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T. W. Anderson and Kai-Tai Fang

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20. Abstract.

A random vector $\boldsymbol{X}$ is said to have an elliptically contoured distribution if it has the distribution of $\boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Y}$, where $\boldsymbol{\mu}$ is a constant vector, $\boldsymbol{A}$ is a constant matrix, and the random vector $\boldsymbol{Y}$ has a spherical distribution; that is, $\boldsymbol{Y}$ is distributed as $\boldsymbol{Q}\boldsymbol{Y}$ for every orthogonal matrix $\boldsymbol{Q}$. This paper surveys recent work on distribution theory and statistical inference for elliptically contoured distributions. It emphasizes the contributions of the authors and of the students and associates of the second author; it gives a picture of this area of multivariate analysis that is particularly active in the People's Republic of China. Included are classification of elliptically contoured and related distributions, distributions of quadratic forms, estimation of parameters, testing hypotheses, and applications. Since the normal distribution is in this class, the properties of elliptically contoured distributions are similar to those of the normal distribution.

Theory and Applications of Elliptically Contoured and Related Distributions

T. W. Anderson and Kai-Tai Fang

1. Introduction.

The multivariate normal distribution has long served as the standard model for the statistical analysis of multivariate observations. Statisticians have been interested in generalizing the model from the normal population to a wider class of distributions that retain the most important properties of the multivariate normal distribution. In the past twenty years it has been found that the class of elliptically contoured distributions (ECD) can be regarded as a suitable extension of the multivariate normal distribution. The class of ECD includes many multivariate distributions, such as the multivariate normal, the multivariate t, the multivariate Cauchy, the multivariate Laplace, the multivariate uniform, mixtures of normal distributions, and the multivariate stable distributions. Many authors have developed the theory and methods of statistical inference for the ECD. Survey papers have been published by Muirhead [70], Chmielewski [19], and Fang [37].

The purpose of this paper is to introduce the contributions of theory and applications of ECD and related distributions, mainly by Chinese statisticians. When the second author visited Stanford University in the academic year 1981-82 to pursue research, the first author suggested ECD as furnishing a fruitful area of investigation; they cooperated in this venture. Upon his return to China the second author directed his doctoral students in conducting research on this subject. Most of the papers were originally published in Chinese journals and collected in the volume [39] in English. Under the influence of this work a number of Chinese authors entered this area and made valuable contributions as listed in the references. We regret any omission of major contributions due to the limitations of our survey.

There are several ways to define ECD and its standard form, spherical distributions (SD), by using different properties of the normal distribution. (See, for example, the preface of [39] and Section 1.1 of [49].) One is the following. The random vector $\boldsymbol{X}$ has the distribution $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ if and only if

$$\boldsymbol{X} \overset{\mathrm{d}}{=} \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Y},$$

where $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$ and $\boldsymbol{Y}$ has the standard normal $N(\boldsymbol{0}, \boldsymbol{I})$. Here $\overset{\mathrm{d}}{=}$ denotes that the two sides of the equality have the same distribution. For this kind of definition of ECD we define the spherical distribution first.

Definition 1.1. An $n \times 1$ random vector $\boldsymbol{X}$ is said to have a spherical distribution if for each $\boldsymbol{Q} \in O(n)$

$$\boldsymbol{Q}\boldsymbol{X} \stackrel{\mathrm{d}}{=} \boldsymbol{X}, \tag{1.1}$$

where $O(n)$ denotes the set of $n \times n$ orthogonal matrices.

The following theorem gives some equivalent definitions of SD.

Theorem 1.1. Let $\boldsymbol{X}$ be an $n \times 1$ random vector. Then the following statements are equivalent:

1) $\boldsymbol{Q}\boldsymbol{X} \stackrel{\mathrm{d}}{=}$ for each $\boldsymbol{Q} \in O(n)$;

2) The c.f. of $\boldsymbol{X}$, $Ee^{i\boldsymbol{t}'\boldsymbol{X}}$, is a function of $\boldsymbol{t}'\boldsymbol{t}$, $\boldsymbol{t} \in R^n$;

3) $\boldsymbol{X}$ has a stochastic representation

$$\boldsymbol{X} \stackrel{\mathrm{d}}{=} R\boldsymbol{U}^{(n)} \tag{1.2}$$

for some $R \geq 0$, where R is independent of $\boldsymbol{U}^{(n)}$ and the latter is uniformly distributed on the unit sphere in R^n;

4) For any $\boldsymbol{a} \in R^n$ we have

$$\boldsymbol{a}'\boldsymbol{X} \stackrel{\mathrm{d}}{=} \|\boldsymbol{a}\| X_1, \tag{1.3}$$

where $\|\boldsymbol{a}\|$ is the Euclidean norm and X_1 is the first component of $\boldsymbol{X}$.

From part 2) of the theorem the c.f. of a SD has the form $\phi(\boldsymbol{t}'\boldsymbol{t})$, where $\phi(\cdot)$ is a scalar function. Therefore, we write $\boldsymbol{X} \sim S_n(\phi)$. The set of all possible ϕ's is denoted by Φ_n; that is,

$$\Phi_n = \left\{\phi : \phi(t_1^2 + \cdots + t_n^2) \quad \text{is an } n\text{-dimensional c.f.}\right\}. \tag{1.4}$$

The probability method that treats models directly with random variates rather than their distribution function or c.f. plays an important role in theory of ECD. In particular, Anderson and Fang [2,3] gave a systematic discussion of the $\stackrel{\mathrm{d}}{=}$ operator. Many results mentioned in this paper were obtained by the probability method and show that the $\stackrel{d}{=}$ operator is a powerful tool. The fact is true in [13], [14], and Zolotarev's book [91]. Therefore, the stochastic representation (1.2) is one of the most important properties of SD which shows the following properties: (a) The set of SD's is equivalent to that of the set of nonnegative random variables. (b) The SD is essentially a function of a random variable R. (c) $\boldsymbol{X}/\|\boldsymbol{X}\|$ and $\|\boldsymbol{X}\|$ are independent, and $R \stackrel{d}{=} \|\boldsymbol{X}\|$ and $\boldsymbol{U}^{(n)} \stackrel{d}{=} \boldsymbol{X}/\|\boldsymbol{X}\|$. (d) $\boldsymbol{X}$ has a density which is of the form $g(\boldsymbol{x}'\boldsymbol{x})$ if and only if R has the density

$$f(r) = \frac{2}{\Gamma(n/2)} r^{n-1} g(r^2). \tag{1.5}$$

(In this case we prefer to write $X \sim S_n(g)$ instead of $\boldsymbol{X} \sim S_n(\phi)$ and g is called the density generating function [49].) (e) Let $t(\boldsymbol{X})$ be a statistic satisfying $t(a\boldsymbol{X}) = t(\boldsymbol{X})$ for any $a > 0$; if $P(\boldsymbol{X} = \boldsymbol{0}) = 0$, then $t(\boldsymbol{X}) \stackrel{d}{=} t(\boldsymbol{Z})$ where $\boldsymbol{Z} \sim N(\boldsymbol{0}, \boldsymbol{I})$, i.e., the distribution of $t(\boldsymbol{X})$ is invariant in the class; for instance, the t-statistic has the same distribution for all members of the class. (f) The marginal distribution of $X_1, \ldots, X_m$ is a SD again which has the stochastic representation (1.2) with m instead of n and RB instead of R, where $B > 0$, $B \sim B(m/2, (n-m)/2)$, and R, B, and $\boldsymbol{U}^{(m)}$ are independent.

