in $s$ and bounded. Q.E.D.

## 2. – Existence theorems.

In this section we state our main existence results. Let us introduce the following symbols:

$$\underline{f}(\pm\infty) = \liminf_{s\to\pm\infty} f(x,s)\,, \qquad \bar{f}(\pm\infty) = \limsup_{s\to\pm\infty} f(x,s)$$

$$\Omega^+(z) = \{x \in \Omega : z(x) > 0\}\,, \qquad \Omega^-(z) = \{x \in \Omega : z(x) < 0\}\,.$$

THEOREM 2.1. *Assume* (I) *and let* $g \in E$ *be given. Then there exists a bounded set* $A_{Qg} \subset V$ *such that* (1.1) *has solution provided* $Pg \in A_{Qg}$. *Moreover* $A_{Qg}$ *contains the set of* $Pg$ *such that either*

$$\int_{\Omega} gz < \int_{\Omega^+(z)} \underline{f}(+\infty)z + \int_{\Omega^-(z)} \bar{f}(-\infty)z\,, \qquad \forall z \in V,\ \|z\|_0 = 1 \tag{2.1}$$

*or*

$$\int_{\Omega} gz > \int_{\Omega^+(z)} \bar{f}(+\infty)z + \int_{\Omega^-(z)} \underline{f}(-\infty)z \qquad \forall z \in V,\ \|z\|_0 = 1 \tag{2.1'}$$

From theorem 2.1 it follows easily the following Corollary, which is related with preceding results (see ref. in [1]).

COROLLARY 2.2. *If, besides the hypothesis above,* $f$ *has limits* $f(\pm\infty) = \lim_{s\to\pm\infty} f(x,s)$, *with* $f(-\infty) < f(+\infty)$, *then*

1) *a sufficient condition for* (1.1) *be solvable is that* (2.1) *holds*;

2) *if* $f(-\infty) < f(x,s) < f(+\infty)$ $\forall (x,s) \in \Omega \times R$, *then* (2.1) *is also a necessary condition for* (1.1) *to be solvable.*

PROOF OF THE THEOREM. – By lemmas 1.1 and 1.2 we have to solve only equation $PF(v + w(v)) = Pg$, where we indicated for brevity $w(v)$ instead of $w_{Qg}(v)$. We set

$$\Gamma(v) = \Gamma_{Qg}(v) = PF(v + w(v)) .$$

The mapping $\Gamma\colon V \to V$ is continuous, and since $f$ is bounded $\Gamma(v)$ is bounded. Set $A_{Qg} = \Gamma(V)$, the first part of the theorem is proven. Next we need a lemma.

LEMMA 2.3. *Let us assume* (I). *If $g \in E$ satisfies* (2.1) (*resp.* (2.1′)) *then* $\exists r$ *such that* $(v, \Gamma(v)) > (v, Pg)$ (*resp.* $(v, \Gamma(v)) < (v, Pg)$) *for* $\|v\|_0 = r$.

PROOF. We suppose (2.1) holds, and argue by contradiction. Then $\exists v_n \in V$ with $t_n = \|v_n\|_0 \to \infty$ and $z_n = t_n^{-n} v_n$ with $z_n \to z$ in $V$, $\|z_n\|_0 = 1$ such that

$$\liminf \Big[\int_\Omega f(t_n z_n + w_n) z_n - \int_\Omega g z_n\Big] \leqslant 0 \qquad (w_n = w(v_n)) . \tag{2.2}$$

Consider

$$\liminf \Big[\int_{\Omega^+(z)} f(t_n z_n + w_n) z_n + \int_{\Omega^-(z)} f(t_n z_n + w_n) z_n\Big] \geqslant \tag{2.3}$$
$$\geqslant \liminf \int_{\Omega^+(z)} f(t_n z_n + w_n) z_n + \liminf \int_{\Omega^-(z)} f(t_n z_n + w_n) z_n .$$

