Theoretical Analysis of Inverse Weibull Distribution

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Abstract: In this study we present the theoretical analysis of Inverse weibull distribution. This paper presents the flexibility of the Inverse weibull distribution that approaches to different distributions. Here we compare the relevant parameters such as shape, scale parameters by using simulation analysis. Here we present the relationship between shape parameter and other properties such as mean, median, mode, variance, coefficient of variation, coefficient of skewness and coefficient of kurtosis models are shown graphically and mathematically presented.

Key Words: Inverse weibull distribution, simulation analysis, graphically analysis

1. Introduction

The Inverse Weibull distribution is another life time probability distribution which can be used in the reliability engineering discipline. The Inverse Weibull distribution can be used to model a variety of failure characteristics such as infant mortality, useful life and wear-out periods. The Inverse Weibull distribution can also be used to determine the cost effectiveness and maintenance periods of reliability centered maintenance activities. This paper focuses on the theoretical analysis of Inverse weibull distribution to model in which some operational time has already been accumulated for the equipment of interest. This paper present the relationship between shape parameter and other properties such as mean, median, mode, var, c.v, c.s, c.k models are shown graphically and mathematically presented. Liu (1997) explain in his work if the item consists of many parts, and each part has the same failure time distribution, and the item falls when the weakest part fails, then the Weibull distribution be an acceptable model of that failure mode (Nelson 1982). Similarly the Inverse Weibull distribution will be suitable for modeling when there are these types of applications of mechanical or electrical components lying in the life testing experiment.

2. Inverse Weibull Models Analysis

2.1 Probability density function

The Inverse Weibull probability distribution has three parameters β,η and t_0. It can be used to represent the failure probability density function (PDF) and is defined as

$$f_{Inw}(t) = \frac{\beta}{\eta}\left(\frac{1}{t-t_0}\right)^{\beta+1} e^{-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)^{\beta}}$$

$$, \beta > 0, \eta > 0, t_0 > 0, -\infty < t_0 < t \qquad (2.1)$$

Where β is the shape parameter representing the different patterns of the inverse Weibull PDF that approaches to different distributions and is positive, η is a scale parameter representing the characteristic life at which 36.78% of the population can be expected to have failed and is also positive, t_0 is a location parameter sometimes called a guarantee time, failure-free time or minimum life, the value of t_0 has a real number.

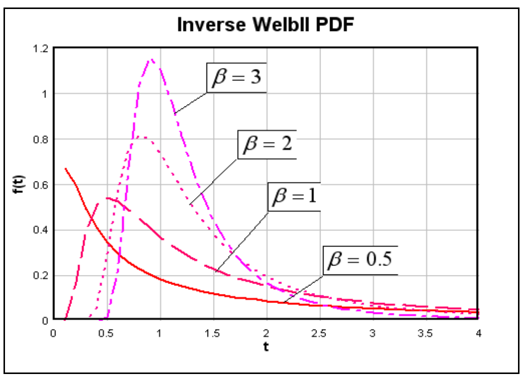

Fig2.1 The Inverse Weibull PDF

The Inverse Weibull distribution is said to be two-parameter when $t_0 = 0$. The pdf of the Inverse Weibull distribution as given in (2.1) becomes identical with the pdf of Inverse Rayleigh distribution for $\beta = 2$, and for $\beta = 1$ it coincides with that of Inverse Exponential distribution. Some works has already been done on Inverse Rayleigh distribution by Voda (1972), Gharraph (1993), and Mukarjee & Mait (1996) and some distributional properties of Inverse Weibull distribution have been studied by Aleem and Pasha (2003). Since the restrictions in (2.1) on the values of β, η and t_0 are always the same for the Inverse Weibull distribution. Fig. 2.1 shows the diverse shape of the Inverse Weibull PDF with β (= 0.5, 1, 2, 3), $\eta = 1$ and the value of $t_0 = 0$. It is important to note that figures 2.1 to 2.5 are all based on the assumption that $t_0 = 0$.