Definition 1.2. An $n \times 1$ random vector $\boldsymbol{X}$ is said to have an elliptically contoured distribution (ECD) with parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}\,(n \times n)$ if

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{\mu} + \boldsymbol{A}\boldsymbol{Y}, \qquad Y \sim S_k(\phi), \tag{1.6}$$

where $\boldsymbol{A} : n \times k$ and $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$ with $\operatorname{rank}(\boldsymbol{\Sigma}) = k$. We write $\boldsymbol{X} \sim EC(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$.

Many properties of ECD can be transferred from those of SD by means of (1.6). The following properties are important and are needed in this paper.

1) A linear transformation of an ECD is again an ECD; in particular, all marginal distributions of an ECD are ECD.

2) All conditional distributions of an ECD are ECD.

3) The c.f. of $EC_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ is $\exp(i\boldsymbol{t}'\boldsymbol{\mu})\phi(\boldsymbol{t}'\boldsymbol{\Sigma}\boldsymbol{t})$.

4) $\boldsymbol{X} \sim EC_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$ with $\operatorname{rank}(\boldsymbol{\Sigma}) = k$ if and only if

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{\mu} + R\boldsymbol{A}\boldsymbol{U}^{(k)}, \tag{1.7}$$

where $R > 0$ is independent of $\boldsymbol{U}^{(k)}$, $\boldsymbol{A} : n \times k$, and $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$.

5) If $\boldsymbol{Y}$ has the density $g(\boldsymbol{x}'\boldsymbol{x})$ and $\boldsymbol{A}$ is square and nonsingular, then $\boldsymbol{X} = \boldsymbol{AY}$ has the density

$$|\boldsymbol{\Sigma}|^{-1/2} g\big[(\boldsymbol{x} - \boldsymbol{\mu})]\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\big], \tag{1.8}$$

where $\boldsymbol{AA}' = \boldsymbol{\Sigma}$, and is denoted by $X \sim EC_n(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$. The contours of constant density are ellipsoids

$$(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{x} - \boldsymbol{\mu}) = \text{const}.$$

This fact leads to the name of ECD. There are various other terms used, such as round distribution, isotropic distribution [24] for SD, and ellipsoidal symmetric distribution in the literature.

More properties and detailed discussions are referred to [49].

This paper is organized as follows. Several types of spherical and elliptical matrix distributions and their relationships are discussed in Section 2. The distributions of their quadratic forms and associated Cochran's theorem are presented there as well. Some results of estimation of parameters of and testing hypotheses about ECD are given in Sections 3 and 4, respectively. The stochastic representation (1.2) and (1.6) gives the structure of SD and ECD. The same idea can be applied to some other distributions and produces other classes of symmetric multivariate distributions; a summary constitutes Section 6. Section 5 collects applications of ECD models in regression analysis, principal component analysis, canonical correlation analysis, discriminant analysis, and econometrics. The last section consists of miscellaneous results.

2. Classes of Distributions and Distributions of Quadratic Forms

A sample of n observations from a multivariate distribution $\boldsymbol{X}_{(1)}, \ldots, \boldsymbol{X}_{(n)}$ can be expressed by an $n \times p$ matrix

$$\boldsymbol{X} = \begin{pmatrix} \boldsymbol{X}'_{(1)} \\ \vdots \\ \boldsymbol{X}'_{(n)} \end{pmatrix} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_p). \tag{2.1}$$

This matrix of observations is the basis of multivariate analysis and data analysis. Therefore, we study its distribution first. If the observation vectors are drawn independently from $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, then the matrix $\boldsymbol{X}$ has a matrix normal distribution $N_{n\times p}(\boldsymbol{M}, \boldsymbol{I} \otimes \boldsymbol{\Sigma})$ with the c.f.

$$\begin{aligned} \psi(\boldsymbol{T}) &= E\big[\exp(i\,\mathrm{tr}(\boldsymbol{T}'\boldsymbol{X}))\big] \\ &= \exp(i\,\mathrm{tr}\,\boldsymbol{T}'\boldsymbol{M})\exp\Big(-\frac{1}{2}\,\mathrm{tr}\,\boldsymbol{\Sigma}\boldsymbol{T}'\boldsymbol{T}\Big), \end{aligned} \tag{2.2}$$

where $\boldsymbol{M} = \mathbf{1}\boldsymbol{\mu}'$, and

$$\boldsymbol{T} = (\boldsymbol{t}_{(1)}, \ldots, \boldsymbol{t}_{(n)})' = (\boldsymbol{t}_1, \ldots, \boldsymbol{t}_p). \tag{2.3}$$

When $\boldsymbol{\mu} = \mathbf{0}$, $\phi(\boldsymbol{T})$ is a function of $\boldsymbol{T}'\boldsymbol{T}$ and is invariant under $n \times n$ orthogonal transformations.

When the parent distribution is more generally $EC_p(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \phi)$, the c.f. of $\boldsymbol{X}$ is

$$E\left(e^{i \operatorname{tr} \boldsymbol{T}'\boldsymbol{X}}\right) = e^{i \operatorname{tr} \boldsymbol{T}'\boldsymbol{M}} \prod_{j=1}^{n} \psi(\boldsymbol{t}'_{(j)} \boldsymbol{\Sigma} \boldsymbol{t}_{(j)}), \tag{2.4}$$

where $\boldsymbol{M}$ is the same matrix as in (2.2). Unfortunately, most results for this model are based on asymptotic theory and numerical evaluation with the exception of $\boldsymbol{X} \sim N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. (See [69] and [70].)

An alternative model for random $\boldsymbol{X}$ is that the columns of $\boldsymbol{X}$ are uncorrelated and each has mean $\boldsymbol{\mu}$ and the covariance matrix $\boldsymbol{\Sigma}$. This model generates various spherical/elliptical matrix distributions.

Corresponding to the invariance of (1.1) Dawid [20–22] proposed two classes of spherical matrix distributions (SMD).

Definition 2.1. Let $\boldsymbol{X}$ be an $n \times p$ random matrix. If $\boldsymbol{Q}\boldsymbol{X} \overset{d}{=} \boldsymbol{X}$ for every $\boldsymbol{Q} \in O(n)$ we call $\boldsymbol{X}$ left-spherical and write $\boldsymbol{X} \in LS$. If $\boldsymbol{X}$ and $\boldsymbol{X}'$ are both LS we call $\boldsymbol{X}$ symmetrically spherical and write $\boldsymbol{X} \in SS$.

In terms of the c.f. Anderson and Fang [3] suggested the following.

Definition 2.2. An $n \times p$ random matrix $\boldsymbol{X}$ is said to have a multivariate spherical distribution if the c.f. of $\boldsymbol{X}$ has the form $\phi(\boldsymbol{t}'_1\boldsymbol{t}_1, \boldsymbol{t}'_2\boldsymbol{t}_2, \ldots, \boldsymbol{t}'_p\boldsymbol{t}_p)$ and is denoted $\boldsymbol{X} \in MS$ or $\boldsymbol{X} \sim MS_{n \times p}(\phi)$.

The most direct extension of spherical distribution to the matrix case is by means of the vector operator vec$(\cdot)$, defined as

$$\operatorname{vec}(\boldsymbol{X}) = \left(\boldsymbol{x}'_1, \ldots, \boldsymbol{x}'_p\right)' \tag{2.5}$$

and considered by many authors, such as Kariya [64], Jensen and Good [62], Fraser and Ng [59], and Anderson and Fang [3,4].

