THEORETICAL LIMITATIONS ON

THE BROADBAND MATCHING OF

ARBITRARY IMPEDANCES

by

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Abstract

The work presented in this paper deals with the general problem of matching an arbitrary load impedance to a pure resistance by means of a reactive network. More precisely, it consists of a systematic study of the origin and nature of the theoretical limitations on the tolerance and bandwidth of match and of their dependence on the characteristics of the given load impedance.

Following a general discussion of the matching problem in the first chapter, the second chapter presents a derivation of the necessary and sufficient conditions for the physical realizability of a function of frequency representing the input reflection coefficient of a matching network terminated in a given load impedance. These conditions of physical realizability are transformed in the third chapter into a set of integral relations which are particularly suitable for the study of the limitations on the bandwidth and tolerance of match. Unfortunately, definite expressions for these quantities could be obtained only in very special cases, because of inherent mathematical difficulties resulting from high-order algebraic equations.

The fourth chapter deals with the practical problem of approaching the optimum theoretical tolerance by means of a network with a finite number of elements. The general solution of this problem was hampered again by mathematical difficulties resulting from the necessity of solving systems of transcendental equations. Design curves are provided, however, for a particularly simple, but very important, type of load impedance. In addition, a very convenient method is presented for computing the values of the elements of the resulting matching network. The whole design procedure is illustrated by numerical examples.

ACKNOWLEDGMENT

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I wish also to express my gratitude to the Directors of the Research Laboratory of Electronics, Professors J. A. Stratton and A. G. Hill, under whose sponsorship this research was carried out, and to the colleagues of the Laboratory who on many occasions gave me so much of their time to discuss my work.

CHAPTER I

The Matching Problem

1.1 Origin and nature of the problem - The transfer of power from a generator to a load constitutes one of the fundamental problems in the design of communication systems. Some generators may be considered as ideal voltage sources in series with a linear impedance; maximum power transfer is then obtained when the load impedance is made equal to the conjugate of the source impedance. This simple representation fails in the case of other generators, but there is still for each generator a definite load impedance which yields maximum transfer of power. Linear amplifiers, for instance, may be included in the first group of generators, while certain power amplifiers and all oscillators belong to the second group. When the load and the generator are physically so distant from each other that an electrically long transmission line has to be employed, it is usually desirable to avoid the presence of standing waves on the line. This requirement is met by making the load impedance equal to the characteristic impedance of the line, which is, in general, a real quantity.

Problems of the types mentioned above involve in every case the design of a non-dissipative coupling network to transform a given load impedance into another specified impedance. One refers to this operation as "impedance matching". The impedance to which the load has to be matched is, in most cases, a pure resistance, or can be made such by means of a separate coupling network. Therefore, from a practical standpoint, little loss of generality results from limiting our discussion to this simpler case, that is, to the case of matching to a pure resistance.

It will be shown later that it is not possible to match an arbitrary impedance to a pure resistance over the whole frequency spectrum, or even at all frequencies within a finite frequency band. On the other hand, it is evidently possible to obtain a match at any desired number of frequencies, provided the load impedance has a finite resistive component at these frequencies. Such a solution, however, has little practical value because it is incorrect to assume that one can obtain a reasonable match over a frequency band by correctly matching at a sufficiently large number of frequencies within the desired band.

It becomes clear at this point that the statement of any matching problem must include the maximum tolerance on the match as well as the minimum bandwidth within which the match is to be obtained. Furthermore, it is reasonable to expect that, for a given load impedance and a given frequency band, there exists a lower limit to the maximum tolerance that can be obtained by means of a physically realizable coupling network. It follows that an investigation of this lower limit should be the first step in any systematic study of matching networks. Before this problem can be stated in a precise manner, however, one must define an appropriate measure of the match so as to give to the tolerance a definite quantitative meaning.

1.2 Quantitative definition of matching - In view of the fact that matching is used to maximize the load power, it appears reasonable to use as a measure of the match the ratio of the actual load power P_L to the maximum power P_o that could be delivered to the load by the same generator. It is more convenient, however, to use instead the per unit value of the power which is not delivered to the load, that is, the quantity

$$|\rho|^2 = \frac{P_o - P_L}{P_o} = 1 - \frac{P_L}{P_o} \tag{1}$$

The meaning of the symbol $|\rho|^2$ will become clear shortly. This quantity can be readily expressed in terms of the characteristics of the generator and of the load when the generator can be represented by means of a constant voltage source in series with a linear impedance. If this is not the case, $|\rho|^2$ becomes a complicated function of the characteristics of the generator and very often cannot be expressed mathematically. For lack of a better method of attack, it will be assumed again that the generator can be represented by a voltage source in series with a linear impedance, which is now chosen equal to the conjugate of the load impedance which yields optimum operation. As pointed out before, this impedance is assumed to be a pure resistance. The resulting quantity $|\rho|^2$ is not simply related to actual load power but is a definite function of the impedance presented to the generator, in terms of which the output characteristics of the generator can be expressed.

For the purpose of analysis it is convenient to normalize all impedances with respect to the internal impedance of the generator. Let Z be the normalized impedance of the load presented to the generator, as shown in Fig. 1.1. One obtains readily:

$$\left|\rho\right|^2 = \left|\frac{Z-1}{Z+1}\right|^2 \tag{2}$$

It is clear at this point that $|\rho|$ is the magnitude of the reflection coefficient

$$\rho = \frac{Z-1}{Z+1} \tag{3}$$

which would be obtained if the generator were connected to the load through a lossless transmission line of unity characteristic impedance. Since ρ is defined as the ratio of the voltage of the reflected wave to the voltage of the incident wave, $|\rho|^2$ is evidently equal to the per unit reflected power, that is, to the power which is not delivered to the load. It will be remembered in this connection that the voltage standing-wave ratio on a transmission line is related to ρ by:

$$\text{VSWR} = \frac{1+|\rho|}{1-|\rho|} \tag{4}$$

It follows that $|\rho|$ is the most appropriate measure of the match when a transmission line is actually present and standing waves are to be minimized. In conclusion the tolerance on the match

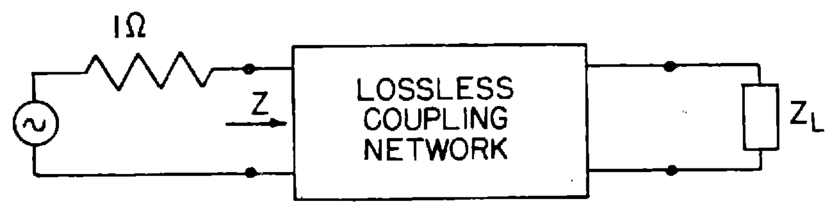

Figure 1.1 - Matching network for an arbitrary load impedance

will be expressed in all cases by the maximum allowable value of the magnitude of the reflection coefficient $|\rho|_{max}$.

1.3 Preliminary statement and analysis of the problem - The problem of the broadband matching of an arbitrary impedance can now be stated more precisely in the light of the previous discussion. With reference to Fig.1.1, Z_L is a given linear, passive impedance normalized with respect to the source resistance. A non-dissipative coupling network must be designed such that, when terminated in Z_L, the magnitude of the reflection coefficient is smaller than or equal to a specified value $|\rho|_{max}$ at all frequencies within a prescribed band.

The impedance Z_L and the coupling network may include, in the most general case, distributed constant elements such as transmission lines, waveguides, cavity resonators, etc. Such a general case, however, is outside the field of application of the available techniques of network analysis and synthesis so that the problem must be limited to the case of impedances realizable by means of a finite number of lumped elements. This limitation is not as serious as it may appear at first because, in many practical cases, the results obtained in the case of lumped element networks can be extended in an approximate fashion to the case of distributed constant systems. For instance, such a technique has been successfully employed by the author in the design of microwave filters.(1)

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(1) See Bibliography.

An additional remark must be made on the fact that the coupling network is assumed to be lossless. In practice, of course, a certain amount of incidental dissipation will be present, which will result in a distortion of the characteristics of the coupling network. This distortion can be computed without difficulty when the dissipation is uniformly distributed.(2) Moreover, in certain cases, it is possible to predistort the characteristics of a lossless network to balance the distortion produced by dissipation,(3) apart, of course, from a constant transmission loss over the pass band which cannot be eliminated. This correction is often carried out experimentally since a small amount of dissipation requires only small readjustments of the element values. Such procedures for taking into account the effect of incidental dissipation have been developed in connection with the design of filters, since the same problem arises there as in the case of matching networks. It seems appropriate, therefore, to neglect the presence of losses in our study, and to rely on the available techniques for any correction that becomes necessary in the final stage of a particular design.

1.4 Previous work on matching - The matching problem is now limited to the design of an appropriate two-terminal-pair reactive network consisting of a finite number of lumped elements. This design problem has been attacked in the past by following a step-by-step procedure leading to the ladder structure of reactances shown in Fig.1.2. These reactances are designed successively in

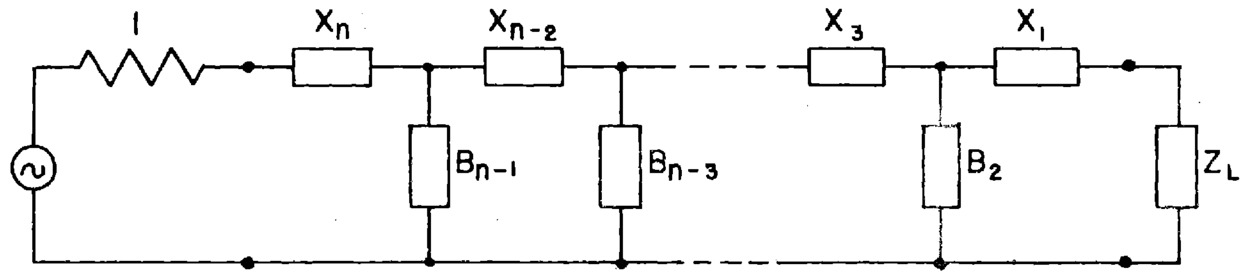


Figure 1.2 - Step-by-step matching of an impedance Z_L

such a way that the resulting impedance (or admittance) measured toward the load at each node of the structure approximates better and better a pure resistance over the prescribed frequency band. This procedure has two main weaknesses. In the first place, the designer does not know whether the requirements on the tolerance and the bandwidth that he is trying to meet are consistent with the given load impedance; for the same reason he cannot decide, at a certain stage of the design, whether any further possible improvement is worth the required additional complexity of the network. In the second place, it is implicitly assumed that the step-by-step procedure converges to the optimum design or at least to a design reasonably close to the optimum. This is not the case in general; moreover, it will be shown later that perfect matching at any frequency is paid for very dearly in terms of maximum possible bandwidth. A procedure for designing the ladder structure as a unit was suggested by Bode [4,5] in 1930. This procedure, however, has still most of the weaknesses of the step-by-step method of design, and has not sufficient bearing on the work presented in this paper to deserve a detailed discussion.

The first step toward a systematic investigation of matching networks was made by Bode[6] some time later, in connection with a very special but important type of load impedance. He considered the case of a load impedance Z_L consisting of a resistance R shunted by a capacitance C, and showed that the fundamental limitation on the matching network takes the form:

$$\int_0^\infty \ln \frac{1}{|\rho|} \, d\omega \leq \frac{\pi}{CR} \tag{4}$$

where ρ is the input reflection coefficient corresponding to the impedance Z in Fig. 1. If $|\rho|$ is kept constant and equal to $|\rho|_{max}$ over a frequency band of width w (in rad. per sec.) and is made equal to unity over the rest of the frequency spectrum, eq. 4 yields:

$$w \ln \frac{1}{|\rho|_{max}} \leq \frac{\pi}{CR} \tag{5}$$

In words, the product of the bandwidth by the maximum value of the "return loss", $\ln \frac{1}{|\rho|_{max}}$, has a maximum limit fixed by the product RC. Eq. 4 indicates also that approaching a perfect match, that is, making $|\rho|$ very small at any frequency, results in an unnecessary waste of the area represented by the integral, and, therefore, in a reduction of the bandwidth. It is also clear that the limitation found by Bode applies to any impedance consisting of a reactive two-terminal-pair network terminated in a parallel RC combination. In this case, however, no assurance is given that the maximum theoretical bandwidth can be approached even in the limit when a very large number of elements is used in the matching network. On the contrary, in the case of a simple RC combination, the matching network can be designed to satisfy eqs. 4 and 5 with the equal sign. This point will be discussed further in Chapter 4.

1.5 Precise statement of the problem - Bode's work discussed in the preceding section indicates the existence of definite limitations on the bandwidth and on the tolerance of match for any given load impedance. These limitations must originate from some conditions of physical realizability of the function representing the input reflection coefficient ρ, conditions which must, in their turn, depend on the load impedance. It is clear at this point that the first step in a systematic solution of the matching problem must be the determination of these theoretical limitations. The development of a design procedure should then follow, whose objective would be to approach the theoretical limit with the smallest number of elements in the matching network. For the purpose of discussion, one can then divide the matching problem in three parts as follows:

I - Given an impedance function Z_L, subject only to the condition of being realizable by means of a finite number of lumped elements, find the conditions of physical realizability for the reflection coefficient function ρ of a reactive, two-terminal-pair network terminated in Z_L (See Fig. 1).

II - From the conditions of physical realizability for ρ, determine the minimum tolerance on the magnitude of the reflection coefficient $|\rho|$ over a prescribed frequency band.

III - Obtain appropriate functions for ρ, which satisfy the conditions of physical realizability and, at the same time, minimize the tolerance over a specified frequency band for a given number of elements in the coupling network.

The work presented in the following chapters will provide a complete solution to the first part of the problem. The solution to the second part will be carried out as far as possible in general terms. Unfortunately, a definite expression for the minimum tolerance could be obtained only in the simplest cases, since the desired answer depends on the solution of a system of high order algebraic equations. However, the conditions of physical realizability will be expressed in the form of integral relations similar to the one obtained by Bode, which will indicate clearly the nature of the limitations imposed by the load impedance on the frequency behavior of $|\rho|$. Very little progress could be made toward a general solution of the third part of the problem, because it involves a system of transcendental equations. Yet a set of design curves will be presented for the simplest case considered by Bode. These curves will show the behavior of the tolerance as a function of the bandwidth and of the number of elements in the matching network.

CHAPTER II

Physical Realizability of the Reflection Coefficient

2.1 Analysis of the problem - This chapter is devoted to the solution of the first part of the matching problem as stated in Sec. 1.5. For the sake of clarity this part of the problem is restated below. With reference to Fig. 1.1, Z_L is given as a function of the complex frequency variable $\lambda = \sigma + j\omega$, subject only to the condition of being physically realizable by means of a finite number of linear passive lumped elements. Z_L is connected at the output terminals of an arbitrary two-terminal-pair reactive network consisting of a finite number of lumped elements; the input terminals of this network are connected to a generator consisting of an ideal voltage source in series with a one ohm resistance. It is desired to determine the restrictions that must be imposed on the function $\rho(\lambda)$ representing the reflection coefficient at the terminals of the generator, in order to insure the physical realizability of the reactive network.

It will be recalled (Sec. 1.4) that, if Z_L consists of a two-terminal-pair reactive network terminated in a parallel RC combination, the limitation found by Bode for this case involves the product RC. Therefore, if such an impedance were given mathematically, one would have to determine its physical structure before Bode's relation could be applied. On the other hand, Darlington has shown (3) that any physically realizable impedance function can be considered as the input impedance of a reactive two-terminal-pair network terminated in a pure resistance. This resistance can be made equal to one ohm in all cases by incorporating an appropriate ideal transformer in the reactive network. The network shown in Fig. 1.1 can then be transformed as indicated in Fig. 2.1.

At this point the problem under consideration takes a form particularly interesting from a general network theory point of view. It will be pointed out in the next section that the over-all characteristics of a two-terminal-pair reactive network are completely specified by the impedance (or the reflection coefficient) measured at one pair of terminals when a one ohm resistance is connected to the other pair of terminals. It follows that the conditions of physical realizability for ρ (See Fig. 2.1) are the same as the conditions that must be satisfied by any other function or set of functions representing the over-all characteristics of the two reactive networks of Fig. 2.1 in cascade. In conclusion the problem can be restated as follows:

Given two reactive two-terminal-pair networks of which one is fixed, the other arbitrary, determine the conditions of physical realizability for the over-all characteristics of the two networks connected in cascade.

Before proceeding to the solution of this problem, it is necessary to review some of the properties of the functions that will be used to represent the characteristics of a two-terminal-pair reactive network.

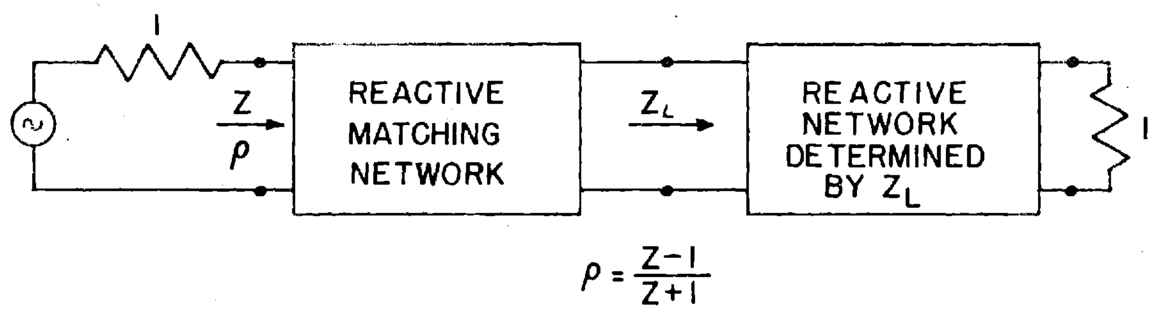


Figure 2.1 - Matching of an arbitrary load impedance

2.2 <u>Reflection and transmission coefficients</u> - The reflection and transmission coefficients of a two-terminal-pair reactive network represent the characteristics of the network when one ohm terminations are connected to both pairs of terminals as shown in Fig. 2.2. The two reflection coefficients are defined by:

$$\rho_1 = \frac{Z_1 - 1}{Z_1 + 1} = \left[\frac{2V_1}{E_1} - 1\right]_{E_2 = 0} \tag{1}$$

$$\rho_2 = \frac{Z_2 - 1}{Z_2 + 1} = \left[\frac{2V_2}{E_2} - 1\right]_{E_1 = 0} \tag{2}$$

where Z_1 and Z_2 are the impedances measured at the two pairs of terminals when the voltage sources are short circuited. The transmission coefficient is defined, with reference to Fig. 2.2, by

$$t = \left[\frac{2V_2}{E_1}\right]_{E_2 = 0} = \left[\frac{2V_1}{E_2}\right]_{E_1 = 0} \tag{3}$$

The physical significance of these coefficients is best understood by inserting two transmission lines of unity characteristic impedance between the network and the terminations. The reflection coefficient ρ_1 is then the ratio of the voltage of the reflected wave to the voltage of the incident wave measured at terminals 1 for $E_2 = 0$; ρ_2 has the same significance for terminals 2. The transmission coefficient t is the ratio of the voltage of the

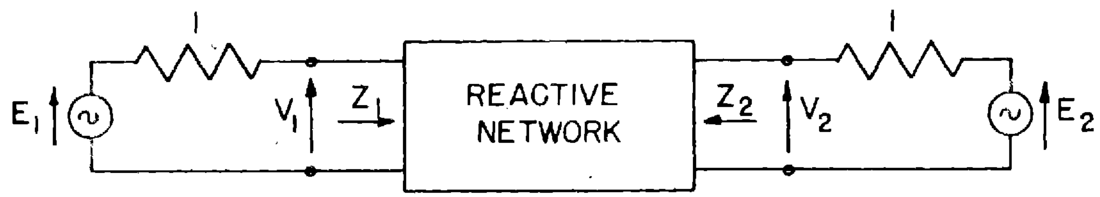


Figure 2.2 - Two-terminal-pair reactive network

transmitted wave at terminals 2 to the voltage of the incident wave at terminals 1, for $E_2 = 0$. Because of the reciprocity theorem, the same value of t is obtained for transmission in the opposite direction.

It is clear from the above definitions that $|\rho_1|^2$ is the per unit power reflected and $|t|^2$ is the per unit power transmitted. Since the network is non-dissipative, one obtains, for $\lambda = j\omega$,

$$|\rho_1|^2 + |t|^2 = 1 \tag{4}$$

A similar relation can be written for transmission in the opposite direction.

$$|\rho_2|^2 + |t|^2 = 1 \tag{5}$$

It follows that, for $\lambda = j\omega$,

$$|\rho_1|^2 = |\rho_2|^2 = 1 - |t|^2 \tag{6}$$

and

$$|\rho_1| = |\rho_2| \leq 1 \; ; \quad |t| \leq 1 \tag{7}$$

Equation 7 is a necessary condition for the physical realizability of a reflection coefficient. Furthermore, since a reflection coefficient is a measure of the voltage of a reflected wave, all the poles of this function must be in the left half of the complex plane, otherwise the network would oscillate without the help of any external generator. It can be shown (3,7) that this condition on the poles together with eq. 7 are sufficient as well as necessary conditions for the physical realizability of a reflection coefficient. It is, of course, understood that the reflection coefficient of a lumped element network must be a ratio of two real polynomials in the complex frequency variable $\lambda = \sigma + j\omega$. In conclusion the reflection coefficient ρ_1 of a reactive network must be of the form:

$$\rho_1(\lambda) = K \frac{(\lambda - \lambda_{o1})(\lambda - \lambda_{o2})\cdots\cdots(\lambda - \lambda_{on})}{(\lambda - \lambda_{p1})(\lambda - \lambda_{p2})\cdots\cdots(\lambda - \lambda_{pn})} \qquad (8)$$

where K is a real number. All λ_p's have a negative real part and all complex λ_p's and λ_o's must be present in conjugate pairs (the polynomials have real coefficients). Furthermore $\rho_1(\lambda)$ must satisfy the condition:

$$\left|\rho_1\right|^2_{\lambda = j\omega} = \Big[\rho_1(\lambda)\,\rho_1(-\lambda)\Big]_{\lambda = j\omega} \leq 1 \qquad (9)$$

It can be shown that the companion reflection coefficient $\rho_2(\lambda)$ must have the form:

$$\rho_2(\lambda) = (-1)^{n+1} K \frac{(\lambda + \lambda_{o1})(\lambda + \lambda_{o2})\cdots\cdots(\lambda + \lambda_{on})}{(\lambda - \lambda_{p1})(\lambda - \lambda_{p2})\cdots\cdots(\lambda - \lambda_{pn})} \qquad (10)$$

where the λ_o's and the λ_p's are the same quantities appearing in the expression for ρ_1.

Consider now the function

$$\rho_1(\lambda)\rho_1(-\lambda) = \rho_2(\lambda)\rho_2(-\lambda) \tag{11}$$

which is equal to $\left|\rho_1\right|^2 = \left|\rho_2\right|^2$ for $\lambda = j\omega$. The poles and zeros of this function are arranged in quadruplets or pairs as shown in Fig. 2.3. Note that all the poles and zeros on the imaginary axis must have even multiplicity. The zeros of (11) are divided between ρ_1 and ρ_2, as indicated by eqs. 8 and 10, without any other restriction but that all of them must appear in either ρ_1 or ρ_2.

The transmission coefficient is related to ρ_1 and ρ_2 through the equation:

$$\left|t\right|^2_{\lambda=j\omega} = \Big[t(\lambda)t(-\lambda)\Big]_{\lambda=j\omega} = 1 - \left|\rho_1\right|^2_{\lambda=j\omega} = 1 - \Big[\rho_1(\lambda)\rho_1(-\lambda)\Big]_{\lambda=j\omega} \tag{12}$$

The poles of t must have a negative real part for the same reason as the poles of ρ_1 and ρ_2. Moreover eq. 12 shows that ρ_1, ρ_2 and t have the same poles and, therefore, the same denominator polynomial. It can be shown [3,7] that the numerator of t must be either an even or an odd polynomial. This requirement implies that the zeros of t must be present in quadruplets or pairs, depending on their location, as indicated in Fig. 2.4; zeros at the origin and at infinity may have odd multiplicity. It follows that

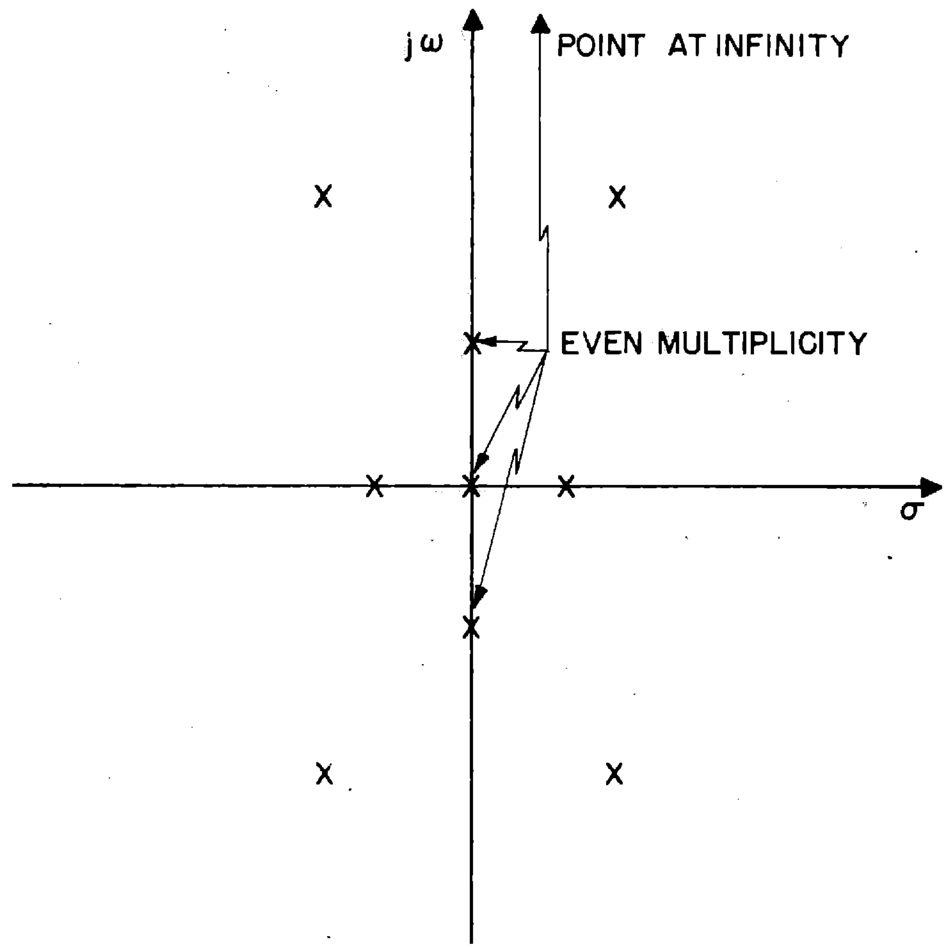


Figure 2.3 - Grouping of the zeros of a typical transmission coefficient

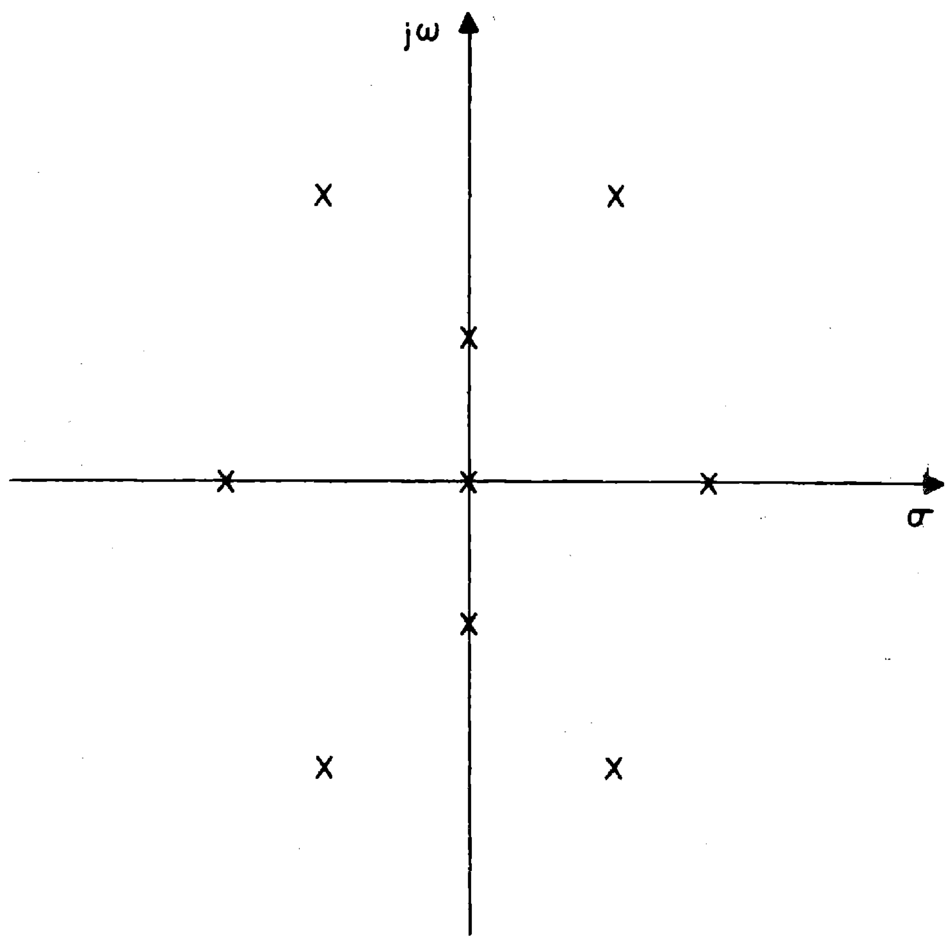


Figure 2.4 - Types of singularity groups for $[\rho(\lambda)\rho(-\lambda)]$

all the zeros of the function $\left[t(\lambda)\ t(-\lambda)\right]$ must have even multiplicity, that is, the function must be the square of either an even or an odd polynomial. The transmission coefficient can then be written in the form:

$$t = \pm\sqrt{1 - K^2}\,\frac{(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots\cdots(\lambda - \lambda_3)}{(\lambda - \lambda_{p1})(\lambda - \lambda_{p2})\cdots\cdots(\lambda - \lambda_{pn})} \tag{13}$$

where the λ_k's are determined by means of eq. 12.