2.2 Cumulative Distribution Function

The cumulative distribution function (CDF) of the Inverse Weibull distribution is denoted by $F_{Inw}(t)$ and is defined as

$$F_{Inw}(t) = e^{-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)^{\beta}} \qquad (2.2)$$

When the CDF of the Inverse Weibull distribution has zero value then it represents no failure components by t_0.

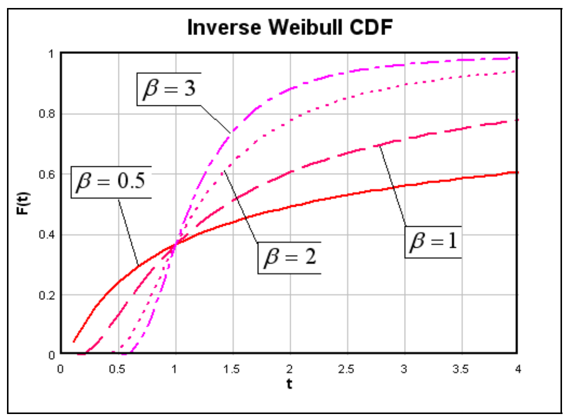

Fig. 2.2 The Inverse Weibull CDF

In the Inverse Weibull CDF t_0 is called minimum life. When $t = t_0 + \eta$ then $F_{Inw}(t_0 + \eta) = e^{-\frac{1}{\eta}\left(\frac{1}{\eta}\right)^{\beta}}$ and for $\eta = 1$ then $F_{Inw}(t_0 + 1) = e^{-1}$ = 0.367879, it represents the characteristic life' or 'characteristic value. Fig. 2.2 shows the special case of Inverse Weibull CDF with t_0 = 0 and for the value of $\eta = 1$ and β (=0.5, 1, 2, 3). It is clear from the Fig. 2.2 that all curves intersect at the point of (1, 0.367879), the characteristic point for the Inverse Weibull CDF. The CDF of the Inverse Weibull as given in (2.2) becomes identical with the CDF of Inverse Rayleigh distribution for $\beta = 2$, and for $\beta = 1$ it coincides with that of Inverse Exponential distribution. For the standard form of the CDF of the Inverse Weibull as given in (2.2) becomes when $\beta = 3$ and when $\beta = 0.5$ then its shape will approximately equal to the inverse Gamma distribution.

2.3 Reliability Function

The reliability function (RF) of the Inverse Weibull distribution is denoted by $R_{Inw}(t)$ also known as the survivor function and is defined as $1 - F_{Inw}(t)$

$$R_{Inw}(t) = 1 - e^{-\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)^{\beta}} \quad (2.3)$$

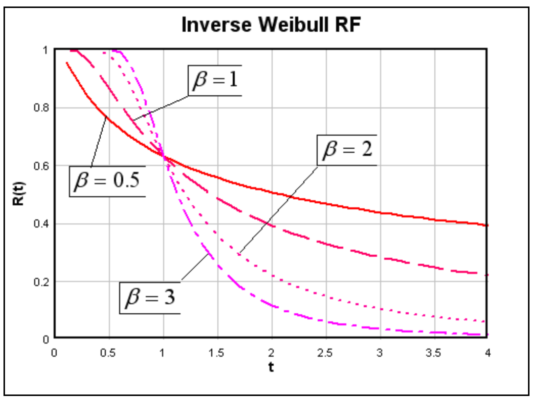

Fig. 2.3 The Inverse Weibull RF

It is important to note that $R_{Inw}(t) + F_{Inw}(t) = 1$. Fig. 2.3 shows the Inverse Weibull RF with $t_0 =$ 0, $\eta = 1$ and β (=0.5, 1, 2, 3). It is clear that all curves intersect at the point of (1, 0.632) the characteristic point for the Inverse Weibull RF. When β = 1, the distribution is the same as the inverse exponential distribution for a constant RF. When $\beta = 2$, it is known as the inverse Rayleigh distribution for the RF.