Definition 2.3. Let $\boldsymbol{X}$ be an $n \times p$ random matrix. If $\mathrm{vec}(\boldsymbol{X})$ is spherical, we call $\boldsymbol{X}$ vector-spherical and write $\boldsymbol{X} \in VS$.

The above four classes of spherical matrix distributions have been studied individually by the above authors. Fang and Chen [42,43] established relationships among them and found more properties as follows. (They used $\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$, and $\mathcal{F}_s$ to denote the classes LS, MS, VS, and SS, respectively).

Theorem 2.1. The c.f. of $\boldsymbol{X}$ has the form

$$\begin{array}{ll} \phi(\boldsymbol{T}'\boldsymbol{T}), & \text{if } \boldsymbol{X} \in LS, \\ \phi\left[\mathrm{diag}(\boldsymbol{T}'\boldsymbol{T})\right], & \text{if } \boldsymbol{X} \in MS, \\ \phi\left[\mathrm{tr}(\boldsymbol{T}'\boldsymbol{T})\right], & \text{if } \boldsymbol{X} \in VS, \\ \phi\left[\mathrm{eig}(\boldsymbol{T}'\boldsymbol{T})\right], & \text{if } \boldsymbol{X} \in SS, \end{array} \tag{2.6}$$

where $\mathrm{diag}(\boldsymbol{A}) = (a_{11}, \ldots, a_{pp})$ and $\mathrm{eig}(\boldsymbol{A})$ = the vector eigenvalues of $\boldsymbol{A}$. As a sequel we have

$$VS \subset MS \subset LS \quad \text{and} \quad VS \subset SS \subset LS. \tag{2.7}$$

Furthermore, $VS = MS \cap SS$.

In the following exposition $\boldsymbol{X} \sim LS(\phi)$ denotes that $\boldsymbol{X} \in LS$ and the c.f. of $\boldsymbol{X}$ is $\phi(\boldsymbol{T}'\boldsymbol{T})$, with similar notations for the other cases.

Theorem 2.2. If the $n \times p$ random matrix $\boldsymbol{X}$ has one of the spherical matrix distributions, then it has one of the following stochastic representations:

LS: $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{U}_1\boldsymbol{A}$, where $\boldsymbol{U}_1 : n \times p$, $\boldsymbol{A} : p \times p$, $\boldsymbol{U}_1 \in LS$, $\boldsymbol{U}_1'\boldsymbol{U}_1 = \boldsymbol{I}_p$, $\boldsymbol{A}'\boldsymbol{A} = \boldsymbol{X}'\boldsymbol{X}$, and $\boldsymbol{A}$ and $\boldsymbol{U}_1$ are independent;

MS: $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{U}_2\boldsymbol{R}$, where $\boldsymbol{U}_2$ has i.i.d. columns, each distributed as $\boldsymbol{U}^{(n)}$, and $\boldsymbol{R} = \mathrm{diag}(R_1, \ldots, R_p) \geq \boldsymbol{0}$ is independent of $\boldsymbol{U}_2$;

SS: $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{U}_1\boldsymbol{\Lambda}\boldsymbol{V}$ is the singular value decomposition, where $\boldsymbol{U}_1$, $\boldsymbol{\Lambda}$, and $\boldsymbol{V}$ are independent, $\boldsymbol{U}_1$ is the same as in LS, $\boldsymbol{V}' \in LS$, $\boldsymbol{V}'\boldsymbol{V} = \boldsymbol{I}_p$, and $\boldsymbol{\Lambda}$ is a diagonal matrix with nonnegative elements,

VS: $\boldsymbol{X} \stackrel{d}{=} R\boldsymbol{U}_3$, where $R \geq 0$ is independent of $\boldsymbol{U}_3$ and $\mathrm{vec}(\boldsymbol{U}_3) \stackrel{d}{=} \boldsymbol{U}^{(np)}$.

Each of the distributions of $\boldsymbol{U}_1$, $\boldsymbol{U}_2$, and $\boldsymbol{U}_3$ is called the uniform matrix distribution with its respective specific meaning. Furthermore, they have the stochastic representations

$$\boldsymbol{U}_1 = \boldsymbol{Y}(\boldsymbol{Y}'\boldsymbol{Y})^{-1/2},\ \boldsymbol{U}_2 = (\boldsymbol{Y}_j/\|\boldsymbol{Y}_j\|, j = 1, \ldots, p),\ \boldsymbol{U}_3 = \boldsymbol{Y}/(\operatorname{tr} \boldsymbol{Y}'\boldsymbol{Y})^{1/2}, \tag{2.8}$$

where $\boldsymbol{Y} = (\boldsymbol{Y}_1, \ldots, \boldsymbol{Y}_p)$ has the standard matrix normal distribution $N(\mathbf{0}, \boldsymbol{I}_n \times \boldsymbol{I}_p)$.

The c.f.'s of $\boldsymbol{U}_2$ and $\boldsymbol{U}_3$ can be found by the result of Schoenberg [75]. Zhang and Fang [89] obtained an expression of the c.f. of $\boldsymbol{U}_1$ in terms of the hypergeometric function.

With the stochastic representations given by Theorem 2.2 many results can be transferred from multivariate normal populations to these wider classes. A number of authors, such as Dawid [20], Chmielewski [18], Fraser and Ng [59], Jensen and Good [62], and Anderson and Fang [3], found invariant statistics in these classes. Fang and Chen [42] obtained necessary and sufficient conditions for invariant statistics in the four classes. For simplicity, we cite only the theorem in the LS case. Let

$$LS^+ = \{\boldsymbol{X} : \boldsymbol{X} \in LS \quad \text{and} \quad P(\boldsymbol{X}'\boldsymbol{X} > \mathbf{0}) = 1\} \tag{2.9}$$

Theorem 2.3. Let $t(\boldsymbol{X})$ be a statistic. Then the distribution of $t(\boldsymbol{X})$ is invariant in LS^+ if and only if $t(\boldsymbol{X}\boldsymbol{A}) \stackrel{d}{=} t(\boldsymbol{X})$ for each $\boldsymbol{A} \in UT$, the set of $p \times p$ upper triangular matrices with positive diagonal elements.

As an application of Theorem 2.3, one can find many useful statistics (such as the Wilks statistic and the Hotelling T^2, and the statistic for testing equality of several covariance matrices) that are invariant in LS^+. This fact shows some overwhelming advantages of spherical matrix distributions and gives the possiblity of extending the multivariate analysis techniques into these wider classes. For more details see Section 6.

A random matrix $\boldsymbol{X}$ with an elliptical matrix distribution (EMD) is the linear transform

$$\boldsymbol{X} = \boldsymbol{M} + \boldsymbol{Y}\boldsymbol{A}, \tag{2.10}$$

where $\boldsymbol{Y}$ has a spherical matrix distribution in any of the above classes and $\boldsymbol{M}$ and $\boldsymbol{A}$ are constant matrices. Thus we have four classes of elliptical matrix distributions; we denote them by LE, SE, ME, and VE, respectively. Zhang, Fang, and Chen [90] gave a comprehensive study of these classes. They found marginal and conditional distributions, stochastic decompositions, moments, and invariant statistics. It should be noted that the matrix normal $\boldsymbol{X}$ with distribution $N(\mathbf{0}, \boldsymbol{I} \otimes \boldsymbol{\Sigma})$ is a member of LS, but the rows are not necessarily spherical.