Since $\|w_n\| \leqslant k$ in virtue of lemma 1.2, passing possibly to a subsequence, we can assume $w_n \to w$ a.e. in $\Omega$; then $\forall x \in \Omega^+(z)$ $t_n z_n + w_n \to +\infty$, while $\forall x \in \Omega^-(z)$ $t_n z_n + w_n \to -\infty$. Therefore by I-i) and the Fatou's lemma, (2.3) implies

$$\liminf \int_{\Omega^+(z) \cup \Omega^-(z)} f(t_n z_n + w_n) z_n \geqslant \int_{\Omega^+(z)} \underline{f}(+\infty) z + \int_{\Omega^-(z)} \bar{f}(-\infty) z . \tag{2.4}$$

On the other hand, since $f$ is bounded and $z_n \to z$ in $V$, it results $\int_{\Omega^0} f(t_n z_n + w_n) z_n \to 0$, where $\Omega^0 = \{x \in \Omega\colon z(x) = 0\}$ and therefore from (2.2) and (2.4) it follows:

$$\int_\Omega g z \geqslant \int_{\Omega^+(z)} \underline{f}(+\infty) z + \int_{\Omega^-(z)} \bar{f}(-\infty) z , \qquad \|z\|_0 = 1$$

which contradicts (2.1). If (2.1′) holds, we use, with obvious modifications, lim sup instead of lim inf. Q.E.D.

PROOF OF THE THEOREM COMPLETED. Let $g$ satisfy (2.1). Then by lemma above $\exists r$ such that $(v, \Gamma(v) - Pg) > 0$ on $\|v\|_0 = r$. Therefore $\deg(\Gamma - Pg, B_r, 0) \neq 0$ and the equation $\Gamma(v) = Pg$ has a solution in $B_r = \{v \in V: \|v\|_0 \leqslant r\}$, as required. The same arguments hold if $g$ satisfies (2.1'). Q.E.D.

## 3. – Multiplicity results.

If $f(+\infty) = f(-\infty) = 0$ theorem 2.1 or corollary 2.2 gives no informations. The following theorem studies the solvability of (1.1) in such a case and shows that multiple solutions occurr. For other related results we refer to [2], [3], [4] and [1-Th. 5.2]. In theorem below we will assume the following unique continuation property holds:

(UCP) *for every* $z \in V$, $z \neq 0$ *the set* $\{x \in \Omega: z(x) = 0\}$ *has zero Lebesgue measure.*

THEOREM 3.1. *Let us assume* (I) *and* (UCP). *Moreover we suppose* $f(+\infty) = f(-\infty) = 0$ *and*

$$\lim_{s \to \pm\infty} s f(x, s) = \mu > 0 . \tag{3.1}$$

*Let* $g \in E$ *be given. Then*:

1) *there exists* $\varepsilon_{Qg} > 0$ *such that* (1.1) *has at least one solution provided* $\|Pg\|_0 < \varepsilon_{Qg}$;
2) *if* $0 < \|Pg\|_0 < \varepsilon_{Qg}$ *then* (1.1) *has at least two distinct solutions.*

PROOF. We follow the same procedure as in sections 1-2. The problem (1.1) is equivalent to (1.2)-(1.3). Solved (1.2) we consider the equation $\Gamma(v) = Pg$. Using (3.1) we will actually show $\exists r$: $(v, \Gamma(v)) \geqslant \text{const.} > 0$ for $\|v\|_0 = r$. In fact, if not, $\exists t_n = \|v_n\|_0 \to \infty$ such that

$$(v_n, \Gamma(v_n)) = \int_\Omega f(v_n + w_n) v_n = \int_\Omega f(v_n + w_n)(v_n + w_n) - \int_\Omega f(v_n + w_n) w_n \leqslant 0$$

Setting, as in lemma 2.3, $z_n = t_n^{-1} v_n$, we can suppose $z_n \to z$; moreover $w_n \to w$ a.e. in $\Omega$. Using (UCP) it follows

$$\int_\Omega f(t_n z_n + w_n)(t_n z_n + w_n) = \int_{\Omega^+(z) \cup \Omega^-(z)} f(t_n z_n + w_n)(t_n z_n + w_n) .$$

Since $t_n z_n + w_n \to +\infty$ (resp. $-\infty$) provided $x \in \Omega^+(z)$ (resp. $x \in \Omega^-(z)$), by I-i) and (3.1) it follows the integral above tends to $\mu|\Omega|$.