2.4 Hazard Function

The hazard function (HF) of the Inverse Weibull distribution also known as instantaneous failure rate denoted by $h_{Inw}(t)$ and is defined as $f_{Inw}(t) / R_{Inw}(t)$

$$h_{Inw}(t) = \frac{\frac{\beta}{\eta}\left(\frac{1}{t-t_0}\right)^{\beta+1}}{Exp\left(\frac{1}{\eta}\left(\frac{1}{t-t_0}\right)^{\beta}\right) - 1} \quad (2.4)$$

It is important to note that the units for $h_{Inw}(t)$ are the probability of failure per unit of time, distance or cycles.

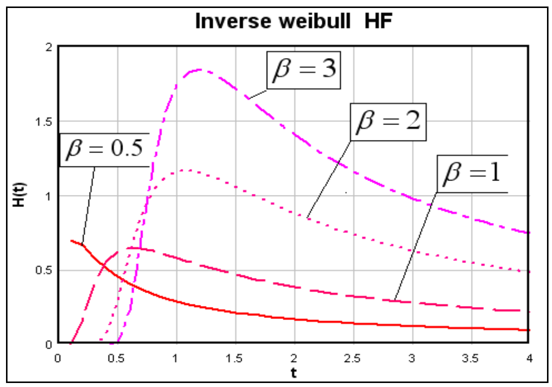

Fig. 2.4 The Inverse Weibull HF

When β = 1, the distribution is the same as the Inverse exponential distribution for a constant HF so the inverse exponential distribution is a special case of the Inverse Weibull distribution and the Inverse Weibull distribution can be treated as a generalization of the inverse exponential distribution. When β < 1, the HF is continually decreasing which represents early failures. When β > 1, the HF is continually increasing which represents wear-out failures. In particular, when $\beta = 2$, it is known as the inverse Rayleigh distribution. So the Inverse Weibull distribution is a very flexible distribution. Fig. 2.4 shows the Inverse Weibull HF with t_0 = 0, $\eta = 1$ and β (=0.5, 1, 2, 3,).

2.5 Cumulative Hazard Function

The Cumulative hazard function (CHF) of the Inverse Weibull distribution is denoted by $H_{Inw}(t)$ and is defined as

$$H_{Inw}(t) = -\ln\left|1 - \exp\left(-\frac{1}{\eta}\left(\frac{1}{t - t_0}\right)^{\beta}\right)\right| \quad (2.5)$$

It is important to note that the units for $H_{Inw}(t)$ are the cumulative probability of failure per unit of time, distance or cycles.

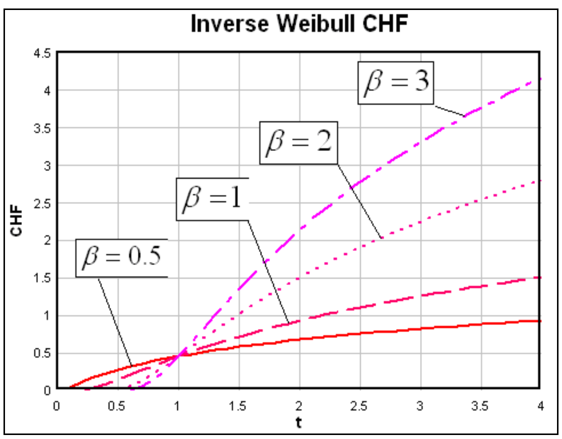

Fig. 2.5 The Inverse Weibull CHF

When β = 1, the distribution is the same as the Inverse exponential distribution for a constant CHF. When β < 1, CHF is continually decreasing which represents early failures. When β > 1, CHF is continually increasing which represents wear-out failures. In particular, when $\beta = 2$, it is known as the inverse Rayleigh distribution. Fig. 2.5 shows the Inverse Weibull CHF with t_0 = 0, $\eta = 1$ and β (=0.5, 1, 2, 3,).