The distributions of quadratic forms and Cochran's theorem play an important role in multivariate analysis. Let the $n \times p$ random matrix $\boldsymbol{X}$ in LS be partitioned into m parts $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ with $n_1, \ldots, n_m$ rows, respectively. When $p = 1$ and $\boldsymbol{X}$ has a density, Kelker [65] obtained the distribution of $\boldsymbol{X}_1'\boldsymbol{X}_1$. Anderson and Fang [2] derived the distribution of $\boldsymbol{X}_j'\boldsymbol{X}_j$, $j = 1, \ldots, m$, without the assumption of $\boldsymbol{X}$ having a density. As a sequel, they [3] obtained the distributions of the sample covariance matrix, the correlation matrix, the multiple correlation coefficient, the generalized variance, the eigenvalues of the sample covariance matrix, etc. Fang and Wu [57] extended their results to the case of LE with $\boldsymbol{M} = \boldsymbol{0}$ in (2.10). When $\boldsymbol{M} \neq \boldsymbol{0}$, Teng, Fang, and Deng [77] obtained the density of $\boldsymbol{X}_1'\boldsymbol{X}_1$ under some regularity conditions, thus extending the result of Cacoullos and Koutras [11] for $p = 1$. Fan [25] obtained the noncentral t-, $F-$, and T^2-distributions by using the method of [11] and gave a detailed discussion of the distributions.

Let $\boldsymbol{X} \sim N(\boldsymbol{1}\boldsymbol{\mu}', \boldsymbol{I}_n \otimes \boldsymbol{I}_p)$. The basic features of Cochran's theorem can be formulated as follows:

1) $\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X} \sim \chi_k^2(\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu})$ (the noncentral chi-square distribution with k degrees freedom and noncentrality parameter $\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}$) if and only if $\boldsymbol{A}^2 = \boldsymbol{A}$ and $\operatorname{rank}(\boldsymbol{A}) = k$;

2) $\boldsymbol{X}'\boldsymbol{A}\boldsymbol{X}$ and $\boldsymbol{X}'\boldsymbol{B}\boldsymbol{X}$ are independent if and only if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{0}$. We shall call the result the central Cochran's theorem if $\boldsymbol{\mu} = \boldsymbol{0}$; otherwise we shall call it the noncentral Cochran's theorem. Anderson and Styan [6] reviewed various extensions of Cochran's theorem for the normal case.

We would here like to mention several contributions to Cochran's theorem for ECD and LS. Kelker [65] extended the central Cochran's theorem to ECD under the condition that $\boldsymbol{X}$ has a density with finite fourth moments. Anderson and Fang [2,3], using the $\stackrel{d}{=}$ operator, gave a new approach to various extensions of Cochran's theorem in ECD without the condition of Kelker. Fang and Wu [57] extended their results to more general quadratic forms. Due to the need in the theory of multivariate analysis, Fang, Fan, and Xu [45] extended the results in [4,5] to the case where the matrix $\boldsymbol{A}$ is random and gave some applications to T^2- and Wilks statistics and Tukey testing.

The noncentral case of Cochran's theorem is much more difficult to handle than the central case. Thus the results for ECD are not as extensive in the noncentral case as in the central case. Under the assumption of finite fourth moments Fan [26] proved the noncentral Cochran's theorem using c.f.'s. Zhang [85] extended the results of Fang and Wu [57] to the noncentral situation, as well as the result of Fan [26], but under the condition of finite

2nth moment.

3. Estimation of Parameters of Elliptically Contoured Distributions.

Estimation theory for the normal distribution is highly developed. Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be a sample of independent observations from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The maximum likelihood estimators of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are the sample mean and the sample covariance matrix

$$\overline{\boldsymbol{X}} = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{X}_i, \qquad \widehat{\boldsymbol{\Sigma}} = \frac{1}{n}\sum_{i=1}^{n} (\boldsymbol{X}_i - \overline{\boldsymbol{X}})(\boldsymbol{X}_i - \overline{\boldsymbol{X}})', \tag{3.1}$$

respectively. Estimation in elliptical populations can be established in parallel fashion.

Let the matrix of observations $\boldsymbol{X}$ have an elliptical matrix distribution (2.10) with $\boldsymbol{M} = \mathbf{1}\boldsymbol{\mu}'$ and $\boldsymbol{A}\boldsymbol{A}' = \boldsymbol{\Sigma}$. We want to estimate the parameters $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. If $\boldsymbol{X}$ has a density and $X \in LE$, the density must have the form

$$|\boldsymbol{\Sigma}|^{-p/2} g\big[(\boldsymbol{X} - \boldsymbol{M})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \boldsymbol{M})\big]. \tag{3.2}$$

When $\boldsymbol{X}$ is from ME, VE, or SE, the density of $\boldsymbol{X}$ has the same form (3.2) with $g\big[\operatorname{diag}(\cdot)\big]$, $g\big[\operatorname{tr}(\cdot)\big]$, and $g\big[\operatorname{eig}(\cdot)\big]$, respectively. We shall write $\boldsymbol{X} \sim LE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, $\boldsymbol{X} \sim ME(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, and so on for these models. In this section some results on maximum likelihood estimates (MLE), minimax estimates, shrinkage estimates, and inadmissibility of the sample mean are mentioned.

For $\boldsymbol{X} \sim VE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$ Anderson and Fang [4] developed a new approach to the MLE's of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. The MLE's are

$$\hat{\boldsymbol{\mu}} = \overline{\boldsymbol{X}}, \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}} = y_g \boldsymbol{S} = y_g(\boldsymbol{X} - \mathbf{1}\overline{\boldsymbol{X}}')'(\boldsymbol{X} - \mathbf{1}\overline{\boldsymbol{X}}'), \tag{3.3}$$

where the constant y_g will be given in Lemma 3.1 below. Later Anderson, Fang, and Hsu [5] established the relationship of the MLE's in normal and elliptical models and therefore gave a unified approach to MLE for EV. Their main result is the following:

Theorem 3.1. Let Ω be a set in the space of $(\boldsymbol{\mu}, \boldsymbol{V})$, $\boldsymbol{V} > 0$, such that if $(\boldsymbol{\mu}, \boldsymbol{V}) \in \Omega$ then $(\boldsymbol{\mu}, c\boldsymbol{V}) \in \Omega$ for all $c > 0$. Suppose g is such that $g(\boldsymbol{x}'\boldsymbol{x})$ is a density in R^N and $y^{N/2}g(y)$ has a finite positive maximum y_g. Suppose that on the basis of an observation $\boldsymbol{X}$ from $|\boldsymbol{V}|^{-1/2} g\big[(\boldsymbol{x} - \boldsymbol{\mu})'\boldsymbol{V}^{-1}(\boldsymbol{x} - \boldsymbol{\mu})\big]$ the MLE's under normality $(\tilde{\boldsymbol{\mu}}, \widetilde{\boldsymbol{V}}) \in \Omega$ exist and are unique and that $\widetilde{\boldsymbol{V}} > 0$ with probability 1. Then the MLE's for g are

$$\hat{\boldsymbol{\mu}} = \tilde{\boldsymbol{\mu}}, \qquad \widehat{\boldsymbol{V}} = (N/y_g)\widetilde{\boldsymbol{V}},$$

and the maximum of the likelihood is $|\boldsymbol{V}|^{-1/2}g(y_g)$.

The existence of y_g mentioned in the theorem may be based on the following lemma.

Lemma 3.1. Suppose that $g(\boldsymbol{x}'\boldsymbol{x})$ is a density in $\boldsymbol{x} \in R^N$ such that $g(y)$ is continuous and decreasing for y sufficiently large. Then the function

$$h(y) = y^{N/2}g(y), \quad y > 0,$$

has a maximum at some finite $y_g > 0$. An alternative condition is that g is continuous and $E(\boldsymbol{X}'\boldsymbol{X}) < \infty$.