The conditions of physical realizability for ρ_1 do not imply that the zeros of $\left[t(\lambda)\ t(-\lambda)\right]$ have necessarily even multiplicity. This difficulty is circumvented by multiplying and dividing the function by the root factors of the numerator that have odd multiplicity. It must be kept in mind, however, that these root factors must be carried back into ρ_1 and ρ_2 as shown in the following example. Suppose the quadruplet of zeros at λ_ν, $\bar{\lambda}_\nu$, $-\lambda_\nu$, $-\bar{\lambda}_\nu$ has odd multiplicity. Then one must multiply the function $\left[t(\lambda)\ t(-\lambda)\right]$ by $\dfrac{(\lambda-\lambda_\nu)(\lambda-\bar{\lambda}_\nu)(\lambda+\lambda_\nu)(\lambda+\bar{\lambda}_\nu)}{(\lambda-\lambda_\nu)(\lambda-\bar{\lambda}_\nu)(\lambda+\lambda_\nu)(\lambda+\bar{\lambda}_\nu)}$ and the function t will, therefore, include the factor

$$\frac{\cancel{(\lambda-\lambda_\nu)}\cancel{(\lambda-\bar{\lambda}_\nu)}(\lambda+\lambda_\nu)(\lambda+\bar{\lambda}_\nu)}{\cancel{(\lambda-\lambda_\nu)}\cancel{(\lambda-\bar{\lambda}_\nu)}}$$

in which two of the roots cancel out (λ_ν has a negative real part), as indicated. It can be seen by inspection that the function $\left[\rho_1(\lambda)\ \rho_1(-\lambda)\right]$ must be multiplied by the same quantity as

$\left[\, t(\lambda)\, t(-\lambda) \right]$. It follows that either ρ_1 or ρ_2 must contain the phase-shift factor

$$\frac{(\lambda + \lambda_\nu)(\lambda + \bar{\lambda}_\nu)}{(\lambda - \lambda_\nu)(\lambda - \bar{\lambda}_\nu)}$$

whose magnitude is unity on the imaginary axis. No λ_ν-root will appear in the other reflection coefficient because the zeros in the left half plane at λ_ν and $\bar{\lambda}_\nu$ will cancel the corresponding poles. It may also happen that t contains a factor of the form

$$\frac{(\lambda + \lambda_\nu)(\lambda + \bar{\lambda}_\nu)\cancel{(\lambda - \lambda_\nu)}\cancel{(\lambda - \bar{\lambda}_\nu)}}{(\lambda - \lambda_\nu)^{\cancel{2}}\,(\lambda - \lambda_\nu)^{\cancel{2}}}$$

with the eliminations shown and ρ_2 contains the same factor squared. In this case it would be impossible to detect the presence of the zeros of t at $-\lambda_\nu$ and $-\bar{\lambda}_\nu$ from a knowledge of ρ_1 alone. The physical significance of this situation is that an all-pass network of unity characteristic impedance is connected in cascade at terminals 2, with the result that a phase shift appears in both ρ_2 and t without their magnitudes at real frequencies being changed. In view of this fact, the statement that either reflection coefficient defines completely a two-terminal-pair reactive network should be modified to read "defines completely the network apart from an arbitrary all-pass phase-shifting network connected in cascade at the opposite terminals".

2.3 Cascade connection of two-terminal-pair reactive networks - The problem stated in Sec. 2.1 involves a cascade connection of two reactive networks. It is, therefore, necessary to develop appropriate relations between the characteristics of the two individual networks and the over-all characteristics of the two networks in cascade.

Consider a reactive network terminated at both ends in lossless transmission lines of unity characteristic impedance. Let $\mathcal{J}_1$ and $\mathcal{R}_1$ be the voltages of the incident and reflected waves at the input terminals, and $\mathcal{J}_2$ and $\mathcal{R}_2$ the corresponding voltages at the output terminals as indicated in Fig. 2.5. On the basis of the definitions of ρ_1, ρ_2 and t given in the preceding sections, one can write the following equilibrium equations:

$$\mathcal{R}_1 = \rho_1 \mathcal{J}_1 + t\mathcal{R}_2 \tag{14}$$

$$\mathcal{J}_2 = t\mathcal{J}_1 + \rho_2 \mathcal{R}_2 \tag{15}$$

Solving for $\mathcal{R}_1$ and $\mathcal{J}_1$ yields

$$\mathcal{J}_1 = \frac{1}{t}\mathcal{J}_2 - \frac{\rho_2}{t}\mathcal{R}_2 \tag{16}$$

$$\mathcal{R}_1 = \frac{\rho_1}{t}\mathcal{J}_2 + \left(t - \frac{\rho_1\rho_2}{t}\right)\mathcal{R}_2 \tag{17}$$

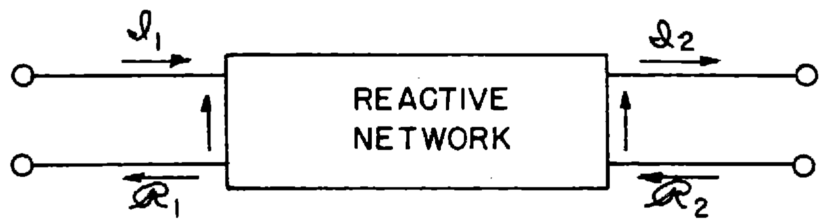


Figure 2.5 - Reactive network with input and output transmission lines

or in matrix form

$$\begin{bmatrix} J_1 \\ \\ R_1 \end{bmatrix} = \begin{bmatrix} T \end{bmatrix} \times \begin{bmatrix} J_{r_2} \\ \\ R_{r_2} \end{bmatrix} \tag{18}$$

where

$$\begin{bmatrix} T \end{bmatrix} = \begin{bmatrix} \frac{1}{t} & -\frac{\rho_2}{t} \\ \\ \frac{\rho_1}{t} & t - \frac{\rho_1\rho_2}{t} \end{bmatrix} \tag{19}$$

Eq. 18 indicates that the matrix $\begin{bmatrix} T \end{bmatrix}$ has the property that the matrix for two networks in cascade is the product of the matrices of the individual networks. With reference to Fig. 2.6 one obtains without difficulty by matrix multiplication

$$t = \frac{t' t''}{1 - \rho_2' \rho_1''} \tag{20}$$

$$\rho_1 = \rho_1' + \rho_1'' \frac{(t')^2}{1 - \rho_2' \rho_1''} \tag{21}$$

$$\rho_2 = \rho_2'' + \rho_2' \frac{(t'')^2}{1 - \rho_2' \rho_1''} \tag{22}$$

These equations are the desired relations in terms of reflection and transmission coefficients. The factor $1 - \rho_2' \rho_1''$ results from the multiple reflection at the junction of the two networks. The other terms appearing in these equations have an obvious physical

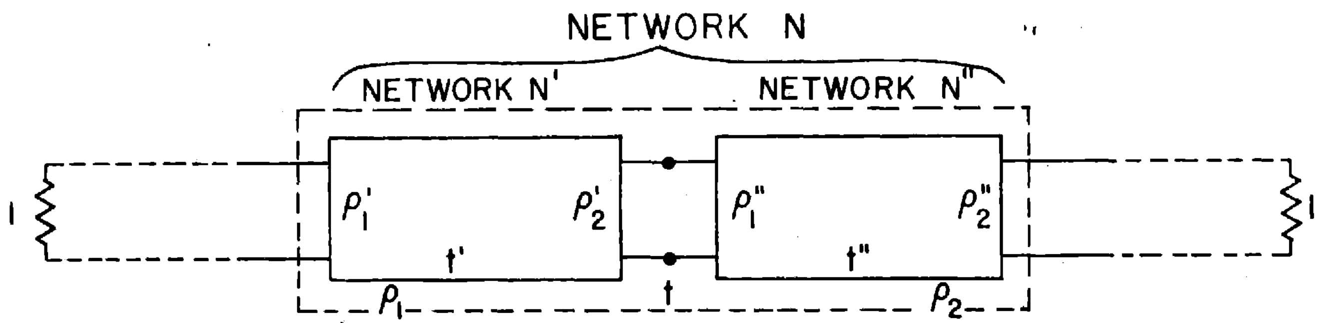


Figure 2.6 - Two reactive networks in cascade

significance from a transmission line point of view.

2.4 Method of attack of the problem - Before proceeding to the mathematical solution of the physical realizability problem, it is desirable to give some thought to the physics of the problem hoping to get some idea of the kind of results that are to be expected, and an indication of the proper mathematical approach. Consider then the system represented in Fig. 2.6, and let N' be the reactive network whose characteristics are fixed by the given impedance Z_2 while N" is the arbitrary matching network (it will be noted that these definitions imply a reversal of the original system as represented in Fig. 2.1). The first question that one is likely to ask himself is: are there any characteristics of N' which must belong also to the whole network N, irrespective of N"? A partial answer to this question is suggested immediately by the physical structure of the system. If t' is zero at a real frequency, that is, at any point of the imaginary axis of the λ-plane, then a wave of that frequency traveling from left to right would be completely stopped by N' so that no part of the wave would come out of N" or even enter it. It follows that any point of the imaginary axis which is a zero of the transmission coefficient t' must necessarily be a zero of transmission for the whole network N, and, therefore, must be a zero of t. Furthermore, the reflected wave at the input terminals cannot depend on N" if no part of the incident power reaches N". Therefore, ρ_1 must be equal to ρ_1' for any value of $\lambda = j\omega$ for which t' is zero.

It is natural at this point to investigate the effect of a zero of t' at real frequencies, whose multiplicity is larger than one. It is hard to make any definite statement in this regard on the basis of simple physical reasoning, however, it is reasonable to expect that the zero will appear with the same multiplicity in t. Moreover, it is also to be expected that, at frequencies in the neighborhood of the zero of t', the behavior of ρ_1 will be, to some extent, independent of N". This idea indicates the possibility that a certain number of derivatives of ρ_1, computed at the zero of t', might be independent of N" and, therefore, might be equal to the corresponding derivatives of ρ_1'.

Suppose, now, that t' has a zero somewhere in the right half of the complex λ-plane. It is clear that the behavior of a network for values of λ having a positive real part σ, is the same as the behavior, for purely imaginary values of λ, of a network obtained from the previous one by adding a resistance σL in series with every inductance L and a conductance σC in parallel with every capacitance C. It follows that if t' has a zero for a value of λ with a positive real part σ_ν, one can make a network consisting of passive elements such that its behavior for $\lambda = j\omega$ is identical to the behavior of the original network for $\lambda = \sigma_\nu + j\omega$. The zero of transmission of this new network lies on the imaginary axis and one can apply to it the results obtained above. These results are thus extended to zeros of t' lying in the right half of the complex λ-plane. If σ_ν were negative, that is, if the zero of t'

were in the left half plane, the above line of thought would be incorrect because one would be dealing, in that case, with an equivalent network containing negative resistance, that is , power sources. It will be shown later that zeros of t' lying in the left half plane are not necessarily zeros of t. One must remember, however, that for a reactive network the presence of a zero of transmission in the left half plane implies the presence of a symmetrical zero in the right half plane (but not vice versa). Therefore, since the zeros of t' in the right half plane must necessarily be zeros of t, the elimination of any zero in the left half plane, which may result from an appropriate design of N", is, in a certain sense, only apparent.

On the basis of the above discussion, one can conclude that any zero of transmission of the original network N', that is, any zero of t', which lies in the right half or on the imaginary axis of the λ-plane must also be a zero of transmission of the whole network N, that is, a zero of t. At any such zero of t, the reflection coefficient ρ_1 is independent of N" and, therefore, is equal to ρ_1'. Furthermore, there is a good indication that, in the case of a zero of t' of multiplicity larger than one, the corresponding zero of t will have the same multiplicity, and that a number of successive derivatives of ρ_1 computed at the zero of t will be equal to the corresponding derivatives of ρ_1'. These conclusions suggest a definite approach to the solution of the problem and promise to be a useful guide in the mathematical analysis that will follow.

2.5 <u>Zeros of transmission</u> - Physical reasoning indicates, as pointed out in the preceding section, that the first step in the mathematical solution of the problem should be a study of the conditions under which the zeros of transmission of the given network N' appear as zeros of transmission of the whole network N. Consider, for this purpose, the expression for the transmission coefficient of N given by eq. 20, which is rewritten below for convenience.

$$t = \frac{t' t''}{1 - \rho_2' \rho_1''} \qquad (23)$$

Suppose, first, that t' has a zero of multiplicity n at some point λ_ν in the right half of the complex λ plane. Since t" is analytic in the right half of the plane, it is clear that t must have a zero of the same multiplicity at λ_ν unless $1 - \rho_2' \rho_1''$ is zero at that point. On the other hand, any reflection coefficient is analytic in the right half plane, that is, for $\sigma \gtreqless 0$ and satisfies the condition $|\rho| \leq 1$ on the imaginary axis. It follows from the maximum modulus theorem [8] that $|\rho| < 1$ for $\sigma > 0$, and, therefore, that the denominator of (23) cannot be zero at any point in the right half plane ($\sigma \neq 0$). In conclusion any zero of t' in the right half plane must necessarily appear in t with at least the same multiplicity.

Consider next a zero of t' located on the imaginary axis at a frequency ω_ν. In this case the denominator of eq. 23 will be zero for $\lambda = j\omega_\nu$ if $\rho_1'' = \frac{1}{\rho_2'}$, that is, since $|\rho_1''| = 1$, if

$\rho_1'' = \overline{\rho_2'}$. On the other hand, if $|\rho_2'| = 1$ for $\lambda = j\omega_\nu$, t" must have a zero at the same point, so that in the end the zero of t' will necessarily appear in t with, at least, the same multiplicity. One might object to this conclusion on the ground that $(1 - \rho_2' \rho_1'')$ could have a zero of higher multiplicity at $\lambda = j\omega_\nu$. This situation, however, is not possible for the following reasons. If $1 - \rho_2' \rho_1''$ had a double zero at $\lambda = j\omega_\nu$ one would have:

$$\left|\rho_2'\right|_{\lambda = j\omega_\nu} = \left|\rho_1''\right|_{\lambda = j\omega_\nu} = 1 \tag{24}$$

$$\left[\frac{1}{\rho_2'} \frac{d\rho_2'}{d\lambda}\right]_{\lambda = j\omega_\nu} = -\left[\frac{1}{\rho_1''} \frac{d\rho_1''}{d\lambda}\right]_{\lambda = j\omega_\nu} \tag{25}$$

$$\left[\frac{d \ln|\rho_2'|}{d\sigma}\right]_{\lambda = j\omega_\nu} = -\left[\frac{d \ln|\rho_1''|}{d\sigma}\right]_{\lambda = j\omega_\nu} \tag{26}$$

On the other hand, both $|\rho_2'|$ and $|\rho_1''|$ must decrease with a positive increment of σ as required by the maximum modulus theorem, and, therefore, eq. 26 cannot be satisfied. In conclusion, if t' has a zero at a point λ_ν with a non-negative real part ($\sigma_\nu \geq 0$), t has a zero at λ_ν with, at least, the same multiplicity.

The case of a zero of t' at any point in the left half plane leads to the opposite result, since the denominator $(1 - \rho_2' \rho_1'')$ can have a zero of higher multiplicity at any such point, without t" being also zero. However, as pointed out in the preceding section, a zero in the left half plane implies the existence of a symmetrical zero in the right half plane which cannot be eliminated. Moreover, since these zeros of transmission must originally appear in quadruplets so that the numerator of t can be either an even or an odd function of λ , the elimination of a pair of zeros in the left half plane at λ_ν and $\overline{\lambda_\nu}$ requires the presence of a corresponding pair of poles in the reflection coefficients as well as in the transmission coefficient. The end result will be, according to the discussion in Sec. 2.2, that a quadruplet of singularities formed by a pair of conjugate poles in the left half plane at λ_ν and $\overline{\lambda_\nu}$ and a pair of conjugate zeros symmetrically located in the right half plane will be present in either one or the other of the reflection coefficients. In the case of multiple zeros of transmission, the multiplicity of the pair of zeros eliminated will be equal to the sum of the multiplicities of the corresponding quadruplets of singularities in the two reflection coefficients.

The above analysis can be extended step by step to the case of any number of two-terminal-pair reactive networks connected in cascade. One begins to suspect at this point that any arbitrary two-terminal-pair reactive network might be realizable in the form of a chain of elementary networks, each of them representing a zero of transmission, a pair of zeros, or a quadruplet of zeros, depending on their location. Darlington[3] showed this to be actually the case, by developing a synthesis procedure which leads to a cascade connection of sections of the four types shown in Fig. 2.7, and their duals. Type A corresponds to a zero of transmission at infinity, type B to a zero at the origin, type C to a pair of

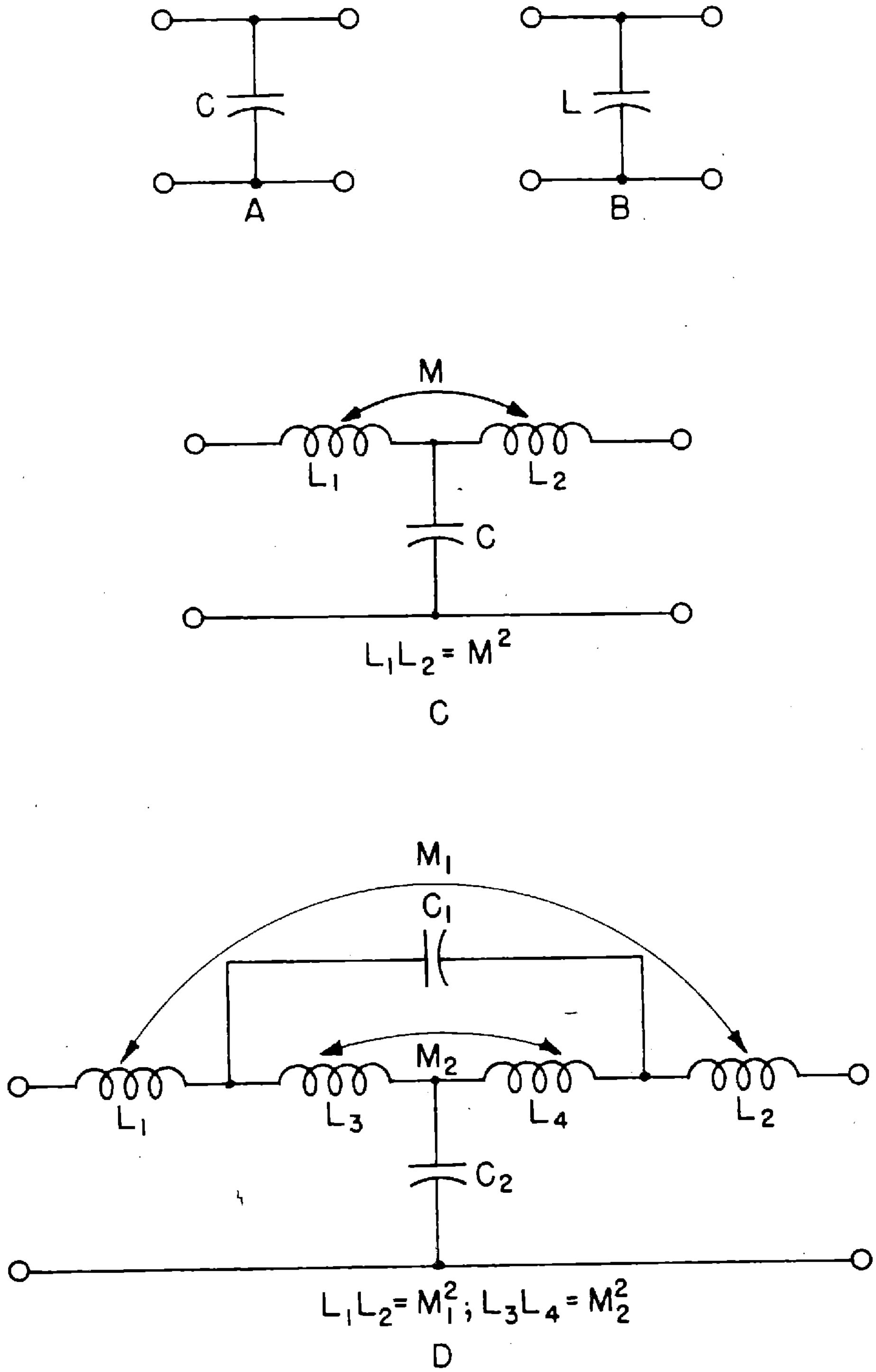


Figure 2.7 - Elementary sections representing different types of zeros of transmission

conjugate zeros on the imaginary axis or a pair of symmetrical zeros on the real axis, type D to a quadruplet of zeros symmetrically located with respect to the origin. The order in which the sections are connected in cascade is immaterial as far as the physical realizability is concerned, but, of course, the values of the elements in each section will vary when the order of the sections is changed. This synthesis procedure provides a further proof for the fact that the zeros of transmission of a reactive network cannot be effectively eliminated by the addition of another network in cascade.

2.6 <u>Behavior of the reflection coefficient in the vicinity of a zero of transmission</u> - It was suggested in Sec. 2.4, on the basis of physical reasoning, that a certain number of successive derivatives of the input reflection coefficient ρ_1 of the network N might be independent of N" for any value of λ at which the transmission coefficient t' of N' has a zero. It is reasonable to expect, in addition, that the number of these derivatives will be proportional to the multiplicity of the zero of t'.

The logical starting point for a mathematical investigation of this question is eq. 21 which is rewritten below for convenience.

$$\rho_1 = \rho_1' + \rho_1'' \frac{(t')^2}{1 - \rho_2' \rho_1''} \qquad (27)$$

If t' has a zero of multiplicity n at λ_ν in the right half plane, the numerator of the second term in the right hand side of eq. 27 will contain a factor $(\lambda - \lambda_\nu)^{2n}$. The denominator of this term must be different from zero and finite at all points in the right half of the λ-plane, as already pointed out in the preceding section. It follows that this term and its first 2n - 1 derivatives must vanish for $\lambda = \lambda_\nu$, so that one obtains:

$$\rho_1 = \rho_1' \tag{28}$$

$$\left[\frac{d^m \rho_1}{d\lambda^m}\right]_{\lambda=\lambda_\nu} = \left[\frac{d^m \rho_1'}{d\lambda^m}\right]_{\lambda=\lambda_\nu} \quad \text{for} \quad m \leq 2n-1 \tag{29}$$

Consider now the case of a zero of t' at a point $j\omega_\nu$ of the imaginary axis. One can follow the same reasoning as in the previous case with the only difference that the denominator $(1 - \rho_2' \rho_1'')$ may have a simple (not a multiple) zero for $\lambda = j\omega_\nu$. This situation leads to what may be called a degenerate case, because t" must then have a zero at the same point which effectively combines with the zero of t'. In fact, the resulting multiplicity of the zero of the over-all transmission coefficient t is, in this case, one less than the sum of the multiplicities of the zeros of t' and t" at the same point, as indicated by eq. 23. The physical significance of this degenerate case and the way of handling it will be discussed later in Sec. 2.9. It will be assumed, for the moment, that eqs. 28 and 29 apply to the case of a zero of t' on the imaginary axis as well as to the case of a zero in the right half plane.

In the case of a zero of transmission in the left half plane ρ_1 and its derivatives will, or will not, be equal to ρ_1' and the corresponding derivatives, depending on the multiplicity of the zero of the denominator $(1 - \rho_2' \rho_1'')$. It is not necessary, however, to investigate this situation in detail, because the behaviors of ρ_1 and ρ_1' at a zero of transmission in the left half plane depend on the behaviors of the same functions at the symmetrically located zero in the right half plane through eq. 12 and the relation $\rho(\bar{\lambda}) = \overline{\rho(\lambda)}$.

It should be noted at this point, that the fact that ρ_1 and its first 2n - 1 derivatives are independent of N'' at a zero of t' of multiplicity n is not a speciality of reflection coefficients. As one would expect, the same is true for any driving-point function and, in particular, impedance. In fact, if z_{11}', z_{22}', z_{12}' are the open circuit impedances of the network N', and Z_1'' is the input impedance of the network N'' (terminated in any arbitrary impedance), one has for the input impedance Z_1 of the whole network:

$$Z_1 = z_{11}' - \frac{(z_{12}')^2}{z_{22}' + Z_1''} \tag{30}$$

This equation has a form very similar to eq. 27 and leads to the same type of results, since it can be shown that all the zeros of t' appear in z_{12}' with the same multiplicity.

2.7 Number of independent conditions imposed by N' on N - The analysis carried out in the preceding sections yields a certain number of necessary conditions that must be satisfied by the function ρ_1 in order to be physically realizable by means of the two networks N' and N". It is desirable at this point to consider the actual number of independent conditions that can be imposed by the fact that N' is specified.

Suppose the sum of the multiplicities of the zeros of t' is m; for this purpose all the zeros in the left half plane are included, even if some of them are eliminated by corresponding roots of the denominator of t'. The network N' is completely specified (Sec. 2.2) by the zeros of t' and the numerator of either reflection coefficient (including the constant multiplier). On the other hand, the degree of the polynomial at the numerator of a reflection coefficient is equal to the total number of zeros of transmission, that is, to m. It follows that m + 1 independent parameters (real numbers) are required to specify completely N', in addition to the knowledge of the zeros of t'.

The above statement can be checked by observing that the sections used in Darlington's synthesis procedure (Fig. 2.7) contain the correct number of elements. In fact, a zero of t' at the origin or at infinity leads to a section of type A or type B employing a single element, that is, it requires a single parameter. A section of type C, which corresponds to a pair of zeros on either axis, contains three elements, specified by the location of one of the zeros and by two additional parameters. A section of type D, which corresponds to a quadruplet of zeros, has six elements specified by the location of one zero (two real numbers) and four additional parameters. The one parameter in addition to the m parameters corresponds to an ideal transformer required by the fact that both resistive terminations are made equal to one ohm. The ratio of this ideal transformer can be changed arbitrarily by simply connecting in cascade with the network an additional ideal transformer or otherwise performing an equivalent operation. It follows that the additional parameter will not impose any limitation upon the characteristics of the network N.

In conclusion the fact that the network N' is fixed will impose m independent conditions upon the characteristics of N in addition to the requirement that t must have all the zeros of t' (the ones in the left half plane potentially at least) with at least the same multiplicities. It appears, therefore, that the requirements on the derivatives of the reflection coefficient derived in Sec. 2.6 lead to imposing on N a number of conditions which exceeds the number of parameters by which N' is completely specified. It should be noted, in this respect, that the derivatives of ρ_1 and ρ_1' are, in general, complex quantities, and, therefore, each derivative yields two separate conditions. It follows that the derivatives considered must depend on one another and also upon the location of the zeros of transmission at which they are computed.

One observes, first of all, in this regard, that any two derivatives computed at conjugate zeros must be conjugates of each other. Furthermore, the derivatives at zeros in the left half plane are related to the derivatives at the corresponding zeros in the right half plane through eq. 12. In the second place, one observes that the magnitude of the reflection coefficient becomes unity at any zero of transmission on the imaginary axis and, moreover, a certain number of its derivatives along the imaginary axis must vanish in the case of a multiple zero. This fact indicates that it might be better to consider the derivatives of $\ln \frac{1}{\rho_1}$, rather than the derivatives of ρ_1.

2.8 <u>The behavior of the function $\ln \frac{1}{\rho_1}$</u> - It is observed, first, that the n^{th} derivative of $\ln \frac{1}{\rho_1}$ can always be expressed in terms of ρ_1 and its first n derivatives. Special care must be used, as it will be shown later, when ρ_1 has a zero at the point considered and, therefore, $\ln \frac{1}{\rho_1}$ is singular at that point. It was shown in Sec. 2.6 that, if the n^{th} derivative of ρ_1 is fixed by N', the values of the preceding n - 1 derivatives and of ρ_1 itself are also fixed. It follows that the n^{th} derivative of $\ln \frac{1}{\rho_1}$ computed at the same point is also fixed by N' and must, therefore, be equal to the corresponding derivative of $\ln \frac{1}{\rho_1'}$. The derivatives of $\ln \frac{1}{\rho_1}$ at any point λ_ν, differ only by a factorial from the coefficients of the Taylor series for $\ln \frac{1}{\rho_1}$ about the point λ_ν. It follows that the first 2n terms of the Taylor series for $\ln \frac{1}{\rho_1}$ about a zero of transmission of multiplicity n are equal to the corresponding terms of the series for $\ln \frac{1}{\rho_1'}$ about the same point. This statement, of course, does not apply to zeros in the left half plane; also the possibility of degenerate zeros on the imaginary axis is still overlooked for the present. Note, also, that the first term of the series is the value of the function, so that the 2n terms considered correspond to derivatives of $\ln \frac{1}{\rho_1}$ up to and including the $(2n - 1)^{th}$.

To determine the 2m independent conditions imposed on ρ_1 (corresponding to the 2m parameters which, in addition to the zeros of transmission, specify N!) one must study in more detail the properties of the coefficients of the Taylor series for $\ln \frac{1}{\rho_1}$ about the five different types of zeros of transmission. Convenient relations will be derived at the same time between these coefficients and the zeros and poles of ρ_1.

Consider first the case of a zero of transmission at the origin, and let its multiplicity be equal to n. The real part of $\ln \frac{1}{\rho_1}$, that is, $\ln \frac{1}{|\rho_1|}$, is an even function of $j\omega$ on the imaginary axis, since $\rho_1(-j\omega) = \overline{\rho_1(j\omega)}$. For the same reason, the imaginary part of $\ln \frac{1}{\rho_1}$, that is, the phase of $\frac{1}{\rho_1}$ is an odd function of $j\omega$ on the imaginary axis. It follows that both the odd and the even derivatives of $\ln \frac{1}{\rho_1}$ at the origin are real. The phase of $\frac{1}{\rho_1}$ at the origin may be either zero or $\pm\pi$ depending on whether the network behaves at zero frequency as a capacitance or as an inductance; the magnitude of ρ_1 at zero frequency is, of course, equal to one. Further information about the derivatives of $\ln \frac{1}{\rho_1}$ can be obtained from eq. 4, keeping in mind that $|t|^2$ has a zero of multiplicity 2n at the origin. One has then

$$\left[\frac{d^k |\rho_1|^2}{d(j\omega)^k}\right]_{\lambda=0} = 0 \qquad k \leq 2n-1 \qquad (31)$$

It follows that

$$\left[\frac{d^k}{d(j\omega)^k} \ln \frac{1}{|\rho_1|}\right]_{\lambda=0} = 0 \qquad k \leq 2n-1 \qquad (32)$$

Since all the even derivatives of the phase of ρ_1 are zero at the origin, it can be concluded that all the even derivatives of $\ln \frac{1}{\rho_1}$ up to and including the $2(n-1)^{th}$ vanish completely at the origin. In other words, the first n - 1 coefficients of the even powers of λ in the Taylor series for $\ln \frac{1}{\rho_1}$ about the origin are identically zero. The coefficients of the odd powers of λ are real numbers since the corresponding derivatives are real, as pointed out above. The Taylor series for $\ln \frac{1}{\rho_1}$ about the origin can then be written in the form

$$\ln \frac{1}{\rho_1} = \begin{Bmatrix} 0 \\ \pm j\pi \end{Bmatrix} + A_1^\circ \lambda + A_3^\circ \lambda^3 \cdots\cdots A_{2n-3}^\circ \lambda^{2n-3} + A_{2n-1}^\circ \lambda^{2n-1} + \cdots\cdots \quad (33)$$

The A°'s must be related to the poles λ_{pi} and zeros λ_{oi} of ρ_1, which can be written in the form

$$\rho_1 = \pm \frac{\left(1-\frac{\lambda}{\lambda_{o1}}\right)\left(1-\frac{\lambda}{\lambda_{o2}}\right)\cdots\cdots\left(1-\frac{\lambda}{\lambda_{om}}\right)}{\left(1-\frac{\lambda}{\lambda_{p1}}\right)\left(1-\frac{\lambda}{\lambda_{p2}}\right)\cdots\cdots\left(1-\frac{\lambda}{\lambda_{om}}\right)} \quad (34)$$

The logarithm of each root factor can be written as a Taylor series about the origin as follows:

$$-\ln\left(1-\frac{\lambda}{\lambda_i}\right) = \frac{\lambda}{\lambda_i} + \frac{1}{2}\left(\frac{\lambda}{\lambda_i}\right)^2 + \frac{1}{3}\left(\frac{\lambda}{\lambda_i}\right)^3 + \cdots\cdots \quad (35)$$

Collecting the terms with the same power of λ in the series for all root factors yields

$$\ln \frac{1}{\rho_1} = \begin{Bmatrix} 0 \\ \pm j\pi \end{Bmatrix} + \left[\sum_i \lambda_{oi}^{-1} - \sum_i \lambda_{pi}^{-1}\right]\lambda + \frac{1}{2}\left[\sum_i \lambda_{oi}^{-2} - \sum_i \lambda_{pi}^{-2}\right]\lambda^2 + \frac{1}{3}\left[\sum_i \lambda_{oi}^{-3} - \sum_i \lambda_{pi}^{-3}\right]\lambda^3 + \cdots\cdots \quad (36)$$

By comparing this equation with eq. 33 one obtains

$$A^{o}_{2k+1} = \frac{1}{2k+1}\left[\sum_i \lambda_{oi}^{-(2k+1)} - \sum_i \lambda_{pi}^{-(2k+1)}\right] \tag{37}$$

$$\sum_i \lambda_{oi}^{-2k} = \sum_i \lambda_{pi}^{-2k} \qquad \text{(integer } k \leq n\text{)} \tag{38}$$

Equation 38 is a consequence of the fact that the network has a zero of transmission at the origin with multiplicity n. Equation 37, on the other hand, yields a set of n equations that must be satisfied by the zeros and poles of ρ_1, since A_1^o, A_3^o, ... A^o_{2n-1} must be the same for ρ_1 and ρ_1'. Since the λ_p's and the λ_o's are present in conjugate pairs, the A^o's are evidently real numbers, as it was pointed out before.