The relationship between the CDF and CHF can also be defined as

$$F_{Inw}(t) = 1 - Exp(-H_{Inw}(t)) \quad \text{or}$$
$$H_{Inw}(t) = -\ln(1 - F_{Inw}(t)) \quad (2.6)$$

Here we see that $f_{Inw}(t), F_{Inw}(t)$, $R_{Inw}(t)$, $h_{Inw}(t)$ and $H_{Inw}(t)$ are expressed in closed form solutions and these models can be solved directly from the equations. This is an important advantage of the Inverse Weibull distribution for modeling RF and HF function.

3. Inverse Weibull Models and Simulation Analysis

Here every Inverse Weibull model is in a form of simulation analysis. The simulation analysis of the Inverse Weibull models is a mathematical process of a real system and then conducting computer-based experiments with these Inverse weibull models to describe, explain and predicting the patterns of the real system over extended periods of real time. Other important properties of the Inverse Weibull distribution are summarized as follows. Note that figures 3.1 to 3.7 are all based on the assumption that $t_0 = 0$. Here we compare the relevant parameters such as shape, scale parameters by using simulation analysis. Here we present the relationship between shape parameter and other properties such as mean, median, mode, var, c.v, c.s, c.k models are shown graphically and mathematically presented.

3.1 Mean Life

The mean life of the Inverse Weibull distribution also known as mean-time-to-failure ($MTTF_{Inw}$) and is defined as

$$Mean_{Inw} = t_0 + \eta^{-1/\beta}\Gamma\left(1 - \frac{1}{\beta}\right) \quad (3.1)$$

From our calculation it is clear that there is no mean life when $0 \le \beta \le 1$. Note that the maximum value of the Mean/η = $\Gamma\left(1-\frac{1}{\beta}\right) \cong$ 10.50587because $\Gamma_{\max}(t)$ =10.50587. Since $\Gamma\left(1-\frac{1}{\beta}\right)$ has the maximum value when $\beta \cong$ 1.1 can be determined. Here the Gamma functions which can be calculated with Lanczos' approximate formula (Lanczos 1964). The relationship between β and the mean life/η is shown in Fig. 3.1.Tables of the gamma function can be found in (Bohoris 1994, Kececioglu 1991). For the convenience of display we substitute this function in this notation γ_k.

$$\gamma_k = \Gamma\left(1 - \frac{k}{\beta}\right), \quad k = 1,2,3,4,...... \quad (3.1a)$$

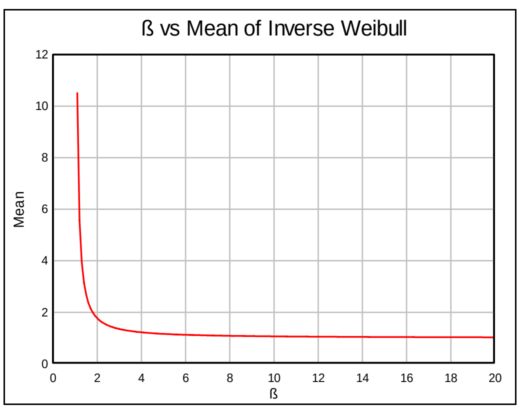

Fig 3.1 β vs mean life/η

3.2 Median Life

The median life or the 50^{th} percentile of the Inverse Weibull distribution is defined as

$$Median_{Inw} = t_0 + \ln(2^{-\eta})^{-\frac{1}{\beta}} \quad (3.2)$$

This is the life by which 50% of the units will be expected to have failed, and so it is also the life at which 50% of the units would be expected to still survive. The maximum value of median life is for β=0.1 we obtain 39.06118.The relationship between β and median life/η is shown in Fig. 3.2. Taking the first derivative of eq. 3.2 and equating it to 0, an extremely small value can be obtained as when $\beta \to \infty$ then median life/$\eta \to$1. The value of β and median life/η have a positive proportion.