Fang and Xu [51] and Fang, Xu, and Teng [54] extended the above results to the case of ES, EM, and EL, respectively. Since the MLE of a function of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ is that same function of the MLE's $\hat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$, we thus obtained the MLE's for the most useful statistics in multivariate analysis.

The usual estimator of $\boldsymbol{\mu}$ in the normal population, namely the sample mean, is inadmissible under a quadratic loss if the dimension of the observations is greater than 2; this result is due to Stein [76]. After improvement of the original proof, several concise proofs have been proposed; see to Anderson [1], for example. Among the many papers on this topic, Brandwein and Strawderman [10] established the inadmissibility of the sample mean for spherical distributions when the dimension is greater than 3. Their proof is very long in comparison to the concise proof for normal case given in [1]. Fan and Fang [31] have given an improved proof which is much shorter than the original one and the conditions are weaker. Let $\boldsymbol{X}$ have an elliptical matrix distribution $LE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$. It is easy to see that $(\overline{\boldsymbol{X}}, \boldsymbol{S})$ is a sufficient statistic for $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ by the Fisher-Neyman factorization theorem. Therefore, the inadmissibility of the mean can be expressed in the following simple statement.

Theorem 3.2. Suppose that $\boldsymbol{X} \sim EC_p(\boldsymbol{\mu}, \boldsymbol{I}_p, \phi)$; that is, $\boldsymbol{X} - \boldsymbol{\mu}$ is spherical. Then the estimate

$$\boldsymbol{\delta}_a(\boldsymbol{X}) = (1 - a/\|\boldsymbol{X}\|^2)\boldsymbol{X} \tag{3.4}$$

is better than the usual estimate $\boldsymbol{X}$ under quadratic loss, provided that $p > 3$ and

$$0 \le a \le \frac{2(p-3)}{(p-1)E_0\|\boldsymbol{X}\|^{-2}}, \tag{3.5}$$

where $E_0\|\boldsymbol{X}\|^{-2}$ is the expected value of $\|\boldsymbol{X}\|^{-2}$ when $\boldsymbol{\mu} = \mathbf{0}$.

This result can be extended to the case where the loss has the form $W\left[(\boldsymbol{\delta}-\boldsymbol{\mu})'(\boldsymbol{\delta}-\boldsymbol{\mu})\right]$ and $W(\cdot)$ is a nonnegative convex function. Furthermore, the estimator

$$\delta_{a,f}(\boldsymbol{X}) = \left(1 - af(\|\boldsymbol{X}\|^2)/\|\boldsymbol{X}\|^2\right)\boldsymbol{X}, \tag{3.6}$$

where $0 \leq f(x) \leq 1$, $f(x)$ is nondecreasing, $f(x)/x$ is nonincreasing for $x > 0$, and $f''(x) \leq 0$ for $x > 0$ [31] and a satisfies (3.5), also dominates $\boldsymbol{X}$. Note that for the inadmissibility of the sample mean under quadratic loss the condition of $p > 2$ in the normal case becomes that of $p > 3$ in the spherical case.

We now consider minimax estimates of $\boldsymbol{\mu}$. Let $\boldsymbol{X}$ have a distribution $VE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, where $g(\cdot)$ is a nonnegative decreasing function. Let $W(\cdot)$ be a nonnegative increasing function. Fan and Fang [29] pointed out that under the loss $W\left[(\boldsymbol{d}-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{d}-\boldsymbol{\mu})\right]$, the sample mean $\overline{\boldsymbol{X}}$ is a minimax estimate for $\boldsymbol{\mu}$. Furthermore, they found that if $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are independently drawn from $EC_p(\boldsymbol{\mu}, \boldsymbol{I}, g)$, then under loss function $W(\|\boldsymbol{d}-\boldsymbol{\mu}\|)$ the mean $\overline{\boldsymbol{X}}$ is a minimax estimate in the class of $\{h(\overline{\boldsymbol{X}}) : h(\cdot) \text{ a real function}\}$. Some sequential minimax properties for the sample mean and Stein's two-stage estimate are also discussed in [30]. In fact, we can find a wider class of minimax estimates of $\boldsymbol{\mu}$, such as $\boldsymbol{\delta}_a(\boldsymbol{X})$ in (3.4) and $\boldsymbol{\delta}_{a,f}(\boldsymbol{X})$ in (3.6) in a certain sense. (See [30].) The estimates (3.4) and (3.6) are shrinkage estimates.

The reader is referred to Section 4.4.2 of [58] for further discussion.

4. Testing Hypotheses about Elliptically Contoured Distributions.

Let the matrix of observations $\boldsymbol{X}$ have an elliptical matrix distribution $LE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$, where $(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \in \Omega$, the parameter space. We want to test

$$H_0 : (\boldsymbol{\mu}, \boldsymbol{\Sigma}) \in \omega \quad \text{vs.} \quad H_1 : (\boldsymbol{\mu}, \boldsymbol{\Sigma}) \in \Omega/\omega. \tag{4.1}$$

Statistics for testing (4.1) can be derived by different principles, among which is the likelihood ratio criteria (LRC). From (3.2) the likelihood function is

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = |\boldsymbol{\Sigma}|^{-n/2} g\left[(\boldsymbol{X} - \mathbf{1}\boldsymbol{\mu}')'\boldsymbol{\Sigma}^{-1}(\boldsymbol{X} - \mathbf{1}\boldsymbol{\mu}')\right]. \tag{4.2}$$

Hence, the LRC of testing (4.1) is

$$T(\boldsymbol{X}) = \max_{\omega} L(\boldsymbol{\mu}, \boldsymbol{\Sigma}) / \max_{\Omega} L(\boldsymbol{\mu}, \boldsymbol{\Sigma}). \tag{4.3}$$

When $\boldsymbol{X} \in EV$, Anderson and Fang [4] obtained many statistics used in multivariate analysis, such as the criterion for testing lack of correlation between sets of variates, testing the hypothesis that a mean vector is equal to a given vector, testing equality of several covariance matrices, testing equality of several means, etc., and found that these statistics have the same form and the same null distribution in these distributions as in the normal distribution. With Theorem 3.1 Anderson, Fang, and Hsu [5] gave a unified approach to LRC's and established the relationship of distributions of the LRC between normal and other elliptical populations. Fang and Xu [51] and Fang, Xu and Teng [54] extended systematically the results to the wider classes ME, SE, and LE. They found that there are some statistics (but not all of those in VE) that have the same form and the same distribution within the entire class. Chmielewski [18] studied invariant statistics for testing equality of k covariance matrices. Chen [16] pointed out that the invariants obtained in [18] are correct only for $k = 2$ and gave the correct invariant statistic for arbitrary k.

A necessary and sufficient condition for a statistic to be invariant in classes of elliptical matrix distributions can be obtained as in Theorem 2.3. Kariya [64] gave an alternative necessary and sufficient condition, but Bian, Wang, and Zhang [8] found that there is a gap in the Kariya's proof. They gave a counter-example to his result, but proved that if the matrix of observations has a density, then Kariya's theorem is true.