Consider next the case of a zero of transmission at infinity with multiplicity equal to n. To obtain the Taylor series for $\ln \frac{1}{\rho_1}$ about the point at infinity one must first make a change of variable, such as $Z = \frac{1}{\lambda}$, which transforms the point at infinity of the plane into the origin of the Z plane and vice versa. Then one proceeds exactly as in the previous case; the final results can be written by inspection.

$$\ln \frac{1}{\rho_1} = \begin{Bmatrix} 0 \\ \pm\pi \end{Bmatrix} + A_1^{\infty} Z + A_3^{\infty} Z^3 + \ldots A^{\infty}_{2n-3} Z^{2n-3} + A^{\infty}_{2n-1} Z^{2n-1} + \ldots \tag{39}$$

$$A^{\infty}_{2k+1} = \frac{1}{2k+1}\left[\sum_i \lambda_{oi}^{2k+1} - \sum \lambda_{pi}^{2k+1}\right] \tag{40}$$

$$\sum_i \lambda_{oi}^{2k} = \sum_i \lambda_{pi}^{2k} \qquad \text{(integer } k < n) \qquad (41)$$

The coefficients of the even powers of Z are again equal to zero up to $A_{2(n-1)}^{\infty}$ included, and all the A^{∞}'s are real numbers. Equation 41 results from the presence of a zero of transmission of multiplicity n at the point at infinity of the λ plane. Equation 40 yields, again, a set of n equations that must be satisfied by the zeros and poles of ρ_1, since A_1^{∞}, A_3^{∞} .....A_{2n-1}^{∞} are the same for ρ_1 and ρ_1'.

Consider now the case of a pair of conjugate zeros of transmission on the imaginary axis at $j\omega_\nu$ and $-j\omega_\nu$. It is not necessary to determine the series about both points because the coefficients of one series are the conjugates of the coefficients of the other series. Consider then the case of the zero at $j\omega_\nu$. One observes, first of all, that eq. 4 yields again:

$$\cdot\left[\ln \frac{1}{|\rho_1|}\right]_{\lambda = j\omega_\nu} = 0 \qquad (42)$$

$$\left[\frac{d^k}{d(j\omega)^k} \ln \frac{1}{|\rho_1|}\right]_{\lambda = j\omega_\nu} = 0 \qquad k \leq 2n - 1 \qquad (43)$$

Therefore, the odd derivatives of $\ln \frac{1}{\rho_1}$ are real while the even derivatives are imaginary up to the order $(2n - 1)$ included. The Taylor series is then written in the form:

$$\ln \frac{1}{\rho_1} = jB_0^{\omega_\nu} + A_1^{\omega_\nu}(\lambda - j\omega_\nu) + jB_2^{\omega_\nu}(\lambda - j\omega_\nu)^2 + A_3^{\omega_\nu}(\lambda - j\omega_\nu)^3 + \cdots\cdots$$

$$+ jB_{2n-2}^{\omega_\nu}(\lambda - j\omega_\nu)^{2n-2} + A_{2n-1}^{\omega_\nu}(\lambda - j\omega_\nu)^{2n-1} + \cdots\cdots \qquad (44)$$

To determine the coefficients of this series in terms of the zeros and poles of ρ_1 one writes $\ln \frac{1}{\rho_1}$ in the form:

$$\ln \frac{1}{\rho_1} = \left[\ln \frac{1}{\rho_1}\right]_{\lambda = j\omega_\nu} + \ln \frac{\left(1 - \frac{\lambda - j\omega_\nu}{\lambda_{p_1} - j\omega_\nu}\right)\left(1 - \frac{\lambda - j\omega_\nu}{\lambda_{p_2} - j\omega_\nu}\right)\cdots\cdots\left(1 - \frac{\lambda - j\omega_\nu}{\lambda_{p_m} - j\omega_\nu}\right)}{\left(1 - \frac{\lambda - j\omega_\nu}{\lambda_{o_1} - j\omega_\nu}\right)\left(1 - \frac{\lambda - j\omega_\nu}{\lambda_{o_2} - j\omega_\nu}\right)\cdots\cdots\left(1 - \frac{\lambda - j\omega_\nu}{\lambda_{o_m} - j\omega_\nu}\right)} \qquad (45)$$

By expanding the logarithm of each root factor in a series, then collecting the terms with the same power of $(\lambda - j\omega_\nu)$ and finally comparing the series resulting with eq. 44, one obtains

$$\left[\ln \frac{1}{\rho_1}\right]_{\lambda = j\omega_\nu} = j\, B_0^{\omega_\nu} \qquad (46)$$

$$\frac{1}{k}\left[\sum_i (\lambda_o - j\omega_\nu)^{-k} - \sum_i (\lambda_p - j\omega_\nu)^{-k}\right] = \begin{cases} j\, B_k^{\omega_\nu} & \text{for } k \text{ even} < 2n \\ A_k^{\omega_\nu} & \text{for } k \text{ odd} < 2n \end{cases} \qquad (47)$$

Equations 46 and 47 yield a set of 2n equations that must be satisfied by ρ_1, since the $A_k^{\omega_\nu}$'s and the $B_k^{\omega_\nu}$'s are the same for ρ_1 and ρ_1' up to and including $k = 2n - 1$.

In considering the case of a pair of zeros of transmission with multiplicity n, symmetrically located on the real axis at $+\sigma_\nu$ and $-\sigma_\nu$, one must remember that the reflection coefficient ρ_1 may have a zero of order n_o at σ_ν, in which case it has also a pole of order n_o at $-\sigma_\nu$. Since the function $\ln \frac{1}{\rho_1}$ is then singular at these two points one considers in its place the function

$$\ln \left[\frac{1}{\rho_1} \left(\frac{\lambda - \sigma_\nu}{\lambda + \sigma_\nu} \right)^{n_o} \right] \tag{48}$$

from which the two singularities have been removed. By multiplying eq. 27 by $\left(\frac{\lambda + \sigma_\nu}{\lambda - \sigma_\nu}\right)^{n_o}$ and following the same line of thought as in the case of $\ln \frac{1}{\rho_1}$, one can show without difficulty that the value of the new function (48) and its first $(2n - n_o - 1)$ derivatives at the point σ_ν are independent of N". Using the Taylor series for this function, $\ln \frac{1}{\rho_1}$ can be written finally in the form:

$$\ln \frac{1}{\rho_1} = n_o \ln \frac{\lambda + \sigma_\nu}{\lambda - \sigma_\nu} + A_o^{\sigma_\nu} + \left\{ \begin{matrix} 0 \\ \pm j\pi \end{matrix} \right\} + A_o^{\sigma_\nu}(\lambda - \sigma_\nu) + A_2^{\sigma_\nu}(\lambda - \sigma_\nu)^2 + \ldots\ldots$$

$$+ A^{\sigma_\nu}_{(2n-n_o-2)}(\lambda - \sigma_\nu)^{2n-n_o-2} + A^{\sigma_\nu}_{2n-n_o-1}(\lambda - \sigma_\nu)^{2n-n_o-1} + \ldots\ldots \tag{49}$$

All the A^{σ_ν}'s are real quantities since ρ_1 is by definition real on the real axis; their expressions in terms of the zeros and poles of ρ_1 can be written by inspection.

$$\ln \left[\frac{1}{\rho_1} \left(\frac{\lambda - \sigma_\nu}{\lambda + \sigma_\nu} \right)^{n_o} \right]_{\lambda = \sigma_\nu} = A_o^{\sigma_\nu} + \left\{ \begin{matrix} 0 \\ \pm j\pi \end{matrix} \right\} \tag{50}$$

$$\frac{1}{k} \left[\sum_i (\lambda_{oi} - \sigma_\nu)^{-k} - \sum_i (\lambda_{pi} - \sigma_\nu)^{-k} \right] = A_k^{\sigma_\nu} \tag{51}$$

It is understood, of course, that the zero at σ_ν and the pole at $-\sigma_\nu$ must be excluded from the summations. Equations 50 and 51 provide a set of $2n - n_o$ equations that must be satisfied by the zeros and poles of ρ_1. In addition to satisfying these equations, ρ_1 must include the factor $\left(\frac{\lambda - \sigma_\nu}{\lambda + \sigma_\nu}\right)^{n_o}$.

The case of a quadruplet of complex zeros of transmission symmetrically located with respect to the origin can be treated in a similar manner. Let the multiplicity of the zeros at λ_ν and $\overline{\lambda}_\nu$ in the right half plane be n, and let n_o be the multiplicity of the pair of zeros of ρ_1 at λ_ν and $\overline{\lambda}_\nu$ and of the symmetrical pair of poles at $-\lambda_\nu$ and $-\overline{\lambda}_\nu$. By operating on the function

$$\ln\left[\frac{1}{\rho 1}\left(\frac{(\lambda-\lambda_\nu)(\lambda-\overline{\lambda}_\nu)}{(\lambda+\lambda_\nu)(\lambda+\overline{\lambda}_\nu)}\right)^{n_o}\right] \tag{51}$$

one obtains

$$\ln\frac{1}{\rho_1} = n_o \ln\frac{(\lambda+\lambda_\nu)(\lambda+\overline{\lambda}_\nu)}{(\lambda-\lambda_\nu)(\lambda-\overline{\lambda}_\nu)} + \left(A_o^{\lambda_\nu} + j B_o^{\lambda_\nu}\right) + \left(A_o^{\lambda_\nu} + j B_1^{\lambda_\nu}\right)(\lambda-\lambda_\nu) +$$

$$+ \left(A_2^{\lambda_\nu} + j B_2^{\lambda_\nu}\right)(\lambda-\lambda_\nu)^2 + \ldots\ldots + \left(A_{2n-n_o-1}^{\lambda_\nu} + j B_{2n-n_o-1}^{\lambda_\nu}\right)(\lambda-\lambda_\nu)^{2n-n_o-1} + \ldots\ldots \tag{52}$$

where:

$$\ln\left[\frac{1}{\rho 1}\left(\frac{(\lambda-\lambda_\nu)(\lambda-\overline{\lambda}_\nu)}{(\lambda+\lambda_\nu)(\lambda+\overline{\lambda}_\nu)}\right)^{n_o}\right]_{\lambda=\lambda_\nu} = A_o^{\lambda_\nu} + j B_o^{\lambda_\nu} \tag{53}$$

$$\frac{1}{k}\left[\sum_i (\lambda_{o2} - \lambda_\nu)^{-k} - \sum (\lambda_{p2} - \lambda_\nu)^{-k}\right] = A_k^{\lambda_\nu} + j B_k^{\lambda_\nu} \tag{54}$$

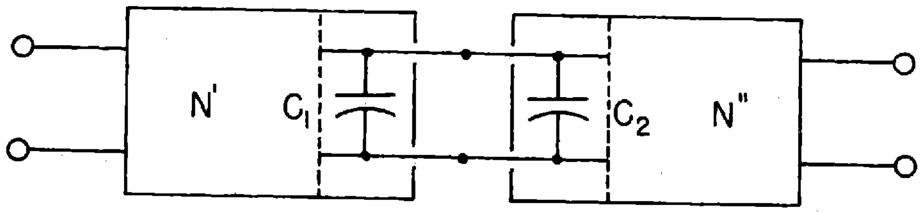
The zeros at λ_ν and $\overline{\lambda}_\nu$ and the poles at $-\lambda_\nu$ and $-\overline{\lambda}_\nu$ are excluded from the summations. The $A_k^{\lambda_\nu}$'s and $B_k^{\lambda_\nu}$'s are the same for ρ_1

and ρ_1' up to and including $k = 2n - n_o - 1$ so that eqs. 53 and 54 yield a set of $(2n - n_o)$ equations which must be satisfied by the zeros and poles of ρ_1. In addition, ρ_1 must include the factor

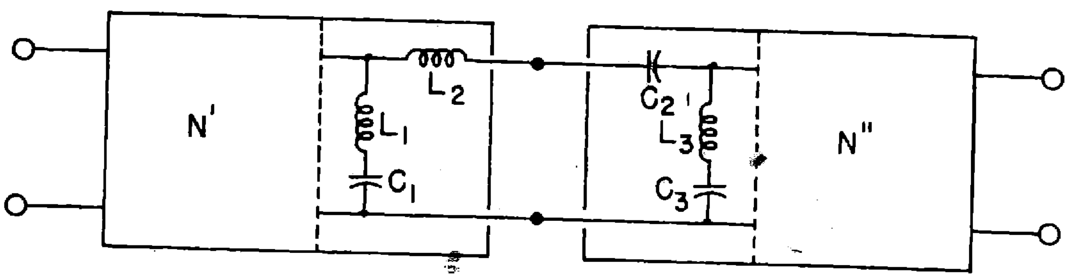
$$\left[\frac{(\lambda - \lambda_\nu)(\lambda - \overline{\lambda_\nu})}{(\lambda + \lambda_\nu)(\lambda + \overline{\lambda_\nu})}\right]^{n_o}.$$

2.9 Degenerate zeros of transmission - It was pointed out in Sec. 2.6 that a zero of transmission of N' on the imaginary axis may effectively combine with a similar zero of N" in such a way that the multiplicity of the zero of t becomes one less than the sum of the multiplicities of the zeros of t' and t" at the same point. This situation arises when the denominator $(1 - \rho_2' \rho_1")$ in eq. 27 has a simple zero coincident with the zero of t'. In other words, the reflection coefficient $\rho_1"$ must be the reciprocal of ρ_2' and, therefore, the impedances in the two directions measured at the common terminals of N' and N" at that frequency must be pure reactances with equal magnitudes and opposite signs.

Two simple examples are shown in Fig. 2.8 for the case of a zero at infinity (a) and a zero at $\lambda = j\omega_\nu$ (b). It is clear from a physical point of view that in these examples the zeros of transmission of N' and N" will combine in such a way that the $(2n - 1)^{th}$ derivative of ρ_1 (n is the multiplicity of the zero of t') will depend on N" as well as on N' and, therefore, will not be equal to the corresponding derivative of ρ_1'. Eq. 27 shows that this is true in the general case of a degenerate zero of transmission. The example of Fig. 2.8 indicates, however, that the change of the derivative of the input reflection coefficient when N" is connected to N'



(a)



$$L_1C_1 = L_2C_2 = L_3C_3 = \frac{1}{\omega_\nu^2}$$

(b)

Figure 2.8 - Examples of degenerate zeros of transmissions

must take place in a particular direction. For instance, in the case of Fig. 2.8 (a) the fact that the total shunt capacitance $C_1 + C_2$ is larger than the capacitance in N' alone must somehow restrict the possible change of the behavior of the input reflection coefficient at infinity.

This situation can be investigated by considering the $(2n - 1)^{th}$ derivative of eq. 30, at the frequency $j\omega_\nu$ where t' and, therefore, z_{12}' have a zero of multiplicity n. One obtains then, neglecting the terms that will obviously vanish,

$$\left[\frac{d^{2n-1} Z_1}{d(j\omega)^{2n-1}}\right]_{\omega=\omega_\nu} = \left[\frac{d^{2n-1} z_{11}'}{d(j\omega)^{2n-1}}\right]_{\omega=\omega_\nu} - \left[\frac{(2n)!\,(z_{12}')^2}{(j\omega - j\omega_\nu)^{2n-1}(z_{22}' + Z_1'')}\right]_{\omega=\omega_\nu} \qquad (55)$$

Since $(z_{22}' + Z_1'')$ has a simple zero at $\omega = \omega_\nu$ the last term of 55 is indeterminate and must be computed by taking the ratio of

$$\left[\frac{d}{dj\omega}\;\frac{(z_{12}')^2}{(j\omega - j\omega_\nu)^{2n-1}}\right]_{\omega=\omega_\nu} = \left[\frac{(z_{12}')^2}{(j\omega - j\omega_\nu)^{2n}}\right]_{\omega=\omega_\nu} \qquad (56)$$

and

$$\left[\frac{d(z_{22}' + Z_1'')}{d(j\omega)}\right]_{\omega=\omega_\nu} \qquad (57)$$

It can be seen by inspection that, since z_{12} is imaginary on the imaginary axis and has a zero of order n at $\omega = \omega_\nu$, the expression (56) is positive real when n is even, and negative real when n is odd. In regard to (57), one observes that Z_1'' is imaginary for $\lambda = j\omega$ and its first derivative at $\lambda = j\omega_\nu$ is independent of the resistive load connected to the output terminals on N", since t" has a zero at that point. It follows that, for our purposes, Z_1'' behaves as a reactance function, and, therefore, has a positive real slope at $\lambda = j\omega_\nu$. Since z_{22}' is actually a reactance function, the whole expression 57 must be positive real. One can then conclude that the $(2n - 1)^{th}$ derivative of Z_1 is always increased by the last term in (55) when n is odd and is always decreased when n is even. On the other hand

$$\left[\frac{d^{2n-1}}{d(j\omega)^{2n-1}} \ln \frac{1}{\rho_1}\right]_{\omega=\omega_\nu} = 2\left[\frac{d^{2n-1}}{d(j\omega)^{2n-1}} \operatorname{ctnh}^{-1} Z_1\right]_{\omega=\omega_\nu} \tag{58}$$

and the only term in (58) that involves the $(2n - 1)^{th}$ derivative of Z_1 is

$$+\left[\frac{2}{1-Z_1^2} \; \frac{d^{2n-1}Z_1}{d(j\omega)^{2n-1}}\right]_{\omega=\omega_\nu} \tag{59}$$

Since Z_1^2 is a negative real quantity for $\lambda = j\omega_\nu$, the expression 58 and, therefore, the value of the real coefficient $A_{2n-1}^{\omega_\nu}$ is always increased by the presence of N" when n is odd, and is always decreased when n is even. The physical significance of this restriction and its practical importance will become clear in the next chapter.

2.10 <u>Necessary and sufficient conditions for the physical realizability of N</u> - The analysis carried out in the preceding sections has led to the formulation of a number of necessary conditions that must be satisfied by the functions representing the network N in order to be physically realizable by means of the given network N' and the arbitrary network N" connected in cascade. These necessary conditions are summarized below for convenience.

With reference to Fig. 2.6 all the zeros of t', that is, the zeros of transmission of the given network N', which lie in the right half of the λ plane or on the imaginary axis must appear as zeros of t, that is, as zeros of transmission of N, with at least the same multiplicity as in t'. Moreover, a certain number of coefficients of the Taylor series for $\ln \frac{1}{\rho_1}$ about each of the zeros of transmission mentioned above must be equal to the corresponding coefficients for $\ln \frac{1}{\rho_1'}$, or, in other words, must be independent of N". The resulting number of real quantities independent of N" is equal to the multiplicity of the corresponding zero of transmission, in the case of a zero located at the origin or at infinity (eqs. 33 and 39), to twice the multiplicity in the case of a pair of zeros on either the imaginary or the real axis (eqs. 44 and 49), to four times the multiplicity in the case of a quadruplet of complex zeros (eq. 52).

If a pair of zeros of t' at $-\lambda_\nu$ and $-\bar{\lambda}_\nu$ in the left half plane is partially or totally eliminated by a pair of poles of t' of multiplicity n_o located at the same points, a number $2n_o$ of these real quantities independent of N" are missing. In this case, however, an equivalent number of conditions are imposed by the fact that ρ_1 must have a pair of poles of multiplicity n_o at $-\lambda_\nu$ and $-\overline{\lambda_\nu}$ and, therefore, a pair of zeros with the same multiplicity at λ_ν and $\overline{\lambda_\nu}$. The case of a degenerate zero on the imaginary axis does not lead to any special difficulty, as pointed out in Sec. 2.9. The total number of conditions imposed by ρ_1' and ρ_1 (in addition to those stated above) is thus equal, in all cases, to the sum of the multiplicities of all the zeros of transmission of t', that is, to m, as defined in Sec. 2.7. These conditions can be shown to be independent as follows.

It was pointed out in Sec. 2.5, that any two-terminal-pair reactive network can be constructed as a chain of sections each representing a simple zero of transmission, a pair of zeros, or a quadruplet of zeros; zeros of multiplicity n are represented by n similar sections. The order in which the sections representing different zeros are connected is immaterial as far as the physical realizability of the network is concerned. One can then divide the network in two separate parts in cascade, of which the second one contains all and only the sections representing a particular zero of transmission. These two parts could be identified, for the purpose of this discussion, with N' and N" of Fig. 2.6. It follows that all the coefficients considered above, which result from derivatives of $\ln \frac{1}{\rho_1}$ evaluated at the zeros of transmission represented by N' are independent of the coefficients resulting from derivatives of $\ln \frac{1}{\rho_1}$ evaluated at the zero represented by N". It can be concluded, thus, that coefficients related to different zeros of transmission (not of the same pair or quadruplet) are independent. A similar procedure can be applied to the coefficients resulting from successive derivatives at a zero of multiplicity n. In fact, the coefficients resulting from the first $2k - 1$ derivatives depend only on the first k section. The zeros and poles of the reflection coefficient which result from the elimination of zeros of transmission in the left half plane, can be taken out first in the form of all pass sections. Finally, the elements of the k^{th} section of a particular zero can be used to vary independently the coefficients resulting from the two corresponding derivatives. In fact a section of type A or B corresponds to one coefficient (A_{2k-1}), a section of type C corresponds to two coefficients (A_{2k-1} and A_{2k-2} or B_{2k-2}), a section of type D corresponds to four coefficients (A_{2k-1}, B_{2k-1}, A_{2k-2}, B_{2k-2}). It was pointed out in Sec. 2.7 that each of these sections has a number of elements just equal to the number of corresponding coefficients plus the number of real quantities (one or two) required to locate the zero of transmission represented by the section. One can show in each of the four cases by means of simple examples that the A's and B's can actually be varied independently by readjusting the values of the elements of the corresponding section.

It can be concluded, therefore, that the conditions for the physical realizability of the network N (when N' is given) which are stated above are independent of one another. On the other hand, these conditions are equal in number to the elements of the network N' and, therefore, specify completely N' apart from the ratio of an ideal transformer and from the sign of the reflection coefficient ρ_1' which simply differentiates N' from the reciprocal network.

To prove that these conditions of physical realizability are sufficient as well as necessary one needs only to observe that the network N defined by ρ_1 can be constructed in two parts, the first of which contains all the sections representing the zeros of t'. If the correct sign of ρ_1 is used, this first part, with an appropriate ideal transformer at the output terminals can be identified with the given network N', because all the sections contained in it are completely specified by the conditions of physical realizability imposed on ρ_1, which on the other hand specify completely N'. The second part of the network is certainly physically realizable because it consists of the sections representing all the zeros of transmission of N which are not zeros of N'.

In practice, it is not necessary to determine the elements of the network N' before proceeding to the synthesis of N". The reflection coefficient ρ_1'' can be determined from ρ_1, ρ_1', ρ_2' and t' (which are known) by means of eq. 27.

$$\rho_1'' = \frac{1}{\rho_2' + \frac{(t')^2}{\rho_1 - \rho_1'}} \tag{60}$$

This equation could be used to prove directly that the above conditions of physical realizability are sufficient as well as necessary, if one could show that, when these conditions are satisfied by ρ_1, $|\rho_1''|$ is smaller than one on the imaginary axis and all the poles of ρ_1'' lie in the left half plane. Such a proof, however, could not be obtained by the author up to this time.

2.11 Final remarks - The preceding sections present a complete solution to the physical realizability problem as stated at the end of Sec. 2.1. This problem was shown to be equivalent to the first part of the matching problem as stated at the beginning of the same section. However, a few remarks should be made in this connection for the sake of completeness and clarity.

In a practical matching problem, the functions ρ_1' and t' are given indirectly, through the reflection coefficient ρ_2' which in turn is specified by the load impedance Z_L, normalized with respect to the source resistance.

$$\rho_2' = \frac{Z_L - 1}{Z_L + 1} \tag{61}$$

The network N', and, therefore, ρ_1' and t' are completely specified by ρ_2', apart from an arbitrary all-pass network connected to terminals (1) of N'. This arbitrary all-pass network, however, can be neglected, because it does not produce any reflection by itself nor does it change the phase of any other reflection when N' is driven from terminals (2). Therefore, for our purposes, N' is completely specified by Z_L. On the other hand, one may observe that the reflection coefficient which is measured in an actual matching problem is not ρ_1 but ρ_2, since the source is connected to terminals 2 of N. This fact, however, is immaterial since only the magnitude of ρ_2 is of importance in most cases and $|\rho_1| = |\rho_2|$ for $\lambda = j\omega$. Moreover if one were interested in the whole function ρ_2, it would be a simple matter to express the conditions of physical realizability in terms of the zeros and poles of ρ_2, since they are very simply related to the corresponding singularities of ρ_1 as indicated in Sec. 2.2.

The fundamental criticism that can be made of the results obtained so far is that the conditions of physical realizability, in the form presented in this chapter, give no indication of the tolerance of match that must be allowed for a given bandwidth. It is the purpose of the next chapter to express these conditions of physical realizability in terms of the behavior of the reflection coefficient on the imaginary axis, that is, at real frequencies. The relations obtained will point out clearly the nature of the limitations on the tolerance and on the bandwidth.

CHAPTER III

Limitations on the Behavior of the Reflection Coefficient at Real Frequencies

3.1 General considerations - This chapter is devoted to the solution of the second part of the matching problem stated in Sec. 1.5. The approach to be followed is based on the transformation of the conditions of physical realizability, derived in the preceding chapter, into a set of relations suitable for the determination of the theoretical limitations on the bandwidth of match and on the minimum tolerance.

The first requirement that these relations must meet in order to be useful in a practical problem is that they should involve the behavior of the magnitude of the reflection coefficient on the imaginary axis of the λ plane, that is, over the real frequency spectrum. The limitation found by Bode satisfies this condition, and, furthermore, indicates that, in general, integral relations involving $\ln \frac{1}{|\rho_1|}$ might be quite appropriate for the desired purpose. One observes, on the other hand, that the conditions of physical realizability derived in the preceding chapter involve the derivatives of $\ln \frac{1}{\rho_1}$ at points in the right half of the λ plane or on the imaginary axis. These two facts indicate that a contour integration of the function $\ln \frac{1}{\rho_1}$ might be the appropriate way of obtaining relations of the type desired. The logical contour of integration for this purpose is that formed by the imaginary axis and a semicircle in the right half plane of radius approaching infinity. As a matter of fact this is just the procedure followed by Bode in deriving his integral relation.

A mathematical difficulty arises as soon as one considers the details of the procedure suggested above, namely that the function $\ln \frac{1}{\rho_1}$ has logarithmic singularities within the region enclosed by the contour of integration whenever ρ_1 has zeros in the right half of the λ plane. One could, of course, modify the contour of integration so as to exclude the singular points and to prevent it from crossing any branch line. A much simpler procedure is again suggested by Bode's work. Since it is expected that the final integral will involve the magnitude of ρ_1 over the imaginary axis, and not its phase (the phase is an odd function of ω) one can substitute for ρ_1 a function ρ_0 which has the same magnitude as ρ_1 over the imaginary axis but whose zeros are all in the *left* half plane. This new function is obtained from ρ_1 by simply moving all the zeros that lie in the right half plane to symmetrical locations in the left half plane. This process does not change the magnitude of the function over the imaginary axis since it is equivalent to multiplying ρ_1 by factors of the type $\frac{\lambda + \overline{\lambda}_r}{\lambda - \overline{\lambda}_r}$ whose magnitude on the imaginary axis is equal to one.

The zeros of ρ_1 on the imaginary axis are not eliminated by the above procedure. They will be considered, however, as limiting cases of zeros located in the left half plane, very close to the imaginary axis. The use of such an artifice can be justified, or better avoided, by following a correct mathematical procedure. On the other hand, the final results themselves will provide a good justification for this artifice, and, in any case, it will be seen that the zeros of ρ_1 must never be placed on the imaginary axis in an optimum design. In view of these facts it seems reasonable to use such an artifice in order to prevent mathematical details from obscuring the main issue.

Another difficulty may arise from the fact that the imaginary part of $\ln \frac{1}{\rho_1}$ jumps from $+\pi$ to $-\pi$ when the real axis is crossed, if ρ_1 is negative at the origin. To avoid this trouble it is sufficient to make the function ρ_0 positive at the origin by changing its sign when so required. The function $\ln \frac{1}{\rho_0}$ is then analytic over the whole right half plane and on the imaginary axis and can be used in the desired contour integrations without further difficulties.