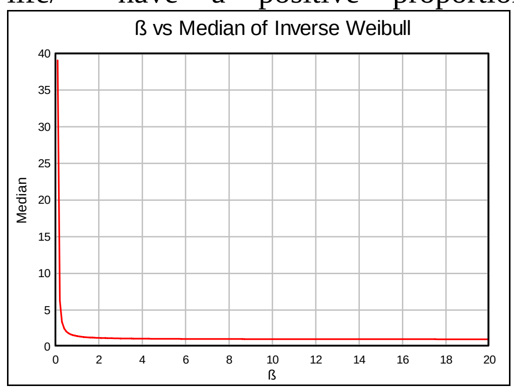

Fig. 3.2 β vs median life/η

3.3 Mode Life

The mode life of the Inverse Weibull distribution is defined as

$$Mode_{Inw} = t_0 + \eta^{-1/\beta}\left(1 + \frac{1}{\beta}\right)^{-1/\beta} \quad (3.3)$$

The relationship between β and the mode life/η is shown in Fig 3.3. We obtain the minimum value of mode life is 3.86E-11 for β=0.1 and obtain the maximum value of mode life is 0.997563465 for β=20. The Mode life/η becomes asymptotic to 1 as $\beta \to \infty$. Again β and mode life/η have a positive proportion when β > 0.

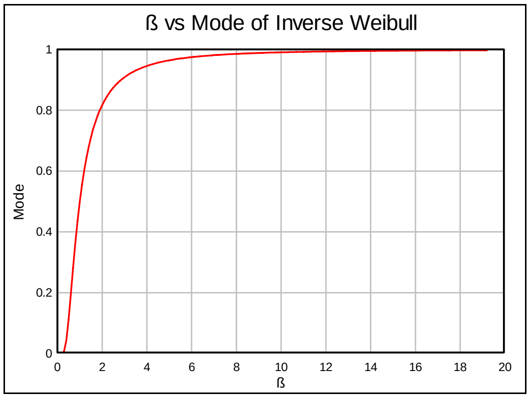

Fig. 3.3 β vs mode life/η

3.4 Variance Life

The variance life of the Inverse Weibull distribution is defined as

$$VAR_{Inw} = \eta^{-2/\beta}\left(\gamma_2 - (\gamma_1)^2\right) \tag{3.4}$$

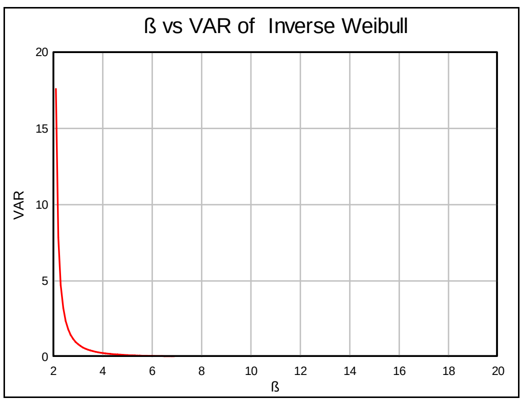

Fig3.4 β vs variance/η^2

From our calculations it is clear that there is no variance life when $0 \le \beta \le 2$. We obtain the maximum value of variance life is 17.59895 for β =2.1. The relationship between β and the variance/η^2 life is shown in Fig 3.4. It is clear that the larger the value of β the smaller the value of variance life/η^2. The relationship between β and the variance/η^2 life shows that it becomes asymptotic to 0 as $\beta \to \infty$. The Inverse Weibull distribution standard deviation SDInw is the measure of spread, and can be obtained by taking the square root of the VAR_{Inw}.