Although the null distribution of an invariant statistic is the same for all elements of the class, the nonnull distribution depends on the specific element of the class. That consideration leads to derivations of noncentral distributions. (See [25], [36], [77].) Let $\boldsymbol{X} \sim EC_n(\boldsymbol{\mu}, \boldsymbol{I}_n, \phi)$. Define the sample mean and standard deviation by

$$\bar{x} = \frac{1}{n}\mathbf{1}'\boldsymbol{X}, \qquad s^2 = \frac{1}{n}\boldsymbol{X}'\left(\boldsymbol{I}_n - \frac{1}{n}\mathbf{1}\mathbf{1}'\right)\boldsymbol{X}.$$

Fang and Yuan [56] studied the power of the t-test in the class of $EC_n(\boldsymbol{\mu}, \boldsymbol{I}, \phi)$ and found that the power can be very different for the different elements of class. They furthermore pointed out the following.

Theorem 4.1. Let

$$\begin{bmatrix} \boldsymbol{X}_1 \\ \boldsymbol{X}_2 \end{bmatrix} \stackrel{d}{=} \begin{bmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu} \end{bmatrix} + \begin{bmatrix} R_1 \boldsymbol{U}^{(n)} \\ R_2 \boldsymbol{U}^{(n)} \end{bmatrix}. \tag{4.4}$$

Let $t_i = \sqrt{n}\,\overline{X}_i / s_i$, where $\overline{X}_i$ and s_i are the sample mean and standard deviation of $\boldsymbol{X}_i$, and let $d(x, y) = |x - y|$ be the L_1-norm distance. Then

$$Ed(t_1, t_2) = cEd(1/R_1, 1/R_2), \tag{4.5}$$

where c is a known constant.

Then $Ed(t_1, t_2)$ can be very large if $Ed(1/R_1, 1/R_2)$ is very large. With this theorem Fang and Yuan [56] obtained the limiting distribution of the t-statistic in some subclasses of $EC_n(\boldsymbol{\mu}, \boldsymbol{I}, g)$. The convergence of the statistic is not only in distribution, but also in density. The same approach can be applied to the F-statistic, T^2-statistic, and so on.

The invariance of a statistic in the class of elliptical distributions can be employed for enlarging the class. For example, let

$$F_t = \{\boldsymbol{X} : \boldsymbol{X} \text{ is exchangeable and } t_x \sim t_{n-1}\} \tag{4.6}$$

be a set of n-dimensional random vectors such that the corresponding t-statistic has the same distribution as in the normal case. ($\boldsymbol{X}$ is exchangeable if $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{P}\boldsymbol{X}$ for every permutation matrix $\boldsymbol{P}$.) Obviously, the set SD belongs to F_t. In fact, the class F_t is much larger than SD. More precisely, let VT denote the class of $\boldsymbol{X}$ that has the stochastic decomposition (1.2) without necessarily independence of R and $\boldsymbol{U}^{(n)}$. Then SD$\subset VT \subset F_t$. This class can serve for deriving Baysian statistics, but its structure has not yet been sufficiently investigated.

Many LRC's in normal populations yield uniformly most powerful (UMP) and unbiased tests. Do those tests retain their optimal properties in elliptical populations. Quan [72] and Quan and Fang [73] investigated this subject for VE and found that many LRC's keep these properties as follows: Let $\boldsymbol{X} \sim VE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$.

1) Partition $\boldsymbol{\mu}$ into two subvectors $\boldsymbol{\mu}_1$ and $\boldsymbol{\mu}_2$. Consider testing

$$H_0 : \boldsymbol{\mu}_1 = 0,\ \boldsymbol{\mu}_2 = 0 \quad \text{vs.} \quad H_1 : \boldsymbol{\mu}_1 \neq 0,\ \boldsymbol{\mu}_2 = 0. \tag{4.7}$$

If the density generating function $g(\cdot)$ is monotonically decreasing and differentiable and $g'(\cdot)$ is increasing, then the LRC test for (4.7) is UMP in the class of tests based on the likelihood ratio statistic.

2) Let $\overline{R}$ be the population multiple correlation in $VE(\boldsymbol{\mu}, \boldsymbol{\Sigma}, g)$ and let R be the sample multiple correlation. If g satisfies the conditions in 1), then the LRC for $H_0 : \overline{R} = 0$ is UMP invariant.

3) The Wilks statistic and the statistics for testing lack of correlation between sets of variates, testing equality of several covariance matrices, testing equality of several mean vectors and covariance matrices simultaneously, and the sphericity test are unbiased if g is decreasing.

The goodness of fit test for elliptical symmetry is a difficult problem. Deng [23] proposed a significance test for elliptical symmetry by use of moment sequence. It is evident that his method requires all moments to be finite.

5. Applications.

Application of the established theory of ECD and EMD shows that many well-known techniques of multivariate analysis, such as regression analysis, multivariate analysis of variance, principal component analysis, canonical correlation analysis, discriminant analysis and econometric methods are valid in these wider classes.

Consider the general regression model

$$\boldsymbol{Y} = \boldsymbol{f}(\boldsymbol{X}, \boldsymbol{B}) + \boldsymbol{E}, \tag{5.1}$$

where $\boldsymbol{Y} : n \times p$, $\boldsymbol{X} : n \times q$, $\boldsymbol{E} : n \times p$, $\boldsymbol{B} : p \times q$, $\boldsymbol{E}$ has a spherical matrix distribution and $\boldsymbol{B}$ is the matrix of undetermined regression coefficients. When

$$\boldsymbol{f}(\boldsymbol{X}, \boldsymbol{B}) = \boldsymbol{X}\boldsymbol{B}, \tag{5.2}$$

(5.1) is the linear model. The least squares estimate (LSE) of $\boldsymbol{B}$ has the same form in general

$$\boldsymbol{B} = (\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{Y} \tag{5.3}$$

as in the case of $\boldsymbol{E}$ having a normal distribution. Here $(\boldsymbol{X}'\boldsymbol{X})^{-}$ denotes a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$. When $\boldsymbol{E} \sim VS(\boldsymbol{0}, \boldsymbol{\Sigma}, g)$ and g is a decreasing function, Anderson and Fang [3] obtained the maximum likelihood estimates of $\boldsymbol{B}$ and $\boldsymbol{\Sigma}$, and their distribution. These results extended some pioneer work of Box, Thomas, and Zellner mentioned in [19]. Bian and Zhang [9] and Fang, Xu and Teng [54] obtained similar results in the classes MS, SS, and LS and gave invariant statistics of testing some hypotheses about $\boldsymbol{B}$. Combining the above results with distributions of quadratic forms and Cochran's theorem for ECD and EMS, we have systematically established the theory and methods of linear models for ECD and EMS.

Fan [27] and Fan and Fang [29] discussed shrinkage estimates, ridge regression, and inadmissibility of estimators of regression coefficients for ECD and EMS. Lin and Gong [68] considered two seemingly unrelated regression models under some regularity conditions. They gave the small sample properties of Zellner's estimator when the disturbances have ECD's. Pan [71] obtained the LSE for the growth curve model and some related invariant statistics for ECD.

When $f(X, B)$ is a nonlinear function of B the model (5.1) is a nonlinear regression model. Wei [81] and Cao, Wei and Qian [15] discussed the nonlinear regression model with errors in ECD and gave an asymptotic expansion and the bias, variance, and skewness of the LSE by a differential geometry approach.

Principal component analysis and the canonical correlation analysis are important techniques of multivariate analysis. When the matrix of observations X is from LS, the algebraic derivations of these two analyses are the same as before. However, the corresponding distribution theory and test of hypotheses may be different. Here we need to find the distributions of eigenvalues and eigenvectors of $X'PX$ or of $X'P_1X$ with respect to $X'P_2X$, where P, P_1, and P_2 are positive definite matrices. The distribution of the eigenvalues and eigenvectors of $X'PX$ for $X \in SS$ were derived by Fang and Zhang in Section 3.5.6 of [58] and [17] which extended the results for $X \in VS$ of Anderson and Fang [2]. From the point of view of spectral decomposition, Fang and Chen [43] studied the spherical matrix distribution and obtained some new subclasses of LS. Their results can be applied to principal component analysis in LS. As the distributions of the eigenvalues and the eigenvectors of $X'P_1X$ with respect to $X'P_2X$ are invariant in the class of VS, canonical correlation analysis can be used in the class.