Using the function $\ln \frac{1}{\rho_0}$ in the contour integrations instead of $\ln \frac{1}{\rho_1}$ will result in relations involving the derivatives of $\ln \frac{1}{\rho_0}$ instead of the derivatives of $\ln \frac{1}{\rho_1}$ in terms of which the conditions of physical realizability are expressed. However, the two sets of derivatives and, therefore, the corresponding coefficients A_k's and B_k's are very simply related through their expressions in terms of the zeros and poles of ρ_0 and ρ_1. Let F_k and G_k be the coefficients for ρ_0 corresponding, respectively, to the coefficients A_k and B_k for ρ_1 and let λ_r be a zero of ρ_1 in the right half plane. One obtains from eqs. 2.37, 40, 46, 47, 50, 51, 53 and 54, noting that the zeros occur in conjugate pairs,

$$F^{o}_{2k+1} = A^{o}_{2k+1} - \frac{2}{2k+1} \sum_i \lambda_{ri}^{-(2k+1)} \tag{1}$$

$$F^{\infty}_{2k+1} = A^{\infty}_{2k-1} - \frac{2}{2k+1} \sum_i \lambda_{ri}^{2k+1} \tag{2}$$

$$G^{\omega_\nu}_{o} = B^{\omega_\nu}_{o} - \frac{1}{j} \sum_i \ln \frac{-\overline{\lambda}_{ri} - j\,\omega_\nu}{\lambda_{ri} - j\,\omega_\nu} - \pi \left(\begin{array}{l}\text{if the sign of } \rho_o\\ \text{has been changed}\end{array}\right) \tag{3}$$

$$F^{\omega_\nu}_{k} = A^{\omega_\nu}_{k} - \frac{2}{k} \,\mathrm{Re.} \sum (\lambda_{ri} - j\,\omega_\nu)^{-k} \qquad \text{(k odd)} \tag{4}$$

$$G^{\omega_\nu}_{k} = G^{\omega_\nu}_{k} - \frac{2}{k} \,\mathrm{Im.} \sum (\lambda_{ri} - j\,\omega_\nu)^{-k} \qquad \text{(k even)} \tag{5}$$

$$F^{\sigma_\nu}_{o} = A^{\sigma_\nu}_{o} - \sum_i \ln \left| \frac{-\overline{\lambda}_{ri} - \sigma_\nu}{\lambda_{ri} - \sigma_\nu} \right| \tag{6}$$

$$F^{\sigma_\nu}_{k} = A^{\sigma_\nu}_{k} - \frac{1}{k}\left[\sum_i (\lambda_{ri} - \sigma_\nu)^{-k} - \sum_i (-\overline{\lambda}_{ri} - \sigma_\nu)^{-k}\right] \tag{7}$$

$$F^{\lambda_\nu}_{o} + jG^{\lambda_\nu}_{o} = A^{\lambda_\nu}_{o} + jB^{\lambda_\nu}_{o} - \sum_i \ln \frac{-\overline{\lambda}_{ri} - \lambda_\nu}{\lambda_{ri} - \lambda_\nu} \underset{\left(\begin{array}{l}\text{if sign of } \rho_o\\ \text{has been changed}\end{array}\right)}{-j\pi} \tag{8}$$

$$F^{\lambda_\nu}_{k} + jG^{\lambda_\nu}_{k} = A^{\lambda_\nu}_{k} + jB^{\lambda_\nu}_{k} - \frac{1}{k}\left[\sum_i (\lambda_{ri} - \lambda_\nu)^{-k} - \sum_i (-\overline{\lambda}_{ri} - \lambda_\nu)^{-k}\right] \tag{9}$$

The following four sections will be devoted to the derivation of appropriate equations relating the behavior of $\ln \frac{1}{|\rho_1|}$ on the imaginary axis to the new F_k and G_k coefficients.

3.2 Multiple zero of transmission at infinity - The coefficients resulting from a multiple zero of transmission at infinity will be considered first. The first equation, involving F_1^{∞} is obtained by integrating the function $\ln \frac{1}{\rho_0}$ over the contour indicated in Fig. 3.1, with the radius r of the semicircle approaching infinity. Since the real part of $\ln \frac{1}{\rho_0}$ is an even function of ω on the imaginary axis, while the imaginary part is an odd function, one obtains for the integral over the imaginary axis:

$$\int_{-j\infty}^{j\infty} \ln \frac{1}{\rho_0}\, d(j\omega) = 2j \int_0^{\infty} \ln \frac{1}{|\rho_0|}\, d\omega = 2j \int_0^{\infty} \ln \frac{1}{|\rho_1|}\, d\omega \qquad (10)$$

To integrate over the semicircle one observes first that $\ln \frac{1}{\rho_0}$ behaves at infinity as $F_1^{\infty} \frac{1}{\lambda}$. Let then

$$\lambda = r\, e^{i\varphi} \qquad (11)$$

from which one obtains over the semicircle

$$d\lambda = jre^{i\varphi} d\varphi \qquad (12)$$

When r approaches infinity, the integral over the semicircle becomes:

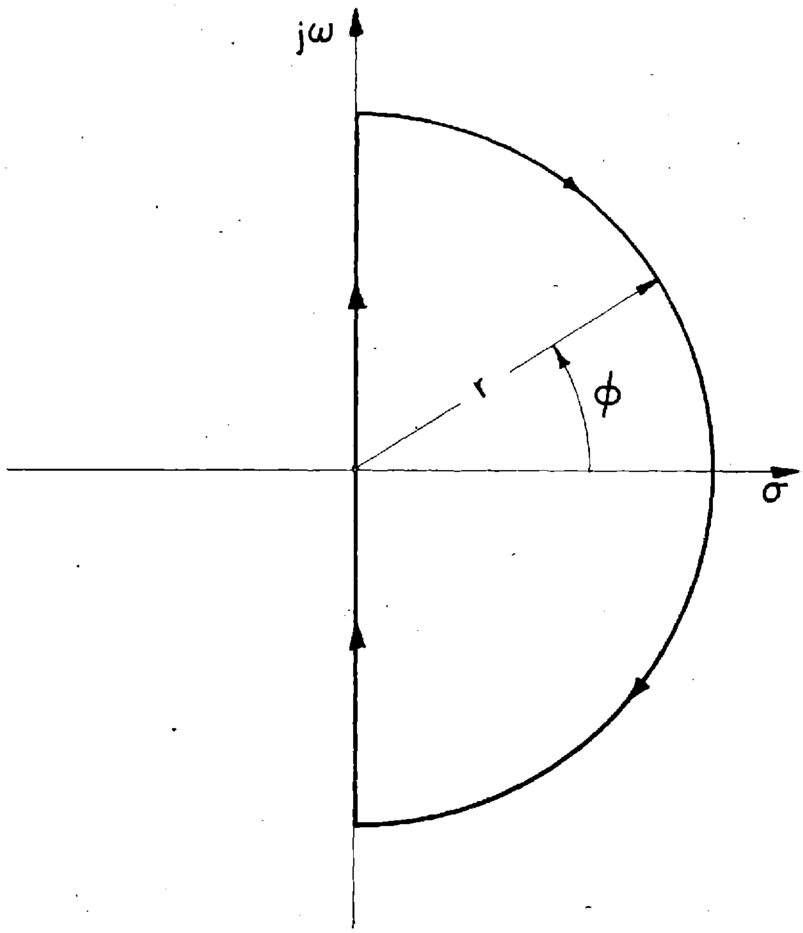


Figure 3.1 - Contour of integration in the λ-plane .

$$\oint_{\omega\to\infty} \ln \frac{1}{\rho_o}\, d\lambda = j \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} F_1^{\infty}\, d\varphi = -j\pi F_1^{\infty} \tag{13}$$

The integral over the whole contour must be zero because $\ln \frac{1}{\rho_o}$ is analytic at all points of the right half plane. Therefore, one obtains from eqs. 10 and 13

$$\int_0^{\infty} \ln \frac{1}{|\rho_1|}\, d\omega = \frac{\pi}{2} F_1^{\infty} = \frac{\pi}{2}\left[A_1^{\infty} - 2\sum_i \lambda_{ri}\right] \tag{14}$$

This equation is identical to that obtained by Bode. A simple computation will show that, when the first element of N' is a shunt capacitance, one has

$$A_1^{\infty} = \frac{2}{C} \tag{15}$$

where C is the value of the capacitance normalized with respect to the terminating resistance.

To obtain a similar equation involving A_3^{∞} one must select an integrand which behaves at infinity as $A_3^{\infty}\frac{1}{\lambda}$. The proper integrand is then

$$\lambda^2 \left[\ln \frac{1}{\rho_o} - F_1^{\infty}\frac{1}{\lambda}\right] \tag{16}$$

Integrating over the imaginary axis yields

$$\int_{-j\infty}^{j\infty} -\omega^2 \left[\ln \frac{1}{\rho_o} + F_1^{\infty}\frac{1}{j\omega}\right] d(j\omega) = -2j\int_0^{\infty} \omega^2 \ln \frac{1}{|\rho_1|}\, d\omega \tag{17}$$

Note that F_1^{∞} is eliminated from the result because the function $j\omega F_1^{\infty}$ is odd. The function (16) behaves at infinity as

$$\lambda^2\left[F_1^{\infty}\frac{1}{\lambda}+F_3^{\infty}\frac{1}{\lambda^3}-F_1^{\infty}\frac{1}{\lambda}\right]=F_3^{\infty}\frac{1}{\lambda} \tag{18}$$

The integration over the semicircle yields then

$$\int_{\omega\to\infty}\lambda^2\left[\ln\frac{1}{\rho_0}-F_1^{\infty}\frac{1}{\lambda}\right]d\lambda = j\int_{\frac{\pi}{2}}^{-\frac{\pi}{2}}F_3^{\infty}\,d\varphi=-j\pi F_3^{\infty} \tag{19}$$

Since the integrand has no singularities in the right half plane one obtains:

$$\int_{-\infty}^{\infty}\omega^2\ln\frac{1}{|\rho_1|}\,d\omega=-\frac{\pi}{2}F_3^{\infty}=-\frac{\pi}{2}\left[A_3^{\infty}-\frac{2}{3}\sum_i\lambda_{ri}^3\right] \tag{20}$$

The equation for the $(2k+1)^{th}$ coefficient is derived in a similar manner by integrating the function

$$\lambda^{2k}\left[\ln\frac{1}{\rho_0}-\sum_{i=0}^{k-1}F_{2i+1}^{\infty}\lambda^{-(2i+1)}\right] \tag{21}$$

One obtains, therefore, in the general case:

$$\int_0^{\infty}\omega^{2k}\ln\frac{1}{|\rho_1|}\,d\omega=(-1)^k\frac{\pi}{2}F_{2k+1}^{\infty}=(-1)^k\frac{\pi}{2}\left[\underbrace{A_{2k\pi}^{\infty}}_{2k+1\,(?)}-\frac{2}{2k+1}\sum_i\lambda_{ri}^{2k+1}\right] \tag{22}$$

3.3 <u>Multiple zero of transmission at the origin</u> – The equations pertaining to a multiple zero of transmission at the origin will be derived next. To obtain these equations a function must be used which has an appropriate singularity at the origin, and whose even part on the imaginary axis is proportional to $\ln \frac{1}{|\rho_o|}$. Because of the singularity at the origin the contour of integration must be modified as shown in Fig. 3.2. Consider then the function

$$\frac{1}{\lambda^2} \ln \frac{1}{\rho_o} \tag{23}$$

When λ approaches infinity, this function approaches zero as a negative power of λ equal, at least, to two. It follows that the integral over the semicircle or radius approaching infinity is zero. When λ approaches zero the function (23) behaves as $\frac{F_1^o}{\lambda}$. Therefore, the integral over the semicircle around the origin yields, when the radius r approaches zero,

$$\oint \frac{1}{\lambda^2} \ln \frac{1}{\rho_o} \, d\lambda = j \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_1^o \, d\varphi = j\pi F_1^o \tag{24}$$

For the integral over the imaginary axis one obtains

$$\int_{-j\infty}^{-j\nu\to 0} -\frac{1}{\omega^2} \ln \frac{1}{\rho_o} d(j\omega) + \int_{j\nu\to 0}^{j\infty} -\frac{1}{\omega^2} \ln \frac{1}{\rho_o} d(j\omega) = -2j \int_0^{\infty} \frac{1}{\omega^2} \ln \frac{1}{|\rho_1|} d\omega \tag{25}$$

Since the whole contour integral must be equal to zero, one obtains from eqs. 24 and 25

$$\int_0^{\infty} \frac{1}{\omega^2} \ln \frac{1}{|\rho_1|} \, d\omega = \frac{\pi}{2} F_1^o = \frac{\pi}{2} \left[A_1^o - 2 \sum_i \lambda_{ri}^{-1} \right] \tag{26}$$

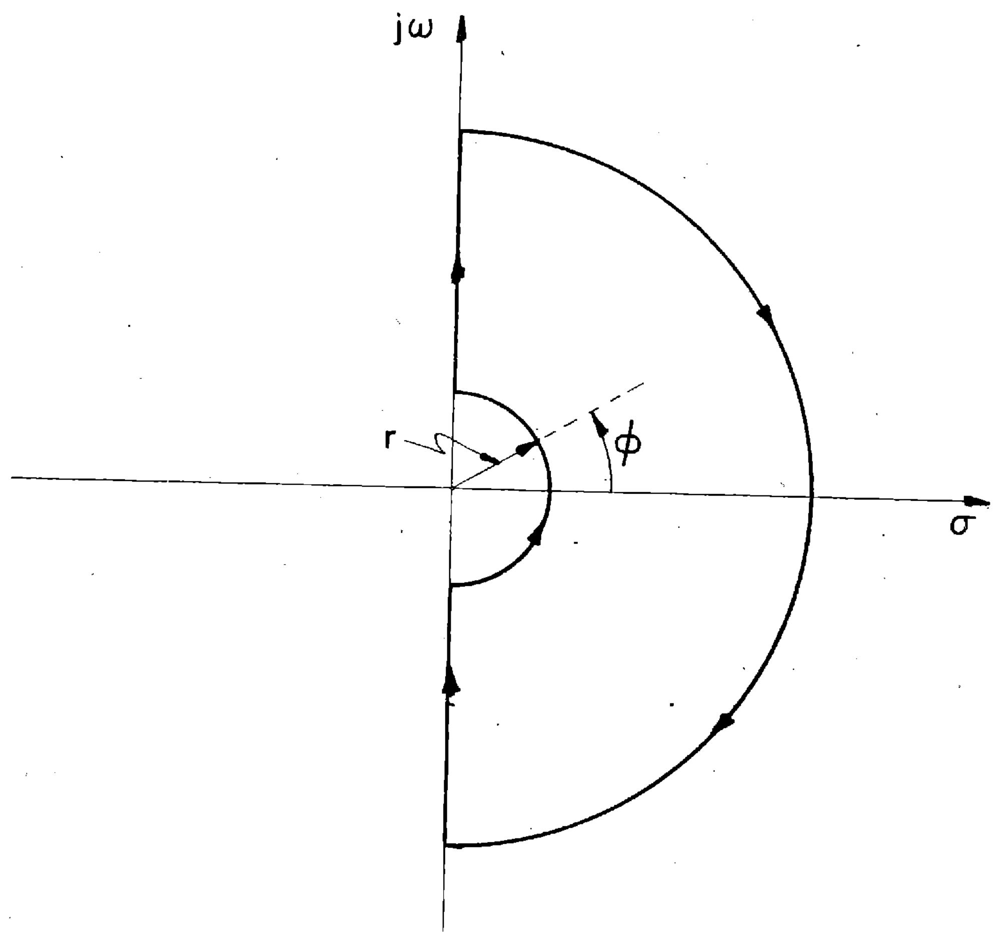


Figure 3.2 - Contour of integration in the λ-plane

Consider next the contour integral of the function

$$\frac{1}{\lambda^4} \ln \frac{1}{\rho_o}$$

The integral over the large semicircle will still vanish and the integral over the imaginary axis will yield

$$2j \int_0^{\infty} \frac{1}{\omega^4} \ln \frac{1}{|\rho_1|} \, d\omega \tag{27}$$

For very small values of λ the integrand behaves as

$$\frac{1}{\lambda^3} F_1^o + \frac{1}{\lambda} F_3^o$$

Integrating over the semicircle about the origin yields

$$\oint \frac{1}{\lambda^4} \ln \frac{1}{\rho_o} d\lambda = j \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_1^o \frac{e^{-j2\varphi}}{r^2} \, d\varphi + j \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} F_3^o \, d\varphi =$$

$$= -jF_1^o \frac{\sin \pi}{r^2} + j\pi F_3^o = j\pi F_3^o \tag{28}$$

and one obtains finally

$$\int_0^{\infty} \frac{1}{\omega^4} \ln \frac{1}{|\rho_1|} \, d\omega = -\frac{\pi}{2} F_3^o = -\frac{\pi}{2} \left[A_3^o - \frac{2}{3} \sum_i \lambda_{ri}^{-3} \right] \tag{29}$$

Operating in a similar manner on the function

$$\frac{1}{\lambda^{2k}} \ln \frac{1}{\rho_o} \tag{30}$$

yields the general equation

$$\int_0^\infty \frac{1}{\omega^{2(k+1)}} \ln \frac{1}{|\rho_1|} d\omega = (-1)^k \frac{\pi}{2} F^o_{2k+1} = (-1)^k \frac{\pi}{2}$$

$$\left[A^o_{2k+1} - \frac{2}{2k+1} \sum_i \lambda_{ri}^{-(2k+1)} \right] \tag{31}$$

3.4 <u>Pair of conjugate zeros of transmission on the imaginary axis</u> - Consider now a pair of zeros of transmission with arbitrary multiplicity, located at $\pm j\omega_\nu$. One must use as integrand, in this case, a function which has poles at $\pm j\omega_\nu$ and whose even part is proportional to $\ln \frac{1}{|\rho_1|}$ on the imaginary axis. The simplest function of this type is:

$$\left[\frac{1}{\lambda - j\omega_\nu} - \frac{1}{\lambda + j\omega_\nu} \right] \ln \frac{1}{\rho_o} = \frac{j2\omega_\nu}{\lambda^2 + \omega_\nu^2} \ln \frac{1}{\rho_o} \tag{32}$$

The contour of integration must be modified accordingly to avoid the singular points as shown in Fig. 3.3.

The integral of 32 over the large semicircle vanishes when the radius approaches infinity, just as in the case treated in the preceding section. In the vicinity of the point $j\omega_\nu$ the integrand behaves as $jG_o^{\omega_\nu} (\lambda - j\omega_\nu)^{-1}$. Letting

$$\lambda - j\omega_\nu = re^{i\varphi} \tag{33}$$

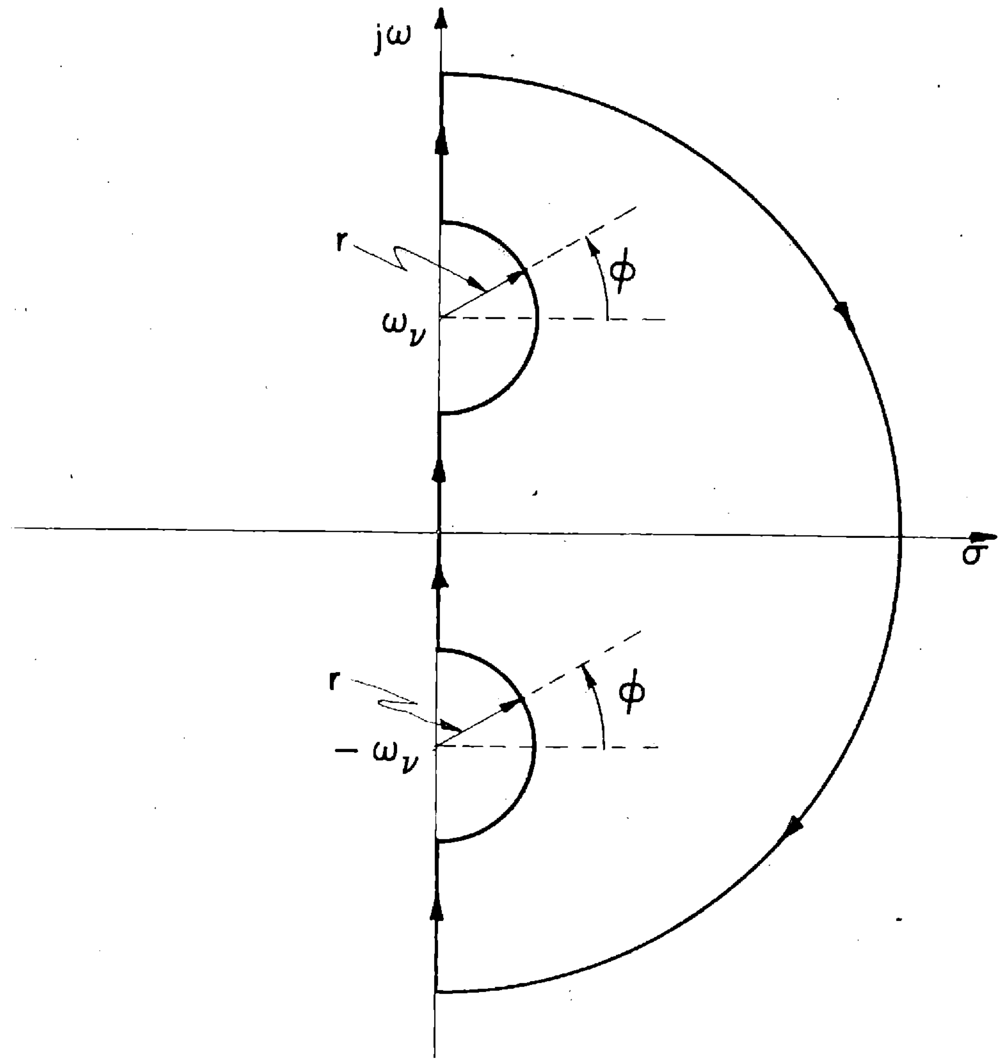


Figure 3.3 - Contour of integration in the λ-plane

one has over the semicircle of radius r

$$d\lambda = jre^{i\varphi}d\varphi \tag{34}$$

The integral over this semicircle becomes then

$$\oint \frac{j2\omega_\nu}{\lambda^2+\omega_\nu^2}\ln\frac{1}{\rho_0}\,d\lambda = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G_0^{\omega_\nu}\,d\varphi = -\pi G_0^{\omega_\nu} \tag{35}$$

The integral over the semicircle about the point $-j\omega_\nu$ yields the same result since the function behaves in the vicinity of that point as $-(-jG_0^{\omega_\nu})(\lambda + j\omega_\nu)^{-1}$. The integral over the imaginary axis will involve only the even part of the integrand, as in the previous cases, yielding thus the value

$$-4\omega_\nu\int_0^\infty \frac{1}{\omega_\nu^2-\omega^2}\ln\frac{1}{|\rho_1|}\,d\omega \tag{36}$$

Since the whole contour integral must be equal to zero, one obtains from 35 and 36

$$\int_0^\infty \frac{1}{1-(\frac{\omega}{\omega_\nu})^2}\ln\frac{1}{|\rho_1|}\,d(\frac{\omega}{\omega_\nu}) = -\frac{\pi}{2}G_0^{\omega_\nu} \tag{37}$$

To derive the equation involving $F_1^{\omega_\nu}$ one integrates over the same contour the function

$$\left[\frac{1}{(\lambda-j\omega_\nu)^2}+\frac{1}{(\lambda+j\omega_\nu)^2}\right]\ln\frac{1}{\rho_0} = 2\,\frac{\lambda^2-\omega_\nu^2}{(\lambda^2+\omega_\nu^2)^2}\ln\frac{1}{\rho_0} \tag{38}$$

The integral over the large semicircle vanishes again, and the integral over the imaginary axis yields:

$$-j4\int_0^{\infty}\frac{\omega_\nu^2+\omega^2}{(\omega_\nu^2-\omega^2)^2}\ln\frac{1}{|\rho_1|}\,d\omega \tag{39}$$

In the vicinity of the point $j\omega_\nu$, the integrand behaves as $jG_o^{\omega_\nu}(\lambda - j\omega_\nu)^{-2} + F_1^{\omega_\nu}(\lambda - j\omega_\nu)^{-1}$. The integral over the semicircle becomes then:

$$2\oint\frac{\lambda^2-\omega_\nu^2}{(\lambda^2+\omega_\nu^2)^2}\ln\frac{1}{\rho_o}\,d\lambda = -G_o^{\omega_\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-j\varphi}}{r}\,d\varphi + jF_1^{\omega_\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi =$$

$$= -\frac{2}{r}G_o^{\omega_\nu}\sin\frac{\pi}{2} + j\pi F_1^{\omega_\nu} \tag{40}$$

One obtains in a similar manner for the integral over the semicircle about the conjugate point $-j\omega_\nu$

$$2\oint\frac{\lambda^2-\omega_\nu^2}{(\lambda^2+\omega_\nu^2)^2}\ln\frac{1}{\rho_o}\,d\lambda = +G_o^{\omega_\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{e^{-j\varphi}}{r}\,d\varphi + jF_1^{\omega_\nu}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\varphi =$$

$$= \frac{2}{r}G_o^{\omega_\nu}\sin\frac{\pi}{2} + j\pi F_1^{\omega_\nu} \tag{41}$$

It will be noted that the terms in eqs. 40 and 41 which are proportional r^{-1} have opposite signs, so that the contribution of both semicircles together remains finite when r approaches zero.

One obtains finally from eqs 39, 40 and 41

$$\int_0^\infty \frac{1+(\frac{\omega}{\omega_\nu})^2}{\left[1-(\frac{\omega}{\omega_\nu})^2\right]^2} \ln\frac{1}{|\rho_1|}\, d\left(\frac{\omega}{\omega_\nu}\right) = \frac{\pi}{2}\omega_\nu F_1^{\omega_\nu} \tag{42}$$

The following equations are derived in a similar manner by operating on functions with higher order poles. One obtains successively

$$\int_0^\infty \frac{1+3(\frac{\omega}{\omega_\nu})^2}{\left[1-(\frac{\omega}{\omega_\nu})^2\right]^3} \ln\frac{1}{|\rho_1|}\, d\left(\frac{\omega}{\omega_\nu}\right) = \frac{\pi}{2}\omega_\nu^2 G_2^{\omega_\nu} \tag{43}$$

$$\int_0^\infty \frac{1+6(\frac{\omega}{\omega_\nu})^2+(\frac{\omega}{\omega_\nu})^4}{\left[1-(\frac{\omega}{\omega_\nu})^2\right]^4} \ln\frac{1}{|\rho_1|}\, d\left(\frac{\omega}{\omega_\nu}\right) = -\frac{\pi}{2}\omega_\nu^3 F_3^{\omega_\nu} \tag{44}$$

and in general

$$\int_0^\infty g_{2k}^{\omega_\nu} \ln\frac{1}{|\rho_1|}\, dx = (-1)^{k+1}\frac{\pi}{2}\omega_\nu^{2k} G_{2k}^{\omega_\nu} \tag{45}$$

$$\int_0^\infty f_{2k+1}^{\omega_\nu} \ln\frac{1}{|\rho_1|}\, dx = (-1)^{k}\frac{\pi}{2}\omega_\nu^{2k+1} F_{2k+1}^{\omega_\nu} \tag{46}$$

where

$$x = \frac{\omega}{\omega_\nu} \tag{47}$$

and

$$g_{2k}^{\omega_\nu} = \frac{1}{2}\,\frac{(1+x)^{2k+1} + (1-x)^{2k+1}}{(1-x^2)^{2k+1}} \tag{48}$$

$$f_{2k\,1}^{\omega_\nu} = \frac{1}{2}\,\frac{(1+x)^{2(k+1)} + (1-x)^{2(k+1)}}{(1-x^2)^{2(k+1)}} \tag{49}$$

The first six "weighing functions" $g_{2k}^{\omega_\nu}$ and $f_{2k\,1}^{\omega_\nu}$ are plotted in Fig. 3.4.

3.5 Zeros of transmission on the real axis - In deriving the equations resulting from a zero of transmission at σ_ν on the real axis, one can use directly Cauchy's integral formula, in conjunction with the contour of integration shown in Fig. 3.1. The first integrand takes the form

$$\left[\frac{1}{\lambda-\sigma_\nu} - \frac{1}{\lambda+\sigma_\nu}\right] \ln\frac{1}{\rho_0} = \frac{2\sigma_\nu}{\lambda^2-\sigma_\nu^2}\ \ln\frac{1}{\rho_0} \tag{50}$$

The integral over the semicircle vanishes when the radius approaches infinity, and the integral over the imaginary axis yields

$$-j\,4\sigma_\nu \int_0^\infty \frac{1}{\sigma_\nu^2+\omega^2}\ln\frac{1}{|\rho_1|}\,d\omega \tag{51}$$

The residue of the pole at the point σ_ν is $F_0^{\sigma_\nu}$. It follows that

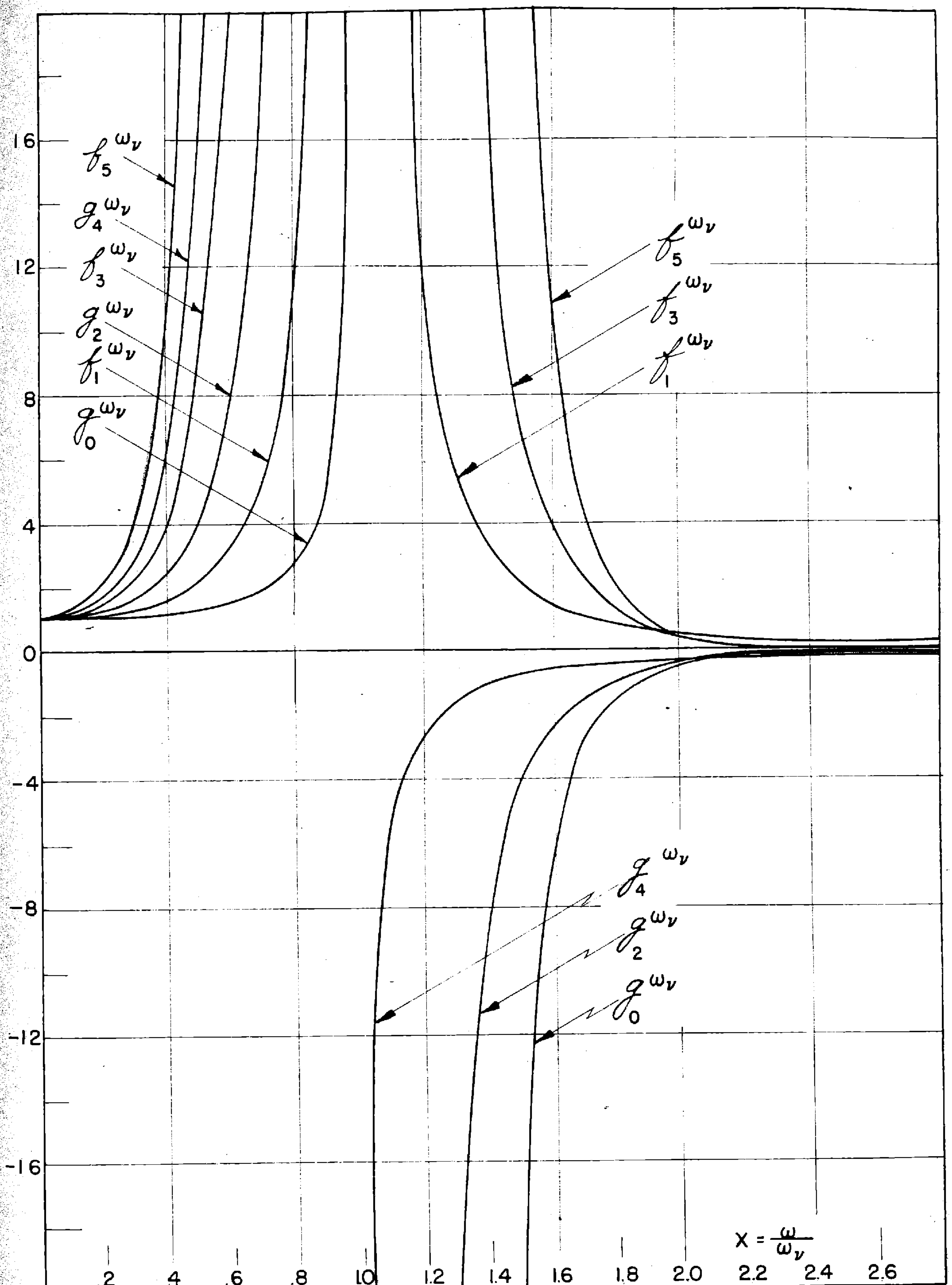


Figure 3.4 - Weighing functions for a zero of transmission at $j\omega_\nu$

$$\int_0^{\infty} \frac{1}{1+\left(\frac{\omega}{\sigma_\nu}\right)^2} \ln \frac{1}{|\rho_1|} \, d\frac{\omega}{\sigma_\nu} = \frac{\pi}{2} F_0^{\sigma_\nu} \tag{52}$$