Moreover $f$ bounded, $f(\mp\infty) = 0$ and $w_n \to w$ in $L^2$ imply $\int_\Omega f(v_n + w_n) w_n$ tends to zero. Then $\limsup (v_n, \Gamma(v_n)) \geq \mu|\Omega|$, a contradiction. Therefore $\exists \varepsilon_{Qg} > 0$ such that for $\|Pg\|_0 < \varepsilon_{Qg}$ it results $\deg(\Gamma, B_r, Pg) = 1$ and 1) follows.

Now we prove 2). Let $\delta$ be such that $0 < \delta < \|Pg\|_0 < \varepsilon_{Qg}$ and denote by $B_\varrho$ the ball in $V$ of radius $\varrho$. Since $f(+\infty) = f(-\infty) = 0$ it follows easily $\exists r' > r$ such that $\Gamma(\partial B_{r'}) \subset B_\delta$. Let us consider $\bar{v} \in V$ with $\bar{v} \notin \Gamma(V)$, which is possible because $\Gamma(V)$ is bounded. Then $\deg(\Gamma, B_{r'}, \bar{v}) = 0$. But, since the topological degree is constant on every connected component of $V - \Gamma(\partial B_{r'})$ (see for ex. [5, pag. 72]), we obtain

$$\deg(\Gamma, B_{r'}, Pg) = \deg(\Gamma, B_{r'}, \bar{v}) = 0 \tag{3.2}$$

provided $\delta < \|Pg\|_0$. Therefore for $\delta < \|Pg\|_0 < \varepsilon_{Qg}$ we obtain

$$\deg(\Gamma, B_{r'} - B_r, Pg) = -1$$

and the equation $\Gamma(v) = Pg$ has at least another solution in $B_{r'} - B_r$. This completes the proof. Q.E.D.

REMARK 3.2. It is possible to show the set $A_{Qg} \subset V$ where the equation $\Gamma(v) = Pg$ has solutions, is closed. In fact 0 is an interior point because 2) of theorem above. Let $a_n \in A_{Qg}$ be such that $a_n \to Pg \neq 0$, and denote by $v_n$ a solution of $\Gamma(v) = a_n$. If $\|v_n\|_0 \to \infty$ we should have $a_n \to 0$; then $v_n$ is bounded and we can extract a subsequence $v_{n_j}$ converging to some $\bar{v}$. Evidently $\Gamma(\bar{v}) = Pg$.

Now we come back to the equation $L_k u + Fu = g$. In what follows we take $g = 0$ and use the variational structure of (1.1). In fact let $w_0(v)$ be the solution of (1.2) with $g = 0$. If we set

$$H(v) = \tfrac{1}{2}\left(L_k w_0(v), w_0(v)\right)_m + \int_\Omega dx \int_0^{v + w_0(v)} f(x, s)\, ds$$

it is easy to show that $\operatorname{grad} H(v) = \Gamma_0(v)$ and thus the solutions of the equation $\Gamma_0(v) = 0$ are the critical points of $H$ on the $p$-dimensional space $V$. We remark that $H$ is even whenever $f$ is odd as function of $s$: in fact in such a case the solution $w_0(v)$ of (1.2) is an odd function of $v$.

In the following proposition we take $f(0) = 0$ and look for nontrivial solutions of (1.1); elementary critical point theory will enable to find them, improving the results of Proposition 5.1 of [1].