Since the distribution of the discriminant function is not invariant in the class of SMD, it is more difficult to establish the theory of discriminant analysis for SMD. Cacoullos and Koutras [11] considered the minimum-distance discrimination for SD. Quand, Fang and Teng [74] employed the information function $I(f, g)$ of f and g defined by

$$I(f,g) = \int f(x) \log \left[f(x)/g(x) \right] dx$$

to discriminant analysis. They proved that under some conditions the information function is a monotonic function of the Mahalanobis distance.

There are some studies of the application of the theory of ECD to econometrics. Teng and Chen [78] and Teng, Fang, and Deng [77] derived the distribution of the instrumental variable (IV) estimator of the coefficients of the endogenous variables in the simultaneous equations with spherical disturbance and some related distributions. The reader is referred to Kunitomo [66] for further results in econometrics.

6. Symmetric Multivariate Distributions.

Why does the class of spherical distributions have so many nice properties? One of the reasons is its special structure (1.2), where $U^{(n)}$ is common to all members of the class. Therefore, a spherical distribution is uniquely determined by the distribution of a scalar variable R, but many properties of SD are independent of R as mentioned before. That fact suggests finding other classes of symmetric multivariate distributions having a structure similar to (1.2) with beautiful properties.

Given an n-dimensional random vector $\boldsymbol{Y}$, we may define a corresponding family of distributions by

$$\mathcal{F}(\boldsymbol{Y}) = \{\boldsymbol{X} : \boldsymbol{X} \stackrel{\mathrm{d}}{=} R\boldsymbol{Y}, R > 0 \text{ is independent of } \boldsymbol{Y}\}, \tag{6.1}$$

and call $\boldsymbol{Y}$ the generating vector of the family $\mathcal{F}(\boldsymbol{Y})$. For simplicity, in this section we always assume $P(\boldsymbol{Y} = \boldsymbol{0}) = 0$ for each generating vector. By choosing different $\boldsymbol{Y}$ we obtain different classes of distributions. The following approach seems to yield useful generating vectors:

1) Take a sample $Z_1, \ldots, Z_n$ from a population with cdf $F(z)$.

2) Let $\boldsymbol{Z} = (Z_1, \ldots, Z_n)'$ and set $\boldsymbol{Y} = (Y_1, \ldots, Y_n)'$ with

$$Y_i = Z_i/||\boldsymbol{Z}||, \quad i = 1, \ldots, n.$$

For example, if $Z_1, \ldots, Z_N$ is from $N(0, \sigma^2)$ and the norm is defined as the Euclidean norm, then $\boldsymbol{Y}$ is simply $\boldsymbol{U}^{(n)}$ and $\mathcal{F}(\boldsymbol{Y})$ is the family of SD's. If $Z_1, \ldots, Z_n$ are sampled from an exponential distribution and the norm is defined as the L_1-norm, then $\boldsymbol{Y}$ is uniformly distributed on the simplex $B_n = \{\boldsymbol{z} : z_i > 0,\ i = 1, \ldots, n,\ \sum_{i=1}^n z_i = 1\}$ and $\mathcal{F}(\boldsymbol{Y})$ is the so-called class of multivariate L_1-norm symmetric distributions that was defined and studied by Fang and Fang [46], [48], [33], [34], [35]. The family $\mathcal{F}(\boldsymbol{Y})$ retains most of the important properties of $\boldsymbol{Z}$. Hence, the family of SD's can be regarded as a multivariate extension of $N(0, \sigma^2)$ and the family of multivariate L_1-norm symmetric distributions as a multivariate extension of the exponential distribution. We can use the same technique in studying these families. The following table gives a brief introduction to this kind of multivariate extensions.

Table 1.

Univariate distribution	Its multivariate extension
normal	spherical
lognormal	logspherical
additive logistic normal	additive logistic spherical
exponential	multivariate L_1-norm symmetric
gamma & beta	multivariate Liouville
Cauchy & stable law	α-symmetric multivariate
Cauchy & stable law	spherical stable law
symmetric gamma	generalized symmetric Dirichlet
Scheidegger-Watson	rotationally invariant

Let $\boldsymbol{W} = (W_1, \ldots, W_n)'$ be a positive random vector. If $\log \boldsymbol{W} = (\log W_1, \ldots, \log W_n)'$ has an ECD we say that $\boldsymbol{W}$ has a logelliptical distribution. Let $\boldsymbol{X}$ be a random vector on the simplex B_{n-1}, and let $\boldsymbol{Y} = \left[\log(X_1/X_n), \ldots, \log(X_{n-1}/X_n)\right]'$. If $\boldsymbol{Y}$ has an ECD we say $\boldsymbol{X}$ has an additive logistic elliptical distribution. These two families were defined and studied by Bentler, Fang, and Wu [7] and Fang, Bentler, and Chou [41]. The reader can refer to Section 2.8 of [49].

Taking $\boldsymbol{Y} \sim D(\alpha_1, \ldots, \alpha_n)$, a Dirichlet distribution, with the norm defined as the L_1-norm, the corresponding family $\mathcal{F}(\boldsymbol{Y})$ is called the family of multivariate Liouville distributions, which can be regarded as an extension of both the gamma and the beta distributions as well as one of the multivariate L_1-norm symmetric distributions. The multivariate Liouville distributions have been discussed by many authors. Gupta and Richards [60] gave a comprehensive study of this family under the assumption of a density. Without this assumption Anderson and Fang [2,3], and Fang, Kotz and Ng [49] gave a parallel discussion with more results. It is worth noting that the structure (1.2) can be applied to a nonsymmetric generating vector $\boldsymbol{Y}$ by the same approach as for symmetric generating vectors.

There is more than one natural way to generalize a univariate distribution to its multivariate extension with structure (1.2). The c.f. of a stable law is

$$\exp\left(-\lambda|t|^{\alpha}\right), \quad 0 < \alpha \leq 2. \tag{6.2}$$

One may rewrite (6.2) as $\exp(-\lambda\|\boldsymbol{t}\|^{\alpha})$, a function of the L_2-norm of $\boldsymbol{t}$, yielding what is called a spherically symmetric stable law. A detailed discussion of this family is given by Zolotarev [91]. Alternatively, one may consider (6.2) as a function of an L_α-norm of t with dimension $n = 1$. This way leads to the α-symmetric multivariate distribution that was defined and thoroughly studied by Cambanis, Keener, and Simons [14]. Zhang [86] obtained the distribution of the sum of squares of independent Cauchy variables and the asymptotic distribution. Zhang [87] generalized α-symmetric multivariate distributions to the matrix case and found its stochastic decomposition for the case of the matrix having infinite rows.

Symmetrizing the Dirichlet distribution about the origin leads to the symmetrized Dirichlet distribution (SDD). Take $\boldsymbol{Y}$ having a SDD; then the corresponding family $\mathcal{F}(\boldsymbol{Y})$ is called one of generalized Symmetrized Dirichlet distributions which contains the family of SD as a special case and retains many properties of $\boldsymbol{Z}$. When the parameters of $\boldsymbol{Y}$ are equal, where $\boldsymbol{Z}$ is sampled from a symmetrized gamma distribution with the degrees of freeedom being the same value as the parameter of $\boldsymbol{Y}$. Fang and Fang [47] made a thorough research of this family.