3.5 Coefficient of Variation

The coefficient of variation CVInw is defined as $SD_{Inw}/(MTTF_{Inw} - t_0)$

$$CV_{Inw} = \sqrt{\frac{\gamma_2}{\gamma_1^{\,2}} - 1} \tag{3.5}$$

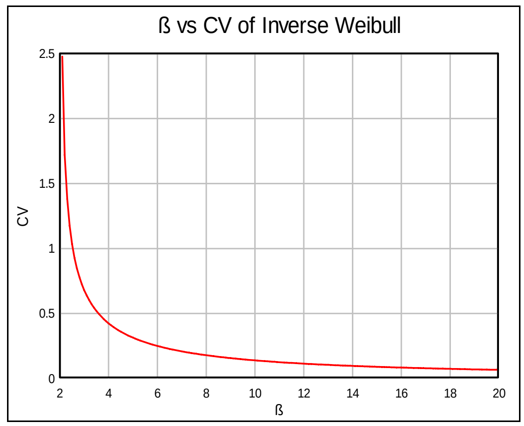

Fig. 3.5 β vs CV_{Inw}

From our calculation it is clear that there is no CVInw life when $0 \le \beta \le 2$. We obtain the maximum value of CVInw life is 2.476739 for β =2.1. The relationship between β and CVInw is shown in Fig 3.5, the larger the value of β the smaller the value of the CVInw.

3.6 Coefficient of Skewness

The coefficient of skewness CS_{Inw} is defined as $E(T - E(T))^3 / \left(E(T - E(T))^2\right)^{\frac{3}{2}}$

$$CS_{Inw} = \frac{\gamma_3 - 3\gamma_2\gamma_1 + 2\gamma_1^{\,3}}{\left(\gamma_2 - \gamma_1^{\,2}\right)^{\frac{3}{2}}} \tag{3.6}$$

Where CS_{Inw} is the quantity used to measure the skewness of the Inverse Weibull distribution, If $CS_{Inw} < 0$ then the PDF of the Inverse Weibull distribution is skewed to the left when (Mean < Median < Mode), if $CS_{Inw} = 0$ then the PDF of the Inverse Weibull distribution shape is symmetrical (Mean = Median = Mode) as in the Normal distribution, and if $CS_{Inw} > 0$ then the PDF of the Inverse Weibull distribution is skewed to the right when (Mean > Median > Mode). The relationship between β and CS_{Inw} is shown in Fig. 3.6. From our calculations it is clear that there is no CS_{Inw} life when $0 \le \beta \le 3$. We obtain the maximum value of CS_{Inw} life is 40.58428 for β =3.1.

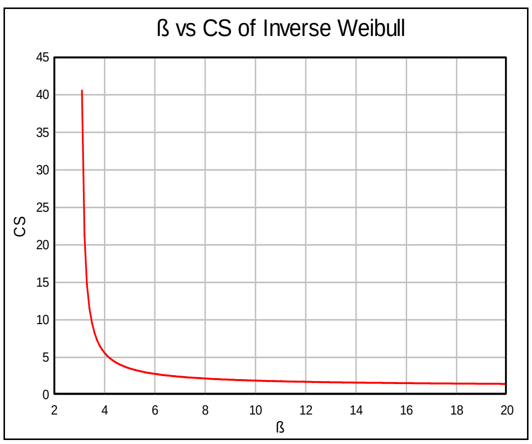

Fig. 3.6 β **vs** CS_{Inw}

3.7 Coefficient of Kurtosis

The coefficient of kurtosis CK_{Inw} is defined as $E(T-E(T))^4 / \left(E(T-E(T))^2\right)^2$

$$CK_{Inw} = \frac{\gamma_4 - 4\gamma_3\gamma_1 + 6\gamma_2\gamma_1^2 - 3\gamma_1^4}{\left(\gamma_2 - \gamma_1^2\right)^2} \quad (3.6)$$