THEOREM 3.3. *Let us assume* (I), *and take $f$ independent of $x$, with* $f(0) = 0$. *Moreover we suppose*:

$$f'(0) < 0 \tag{3.3}$$

$$\int_{\Omega^+(z)} \underline{f}(+\infty)z + \int_{\Omega^-(z)} \bar{f}(-\infty)z > 0 \qquad \forall z \in V,\ \|z\|_0 = 1\,. \tag{3.4}$$

*Then the equation $L_k u + Fu = 0$ has at least two nontrivial solutions. If $f$ is odd, then such equation has at least $p$ pairs of nontrivial solutions.*

PROOF. By (3.4) the functional $H$ satisfies, as in Lemma 2.3, $(v, \operatorname{grad} H(v)) = (v, \Gamma_0(v)) > 0$ for $\|v\|_0$ sufficiently large and thus $H$ possesses a minimum $v_m$ interior to some $B_r$. Since $g = 0$, we obtain $w_0(0) = 0$ and $H(0) = \Gamma_0(0) = 0$. Moreover since it follows easily $H''(0) = f'(0)I$, $I$ identity in $V$, by (3.3) $v = 0$ is a maximum for $H$ and $v_m \neq 0$. Since the index of $v_m$ is 1 while that of 0 is $\pm 1$, and the index of $H$ on $B_r$ is 1, $H$ must have at least another critical point in $B_r$, which leads to the second nontrivial solutions.

Now we take $f$ odd; then $H$ is even and the same arguments above show $(v, \Gamma_0(v)) < 0$ if $\|v\|_0 = \delta$, $\delta$ sufficiently small. Then an application of the Lusternik-Schnierelman theory of critical points in finite dimensional space leads to obtain $p$ pairs of critical points of $H$ in $B_r - B_\delta$. This completes the proof. Q.E.D.

REMARK 3.4. The same results are true if $f'(0) > 0$ and

$$\int_{\Omega^+(z)} \bar{f}(+\infty)z + \int_{\Omega^-(z)} \underline{f}(-\infty)z < 0\,.$$

REMARK 3.5. In the case of odd $f$ our results are related with those of [2]. Moreover using variational arguments as above, it is possible to study equation (1.3) assuming on $f$ conditions like those of papers [6], [2]. For ex., for general $f(x, s)$, if we assume

$$\int_0^\xi f(x, s)\,ds \to +\infty \qquad \text{as } |\xi| \to \infty$$

then the functional $H(v)$ takes arbitrary large values for large $\|v\|_0$, and $H$ possesses a minimum which leads to a solution of (1.3). But we do not carry into details such arguments.

## 4. – Existence and multiplicity results for unbounded nonlinearities.

In this section we deal with problem (1.1) with $\lambda_k = \lambda_1$ and $f$ unbounded. We will take $\lambda_1 = \ldots = \lambda_p < \lambda_{p+1}$ and assume

HYPOTHESIS II:

i) $|f(x, s)| \leqslant c_1 + c_2|s|^\varrho$, $|f'(x, s)| \leqslant c_3 + c_4|s|^{\varrho-1}$
*with* $1 \leqslant \varrho < (N + 2m)/(N - 2m)$ *if* $N > 2$, *and any* $\varrho$ *if* $N \leqslant 2$.

ii) $f'_s(x, s) + \lambda_1 \leqslant \lambda_{p+1} - \varepsilon$, for some $\varepsilon > 0$.

As in section 1, it is possible to define, using (II), a mapping $F: E \to E$ putting $(Fu, \psi)_m = (f(x, u), \psi)$ $\forall \psi \in E$ and $F$ is $C^1$.

Even if in [1] we consider several cases concerning the asymptotic behaviour of $f$ at $\infty$, we will limit ourselves to consider the case $f(-\infty) = +\infty$, $f(+\infty) = -\infty$.