The next equation is derived by operating on the function:

$$\left[\frac{1}{(\lambda-\sigma_\nu)^2} + \frac{1}{(\lambda+\sigma_\nu)^2}\right] \ln \frac{1}{\rho_0} = 2\, \frac{\lambda^2+\sigma_\nu^{\,2}}{(\lambda^2-\sigma_\nu^{\,2})^2} \ln \frac{1}{|\rho_1|} \tag{53}$$

One obtains by means of Cauchy's integral formula for the first derivative of an analytic function

$$\int_0^{\infty} \frac{1-\left(\frac{\omega}{\sigma_\nu}\right)^2}{\left[1+\left(\frac{\omega}{\sigma_\nu}\right)^2\right]^2} \ln \frac{1}{|\rho_1|} \, d\frac{\omega}{\sigma_\nu} = -\frac{\pi}{2} \sigma_\nu F_1^{\sigma_\nu} \tag{54}$$

The other equations can be derived in a similar manner using functions with higher order poles at σ_ν. The general form of these equations is:

$$\int_0^{\infty} f_k^{\sigma_\nu} \ln \frac{1}{|\rho_1|} \, dx = (-1)^k \frac{\pi}{2} \sigma_\nu^{\,k} F_k^{\sigma_\nu} \tag{55}$$

where

$$x = \frac{\omega}{\sigma_\nu} \tag{56}$$

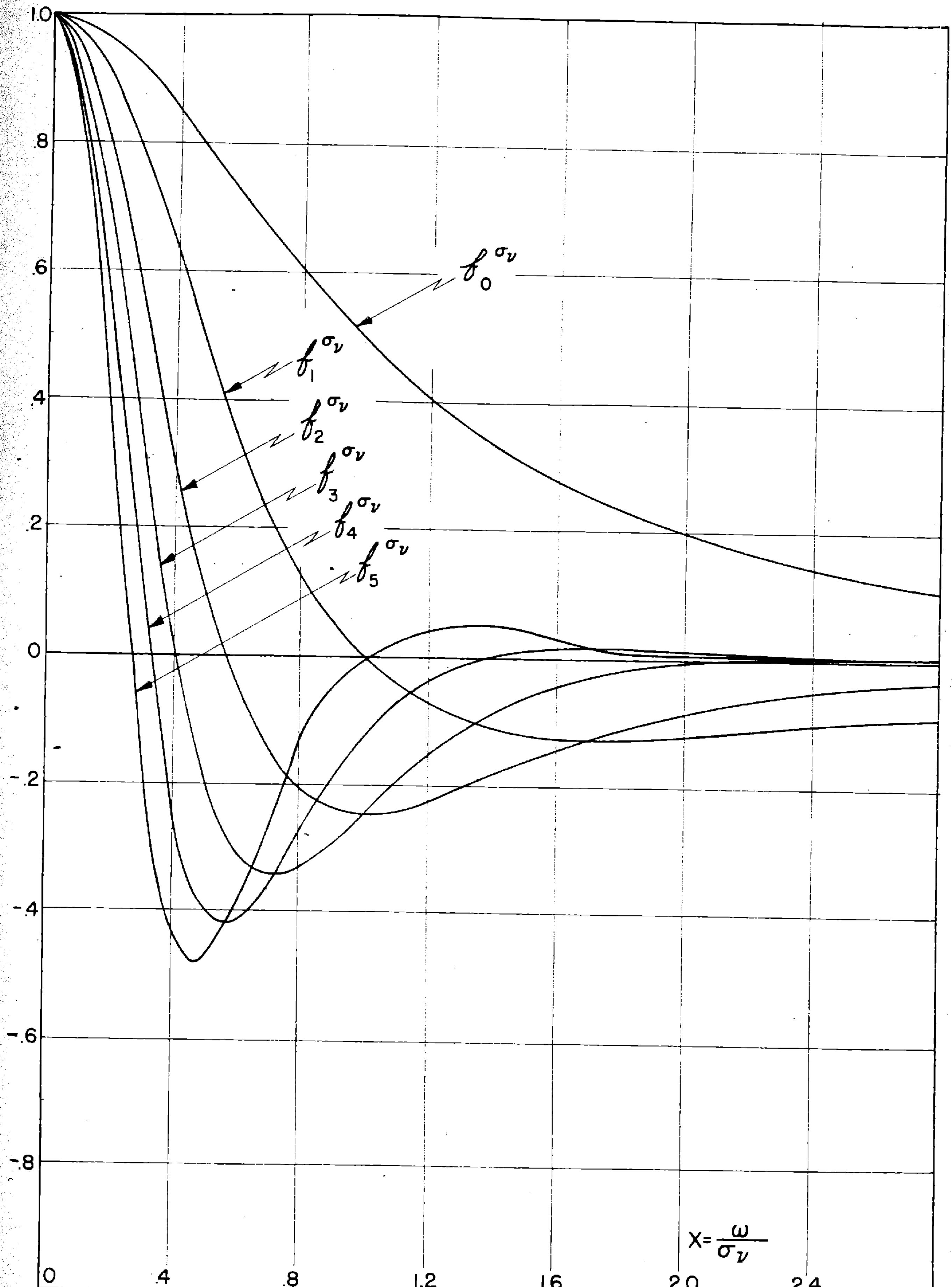


Figure s.5 - Weighing functions for a zero of transmission at σ_ν

and

$$f_k^{\sigma_\nu} = \frac{1}{2}\,\frac{(1 + jx)^{k+1} + (1 - jx)^{k+1}}{(1 + x^2)^{k+1}} \tag{57}$$

The first six weighing functions $f_k^{\omega_\nu}$ are plotted in Fig. 3.5.

3.6 Pair of conjugate zeros of transmission in the right half plane – The case of a pair of conjugate zeros of transmission in the right half plane at λ_ν and $\overline{\lambda}_\nu$ is treated just as the case of a zero on the real axis, with the only difference that there are two conjugate poles within the contour of integration. The first equation is obtained by integrating the function

$$\left[\left(\frac{1}{\lambda - \lambda_\nu} - \frac{1}{\lambda + \lambda_\nu}\right) + \left(\frac{1}{\lambda - \overline{\lambda}_\nu} - \frac{1}{\lambda + \overline{\lambda}_\nu}\right)\right] \ln \frac{1}{\rho_o} =$$

$$= \left[\frac{2\lambda_\nu}{\lambda^2 - \lambda_\nu^2} + \frac{2\overline{\lambda}_\nu}{\lambda^2 - \overline{\lambda}_\nu^2}\right] \ln \frac{1}{\rho_o} = 2\,\frac{\lambda_\nu(\lambda^2 - \overline{\lambda}_\nu^2) + \overline{\lambda}_\nu(\lambda^2 - \lambda_\nu^2)}{\lambda^4 - \lambda^2(\lambda_\nu^2 + \overline{\lambda}_\nu^2) + |\lambda_\nu|^4} \ln \frac{1}{\rho_o} \tag{58}$$

The residues of the integrand at λ_ν and $\overline{\lambda}_\nu$ are, respectively, $F_o + jG_o$ and $F_o - jG_o$. One has then:

$$\int_0^\infty \frac{\delta\,[1 + x^2]}{1 - 2(1 - 2\delta^2)x^2 + x^4} \ln \frac{1}{|\rho_1|}\, dx = \frac{\pi}{2} F_o^{\lambda_\nu} \tag{59}$$

where

$$x = \frac{\omega}{|\lambda_\nu|} \qquad \delta = \frac{1}{2}\,\frac{\lambda_\nu + \overline{\lambda}_\nu}{|\lambda_\nu|} \tag{60}$$

The second equation is derived by integrating the function

$$\left[\left(\frac{1}{\lambda-\lambda_\nu}-\frac{1}{\lambda+\lambda_\nu}\right)-\left(\frac{1}{\lambda-\overline{\lambda}_\nu}-\frac{1}{\lambda+\lambda_\nu}\right)\right] \ln\frac{1}{\rho_o} =$$

$$= 2\,\frac{\lambda_\nu(\lambda^2-\overline{\lambda}_\nu)-\overline{\lambda}_\nu(\lambda^2-\overline{\lambda}_\nu)}{\lambda^4-\lambda^2(\lambda_\nu^{\,2}+\overline{\lambda}_\nu^{\,2})+|\lambda_\nu|^4}\ln\frac{1}{\rho_o} \qquad (61)$$

In this case, the residues of the integrand at λ_ν and $\overline{\lambda}_\nu$ are respectively, $F_o + jG_o$ and $-F_o + jG_o$. One obtains then:

$$\int_0^\infty \frac{\sqrt{1-\delta^2}\,(1-x^2)}{1-2(1-2\delta^2)x^2+x^4}\ln\frac{1}{|\rho_1|}\,dx = -\frac{\pi}{2}G_o^{\lambda_\nu} \qquad (62)$$

The next two equations are derived by means of the functions

$$\left[\left(\frac{1}{(\lambda-\lambda_\nu)^2}+\frac{1}{(\lambda+\lambda_\nu)^2}\right)+\left(\frac{1}{(\lambda-\overline{\lambda}_\nu)^2}+\frac{1}{(\lambda+\overline{\lambda}_\nu)^2}\right)\right]\ln\frac{1}{\rho_o} \qquad (63)$$

$$\left[\left(\frac{1}{(\lambda-\lambda_\nu)^2}+\frac{1}{(\lambda+\lambda_\nu)^2}\right)-\left(\frac{1}{(\lambda-\overline{\lambda}_\nu)^2}+\frac{1}{(\lambda+\overline{\lambda}_\nu)^2}\right)\right]\ln\frac{1}{\rho_o} \qquad (64)$$

By proceeding in the same manner one obtains

$$\int_0^\infty \frac{(1-2\delta^2)-(1+8\delta^2-8\delta^4)x^2-(1-2\delta^2)x^4+x^6}{(1-2(1-2\delta^2)x^2+x^4)^2}\ln\frac{1}{|\rho_1|}\,dx =$$

$$= \frac{\pi}{2}|\lambda_\nu|F_1^{\lambda_\nu} \qquad (65)$$

$$\int_0^\infty \frac{2\delta\sqrt{1-\delta^2}\,\left[1+2(1-2\delta^2)x^2-3x^4\right]}{\left(1-2(1-2\delta^2)x^2+x^4\right)^2}\ln\frac{1}{|\rho_1|}\,dx = \frac{\pi}{2}\left|\lambda_\nu\right| G_1^{\lambda\nu} \tag{66}$$

$$\int_0^\infty f_{2k}^{\lambda\nu}\ln\frac{1}{|\rho_1|}\,dx = (-1)^k\,\frac{\pi}{2}\left|\lambda_\nu\right|^{2k} F_{2k}^{\lambda\nu} \tag{67}$$

$$\int_0^\infty g_{2k}^{\lambda\nu}\ln\frac{1}{|\rho_1|}\,dx = (-1)^{k+1}\,\frac{\pi}{2}\left|\lambda_\nu\right|^{2k} G_{2k}^{\lambda\nu} \tag{68}$$

$$\int_0^\infty f_{2k+1}^{\lambda\nu}\ln\frac{1}{|\rho_1|}\,dx = (-1)^k\,\frac{\pi}{2}\left|\lambda_\nu\right|^{2k+1} F_{2k+1}^{\lambda\nu} \tag{69}$$

$$\int_0^\infty g_{2k+1}^{\lambda\nu}\ln\frac{1}{|\rho_1|}\,dx = (-1)^k\,\frac{\pi}{2}\left|\lambda_\nu\right|^{2k+1} G_{2k+1}^{\lambda\nu} \tag{70}$$

where:

$$\left.\begin{matrix} f_{2k}^{\lambda\nu} = (-1)^k \\ g_{2k}^{\lambda\nu} = (-1)^{k+1} \end{matrix}\right\} \Bigg[\left\{\left[\delta+j(x+\sqrt{1-\delta^2})\right]^{-(2k+1)} - \left[-\delta+j(x-\sqrt{1-\delta^2})\right]^{-(2k+1)}\right\} + \left\{\begin{matrix}+\\-\end{matrix}\right\}\left\{\left[\delta+j(x-\sqrt{1-\delta^2})\right]^{-(2k+1)} - \left[-\delta+j(x+\sqrt{1-\delta^2})\right]^{-(2k+1)}\right\}\Bigg] \tag{71}$$

$$\left.\begin{matrix} f_{2k+1}^{\lambda\nu} = \\ g_{2k+1}^{\lambda\nu} = \end{matrix}\right\} (-1)^k \Bigg[\left\{\left[\delta+j(x+\sqrt{1-\delta^2})\right]^{-2(k+1)} + \left[-\delta+j(x-\sqrt{1-\delta^2})\right]^{-2(k+1)}\right\} + \left\{\begin{matrix}+\\-\end{matrix}\right\}\left\{\left[\delta+j(x-\sqrt{1-\delta^2})\right]^{-2(k+1)} + \left[-\delta+j(x+\sqrt{1-\delta^2})\right]^{-2(k+1)}\right\}\Bigg] \tag{72}$$

The first three weighing functions of the f and g type are plotted in Figs. 3.6 to 3.11 for $\delta = 0.5$ and $\delta = 0.05$.