Where CK_{Inw} the quantity is used to measure the kurtosis or peaked ness of the distribution. If $CK_{Inw} = 3$ it represent the peaked ness of the Normal distribution. The Inverse Weibull PDF shape is more peaked than the Normal PDF when the value of $CK_{Inw} > 3$. The Inverse Weibull PDF shape is flatter than the Normal PDF when the value of $CK_{Inw} < 3$. The relationship between β and CK_{Inw} is shown in Fig. 3.7. From our calculation it is clear that there is no CK_{Inw} life when $0 \le \beta \le 4$. It is important to note that we obtain the maximum value of CK_{Inw} life is 529.5914 at β =4.1.

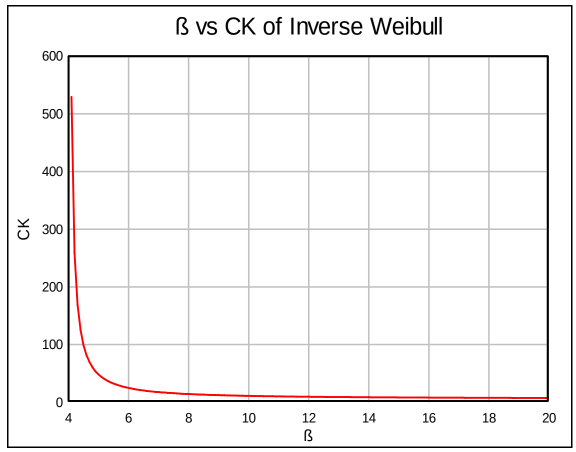

Fig 3.7 β **vs** CK_{Inw}

These Properties mean, median, mode, SDInw, VARInw, $F_{Inw}(t), R_{Inw}(t), h_{Inw}(t)$ and $H_{Inw}(t)$ are frequently used to measure the life of system or process. It is important to note that VARInw, CVInw, CSInw and CKInw are independent of to and CVInw, CSInw and CKInw are not dependent on η. It is worth mentioning that when η = 1 and t_0 = 0, the distribution is sometimes called the standard Inverse Weibull distribution under the relationship between β and these properties.

4. Area of applications of the Inverse Weibull distribution

In the life testing experiment reliability is the probability that a device, system, or process will perform its prescribed duty without failure for a given time when operated correctly in a specified environment. Reliability is used to find the life span of mechanical and electrical components while survival is the probability used to find the life span of biological events associated with the human study. The Inverse Weibull distribution model can be used in reliability analysis. It can be successful in modeling life for many devices and variables such as relays, ball bearings, electron tubes, capacitors, germanium transistors, photo-con due Live cells, motors, automotive radiators, regulators, generators, turbine blades, fatigue in textiles, corrosion resistance, leakage of dry batteries, return of products after-shipment, marketing life expectancy of drugs, the number of downtimes per shift, solids subjected to fatigue stresses etc.

5. Summary and conclusions

In this study we have seen that the Inverse Weibull distribution is the flexible distribution model that approaches to different distributions when its shape parameter changes. The comparative comprehensive study of the reliability modeling is predicted from hazard analysis. The Properties of $f_{Inw}(t), F_{Inw}(t), R_{Inw}(t)$, $h_{Inw}(t)$, $H_{Inw}(t)$, mean, median, mode, SDInw, VARInw, CVInw, CSInw and CKInw can be used to measure life data. When β = 1, the distribution is the same as the inverse exponential distribution for a constant hazard function so the inverse exponential distribution is a special case of the Inverse Weibull distribution and the Inverse Weibull distribution can be treated as a generalization of the inverse exponential distribution. When β < 1, the hazard function is continually decreasing which represents early failures and follow the inverse gamma distribution. When β > 1, the hazard function is continually increasing which represents wear-out failures. In particular, when $\beta = 2$, it is known as the inverse Rayleigh distribution. So it concludes that the Inverse Weibull distribution is a very flexible reliability model that approaches to different distributions.

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