THEOREM 4.1. *Let us assume* (II), (UCP) *and* $f(-\infty) = +\infty$, $f(+\infty) = -\infty$. *Then for every* $g \in E$ *the problem*

$$L_1 u + Fu = g \qquad u \in E \tag{4.1}$$

*has at least one solution.*

PROOF. Since the proof is essentially the same as in previous theorems, we will give only the outline. The system (1.2)-(1.3) with $k = 1$ is equivalent to (4.1). Equation (1.2) can be uniquely solved by monotonicity arguments as in [1, lemma 7.1]. Equation (1.3) leads to a finite dimensional equation $\Gamma(v) = Pg$. Also in this case it is possible to show that $\forall g \in E$ $\exists r$ such that $(v, \Gamma(v) - Pg) < 0$ for $\|v\|_0 = r$. In the present case we will use the arguments of lemmas 8.3 and 8.5-i) of [1]. In conclusion $\forall g$ $\Gamma(v) = Pg$ has at least one solution, as required. Q.E.D.

The following is a multiplicity result about (4.1). For the proof we use the same arguments of 3.3 and 3.4

THEOREM 4.2. *Let us assume* (II), (UCP) *and* $f(-\infty) = +\infty$, $f(+\infty) = -\infty$. *Moreover take* $g = 0$, $f$ *independent of* $x$, *with* $f(0) = 0$. *If* $f'(0) > 0$ *then* (4.1) *has at least two nontrivial solutions. If* $f$ *is odd then* (4.1) *has at least* $p$ *pairs of nontrivial solutions.*

## 5. – Further remarks on existence.

In sections above we have assumed some conditions on $f'$, namely hypothesis I-ii) and II-ii). We will show here how it is possible to eliminate such assumptions.

THEOREM 5.1. *Let us assume* (UCP), I-i) *and that* $\exists c$ *such that* $|f_s'(x, s)| \leqslant c$ $\forall (x, s) \in \Omega \times R$. *Then the same conclusions of theorem* 2.1 *are true.*

PROOF. Let $m$ be such that $f_s'(x, s) + \lambda_k \leqslant \text{const.} < \lambda_{m+1}$, and set $\tilde{V} = \text{span}\{\varphi_1, \ldots, \varphi_m\}$. Denoted by $\tilde{P}$ the projection on $\tilde{V}$ and by $\tilde{Q} = I - \tilde{P}$ the projection on $\tilde{V}^\perp$, we pose $u = \tilde{v} + \tilde{w}$, with $\tilde{v} \in \tilde{V}$ and $\tilde{w} \in \tilde{V}$. We project equation (1.1) on $\tilde{V}$ and $\tilde{V}^\perp$ and obtain the following equivalent system:

$$L_k \tilde{w} + \tilde{Q}F(\tilde{v} + \tilde{w}) = \tilde{Q}g \tag{5.1}$$

$$L_k \tilde{v} + \tilde{P}F(\tilde{v} + \tilde{w}) = \tilde{P}g\,. \tag{5.2}$$

For $\tilde{v} \in \tilde{V}$ we can solve (5.1) exactly in the same way as in Lemma 1.2 and we denote by $\tilde{w}(\tilde{v}) = \tilde{w}_{\tilde{Q}g}(\tilde{v})$ such a solution. We put $\tilde{w}(\tilde{v})$ in (5.2) and obtain the finite dimensional equation $\tilde{\Gamma}(\tilde{v}) = \tilde{P}g$, where $\tilde{\Gamma}(\tilde{v}) = L_k\tilde{v} + \tilde{P}F(\tilde{v} + \tilde{w}(\tilde{v}))$. To solve this equation we need a modification of lemma 2.3.

LEMMA 5.2. *Let us denote by* $\pi$ *the projection of* $\tilde{V}$ *on* $V = \operatorname{Ker} L_k$. *Then* $\exists r$ *such that if* $g$ *satisfies* (2.1) *then*

$$\deg(\tilde{\Gamma} - \tilde{P}g, B_r, 0) = \deg(\tilde{L} + \pi, B_r, 0) \tag{5.3}$$

*where* $\tilde{L} = L_k/\tilde{V}$, and $B_r = \{\tilde{v} \in \tilde{V} : \|\tilde{v}\|_0 \leqslant r\}$.