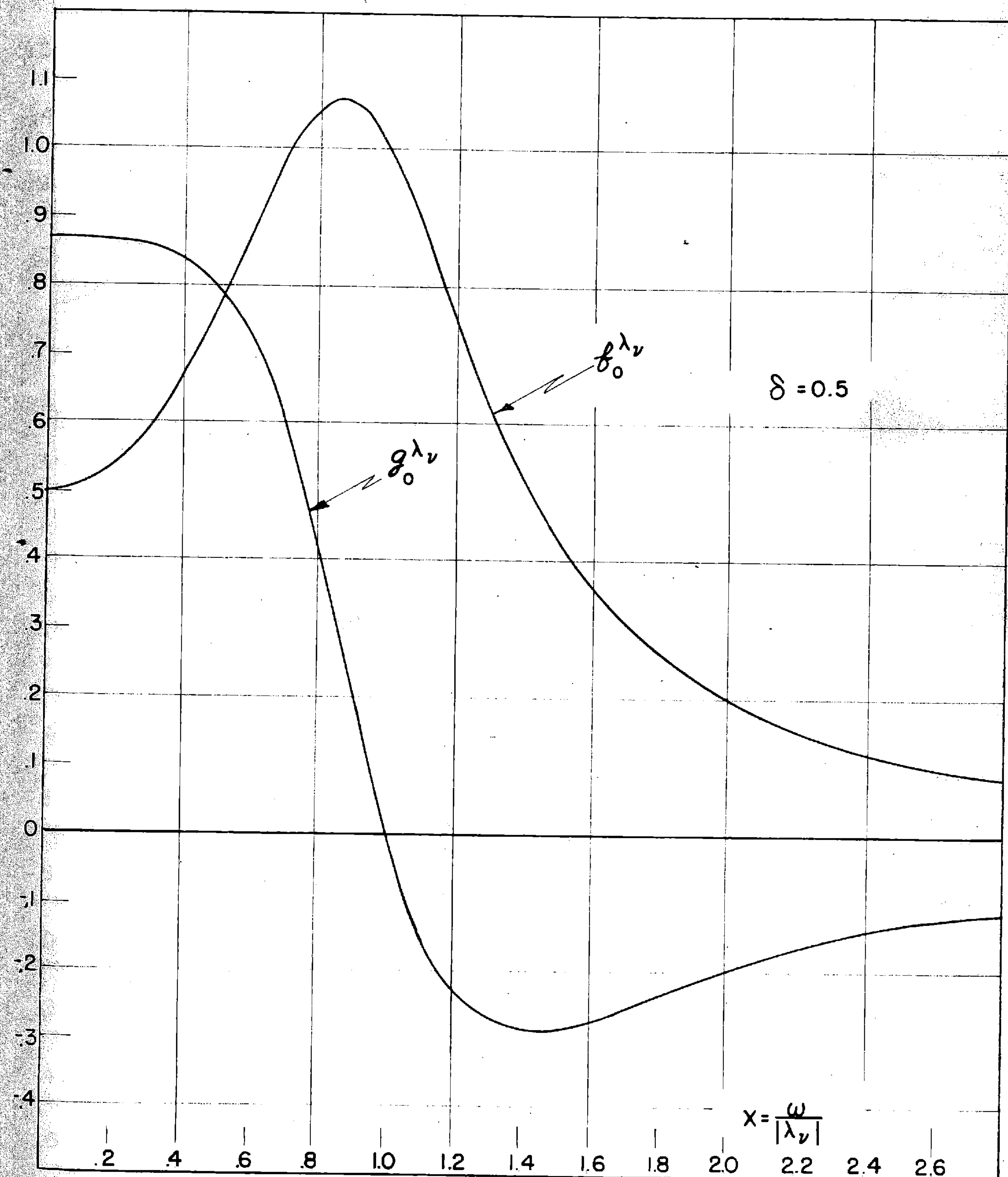


Figure 3.6 – Weighing functions for a zero of transmission at λ_ν

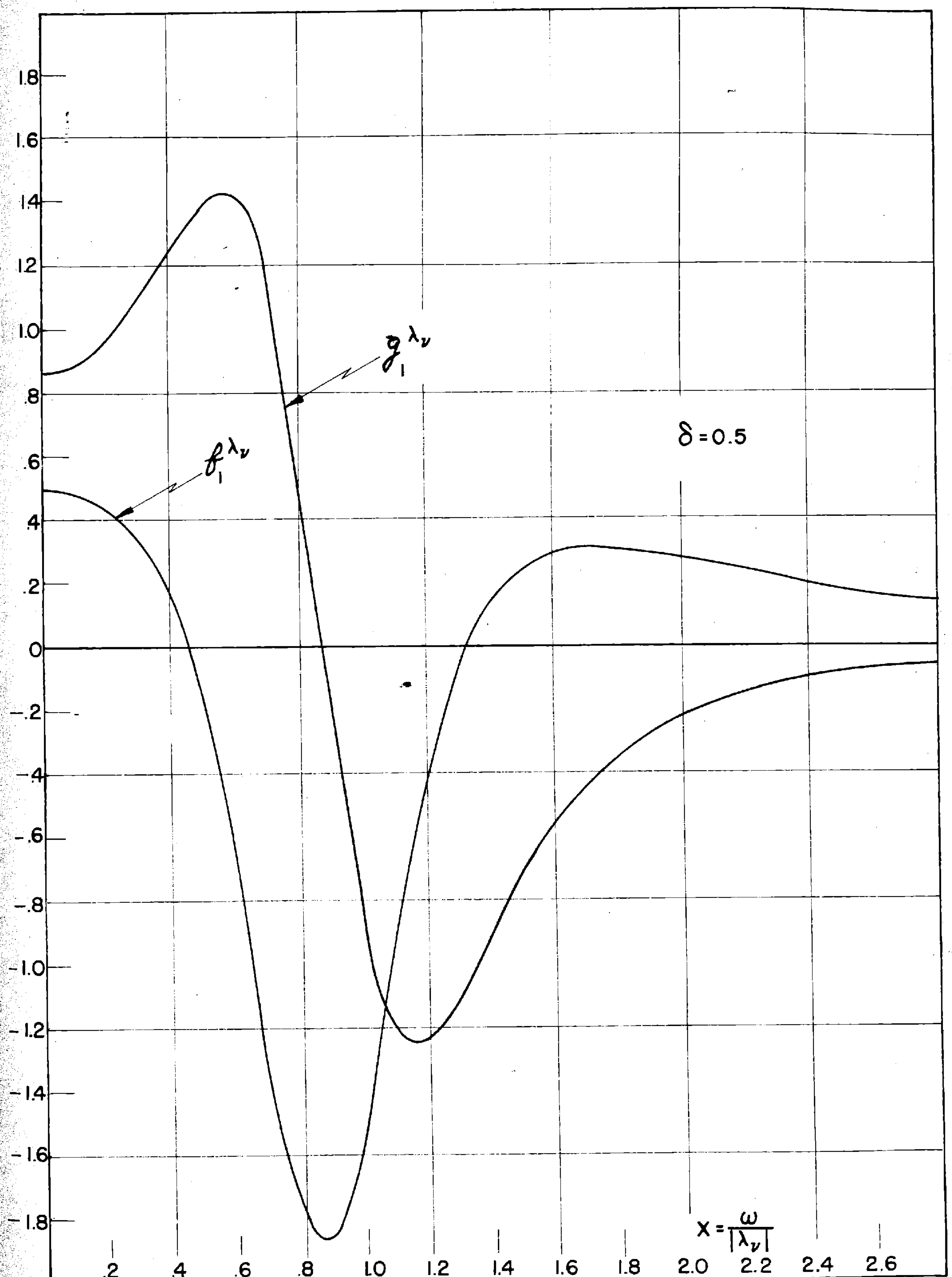


Figure 3.7 – Weighing functions for a zero of transmission at λ_ν

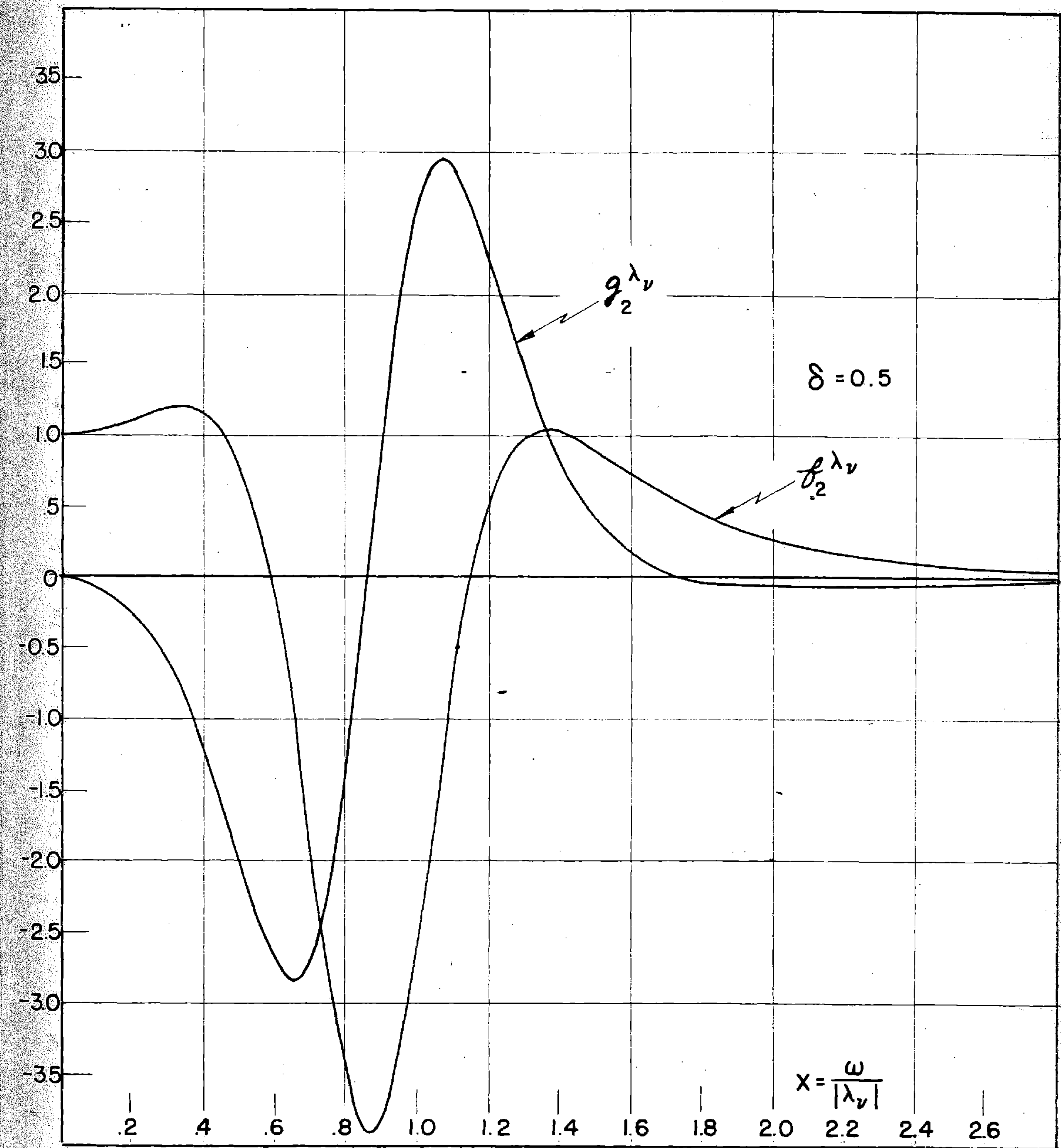


Figure 3.8 - Weighing functions for a zero of transmission at λ_ν

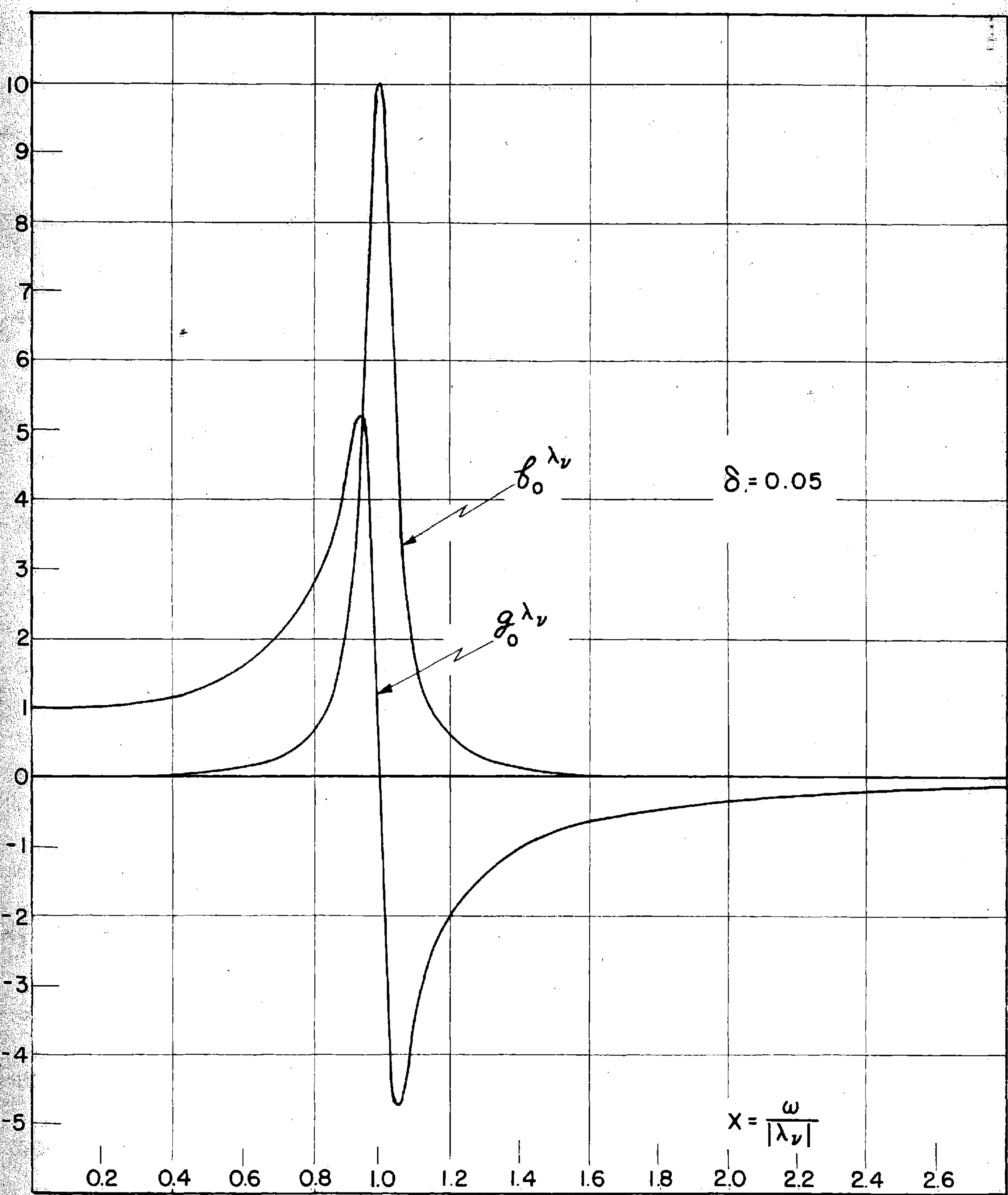


Figure 3.9 - Weighing functions for a zero of transmission at λ_ν

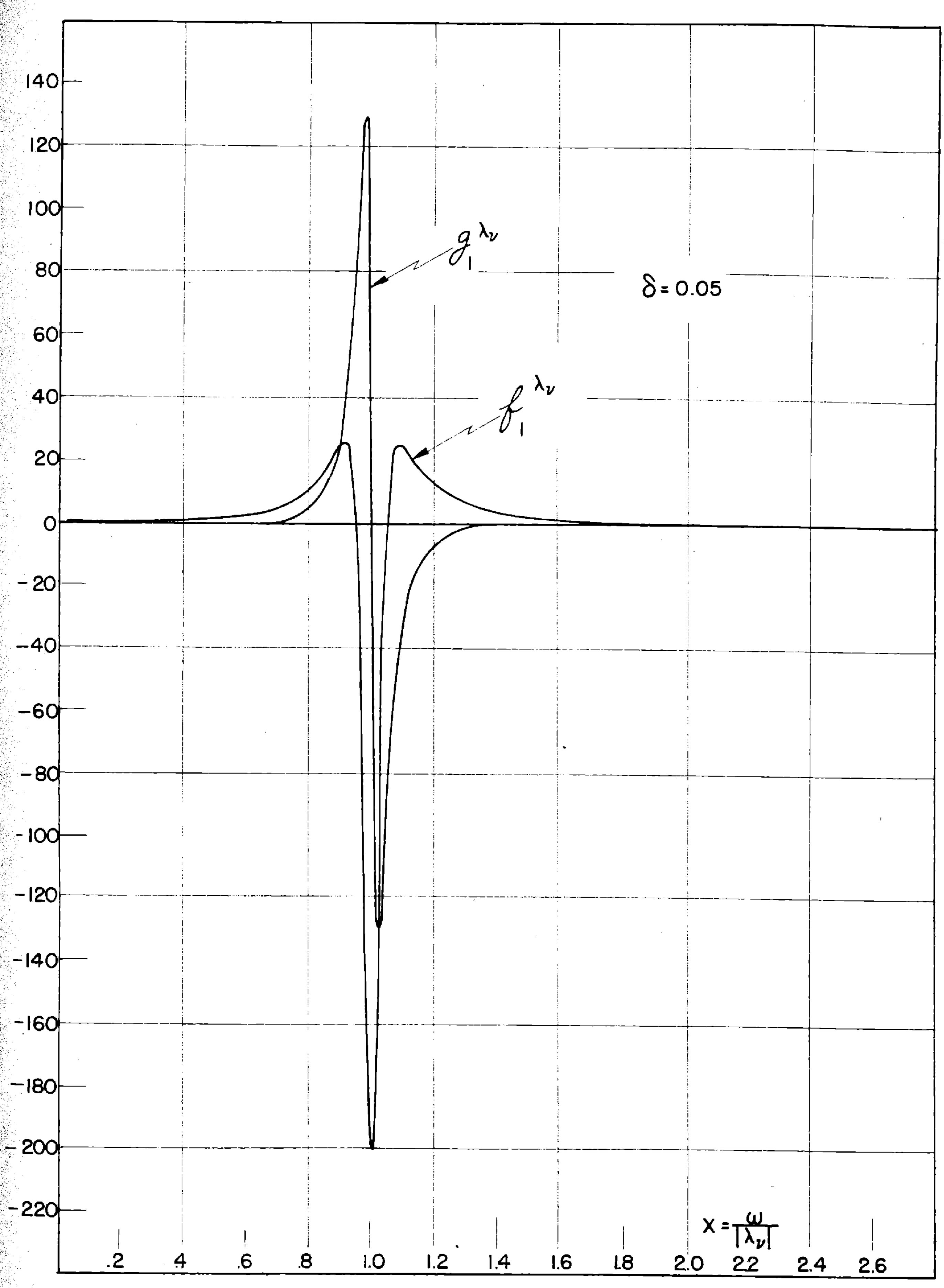


Figure 3.10 - Weighing functions for a zero of transmission at λ_ν

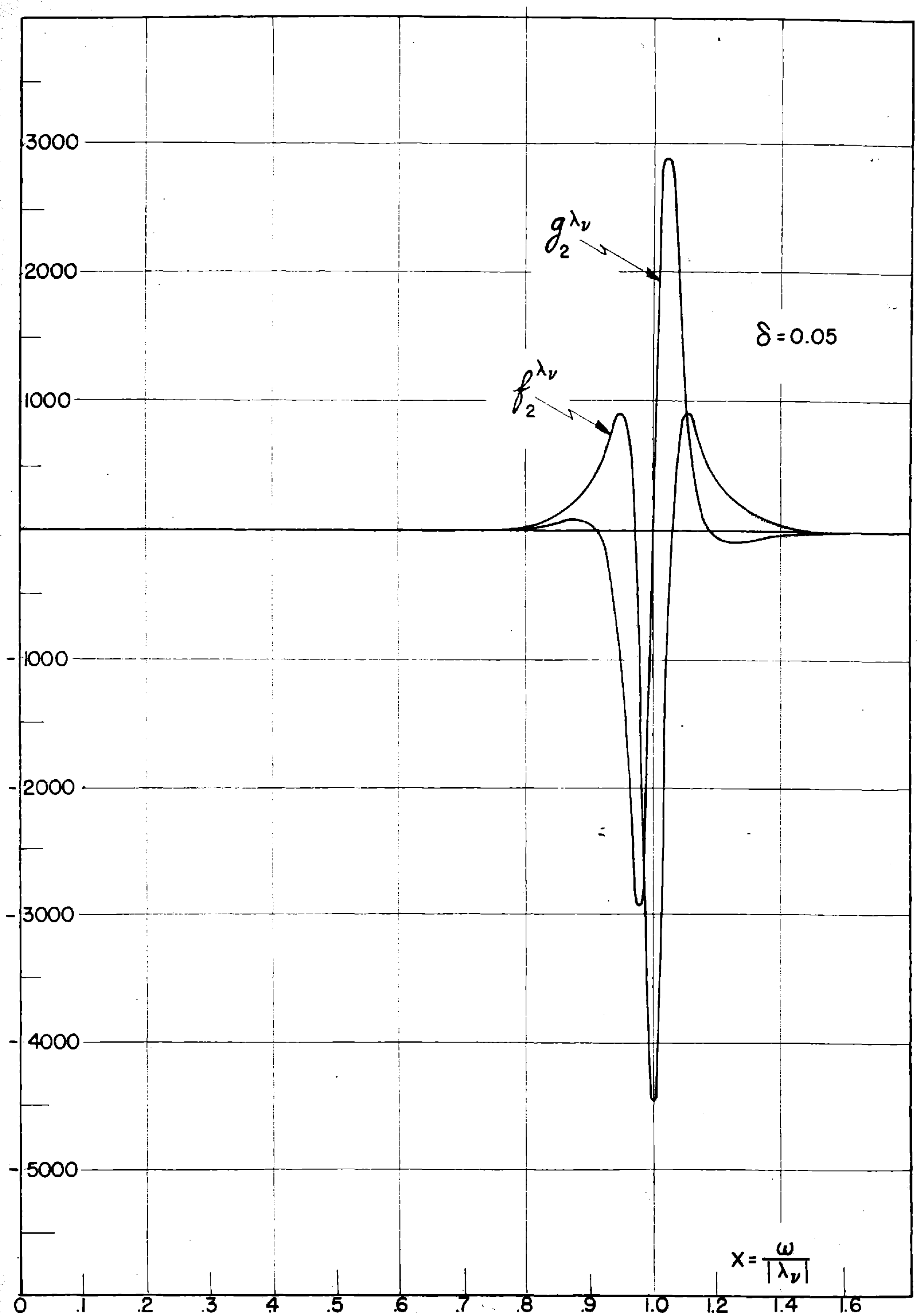


Figure 3.11 – Weighing functions for a zero of transmission at λ_ν

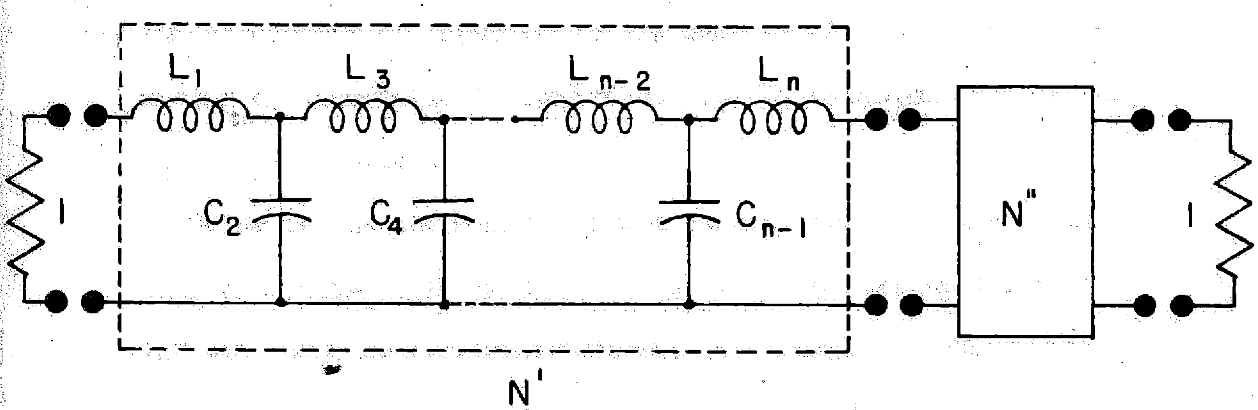


Figure 3.12 - Network with a multiple zero of transmission at infinity

3.7 <u>Interpretation of the results when all the zeros of transmission of N' are at infinity</u> - The conditions of physical realizability have been transformed, in the preceding sections, into equations involving the integral over the frequency spectrum of the return loss ($\ln \frac{1}{|\rho_1|}$) multiplied by some weighing function, the A and B coefficients and the zeros of ρ_1 that lie in the right half plane. The following sections will be devoted to the physical interpretation of these equations ~~and~~ for the purpose of determining the theoretical limitations on the bandwidth and on the tolerance of match.

Owing to the complexity of the general problem, the special case of a network N' having all the zeros of transmission at infinity will be considered first. It will be remembered that a network of this type can always be realized as a low-pass ladder structure of the type shown in Fig. 3.12, or as its dual (starting with a shunt capacitance). The coefficient A_1^∞ depends on L_1 alone, A_3^∞ depends on L_1 and C_2, A_5^∞ depends on L_1, C_2 and L_3, and so forth. The equations derived in Sec. 3.2 are rewritten below for convenience.

$$\int_0^\infty \ln \frac{1}{|\rho_1|} d\omega = \frac{\pi}{2}\left[A_1^\infty - 2 \sum_i \lambda_{ri}\right] \tag{73}$$

$$\int_0^\infty \omega^2 \ln \frac{1}{|\rho_1|} d\omega = -\frac{\pi}{2}\left[A_3^\infty - \frac{2}{3} \sum_i \lambda_{ri}^3\right] \tag{74}$$

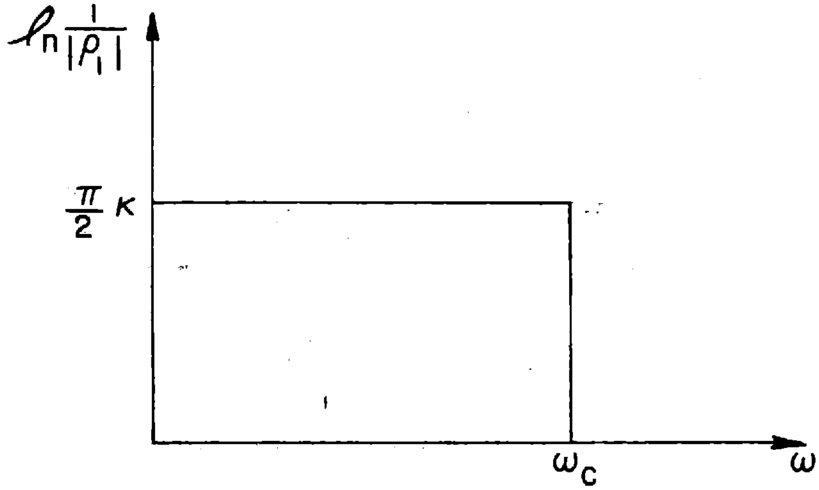


Figure 3.13 - Optimum frequency response

$$\int_0^\infty \omega^4 \ln \frac{1}{|\rho_1|}\, d\omega = \frac{\pi}{2}\left[A_5^\infty - \frac{2}{5}\sum_i \lambda_{ri}^5\right] \tag{75}$$

The λ_{ri}'s are the zeros of ρ_1 that lie in the right half plane.

The left hand side of the first equation represents the area under the curve $\ln \frac{1}{|\rho_1|}$ versus frequency. The coefficient A_1^∞ is fixed by the first element L_1 of the given network N'. The λ_{ri}'s are arbitrary quantities subject only to two restrictions, namely, that their real parts must be positive and that they must be present in conjugate pairs. It follows that the summation in eq. 73 is always real and positive so that A_1^∞ sets an upper limit to the area represented by the integral. Bode argued, in this regard, that the best possible utilization of the area available is obtained when $\ln \frac{1}{|\rho_1|}$ is kept constant over the frequency spectrum. This situation is illustrated for the low-pass case in Fig. 3.13. If w is the desired bandwidth ($w = \omega_c$ in Fig. 3.13) the best possible tolerance is given by:

$$\left[\ln \frac{1}{|\rho_1|}\right]_{max} = \frac{\pi}{2w} A_1^\infty \tag{76}$$

This optimum tolerance can be approached indefinitely when L_1 is the only element of the given network N', that is, when eq. 73 is the only equation to be satisfied. However, if more elements are present in N', all the corresponding equations will have to be satisfied simultaneously. It is evident that the function

$\ln \frac{1}{|\rho_1|}$ which yields the best tolerance will not, in general, meet these requirements. Suppose, for instance, that N' has two elements, L_1 and C_2, that is, has a zero of transmission of multiplicity equal to two. In this case eqs. 73 and 74 must be satisfied simultaneously. If the rectangular function which yields the optimum tolerance is used for $\ln \frac{1}{|\rho_1|}$ in eq. 74, the value of the integral may be larger or smaller than $-\frac{\pi}{2} A_3^{\infty}$ (A_3^{∞} is in general negative). If it is smaller, it is a simple matter to reduce the magnitude of A_3^{∞}. In fact it was found in Sec. 2.10 that in the case of a degenerate zero of transmission the value of A_3^{∞} can be increased, that is, its magnitude can be decreased. Physically, this operation amounts to starting the matching network N" with a shunt capacitance which has the effect of increasing the value of the capacitance C_2 in N'. In other words, when the integral in eq. 74 is smaller than $-\frac{\pi}{2} A_3^{\infty}$, the capacitance C_2 is smaller than the capacitance that would have to be placed across the terminals of N" to obtain the optimum match if L_1 were the only element of N'.

If the value of the integral in eq. 74 is larger than $-\frac{\pi}{2} A_3^{\infty}$, that is, if C_2 is too large, the optimum tolerance determined by A_1^{∞} cannot be reached, even theoretically. One observes, then, that the value of the summation in eq. 74 can be either positive or negative. It follows that the magnitude of A_3^{∞} can be increased by introducing appropriate zeros of ρ_1 in the right half plane. These zeros, however, reduce necessarily the value of A_1^{∞}, so that the area represented by the integral in eq. 74 is increased at the expense of the area represented by the integral in eq. 73. The optimum behavior for $\ln \frac{1}{|\rho_1|}$ is, in any case, of the rectangular type.

Using the case of Fig. 3.13 as an example, let the maximum value of $\ln \frac{1}{|\rho_1|}$ be equal to $\frac{\pi}{2}$ K. It will be shown later that such an ideal low-pass behavior can be approached in the limit when the number of elements in the matching network approaches infinity. One obtains from eqs. 73 and 74

$$\omega_c K = A_1^{\infty} - 2 \sum_i \lambda_{ri} \tag{76}$$

$$\omega_c^{\,3} K = -3A_3^{\infty} + 2 \sum_i \lambda_{ri}^3 \tag{77}$$

The λ_{ri}'s must be selected in such a way as to maximize the value of K for given ω_c, A_1^{∞} and A_3^{∞}. It will be observed, first of all, that ρ_1 can be multiplied by any factor of the type $\frac{\lambda - \lambda_r}{\lambda + \bar{\lambda}_r}$ without changing the value of $\ln \frac{1}{|\rho_1|}$ on the imaginary axis. The behavior of $\ln \frac{1}{|\rho_1|}$ and the values of the λ_{ri}'s can thus be controlled independently.

One observes next that, since the summation in eq. 77 must be made positive, both equations can be satisfied by using a single zero located at some point $\lambda_r = \sigma_r$ of the positive real axis. On the other hand, maximizing K means making $\sum_i \lambda_{ri}^3$ as large as possible while keeping $\sum_i \lambda_{ri}$ as small as possible. Moreover, if $\mathcal{R}e\,\lambda_{ri}^3 > 0$, $\mathcal{R}e\,\lambda_{ri}^3 \leq \sigma_{ri}^3$, and $\sum_i \mathcal{R}e\,\lambda_{ri}^3 \leq (\sum_i \sigma_{ri})^3$. It follows that the maximum value of K is obtained by using a single zero located at σ_r. Eqs. 76 and 77 become then:

$$\omega_c K = A_1^{\infty} - 2\sigma_r \tag{78}$$

$$\omega_c^{\,3} K = -3A_3^{\infty} + 2\sigma_r^{\,3} \tag{79}$$

Eliminating σ_r and solving the resulting cubic yields the maximum theoretical value of K as a function of the cut-off frequency ω_c. The maximum pass-band value of $\ln \frac{1}{|\rho_1|}$ is plotted in Fig. 3.14 as a function of $\frac{\omega_c}{A_1^{\infty}}$ for different values of the parameter $-\frac{A_3^{\infty}}{(A_1^{\infty})^3}$. The curve $K = \frac{A_1^{\infty}}{\omega_c}$ forms the boundary of the region in which the optimum design is obtained by simply increasing the value of the second element.

When the network N' consists of three or more elements the problem of determining the actual value of the optimum tolerance of match becomes much more difficult and no general solution could be obtained. However, a few general considerations can be made in this regard. In the first place, the rectangular form of frequency behavior for $\ln \frac{1}{|\rho_1|}$ yields the optimum tolerance in all cases, since it provides the best utilization of the areas represented by the successive integrals. In the case of a pass-band extending from zero frequency to ω_c, the equations to be

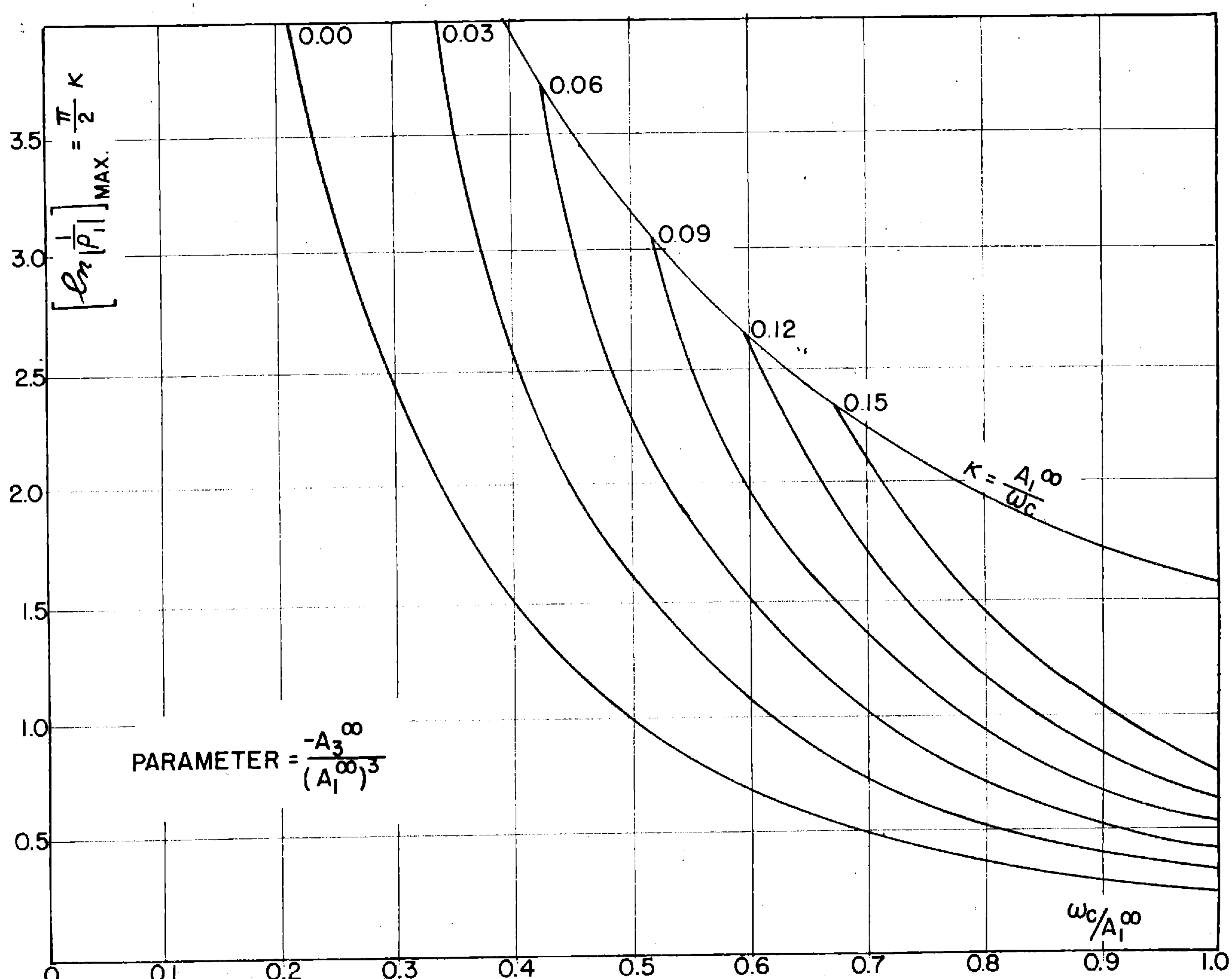


Figure 3.14 - Optimum tolerance of match for an L-C-R impedance

satisfied take the general form:

$$\omega_c^{2k+1} K = (-1)^{k+1}\left[(2k+1)A_{2k+1}^{\infty} - 2\sum_i \lambda_{ri}^{2k+1}\right] \tag{80}$$

The parameter k in the last equation of the set is equal to the number of elements in the network N'. It seems reasonable to expect that the number of λ_{ri}'s for which K is a maximum will be equal, in general, to the minimum number of λ_{ri}'s required for the solution of the set of equations. The reasoning followed in the case of two elements, however, could not be extended rigorously to the case of n elements. Moreover the solution of a given set of n equations, similar to eq. 80, with a number of λ_{ri}'s equal to (n-1), may yield values of the λ_{ri}'s with a negative real part. Such a solution would not be acceptable because the λ_{ri}'s must represent zeros in right half plane. In this case more λ_{ri}'s would have to be used, and their values would have to be determined by maximizing the value of K.

The A_{2n+1}^{∞} in the last equation of any given set can be changed, but only in one direction, by combining one zero of transmission of N' with a similar zero of transmission of N". It will be observed, on the basis of the results obtained in Sec. 2.9, that the direction in which A_{2n+1}^{∞} can be changed is in all cases the one which results in a decrease of the area of the corresponding integral. It follows that one must determine first the optimum tolerance that can be obtained neglecting the last element of the network, as it was done in the case of two elements, to check whether the same tolerance could be obtained by simply increasing the value of this last element.

It is advisable, in general, before proceeding to any numerical computation of the optimum tolerance, to determine the maximum value of K from the first equation alone, and substitute it in the left hand side of the other equations. This procedure will indicate whether each particular A_{2k+1}^{∞} is too large or too small and how much the right hand side of the corresponding equation must be changed by means of the λ_{ri}'s. It will be noted, in this respect, that the magnitudes of some of the A's may be too large, and that it is necessary then to "spend" part of the area represented by A_1^{∞} in order to decrease the value of the right hand side of the corresponding equations.

When the numerical computation of the optimum tolerance becomes so laborious as to be impractical, one can determine without difficulty an upper limit to the value of K (which, however, cannot be approached) by computing the maximum values of K which satisfy the first equation and each one of the other equations, separately. The smallest of the values of K thus obtained sets an upper limit to the value of K that satisfies all the equations simultaneously. The proper use of this and other similar artifices, may lead to reasonable estimate of the optimum tolerance without requiring very laborious computations. It is hardly necessary to point out that, when the network N' contains three or more elements, the actual determination of the optimum tolerance requires the solution of a system of algebraic equation of degree equal or larger than the fifth. This difficulty cannot be avoided as long as the mathematical formulation of the problem remains the same. It is quite possible, however, that a different physical approach, such as, for instance, one based on the time response of the network rather than on the frequency response, might avoid this difficulty and be more successful from a practical point of view.

3.8 Application of the results of Sec. 3.7 to other cases.- The results of Sec. 3.7 can be applied directly to a number of networks obtainable from the low pass ladder structure of Fig. 3.12, by means of appropriate transformations of the frequency variable [9,10]. The simplest type of transformation is the low-pass to high-pass transformation, which interchanges the origin of the λ plane with the point at infinity, and correspond to the change of variable $\lambda' = \frac{1}{\lambda}$. The corresponding network is a high pass ladder structure with series condensers and shunt inductances, that is, a network with a multiple zero of transmission at the origin. It follows that a detailed study of such a network is unnecessary because it would be a mere repetition of Sec. 3.7. This fact can be readily checked by comparing the results of Sec. 3.3 with the corresponding results presented in Sec. 3.2.

The low-pass to band-pass transformation leads to another type of network to which the results of Sec. 3.7 can be applied, namely the band-pass ladder structure. This structure consists of resonant branches tuned to a given frequency ω_0, and can be obtained from the structure of Fig. 3.12 by simply tuning to this frequency every inductance with a series capacitance, and every capacitance with a shunt inductance. If this procedure is applied to a low-pass structure with a low-pass band extending from zero frequency to ω_c, the resulting band-pass structure will have a bandwidth $w = \omega_c$ and a mean frequency ω_0. It will be noted that the band-pass structure is a special case of a network with two zeros of transmission of the same multiplicity located one at the origin and one at infinity. This application of the results of Sec. 3.7 will be illustrated with a numerical example in the following chapter.

An additional remark is in order with regard to networks with zeros of transmission at both the origin and infinity. If the multiplicity of the zero at the origin is n_0 and the multiplicity of the zero at infinity is n_∞, the conditions of physical realizability for the matching network will yield $n_0 + n_\infty$ equations of the types derived in Secs. 3.2 and 3.3. By using a rectangular shaped function for $\ln \frac{1}{|\rho_1|}$ these equations take the forms

$$(\omega_2^{2k+1} - \omega_1^{2k+1})\, K = (-1)^{k+1} \left[(2k+1)A^{\infty}_{2k+1} - 2 \sum_i \lambda_{ri}^{2k+1}\right] \tag{81}$$

$$(\omega_1^{-(2k+1)} - \omega_2^{-(2k+1)})\, K = (-1)^{k+1} \left[(2k+1)A^{o}_{2k+1} - 2 \sum_i \lambda_{ri}^{-(2k+1)}\right] \tag{82}$$

where ω_1 and ω_2 are, respectively, the low frequency and high frequency ends of the pass band and K is the pass-band value of

$\ln \frac{1}{|\rho_1|}$ divided by $\frac{\pi}{2}$. To determine the maximum value of K one must solve simultaneously the whole set of equations. However, if $\omega_2 >> \omega_1$ the two sets of equations relating to the two zeros of transmission can be solved separately. One may observe in this regard that even if ω_1 appears only in one set of equations and ω_2 only in the other set, the λ_{ri}'s appear in both sets. If, however, the problem is considered from a physical point of view, it is clear that the condition $\omega_2 >> \omega_1$ implies that the high frequency response of the network is independent of the low frequency response. Therefore, the λ_{ri}'s which are sufficiently large to affect the high frequency response will be too large to affect the low frequency response, and vice versa. The two sets of equations will yield different values of K for given ω_1 and ω_2, and the smaller of the two will represent the optimum tolerance of match.

3.9 General discussion of the results - All the integral relations derived in this chapter have the same general form, irrespective of the location of the zero of transmission to which they refer. The integrand in the left hand side consists, in all cases, of the function $\ln \frac{1}{|\rho_1|}$ multiplied by a weighing function which depends on the location of the zero of transmission. The right hand side of each equation consists of the difference between a coefficient specified by the network N' and a summation involving the zeros of ρ_1 in the right half plane and the location of the zero of transmission.

In the two simple cases concerning zeros of transmission at infinity and at the origin, the weighing functions are the even powers, positive and negative respectively, of the frequency ω. These functions have the effect of preventing the arbitrary distribution over the frequency spectrum of the area under the $\ln \frac{1}{|\rho_1|}$ - versus - frequency curve. In particular they prevent the value of $\ln \frac{1}{|\rho_1|}$ from remaining large when the frequency approaches infinity, in one case, and zero, in the other case.

The weighing functions have similar properties in the case of a zeros of transmission on the imaginary axis. In the first place the area represented by the integral in the equation involving $A_1^{\omega_\nu}$ can be equal, at most to $\frac{\pi}{2} A_1^{\omega_\nu}$ because the summation in the right hand side of the equation is always positive (See eqs. 4 and 42). The corresponding weighing function $f_1^{\omega_\nu}$, plotted in Fig. 2.4, has a sort of even symmetry with respect to the point $\omega = \omega_\nu$. This would indicate that the area represented by $A_1^{\omega_\nu}$ can be arbitrarily divided between the two sides of the frequency ω_ν. Such an arbitrary division, however, is not possible, because the weighing function $g_0^{\omega_\nu}$ in the first equation of the set has a sort of odd symmetry with respect to the point $\omega = \omega_\nu$. The division of the area is thus limited by the value of B_0 and by the fact that the use of any zero of ρ_1 in the right half plane to modify B_0 results in a decrease of the original area represented by $A_1^{\omega_\nu}$. The weighing functions of higher order are, alternatively, of the even symmetry and odd symmetry types and rise faster and faster when ω approaches ω_ν. Weighing functions with odd symmetry are not present in the case of zeros of transmission at the origin and at infinity, because $\ln \frac{1}{|\rho_1|}$ is by definition an even function of frequency. Apart from this difference, the weighing functions have the same type of effects in the two cases.

Consider next the equations resulting from a zero of transmission on the real axis. In this case the value of the integral in the first equation (52) can never be larger than $\frac{\pi}{2} A_0^{\sigma_\nu}$ and in addition, the value of the integral in the second equation (54) can never be smaller than $(-\frac{\pi}{2}\sigma_\nu A_1^{\sigma_\nu})$. In fact the summations in equations 6 and 7 (for $k = 1$) are always positive. It will be noted, in this regard, that the weighing function f_0 is positive for all values of ω, while the function $f_1^{\sigma_\nu}$ is positive for $\omega < \sigma_\nu$ and negative for $\omega > \sigma_\nu$ (See Fig. 2.5). It follows that, roughly speaking, the value of $\ln \frac{1}{|\rho_1|}$ is limited at low frequencies by the first equation, and at high frequencies by the second equation. If the multiplicity of the zero of transmission is larger than one, the areas represented by the integrals in these first two equations are prevented from being distributed arbitrarily over the frequency spectrum by equations of higher order. The first six weighing functions, corresponding to a zero of transmission of multiplicity equal to three, are plotted in Fig. 2.5.

The last set of equations to be discussed results from zeros of transmission at complex frequencies. The weighing functions of the $f_k^{\lambda_\nu}$ type (See eqs 59, 65, 71 and 72) lead to limitations similar to the ones just discussed in the case of zeros of transmission on the real axis; they become actually equal to the corresponding functions of the $f_k^{\sigma_\nu}$ type when the parameter δ approaches unity ~~zero~~. On the other hand, the $f_{2k+1}^{\lambda_\nu}$'s reduce to the $f_{2k+1}^{\omega_\nu}$ when δ approaches zero. This fact is indicated graphically by the curve of Fig. 2.10, which is computed for $\delta = \frac{1}{20}$. The coefficient $A_o^{\lambda_\nu}$ sets an upper limit to the corresponding integral because the real part of the summation in eq. 8 is always positive. Similarly, eq. 9 shows that $F_1^{\lambda_\nu}$ cannot be larger than $A_1^{\lambda_\nu}$, and therefore $-A_1^{\lambda_\nu}$ sets a lower limit to the value of the corresponding integral.

The curves of Figs. 3.6 to 3.11 show that, while the weighing functions of the $f_k^{\lambda_\nu}$ type have a sort of even symmetry with respect to some frequency in the neighborhood of $|\lambda_\nu|$, the functions of the $g_k^{\lambda_\nu}$ type (See eqs. 52, 66, 71 and 72) have a sort of an odd symmetry. The latters perform in this case a function similar to that performed by the $g_{2k+1}^{\omega_\nu}$'s in the case of zeros of transmission on the imaginary axis. In other words, they prevent the areas limited by $A_o^{\lambda_\nu}$ and $A_1^{\lambda_\nu}$ from being shifted arbitrarily from low frequencies to high frequencies or vice versa. These functions do not appear in the case of zeros of transmission on the real axis because $\ln \frac{1}{|\rho_1|}$ is by nature an even function of frequency. It will be noted, in addition, that the $g_{2k}^{\lambda_\nu}$'s reduce to the $g_{2k}^{\omega_\nu}$'s when δ approaches zero. This fact is again indicated graphically by the curves in Figs. 3.9 and 3.11.