Let V be a linear subspace of R^n and let $\boldsymbol{P}_v$ and $\boldsymbol{P}_{v^\perp}$ be projection matrices into the subspaces V and $V^\perp$, respectively. In the statistics of directional data the Scheidegger-Watson distribution serves as the standard model and is defined by its density

$$g(\boldsymbol{x}'\boldsymbol{P}_v\boldsymbol{x}), \quad \|\boldsymbol{x}\| = 1, \tag{6.3}$$

where g is a scalar function. Fan [28] suggested a family of distributions whose densities have the form $g(\boldsymbol{x}'\boldsymbol{P}_v\boldsymbol{x}, \boldsymbol{x}'\boldsymbol{P}_{v^\perp}\boldsymbol{x})$; the family includes both the family of ECD and the family of $S - W$ distributions. Fan called it the family of rotationally symmetric distributions. Later Fang and Fan [44] discussed asymptotic properties of estimation and hypothesis testing for the class. Let $X_1, \ldots, X_n$ be i.i.d. from a rotaitonally symmetric distribution. They [32] found the MLE of $\boldsymbol{P}_v$ and its maximum likelihood characterization which is defined as follows. Given an intuitive estimator for some parameters, we may be interested in finding the parent distributions such that the estimator is the MLE. This kind of problem is usually called the maximum likelihood characterization of the distribution.

Take $\boldsymbol{Y}$ having i.i.d. components with cdf $F(x)$; then the class $\mathcal{F}(\boldsymbol{Y})$ consists of mixtures of $F(x)$. In the early stage of the study of SD researchers found may properties of mixtures of normal distribution. Later they found that most of those properties can be extended to the class of SD. A natural question is can we extend those properties to some wider class than that of SD?

A largest characterization of SD is a demonstration that there is no generating vector $\boldsymbol{Y}$ such that the family of SD is a proper subfamily of $\mathcal{F}(\boldsymbol{Y})$. This was proved by Fang and Bentler [40]. They pointed out that this largest characterization can be extended to the family of multivariate Liouville distributions.

With the Dirichlet distribution Fang and Xu [55] defined a class of multivariate distributions including the multivariate logistic and Gumbel Type I distributions.

7. Miscellaneous

In this section we include some work not cited above. First of all, we shall introduce some characterizations of multivariate symmetric and related distributions.

Let $\boldsymbol{X}$ be an $n \times p$ random matrix. In general, the (marginal) normality of elements, rows, and/or columns does not imply the multinormality of $\boldsymbol{X}$. Zhang and Fang [88] pointed out that the normality of $\boldsymbol{X}$ can be determined by the normality of 1) any element of $\boldsymbol{X}$ if $X \in VS$; 2) any row of $\boldsymbol{X}$ if $\boldsymbol{X} \in MS$; 3) $X_{11}, \ldots, X_{pp}$ if $\boldsymbol{X} \in SS$; and 4) the upper triangular elements of $\boldsymbol{X}$ if $\boldsymbol{X} \in LS$. They furthermore discussed relationships between the normality of $\boldsymbol{X}$ and the normality of linear transformations of $\boldsymbol{X}$.

Since the order statistics of the exponential distribution have many nice properties, Fang and Fang [34] derived various distributions and moments of the order statistics of a multivariate L_1-norm symmetric distribution. Let $\boldsymbol{Z}$ be an n-dimensional interchangeable random vector; let $Z_{(1)} < \cdots < Z_{(n)}$ be its order statistics; and define the normalized spacings of $\boldsymbol{Z}$ as $U_i = (n - i - 1)(Z_{(i)} - Z_{(i-1)}, i = 1, \ldots, n$, with $Z_{(0)} = 0$. It is known that $\boldsymbol{Z} \stackrel{d}{=} \boldsymbol{U}$ if $Z_1, \ldots, Z_n$ are i.i.d. and Z_1 has an exponential distribution. Fang and Fang [35] extended this property to the class of multivariate L_1-norm symmetric distributions and gave the characterization that if $\boldsymbol{Z}$ is an interchangeable random vector, then $\boldsymbol{Z} \stackrel{d}{=} \boldsymbol{U}$ if and only if $\boldsymbol{Z}$ is a multivariate L_1-norm symmetric distribution.

Let S be a connected set on the unit sphere in R^n and let C be a cone associated with S defined by

$$C = \{\boldsymbol{x} : \boldsymbol{x} \in R^n,\ \boldsymbol{x}/\|\boldsymbol{x}\| \in S\} \cup \{\boldsymbol{0}\}.$$

If $\boldsymbol{X} \in$ SD and $P(\boldsymbol{X} = \boldsymbol{0}) = 0$, then $P(X \in C)$ has the same value for all distributions in the SD. This fact can be used for a characterization of the uniform distribution on a sphere and for spherical distributions if $\boldsymbol{X}$ and $\boldsymbol{X}/\|\boldsymbol{X}\|$ are independent. The result is due to Wang [80] and referred to in [49], pp. 163-165.

Let $X_1, \ldots, X_n$ be exchangeable normal variables with a common correlation ρ, and let $X_{(1)}, \dots, X_{(n)}$ be their order statistics. The random variable $G = X_{(k)} + \cdots + X_{(n)}$ is called the selection differential by geneticists and is of particular interest in genetic selection. Fang and Liang [50] gave results concerning a conjecture of Tong [79] on the distribution of this random variable as a function of ρ. The same technique can be applied to yield general results for linear combinations of order statistics of ECD.

Let $\phi(x)$ and $\Phi(x)$ be the p.d.f. and the c.d.f. of $N(0,1)$, respectively. Mills' ratio, defined by

$$M(x) = [1 - \Phi(x)]/\phi(x),$$

has been studied thoroughly. One can define similarly the Mills' ratio $M(\boldsymbol{x}, \boldsymbol{\Sigma})$ for $N_n(\mathbf{0}, \boldsymbol{\Sigma})$ and $EC_n(\mathbf{0}, \boldsymbol{\Sigma}, g)$. Fang and Xu [53] gave a detailed discussion of these Mills' ratios. They [84] obtained results on the expected values of zonal polynomials of EMD also.

The inverted Wishart distribution has been used in Bayesian statistics. Many inverted matrix distributions related to SMD can be similarly defined. Xu [83] studied the inverted beta/Dirichlet distributions and gave some applications to Bayesian statistics.

There are several studies of the moments of a multivariate distribution. Li [67] had a new approach on this subject. Let $\boldsymbol{X}$ be an $n \times 1$ random vector. The k-th moment of $\boldsymbol{X}$ is defined as

$$\Gamma_k(\boldsymbol{X}) = \begin{cases} E(\boldsymbol{X} \otimes \boldsymbol{X}' \otimes \cdots \otimes \boldsymbol{X} \otimes \boldsymbol{X}'), & \text{if } k \text{ is even} \\ E(\boldsymbol{X} \otimes \boldsymbol{X}' \otimes \cdots \otimes \boldsymbol{X}' \otimes \boldsymbol{X}), & \text{if } k \text{ is odd.} \end{cases}$$

Li gave the relationship between $\Gamma_k(\boldsymbol{X})$ and all the k-th mixed moments of $\boldsymbol{X}$ and a simple formula for the moments of a quadratic form of $\boldsymbol{X}$ as a function of $\Gamma_k(\boldsymbol{X})$. As an application, he gave moments of ECD and its quadratic forms.

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