PROOF OF THE LEMMA. Let us consider the homotopy $T: \tilde{V} \times [0, 1] \to \tilde{V}$ defined by

$$T(\tilde{v}, t) = t(\tilde{\Gamma}(\tilde{v}) - \tilde{P}g) + (1 - t)(\tilde{L}\tilde{v} + \pi\tilde{v})\,.$$

We claim that $\exists r$ such that $T(\tilde{v}, t) \neq 0$ for $\|\tilde{v}\|_0 = r$ and $0 \leqslant t \leqslant 1$. In fact, if not, there are sequences $\|\tilde{v}_n\|_0 \to \infty$, $t_n \to \bar{t} \in [0, 1]$ with $T(\tilde{v}_n, t_n) = 0$, namely (denoted $\tilde{w}_n = \tilde{w}(\tilde{v}_n)$):

$$L_k \tilde{v}_n + t_n \tilde{P}F(\tilde{v}_n + \tilde{w}_n) - t_n \tilde{P}g + (1 - t_n)\pi\tilde{v}_n = 0\,. \tag{5.4}$$

From (5.4) it follows

$$L_k z_n + (1 - t_n)\pi z_n = t_n \frac{\tilde{P}g}{\|\tilde{v}_n\|_0} - t_n \frac{\tilde{P}F(\tilde{v}_n + \tilde{w}_n)}{\|\tilde{v}_n\|_0} \tag{5.5}$$

where $z_n = \|\tilde{v}_n\|_n^{-1}\tilde{v}_n$ can be assumed to converge to some $z \neq 0$.

But the second term in (5.5) tends to 0 because $F$ is bounded and then both $L_k z_n$ and $(1 - t_n)\pi z_n$ tend to zero because they are orthogonal. Therefore it follows $L_k z = 0$, namely $z \in V$. As consequence $\pi z_n \to \pi z = z \neq 0$ and thus $\bar{t} = 1$. Now we multiply (5.4) by $\|\tilde{v}_n\|_0^{-1}\pi\tilde{v}_n$ and obtain

$$t_n \int_\Omega f(\tilde{v}_n + \tilde{w}_n)\|\tilde{v}_n\|_0^{-1}\pi\tilde{v}_n - t_n \int_\Omega g\|\tilde{v}_n\|_0^{-1}\pi\tilde{v}_n \leqslant 0\,. \tag{5.6}$$

Passing to the limit in (5.6), since $\|\tilde{v}_n\|_0^{-1}\pi\tilde{v}_n \to z$ and $t_n \to 1$, we obtain in the same way as in Lemma 2.3

$$\int_{\Omega^+(z)} \underline{f}(+\infty)z + \int_{\Omega^-(z)} \bar{f}(-\infty)z - \int_\Omega gz \leqslant 0$$

which contradicts (2.1). Then the claim is true and $T$ is an admissible homotopy on $B_r$ with $T(\cdot, 0) = \tilde{L} + \pi$, $T(\cdot, 1) = \tilde{\Gamma} - \tilde{P}g$. Therefore (5.3) follows. Q.E.D.

PROOF OF THE THEOREM 5.1 COMPLETED. Since $\deg(\tilde{L} + \pi, B_r, 0) \neq 0$, lemma 5.2 implies $\deg(\tilde{\Gamma} - \tilde{P}g, B_r, 0) \neq 0$ and then the equation $\tilde{\Gamma}(\tilde{v}) = \tilde{P}g$ has a solution, as required. Q.E.D.

REMARK 5.3. The same arguments apply to the case of unbounded nonlinearities, with the same modifications needed in theorem 4.1.

With respect to the multiplicity results the same improvements hold. We will limit ourselves to consider theorems 3.3-4.2, showing that actually the number of nontrivial solutions depend on the intertwinning of $f'$ with the spectrum of $L$. The following improves theorem 3.3.