3.10 Concluding remarks - The results presented in this chapter are obviously insufficient to provide a general solution to the second part of the matching problem as stated in Sec. 1.5. Only in a particularly simple case a numerical answer for the optimum tolerance could be obtained, or a practical method for computing it could be indicated. It seems rather doubtful that a practical method of determining the optimum tolerance can be developed for the most general case, because of the inherent difficulty presented by the required solution of high order algebraic equation. On the other hand, the work presented in this chapter forms a general theoretical basis for further investigation of special cases arising in practice. Moreover, the results obtained indicate clearly the nature of the limitations on the matching of arbitrary impedances, and, therefore, may be useful as a guide in practical design problems, even when a cut-and-try procedure must be followed.

CHAPTER IV

The Design of Simple Matching Networks

4.1 General considerations - The integral relations derived in Ch. III indicate that the ideal type of behavior for the return loss at the input terminals of a matching network is represented by the rectangular shaped function used in the determination of the optimum tolerance. Such a behavior cannot be obtained in practice because it would require an infinite number of elements in the matching network, but can be approximated sufficiently well for practical purposes by means of a reasonably small number of elements. In other words, the function representing $|\rho_1|$ must be selected in such a way as to approximate a small constant over the pass band and unity over the rest of the frequency spectrum, just as in the case of conventional filters. It must be pointed out, however, that filters are designed in most cases to provide a perfect match at a finite number of frequencies in the pass band, while this situation has to be avoided in the case of matching networks. In fact, making $\ln \frac{1}{|\rho_1|}$ very large at any point of the pass band leads to an inefficient use of the area represented by the integrals discussed in the preceding chapter, and results, therefore, in a reduction of the bandwidth of match.

In spite of this essential difference between the characteristics of filters and of matching networks, the same techniques can be used in both cases for the solution of the approximation problem. This point will be made clearer by the illustrative examples discussed in the following sections.

4.2 Matching of series RL and shunt RC impedances - A very simple type of matching problem is presented by the case of a load impedance consisting of a resistance in series with an inductance, or by the dual case of a resistance shunted by a condenser. Practical problems of this type arise, for instance, in connection with the broadbanding of the high frequency response of matching transformers, or when the load resistance is shunted by a stray capacitance. A method of designing appropriate matching networks for a series RL impedance is developed below. The same method will be directly applicable to the dual case of a shunt RC impedance.

The pass band desired in most of the cases mentioned above extends from zero frequency to a frequency ω_c; the ideal behavior for $\ln \frac{1}{|\rho_1|}$ is therefore that illustrated in Fig. 3.13. Let L_1 be the value of the inductance normalized with respect to the series resistance, that is, divided by it. The coefficient A_1^∞, which represents the only condition of physical realizability to be satisfied by the matching network, is, by definition:

$$A_1^\infty = \left[\frac{d}{d\frac{1}{\lambda}} \ln \frac{2+\lambda L_1}{\lambda L_1}\right]_{\frac{1}{\lambda}=0} = \frac{2}{L_1} \tag{1}$$

The maximum theoretical pass band value of $\ln \frac{1}{|\rho_1|}$ is, therefore, according to eq. 3.76:

$$\left[\ln \frac{1}{|\rho_1|}\right]_{max} = \frac{\pi}{\omega_c L_1} \tag{2}$$

The problem consists then of approaching this theoretical limit by means of a matching network involving a finite number of elements. It is self evident that the larger is the number of elements used, the closer the theoretical limit can be approached. Therefore, the practical problem is actually that of obtaining the best tolerance of match with a given number of elements.

The general remarks made in the preceding section indicate that the inductance L_1, which forms the network N' of Fig. 2.6, may be considered as the first element of a low-pass filter, the network N, whose input reflection coefficient is ρ_1. This reflection coefficient cannot be measured in practice, because the inductance L_1 is assumed to be inseparable from the one ohm termination; its magnitude, however, is equal to the magnitude of the reflection coefficient ρ_2 at the other terminals of the filter to which the generator will be connected in actual operation.

Two types of functions are used for the solution of the approximation problem in the case of low-pass filters(3,11). The first type of function is the Tchebysheff polynomial $T_n\left(\frac{\omega}{\omega_c}\right)$, which leads to a function $|\rho_1|$ which oscillates between two given values in the pass band and approaches asynthotically unity in the attenuation band, as illustrated in Fig. 4.1. The maximum pass-band value $|\rho|_{max}$ represents the tolerance of match which must be larger than the theoretical limit given by eq. 2. The second type of function is the Jacobian elliptic function which leads to an oscillatory behavior of $|\rho_1|$ in both the pass band and the attenuation band. In the first case all the zeros of transmission are

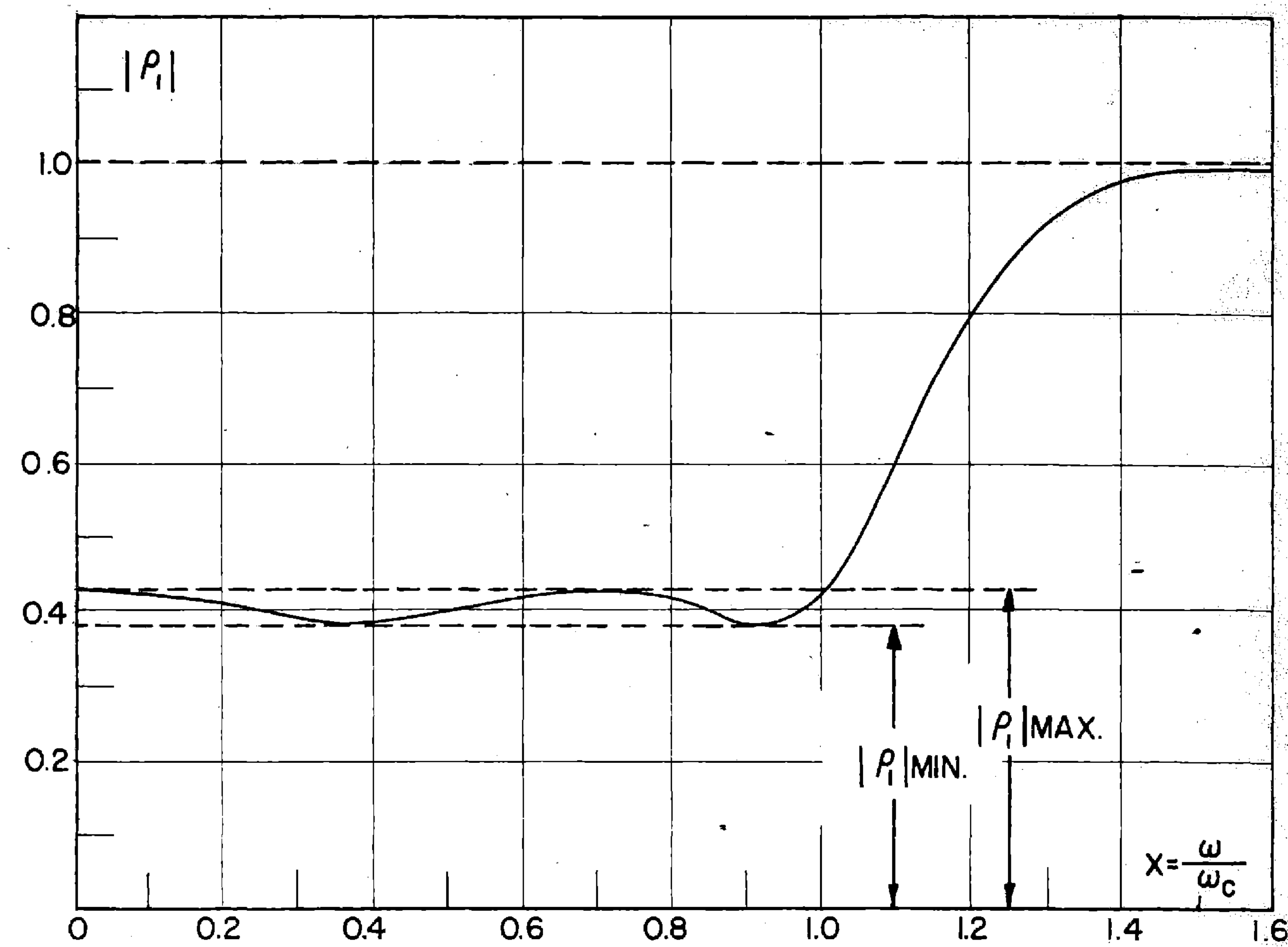


Figure 4.1 – Frequency behavior of $|\rho_1|$ for the matching network of Figure 4.5

at infinity so that the network will consist of a simple ladder structure with series inductances and shunt capacitances. In the second case zeros of transmission are present at finite frequencies as well as at infinity, and, therefore, the network will involve sections of type C (See Fig. 2.7). The filtering characteristics in this second case are somewhat better than in the first case for the same number of elements, because a sharper cut-off can be obtained. In matching problems, on the other hand, the design involving elliptic functions may lead to a slightly better tolerance, but the difference does not seem to be worth the resulting theoretical and practical complications.

4.3 Determination of ρ_1 - The Tchebysheff polynomial of the first kind and order n is defined by

$$T_n(x) = \cos\,(n \cos^{-1} x) \tag{3}$$

In polynomial form one obtains

$$T_1(x) = x \tag{4}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{5}$$

It is clear that $T_n^2(x)$ oscillates between zero and one for $x < 1$ and approaches infinity as $(2^{n-1}x^n)^2$ for $x > 1$. In order to obtain the function $|\rho_1|^2$ one constructs first a function $|t|^2$ which has an oscillatory behavior in the pass band, is smaller than unity, and has all its zeros at infinity. Letting $x = \frac{\omega}{\omega_c}$, $|t|^2$ can be written in the form

$$|t|^2 = \frac{1}{(1+K^2) + \varepsilon^2 T_n^2(x)} \tag{6}$$

where K^2 and ε^2 are arbitrary constants. The corresponding magnitude of the reflection coefficient is

$$|\rho_1|^2 = 1 - |t|^2 = \frac{K^2 + \varepsilon^2 T_n^2(x)}{(1+K^2) + \varepsilon^2 T_n^2(x)} \tag{7}$$

The degree of both the numerator and the denominator polynomials is equal to 2n, and, therefore, the corresponding network will have n elements. With reference to Fig. 4.1, the pass-band tolerance is given by

$$|\rho_1|_{max} = \sqrt{\frac{K^2 + \varepsilon^2}{1+K^2 + \varepsilon^2}} \tag{8}$$

For a given value of n, the constants K^2 and ε^2 must be adjusted to minimize $|\rho_1|_{max}$ and to satisfy, at the same time, the condition of physical realizability

$$\left[\frac{d}{d\frac{1}{\lambda}} \ln \frac{1}{\rho_1}\right]_{\frac{1}{\lambda} = 0} = \sum_{i=0}^{n} (\lambda_{oi} - \lambda_{pi}) = A_1^{\infty} = \frac{2}{L_1} \tag{9}$$

The first step in the design procedure must be, therefore, the determination of the zeros, λ_{oi}, and poles λ_{pi}, of ρ_1. For this purpose the function $|\rho_1|^2$ is rewritten in the form

$$|\rho_1|^2 = \left[\rho_1(Z)\rho_1(-Z)\right]_{Z=jx} = \begin{cases} \left[\dfrac{\sinh^2(nb) + \cosh^2(n \sinh^{-1}Z)}{\sinh^2(na) + \cosh^2(n \sinh^{-1}Z)}\right]_{Z=jx} & (n \text{ even}) \\ \left[\dfrac{\sinh^2(nb) - \sinh^2(n \sinh^{-1}Z)}{\sinh^2(na) - \sinh^2(n \sinh^{-1}Z)}\right]_{Z=jx} & (n \text{ odd}) \end{cases} \tag{10}$$

where

$$Z = \frac{\lambda}{\omega_c} = \frac{\sigma}{\omega_c} + j\frac{\omega}{\omega_c} \tag{11}$$

and

$$\sinh^2 nb = \frac{K^2}{\epsilon^2} \quad ; \quad \sinh^2 na = \frac{1+K^2}{\epsilon^2} \tag{12}$$

Further transformation of this equation yields

$$\rho_1(Z)\rho_1(-Z) = \begin{cases} \dfrac{\cosh\left[n(\sinh^{-1}Z-b)\right]\cosh\left[n(\sinh^{-1}Z+b)\right]}{\cosh\left[n(\sinh^{-1}Z-a)\right]\cosh\left[n(\sinh^{-1}Z+a)\right]} & (n \text{ even}) \\ \dfrac{\sinh\left[n(\sinh^{-1}Z-b)\right]\sinh\left[n(\sinh^{-1}Z+b)\right]}{\sinh\left[n(\sinh^{-1}Z-a)\right]\sinh\left[n(\sinh^{-1}Z+a)\right]} & (n \text{ odd}) \end{cases} \tag{13}$$

The zeros of this function are evidently given by:

$$Z_o = \begin{cases} \sinh\left[\pm b \pm j \frac{\pi}{n}\left(\frac{1}{2}+m\right)\right] & \text{(n even)} \\ \sinh\left[\pm b \pm j \frac{\pi}{n} m\right] & \text{(n odd)} \end{cases} \tag{14}$$

where m is an integer or zero. The poles are given by a similar expression.

$$Z_p = \begin{cases} \sinh\left[\pm a \pm j \frac{\pi}{n}\left(\frac{1}{2}+m\right)\right] & \text{(n even)} \\ \sinh\left[\pm a \pm j \frac{\pi}{n} m\right] & \text{(n odd)} \end{cases} \tag{15}$$

It will be noted that the poles lie on an ellipse centered at the origin with semiaxes equal to cosh a and sinh a, as indicated in Fig. 4.2 for the two cases of $n = 3$ and $n = 4$. The zeros lie, similarly, on an ellipse with semiaxes equal to cosh b and sinh b.* The poles of ρ_1 are necessarily those poles of $\rho_1(\lambda)\rho_1(-\lambda)$ which lie in the right half plane, that is, a in eq. 5 must be taken with the negative sign. The zeros of ρ_1 can be located anywhere in the complex λ plane. It was shown in the preceding chapter, however, that

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* A method has been developed by the author for deriving the approximation function (10) directly from the location of its zeros and poles without any reference to the Tchebysheff polynomials. This method of solving the approximation problem, which finds application in many other cases, will be discussed in a forthcoming report of the Research Laboratory of Electronics.

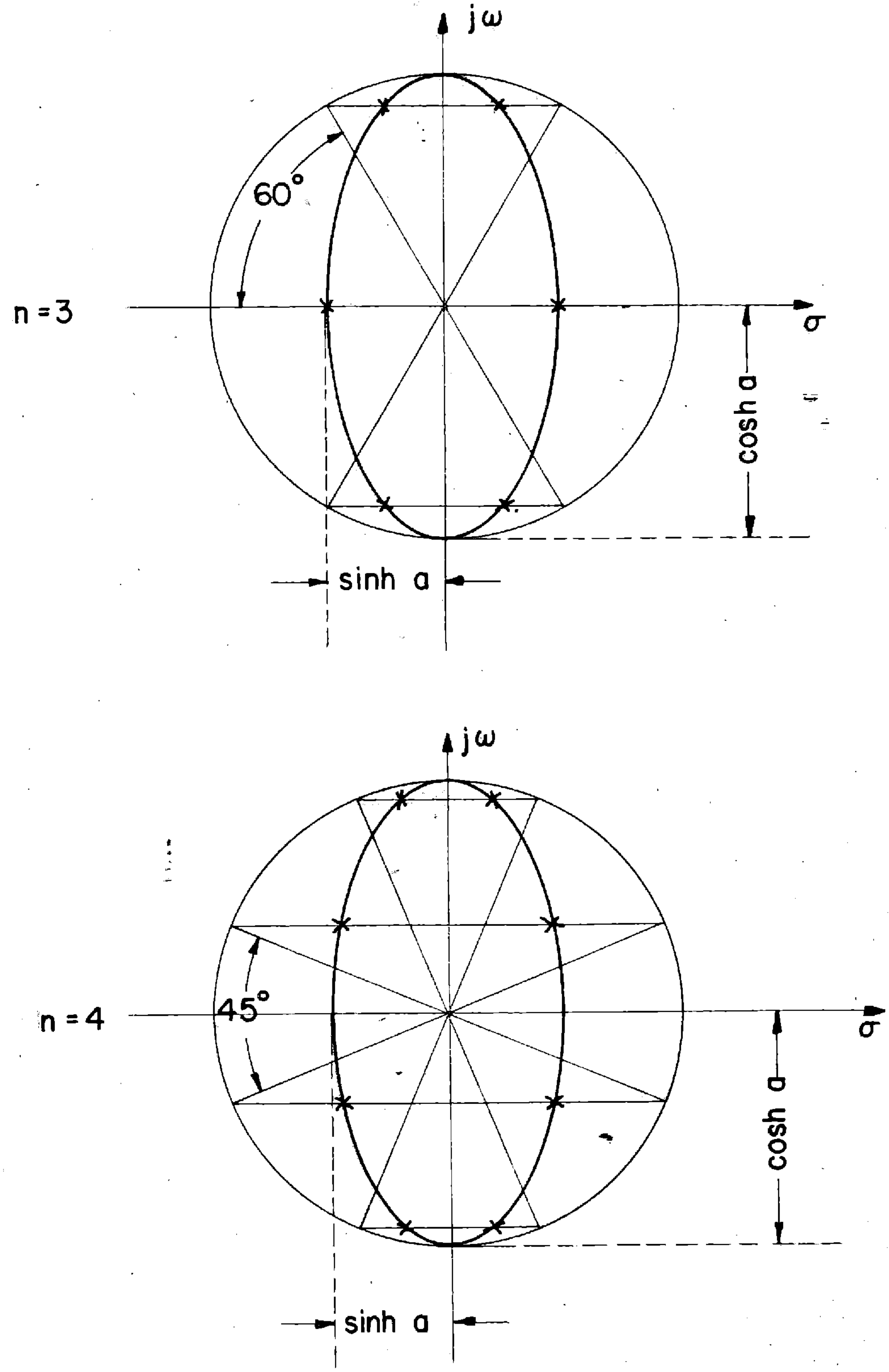

Figure 4.2 - Location of the poles of $|\rho_1|^2$ for networks with 3 and 4 elements

the area represented by $\int_0^\infty \ln \frac{1}{|\rho_1|}\, d\omega$ is a maximum for a given A_1^∞ when all the zeros of ρ_1 are in the left half plane. Therefore, the appropriate λ_{oi}'s are given by eq. 14 in which b is taken with the negative sign.

The evaluation of the summation in eq. 9 is carried out as follows. For n even one has

$$\frac{1}{\omega_c} \sum_i \lambda_{pi} = -2\sinh a \sum_{m=0}^{\frac{n}{2}-1} \cos \frac{\pi}{n}\left(\tfrac{1}{2}+m\right) =$$

$$= -\sinh a \left[e^{j\frac{\pi}{2n}} \sum_{m=0}^{\frac{n}{2}-1} e^{j\frac{\pi}{n}m} + e^{-j\frac{\pi}{2n}} \sum_{m=0}^{\frac{n}{2}-1} e^{-j\frac{\pi}{n}m} \right] =$$

$$= -\sinh a \left[e^{j\frac{\pi}{2n}} \frac{1-e^{j\frac{\pi}{2}}}{1-e^{j\frac{\pi}{n}}} + e^{-j\frac{\pi}{2n}} \frac{1-e^{-j\frac{\pi}{2}}}{1-e^{-j\frac{\pi}{n}}} \right] =$$

$$= -\frac{\sinh a}{\sin \frac{\pi}{2n}} \qquad (16)$$

$$\sum_i (\lambda_{oi} - \lambda_{pi}) = \omega_c \frac{\sinh a - \sinh b}{\sin \frac{\pi}{2n}} \qquad (17)$$

Exactly the same result is obtained for n odd. Substituting eq. 17 for the summation in eq. 9 yields

$$\sinh a - \sinh b = \left(\sin \frac{\pi}{2n}\right) \frac{A_1^{\infty}}{\omega_c} \tag{18}$$

On the other hand, the tolerance of match can be expressed in terms of the design parameters a and b, with the help of eqs. 8 and 12, as follows.

$$|\rho|_{max} = \sqrt{\frac{1 + \sinh^2 nb}{1 + \sinh^2 na}} = \frac{\cosh nb}{\cosh na} \tag{19}$$

These two design parameters are then determined in such a way as to make $|\rho|_{max}$ a minimum and satisfy, at the same time, eq. 18. Using for this purpose the method of indeterminate multipliers, and differentiating eqs. 18 and 19 with respect to a and b, one obtains

$$\left.\begin{aligned} &\cosh a - \alpha \cosh b = 0 \\ &\frac{\sinh na \cosh nb}{\cosh^2 na} - \alpha \frac{\sinh nb}{\cosh na} = 0 \end{aligned}\right\} \tag{20}$$

Eliminating the multiplier α from these two equations yields

$$\frac{\tanh na}{\cosh a} = \frac{\tanh nb}{\cosh b} \tag{21}$$

The parameters a and b which yield the minimum value of $|\rho|_{max}$ are then obtained by solving simultaneously eqs. 18 and 21. These transcendental equations may be solved graphically by plotting the master function $\frac{\tanh nx}{\cosh x}$ versus sinh x, and determining the two values of the variable which differ by $\left(\sin\frac{\pi}{2n}\right)\frac{A_1^{\infty}}{\omega_c}$ and yield the same value for the function. The resulting optimum value of sinh a is plotted in Fig. 4.3 for different values of n up to and including 8.* The corresponding value of sinh b can be obtained by means of eq. 18.

The optimum value of the return loss $\ln \frac{1}{|\rho_1|_{max}}$ is plotted in Fig. 4.4 as a function of $\frac{A_1^{\infty}}{\omega_c}$ for different values of n. The curve indicated by $n = \infty$ is the straight line of slope equal to $\frac{\pi}{2}$ which represents the limiting value of the return loss given by eq. 2. It will be noted that this limiting value is approached reasonably well with a relatively small number (n - 1) of elements in the matching network. In the limit, when n approaches infinity and both a and b approach zero as $\frac{1}{\sqrt{n}}$, one obtains from eqs. 18 and 19

$$\ln \frac{1}{|\rho_1|_{max}} = \frac{\pi}{2}\frac{A_1^{\infty}}{\omega_c}$$

which is the equation derived in Sec.3.7 for the optimum theoretical tolerance.

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* These curves are obtained from computations made by Dr. Cerrillo of the Research Laboratory of Electronics.

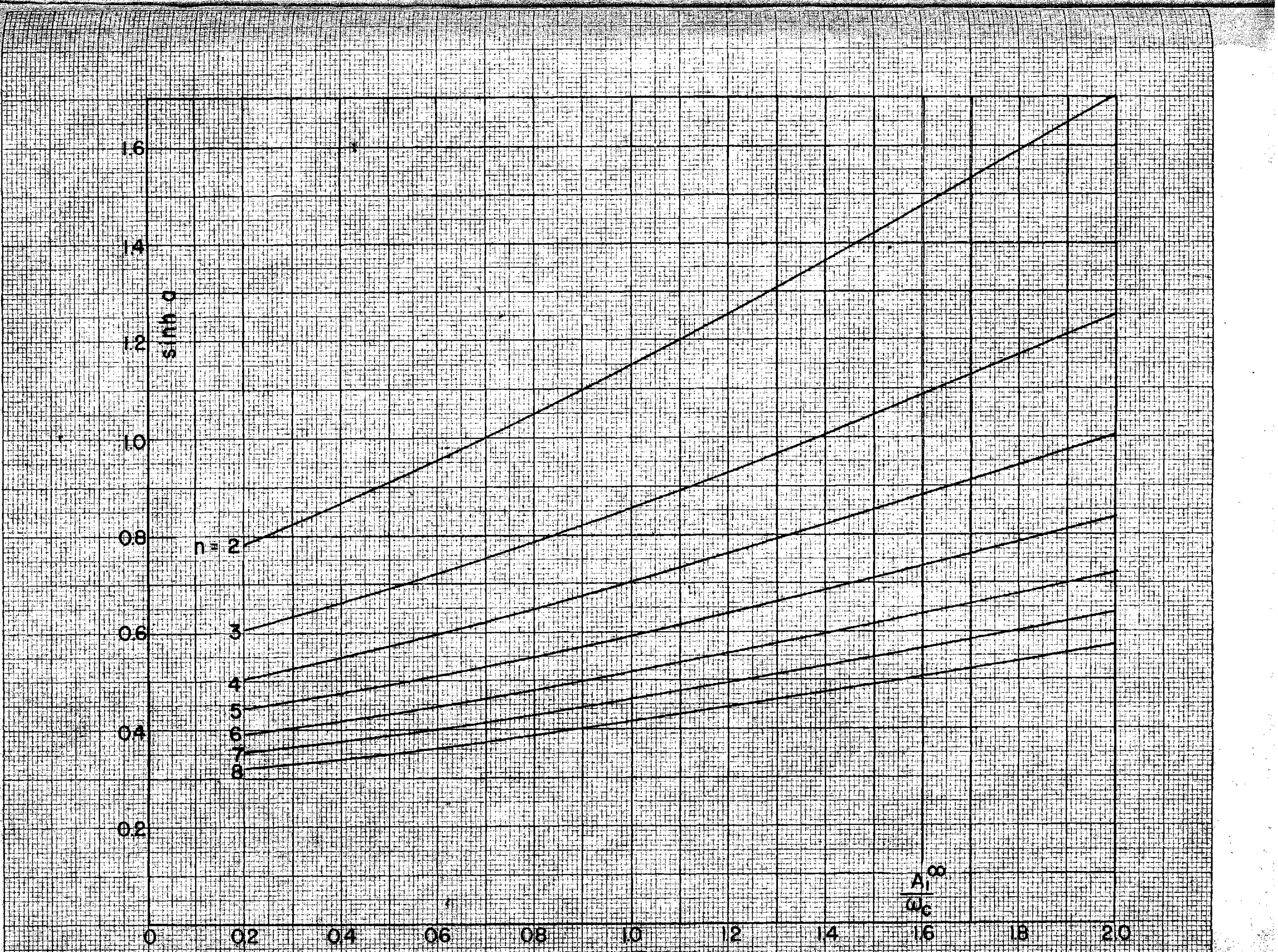


Figure 4.3 - Design curves for a ladder network with n elements

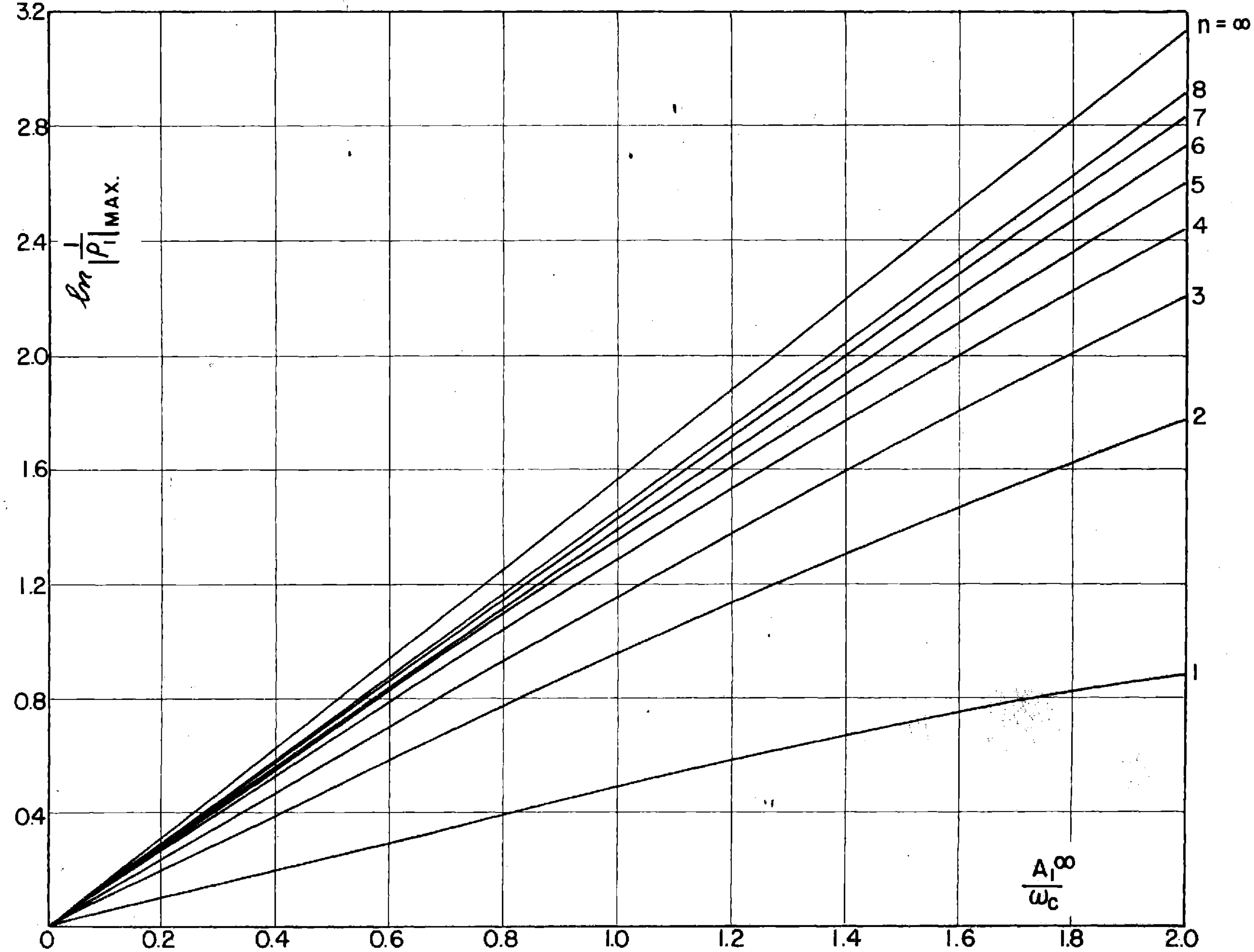


Figure 4.4 - Tolerance of match for a low-pass ladder structure with n elements

4.4 Computation of the element values - The next step in the design procedure, after having determined the function ρ_1, is the actual computation of element values for the matching network. For this purpose one can follow any one of the available synthesis procedures (3,12), a discussion of which is beyond the scope of this presentation. It seems appropriate, on the other hand, to mention a method of computing the element values developed by the author in connection with the matching problem. This method has the advantage of permitting the direct and independent computation of the individual elements from the values of a and b obtained in the preceding section.