THEOREM 5.4. *Let us assume* I-i), *$f$ independent of $x$, $f$ odd and $|f'(s)| \leqslant c$. Moreover we suppose* (3.4) *holds and*

$$f'(0) + \lambda_k < \lambda_{k-l} \qquad l \geqslant 0 \tag{5.7}$$

*Then the equation*

$$L_k u + Fu = 0 \tag{5.8}$$

*has at least $l + p$ pairs of nontrivial solutions.*

PROOF. Let us denote, with the same symbols of theorem 5.1,

$$h(\tilde{v}) = \tfrac{1}{2}(L_k\tilde{v}, \tilde{v}) + \tfrac{1}{2}(L_k\tilde{w}_0(\tilde{v}), \tilde{w}_0(\tilde{v})) + \int dx \int_0^{\tilde{v}+\tilde{w}_0(\tilde{v})} f(s)\, ds\,.$$

To study $h$ we need a lemma, which is essentially the same of th. 1.9 of [2] but in a finite dimensional setting.

LEMMA 5.5. *Let $h\colon R^m \to R$ be a $C^2$ even functional with $h(0) = h'(0) = 0$. Let $X_0$, $X_1$ be subspaces of $R^m$ with $\dim X_0 = m - j$, $\dim X_1 = l$, $l > j$. If $h$ satisfies $(P - S)$ (*) and*

i) $\exists r > 0$: $\operatorname{grad} h(a)\cdot a > 0$ *for* $a \in X_0$, $|a| = r$ *($\cdot$ denotes the scalar product and $|\cdot|$ the euclidean norm)*;

ii) $h''(0)a\cdot a < 0$ $\forall a \in X_1 - \{0\}$;

*then $h$ has at least $l - j$ pairs of critical points.*

PROOF OF THE THEOREM COMPLETED. Set $X_0 = \operatorname{span}\{\varphi_1, \ldots, \varphi_{k+p-1}\}$, $X_1 = \operatorname{span}\{\varphi_{k-l}, \ldots, \varphi_m\}$, we prove Lemma 5.5 is applicable. Let $\tilde{v}_n$ be such that $h'(\tilde{v}_n) = L_k\tilde{v}_n + \tilde{P}F(\tilde{v}_n + \tilde{w}_n) \to 0$; if $\|\tilde{v}_n\|_0 \to \infty$ then, since $f$ is bounded, $L_k z_n \to 0$ with $z_n = \|\tilde{v}_n\|_0^{-1}\tilde{v}_n \to z$. Therefore $z \in \operatorname{Ker} L_k$ and so

$$\int f(\|\tilde{v}_n\|_0 z_n + \tilde{w}_n)z = (h'(\tilde{v}_n), z) \to 0$$

which contradicts (3.4). Then $\tilde{v}_n$ is bounded and $h$ satisfies $(P - S)$. Now we prove 5.5-i) holds. If not $\exists v_n \in X_0$, $t_n = \|v_n\|_0 \to \infty$, $z_n = t_n^{-1}v_n \to z \in X_0$, such that $\liminf \left[t_n(L_k z_n, z_n) + \int f(t_n z_n + w_n)z_n\right] \leqslant 0$. If $z \in \operatorname{Ker} L_k$ we should have, by (3.4) $\liminf \int f(t_n z_n + w_n)z_n > 0$, which jointly with $(L_k z_n, z_n) \geqslant 0$ leads to a contradiction. If $z \notin \operatorname{Ker} L_k$ we have

$$(L_k z_n, z_n) \to \sum_{i=1}^{k-1} (\lambda_k - \lambda_i)(z, \varphi_i)^2 > 0\,.$$

Since $f$ is bounded and $t_n \to \infty$ we obtain again a contradiction and 5.5-i) follows. Moreover $h''(0) = L_k + f'(0)\tilde{I}$, $\tilde{I}$ identity on $\tilde{V}$, and then $\forall v \in X_1$ (5.7) implies

$$(h''(0)v, v) = \sum_{k-l}^{m} (v, \varphi_i)^2(\lambda_k - \lambda_i + f'(0)) \leqslant (\lambda_k - \lambda_{k-l} + f'(0))\|v\|_0^2 < 0$$

which proves 5.5-ii). Thus lemma 5.5 leads to the required result. Q.E.D.