Consider then a ladder structure consisting of series inductances and shunt capacitances. The values of the A^{∞}_{2k+1} coefficients can be computed from the poles and zeros of ρ_1 by means of the equation

$$A^{\infty}_{2k+1} = \frac{1}{2k+1} \sum_{i=0}^{n} \left(\lambda_{oi}^{2k+1} - \lambda_{pi}^{2k+1} \right) \tag{22}$$

derived in Chapter II. It was pointed out, at that time, that the coefficient A^{∞}_{2k+1} depends only on the first k elements of the ladder structure. It follows that the value of the k^{th} element must depend only on the A^{∞} coefficients with a subscript equal to or smaller than 2k + 1, and, therefore, it should be possible to compute it directly from them. Appropriate equations have been derived for $k \leq 4$, by computing successively the A coefficients for a ladder structure with 1, 2, 3, and 4 elements, and then solving the resulting set of equations for the element values. The procedure is straight forward but very laborious, and, therefore, only the final results are given here. Let the successive elements of the ladder be L_1, C_2, L_3, etc., and let also:

$$\alpha_3 = \left(2^2 \frac{A_3^{\infty}}{(A_1^{\infty})^3} - \frac{1}{3}\right) \tag{23}$$

$$\alpha_5 = \left(2^4 \frac{A_5^{\infty}}{(A_1^{\infty})^5} - \frac{1}{5}\right) \tag{24}$$

$$\alpha_7 = \left(2^6 \frac{A_7^{\infty}}{(A_1^{\infty})^7} - \frac{1}{7}\right) \tag{25}$$

One has for the elements

$$L_1 = \frac{2}{A_1^{\infty}} \tag{26}$$

$$C_2 = \frac{L_1}{\alpha_3} \tag{27}$$

$$L_3 = -\frac{\alpha_3 L_1}{1 + \alpha_3 - \frac{\alpha_5}{\alpha_3}} \tag{28}$$

$$C_4 = \frac{\left(1+\alpha_3 - \frac{\alpha_5}{\alpha_3}\right)^2 L_1}{\alpha_3\left[1+\alpha_3 - \frac{\alpha_5}{\alpha_3} + \left(\frac{\alpha_5}{\alpha_3}\right)^2 - \frac{\alpha_7}{\alpha_3}\right]} \tag{29}$$

It was hoped that these equations would indicate a recurrence formula for the following element, but it did not turn out to be the case. In the particular case of the functional form for ρ_1 discussed in the preceding section, the A coefficients can be expressed very simply in terms of the parameters a, b and n. Using eq. 22, and proceeding in the same manner as in the case of A_1^∞, one obtains without difficulty

$$A_3^\infty = -2^{-2}\,\omega_c^{\,3}\left[\frac{\sinh 3a - \sinh 3b}{3\sin\frac{3\pi}{2n}} + \frac{\sinh a - \sinh b}{\sin\frac{\pi}{2n}}\right] \tag{30}$$

$$A_5^\infty = 2^{-4}\,\omega_c^{\,5}\left[\frac{\sinh 5a - \sinh 5b}{5\sin\frac{5\pi}{2n}} + \frac{\sinh 3a - \sinh 3b}{\sin\frac{3\pi}{2n}} + 2\,\frac{\sinh a - \sinh b}{\sin\frac{\pi}{2n}}\right] \tag{31}$$

$$A_7^\infty = -2^{-6}\,\omega_c^{\,7}\left[\frac{\sinh 7a - \sinh 7b}{7\sin\frac{7\pi}{2n}} + \frac{\sinh 5a - \sinh 5b}{\sin\frac{5\pi}{2n}} + 3\,\frac{\sinh 3a - \sinh 3b}{\sin\frac{3\pi}{2n}} + 5\,\frac{\sinh a - \sinh b}{\sin\frac{\pi}{2n}}\right] \tag{32}$$

A numerical example is carried out in the following section to illustrate this method of design.

4.5 Broadbanding of a matching transformer - A convenient example for illustrating the method of design discussed above is the high frequency broadbanding of a matching transformer. Suppose a transformer is to be used to match a low impedance resistive load to a generator, a power amplifier, for instance. The transformer behaves at high frequencies as an inductance L in series with the load resistance R_L. Let

$$\omega_h = \frac{2R_L}{L} \tag{33}$$

be the half-power angular frequency of the transformer when the load is matched to the generator at low frequencies. It is desired to broadband this transformer so that the loss will be smaller or equal to 1 db up to a frequency $\omega_c = \frac{3}{2}\omega_h$. Incidental dissipation is neglected in this preliminary design, and can be taken into account later, if necessary, as indicated in Sec. 1.3.

It is convenient to normalize the network to 1 ohm impedance level, in which case the normalized inductance becomes

$$L_1 = \frac{L}{R_L} \tag{34}$$

One has then from eqs. 9 and 33

$$\frac{A_1^{\infty}}{\omega_c} = \frac{2}{\omega_c L_1} = \frac{2R_L}{1.5\,\omega_h L} = \frac{2}{3} \tag{35}$$

A transmission loss of 1 db corresponds to a return loss equal to 0.79. Fig. 4.4 shows that the optimum theoretical value of $\ln \frac{1}{|\rho_1|_{max}}$ for $\frac{A_1^{\infty}}{\omega_c} = 0.66$ is 1.04 and that a value of 0.86 can be obtained with $n = 4$, that is with a matching network consisting of two capacitances and one inductance. The corresponding value of $|\rho_1|_{max}$ is 0.424 and the resulting transmission loss is 0.86 db. One has then, for design data

$$\frac{A_1^{\infty}}{\omega_c} = 0.66 \;;\; n = 4 \;;\; |\rho_1|_{max} = 0.424 \tag{36}$$

The first step in the design is the determination of the parameters a and b for the function $|\rho_1|$. One obtains from Fig. 4.3

$$\sinh a = 0.615 \;;\; a = 0.582 \tag{37}$$

and from eq. 18

$$\sinh b = 0.363 \;;\; b = 0.356 \tag{38}$$

The corresponding function $|\rho_1|$ is plotted in Fig. 4.1 versus the normalized frequency variable $x = \frac{\omega}{\omega_c}$. Using the above values in eqs. 23, 24 and 25 yields

$$\alpha_3 = -4.493 \; ; \; \alpha_5 = 34.05 \; ; \; \alpha_7 = -435.1 \qquad (39)$$

The values of the elements of the matching network are then computed by means of eqs. 27, 28 and 29.

$$\frac{C_2}{L_1} = 0.2225 \qquad (40)$$

$$\frac{L_3}{L_1} = 1.10 \qquad (41)$$

$$\frac{C_4}{L_1} = 0.1043 \qquad (42)$$

The turns ratio of the ideal transformer is specified by the zero frequency value of $|\rho_1|$, that is, in this case, by $|\rho_1|_{max}$. One has then

$$\text{turns ratio} = \sqrt{\frac{1 \quad 0.424}{1 - 0.424}} = \sqrt{2.47} = 1.57 \qquad (43)$$

The resulting network, for a 1 ohm impedance level, is shown in Fig. 4.5(a).

The ideal transformer is combined, in practice, with the matching transformer by performing a suitable change of impedance level. Suppose, for instance, that the actual load resistance is equal to

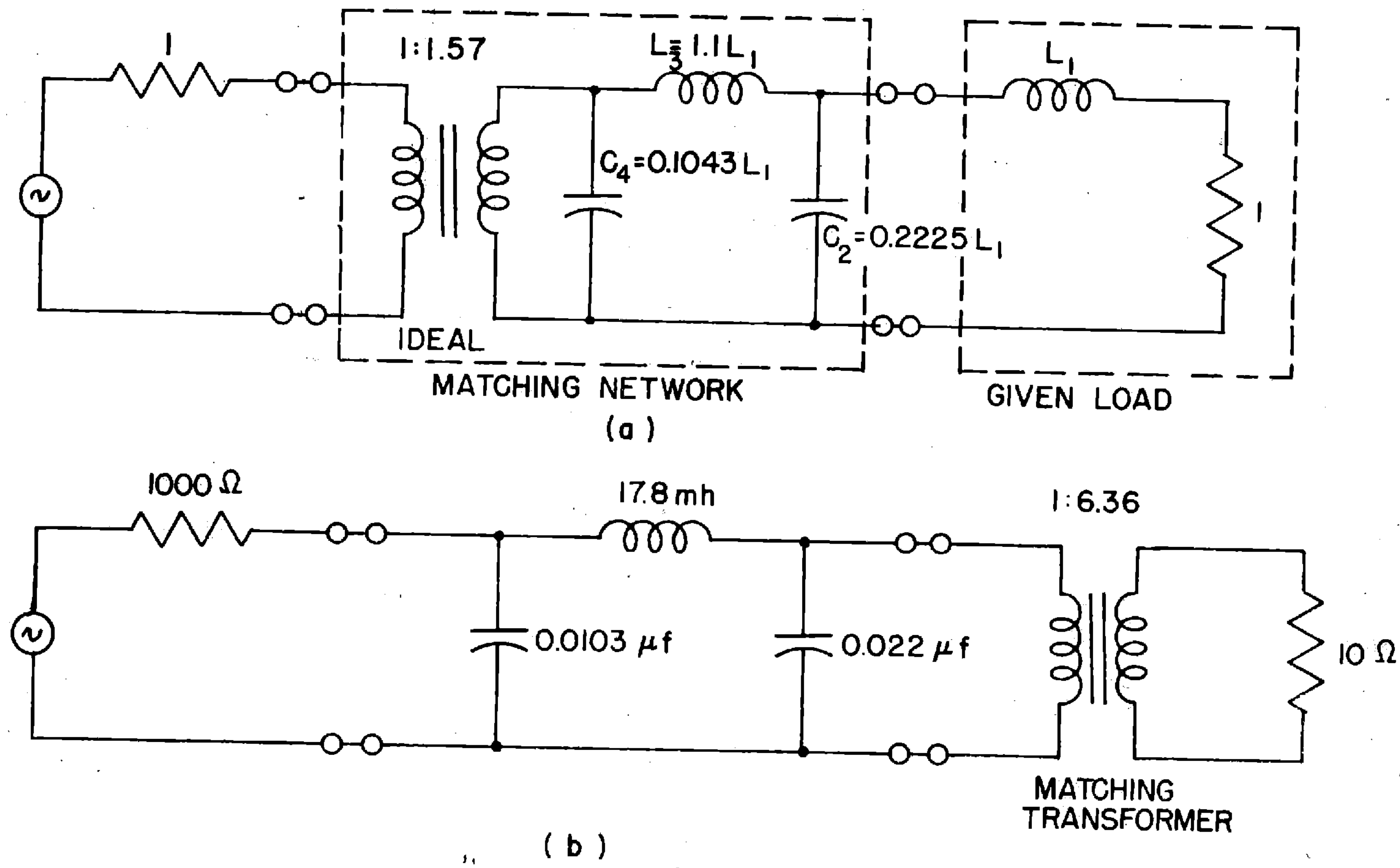


Figure 4.5 - Network for the high-frequency broadbanding of a matching transformer

10 ohms, the source resistance to 1000 ohms, and that the half power frequency is $\omega_h = 50{,}000$ rad/sec. The turns ratio of the matching transformer is then equal to $\frac{10}{1.57} = 6.36$, and the value of leakage inductance referred to the primary side becomes 16.2 mh. The resulting network is shown in Fig. 4.5(b).

The values of the elements can be checked by designing the network from the opposite end, that is, from the reflection coefficient ρ_2. The poles of ρ_2 are the same as the poles of ρ_1, and the zeros of ρ_2 can be obtained from the zeros of ρ_1 by changing the sign of the real part, that is, the sign of b. One must keep in mind, however, that the presence of the ideal transformer modifies the values of the elements when the network is considered from the other end. The same procedure can be followed in designing networks with more than 4 and less than 9 elements. In this case, one determines 4 of the elements by operating on ρ_1 and the rest of them by operating on ρ_2.

4.6 Matching of a resonant antenna - The method of design developed in Sec. 4.4 can be applied also to the case of a load consisting of a series (or parallel) tuned circuit, as discussed in Sec. 3.8, if the frequency band over which the load is to be matched is centered at the resonance frequency ω_o of the tuned circuit. A practical example of this type is offered by the broadbanding of a half-wave antenna, which behaves, to a first approximation, as a series tuned circuit.

Consider then, a quarter-wave grounded antenna (whose impedance is just half of the impedance of the corresponding half wave antenna in free space) with a resonance frequency of 10 Mc; a radiation resistance of 30 ohms and a Q of 10. It is desired to match this antenna to a 50 ohm coaxial line over a 3 Mc. band with a voltage standing-wave ratio on the line smaller than 2.5.

One begins the design by transforming the resonant circuit into the corresponding low pass impedance; that is, into a series resistance-inductance combination. One obtains for the elements

$$R = 30 \text{ ohms} \qquad L = \frac{QR}{\omega_o} = \frac{300}{2 \times 10^7} = 4.77\ 10^{-6}\ \text{h} \tag{44}$$

A matching network is then designed for this impedance following the procedure developed in Secs. 4.3 and 4.4. The required cut-off frequency for this equivalent low-pass network is

$$\omega_c = 2\pi \times 3\ 10^6 = 1.88 \times 10^6 \text{ rad/sec.} \tag{45}$$

Therefore, the basic design data on a 1 ohm impedance level are

$$L_1 = \frac{4.77 \times 10^{-6}}{30} = 1.59 \times 10^{-7} \tag{46}$$

$$\frac{A_1^{\infty}}{\omega_c} = \frac{2}{L_1 \omega_c} = \frac{2}{3} \tag{47}$$

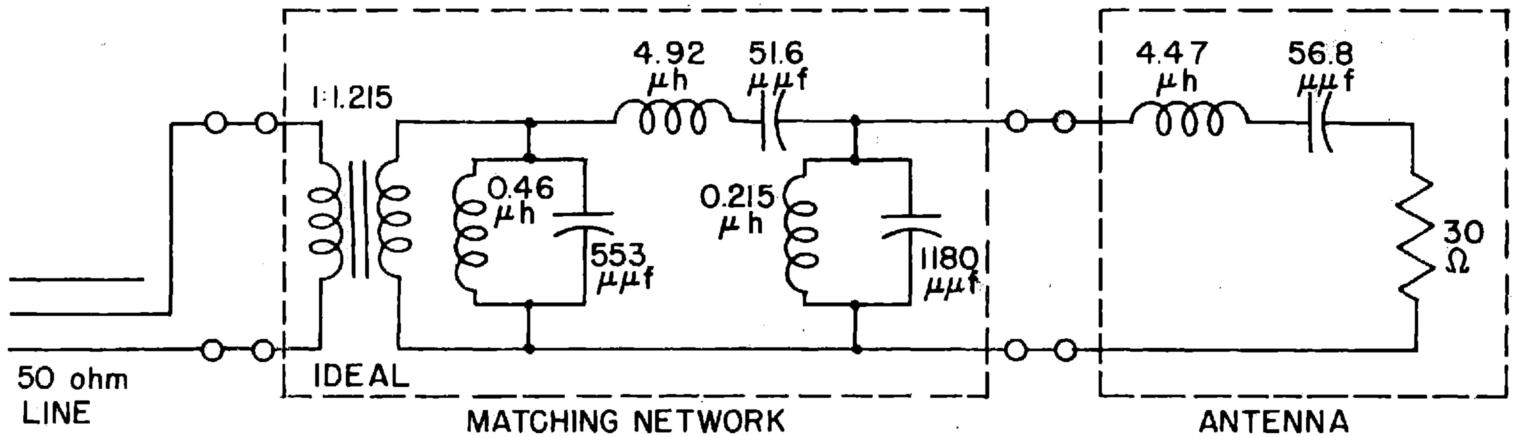


Figure 4.6 - Matching network for an antenna

The maximum tolerable value of the reflection coefficient is $\frac{2.5-1}{2.5+1} = 0.429$. It follows that the basic, low-pass design obtained in the preceding section (See Fig. 4.5-a) can be used also in this case, provided the ratio of the ideal transformer is properly changed to take into account the ratio of the radiation resistance to the characteristic impedance of the line. The impedance level of this network is then increased by a factor of 30, and finally the band-pass equivalent is obtained by tuning to the mean frequency ω_0 all the capacitances by means of shunt inductances and all the inductances by means of series capacitances. The resulting network is shown in Fig. 4.6. This network may be transformed further into a more convenient structure involving tuned coupled coils by means of which it will be possible to eliminate the ideal transformer. The details of such transformation are well known and do not require any further discussion.

The design procedure discussed in this section can be extended to the case of microwave networks, as pointed out in Sec. 1.3 by means of an approximate technique successfully employed by the author in connection with microwave filters (1).

4.7 Matching of an R-C-L impedance - Consider now the matching of a load impedance consisting of an inductance L in series with a parallel RC combination. A problem of this type may arise in connection with the high frequency response of step-up matching transformers. In this case R is the load resistance, C represents the stray capacitance of the secondary coil and L is the total leakage inductance, all of them referred to the primary side of the transformer.

The theoretical limitations for this matching problem have been discussed in Sec. 3.7 and curves for the optimum tolerance are plotted in Fig. 3.14. It was shown there that the optimum tolerance is obtained by introducing a zero of ρ_1 at a point σ_r of the positive real axis together with a symmetrical pole on the negative real axis. The same technique will be used in obtaining the appropriate function for ρ_1 when the matching network contains a finite number of elements. The same approximation function can be used for $|\rho_1|$ as in the case discussed in the preceding section, because the addition of a zero and a pole symmetrically located with respect to the imaginary axis leaves the value of $|\rho_1|$ unchanged for imaginary values of λ. Using eqs. 17 and 30 the conditions of physical realizability can then be written in the form

$$\frac{\sinh a - \sinh b}{\sin \frac{\pi}{2n}} + 2\frac{\sigma_r}{\omega_c} = \frac{A_1^{\infty}}{\omega_c} \tag{48}$$

$$2^{-2}\left[\frac{\sinh 3a - \sinh 3b}{3\sin\frac{3\pi}{2n}} + \frac{\sinh a - \sinh b}{\sin\frac{\pi}{2n}}\right] - \frac{2}{3}\left(\frac{\sigma_r}{\omega_c}\right)^3 = -\frac{A_3^{\infty}}{\omega_c} \tag{49}$$

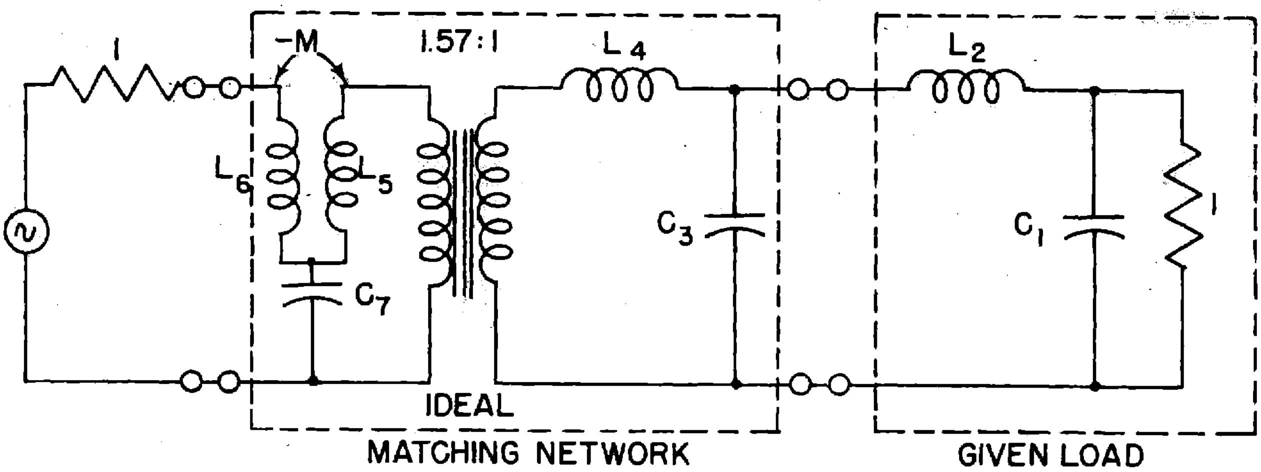
The maximum pass-band value of $|\rho_1|$ is still given by

$$|\rho_1|_{max} = \frac{\cosh nb}{\cosh na} \tag{50}$$

The parameters a, b and σ_r appearing in the above equation must be determined in such a way as to minimize the value of $|\rho_1|_{max}$ and satisfy, at the same time, eqs. 48 and 49. This minimization process involves the solution of a system of transcendental equations, as in the simpler case of a series RL impedance. A convenient graphical procedure could not be developed in this case, so that one must resort to some kind of laborious cut-and-try procedure. If, however, an optimum design is not required, reasonably good results can be obtained by using the same relation between a and b that was determined for the matching of a simple RL impedance. A considerable amount of numerical work is eliminated by following this procedure. Once a, b and σ_r are known, the computation of the element values for the matching network becomes a straight forward problem of network synthesis. The equations developed in Sec. 4.4 can still be used; but, in this case, a section of type C will also be present. In fact, the zero of ρ_1 at σ_r together with the pole at $-\sigma_r$ require the presence of an identical pair of singularities in both ~~the~~ ρ_2 and t. A convenient procedure for computing the values of all the elements, including those representing the zero of transmission at σ_r, is illustrated below by means of a numerical example.

With reference to Fig. 4.7, suppose the normalized values of the elements forming the given load impedance are given by

$$C_1 = \frac{1.205}{\omega_c} \quad ; \; L_2 = \frac{2.3}{\omega_c} \tag{51}$$



$$C_1 = \frac{1.205}{\omega_c} \; ; \; L_2 = \frac{2.3}{\omega_c} \; ; \; C_3 = \frac{0.813}{\omega_c} \; ; \; L_4 = \frac{3.37}{\omega_c}$$

$$L_5 = \frac{0.66}{\omega_c} \; ; \; L_6 = \frac{1.135}{\omega_c} \; ; \; M = \sqrt{L_5 L_6} \; ; \; C_7 = \frac{4.63}{\omega_c}$$

Figure 4.7 - High-frequency broadbanding of a matching transformer with stray capacitance loading

where ω_c is the upper limit of the frequency band over which the impedance is to be matched. For the sake of simplicity these figures have been selected to yield the same values of a and b as in the problem discussed in Sec. 4.5. One obtains from eqs. 23, 26 and 27

$$\frac{A_1^\infty}{\omega_c} = 1.66 \tag{52}$$

$$-\frac{A_3^\infty}{(A_1^\infty)^3} = 0.0475 \tag{53}$$

$$-\frac{A_3^\infty}{(\omega_c)^3} = 0.217 \tag{54}$$

If the approximation function with $n = 4$ is used and the same relation between a and b is assumed which yields the optimum tolerance in the case of an R-L impedance, one obtains

$$\begin{aligned} \sinh a &= 0.615 \\ \sinh b &= 0.363 \\ \sigma_r &= 0.5 \end{aligned} \tag{55}$$

Since a and b are the same as in the numerical case discussed in Sec. 4.5, the maximum value of $|\rho_1|$ is still 0.424, and the corresponding return loss is $\ln \frac{1}{|\rho_1|_{max}} = 0.86$. To see how close to the theoretical limit is this value of the tolerance, one enters the plot of Fig. 3.14 with (52) and (53). The maximum theoretical value of $\ln \frac{1}{|\rho_1|_{max}}$ is found to be equal to 1.32, and the corresponding value of $|\rho_1|_{max}$ is 0.275.

The next step in the design is the computation of the quantities α_3, α_5, and α_7 by means of eqs. 24, 25, 31, 32 and 53. In this case the quantities $\frac{2}{5}(\frac{\sigma_r}{\omega_c})^5$ and $\frac{2}{r}(\frac{\sigma_r}{\omega_c})^7$ must be added to the right-hand sides of eqs. 31 and 32 to take into account the zero of ρ_1 at σ_r and the pole at $-\sigma_r$. One obtains

$$\alpha_3 = -0.523 \quad ; \quad \alpha_5 = 0.156 \quad ; \quad \alpha_7 = -0.666 \tag{56}$$

The values of C_3 and L_4 are then computed by means of eqs. 28 and 29, with due regard to the fact that C and L must be interchanged, because the first element of the ladder is, in this case, a capacitance instead of an inductance.

$$C_3 = \frac{0.813}{\omega_c} \quad ; \quad L_4 = \frac{3.37}{\omega_c} \tag{57}$$

The ratio of the ideal transformer is still 1.57 as in the case considered in Sec. 4.5, but is reversed in direction because the dual network is being designed, that is, inductances and capacitances have been interchanged.

To determine the value of L_5, L_6 and C_7, forming the section of type C (See Fig. 4.7), it is convenient to operate on the reflection coefficient ρ_2, that is, from the opposite end of the network.

The section of type C can be represented as shown in Fig. 4.8. It can be seen by inspection that, if M is a positive quantity,

$$MC_7 = \frac{1}{\sigma_r^2} = \frac{1}{4} \tag{58}$$

At the same time, the reflection coefficient of ρ_2 must have a zero at σ_r and, therefore, the impedance from the L_6 terminals must be equal to 1 for $\lambda = \sigma_r$. It follows that

$$\sigma_r L_6 + \frac{1}{\sigma_r C_7} = 1 \tag{59}$$

$$\sigma_r = \frac{1}{2L_6}\left(1 \pm \sqrt{1 - \frac{4L_6}{C_7}}\right) \tag{60}$$

The + sign must be used when $\rho_2 \dfrac{\lambda + \sigma_r}{\lambda - \sigma_r}$ is positive for $\lambda = \sigma_r$. The third equation required for the determination of the elements is obtained by considering the quantity

$$A_o^{\sigma_r} = \ln\left|\frac{1}{\rho_2}\,\frac{(\lambda - \sigma_r)}{(\lambda + \sigma_r)}\right| \tag{61}$$

which, according to the analysis of Ch. II is completely determined by L_6, M and C_7. One obtains from Fig. 4.8

$$A_o^{\sigma_r} = -\frac{1}{2}\ln\left(1 - \frac{4L_6}{C_7}\right) \tag{62}$$

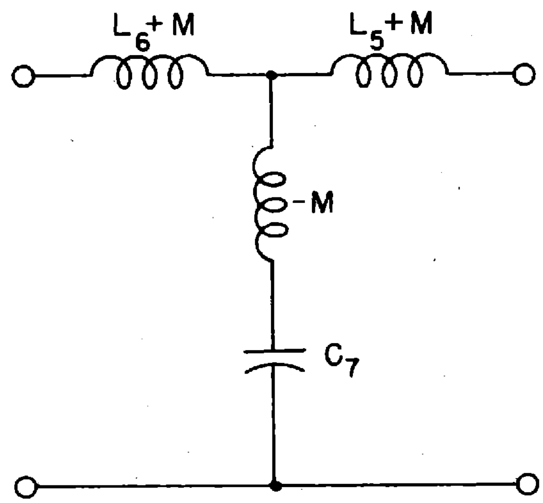


Figure 4.8 - Section of type C for a zero of transmission on the real axis

The numerical value of $\tilde{A}_o$ is found to be equal to -2. One has then, from 58, 60 and 62

$$L_6 = 1.135 \quad ; \quad M = 0.865$$

$$L_5 = 0.66 \quad ; \quad C_7 = 4.63 \tag{63}$$

The ideal transformer can be moved to the end of the structure and combined with the actual transformer, so that the load resistance measured from the primary side and normalized with respect to the source resistance will be equal to 2.47 ohms. Finally, the coupling coefficient of the transformer in the section of type C is made smaller than unity by combining the transformer with the adjacent inductance L_4. The final network is shown in Fig. 4.9, in which the values of all the elements are normalized with respect to the source resistance.

4.8 Concluding remarks - The design of matching network for impedances of a more complex nature than those considered above, is hampered in most cases by mathematical difficulties which lead to laborious numerical and graphical computations. It must be said, however, that most of the matching problems of practical interest are of the types discussed in the preceding sections, or can be reduced to those types by simple changes of the frequency variable. In addition, a rigorous method of design may be effectively combined

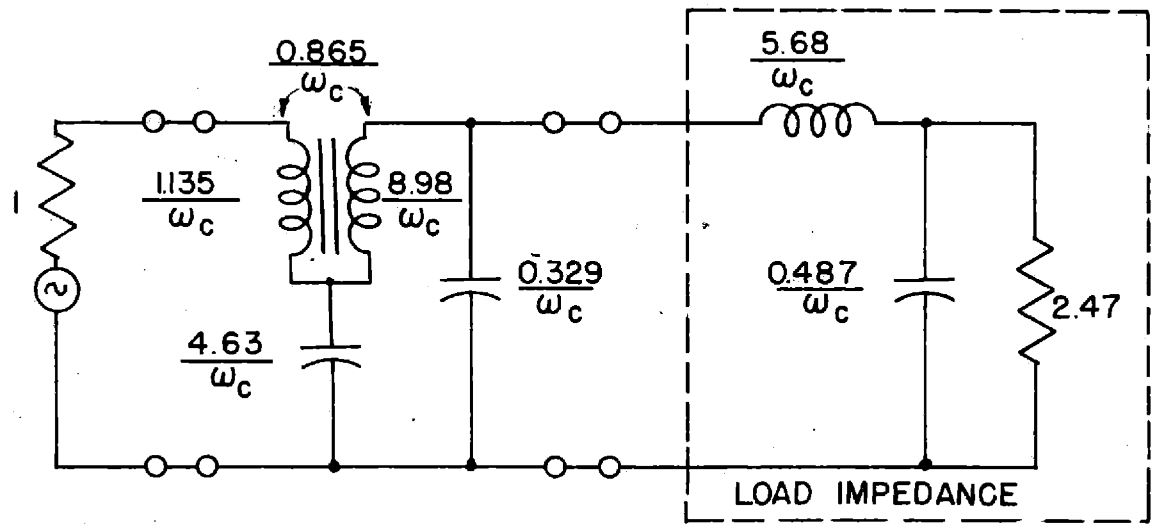


Figure 4.9 - Transformation of the network shown in Figure 4.7

at times with a cut-and-try procedure. For instance, the frequency behavior of a given load impedance might be modified first empirically, in such a way as to approximate, over the desired frequency band, the behavior of a simpler impedance function for which a rigorous design procedure is available. In such cases the ingenuity of the designer becomes of primary importance, since the technique to be used may vary considerably from one type of problem to another.

In regard to further research on the theory of matching networks, there seems to be little hope of developing a general as well as practical design procedure because of the inherent mathematical difficulties. On the other hand, a different physical approach to the matching problem might be more successful. In particular, it might be worth while to investigate the problem from the transient point of view, that is, in the time domain, rather than in the frequency domain. Such an investigation would be of great practical value in any case, since in many problems the transient response, rather than the steady-state amplitude response, is of primary importance. It must be said, however, that little progress has been made up to this time in the synthesis of networks with a prescribed transient response, so that the latter problem would have to be solved first before the matching problem could be attacked with a reasonable chance of success.

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APPENDIX

BIOGRAPHICAL NOTE

The author was born in Torino, Italy on November 11, 1917. After receiving the "Maturita Classica" from the Liceo-Gimnasio Massimo D'Azeglio in 1935, he attended the Politecnico of Torino from 1935 to 1939. He later completed his undergraduate work in Electrical Engineering at the Massachusetts Institute of Technology from which he received his Bachelor's degree in June 1941.

After spending six months as a student engineer at the Grand Rapids, Michigan plant of General Motors, he was appointed Assistant in the Department of Electrical Engineering, M. I. T., in the Fall of 1941. In 1943, he was promoted to the rank of Instructor and the following year, he joined the staff of the Radiation Laboratory, where he worked on the design of microwave components. At the end of the war he became Research Associate in the Department of Electrical Engineering, M. I. T., and joined the staff of the Research Laboratory of Electronics. He was recently promoted to the rank of Assistant Professor of Electrical Communications.

The author wrote, in collaboration with Dr. A. W. Lawson, Jr., two chapters on the theory and design of microwave filters for one of the volumes of the Radiation Laboratory series. Also, while a member of the Radiation Laboratory, he wrote a number of reports on the general subjects of microwave devices. A paper written in collaboration with Dr. Lawson on "Quarter-Wave-Coupled Microwave Filters" has been accepted for publication in the Proceedings of the I.R.E..

The author is a member of the Society of the Sigma Xi and an associate member of the I.R.E. and A.I.E.E.