(*) It is well known $(P - S)$ means: every $a_n$ with $h(a_n)$ bounded and $h'(a_n) \to 0$ has a converging subsequence.

REMARK 5.6. Theorem 5.4 is related to theorem 2.22 of [2] where $l = 0$ is taken.

If $f$ is unbounded we can obtain the following improvement of theorem 4.2. Below we will assume, to avoid technical difficulties, (UCP) holds for every $\tilde{v} \in \tilde{V}$.

THEOREM 5.7. *Let us assume* II-i), (UCP) $\forall \tilde{v} \in \tilde{V}$, $f$ *independent of* $x$, *odd and with* $f'(s) < c$. *Moreover we suppose* $f(-\infty) = \infty$, $f(+\infty) = -\infty$, *and*

$$f'(0) + \lambda_1 > \lambda_{l+p} \qquad l \geq 0 \tag{5.9}$$

*Then the equation* $L_1 u + Fu = 0$ *has at least* $l + p$ *pairs of distinct nontrivial solutions.*

PROOF. We will indicate only the differences with respect theorem 4.2. Let $m$ be such that $f'(s) + \lambda_1 < \lambda_{m+1}$ and $\tilde{V} = \operatorname{span}\{\varphi_1, \ldots, \varphi_m\}$ as in theorem 5.1. In view of the application of lemma 5.5 we set $X_0 = \tilde{V}$, $X_1 = \operatorname{span}\{\varphi_1, \ldots, \varphi_{l+p}\}$ and

$$h(\tilde{v}) = -\tfrac{1}{2}(L_1\tilde{v}, \tilde{v}) - \tfrac{1}{2}(L_1\tilde{w}_0(\tilde{v}), \tilde{w}_0(\tilde{v})) - \int dx \int_0^{\tilde{v}+\tilde{w}_0(\tilde{v})} f(s)\, ds\, .$$

The 5.5-ii) can be proven exactly in the same way as in theorem before. To show i) we assume by contradiction that $\exists t_n \to \infty$ and $z_n \to z$, $z \in \tilde{V}$, $\|z_n\|_0 = \|z\|_0 = 1$ with

$$t_n(L_1 z_n, z_n) + \int f(t_n z_n + w_n) z_n \geq 0\, .$$

Now $(L_1 z_n, z_n) \leq 0$ while the (UCP) on all $\tilde{V}$ and $f(-\infty) = +\infty$, $f(+\infty) = -\infty$, imply $\int f(t_n z_n + w_n) z_n \to -\infty$, a contradiction. The same arguments show $(P - S)$ holds and then the conclusion follows from lemma 5.5. Q.E.D.

REMARK 5.8. Let us consider the problem $\mathcal{L}u + \lambda_k u + f(u) = 0$, $u = 0$ on $\partial\Omega$. Setting $f_0(u) = (\lambda_k - \lambda_1)u + f(u)$, we can apply theorem 5.7 to equation $\mathcal{L}u + \lambda_1 u + f_0(u) = 0$ provided $f(u)/u \to -\infty$ as $|u| \to \infty$ and $f'(0) + \lambda_k > \lambda_{l+k+p}$. In this way we can obtain multiplicity results for equation above according to the statements of [7]. However theorem 5.7 improves (for $\lambda_k = \lambda_1$) the results of that paper.

*Added in proofs.* Prof. L. Nirenberg pointed out our existence theorems are related to some results of a recent paper by H. BREZIS and L. NIRENBERG, *Characterizations of the ranges of some nonlinear operators and applications to boundary value problems*, to appear on *Ann. Scuola Norm. Sup. Pisa*.

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