Theory and Design of Uniform DFT, Parallel, Quadrature Mirror Filter Banks

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Abstract—**In this paper, the theory of uniform DFT, parallel, quadrature mirror filter (QMF) banks is developed. The QMF equations, i.e., equations that need to be satisfied for exact reconstruction of the input signal, are derived. The concept of decimated filters is introduced, and structures for both analysis and synthesis banks are derived using this concept. The QMF equations, as well as closed-form expressions for the synthesis filters needed for exact reconstruction of the input signal $x(n)$, are also derived using this concept. In general, the reconstructed signal $\hat{x}(n)$ suffers from three errors: aliasing, amplitude distortion, and phase distortion. Conditions for exact reconstruction (i.e., all three distortions are zero, and $\hat{x}(n)$ is equal to a delayed version of $x(n)$) of the input signal are derived in terms of the decimated filters. Aliasing distortion can always be completely canceled. Once aliasing is canceled, it is possible to completely eliminate amplitude distortion (if suitable IIR filters are employed) and completely eliminate phase distortion (if suitable FIR filters are employed). However, complete elimination of all three errors is possible only with some simple, pathalogical stable filter transfer functions. In general, once aliasing is canceled, the other distortions can be *minimized* rather than completely eliminated. Algorithms for this are presented. The properties of FIR filter banks are then investigated. Several aspects of IIR filter banks are also studied using the same framework.**

I. Introduction

THE DECOMPOSITION of a signal into contiguous frequency bands and reconstruction of the signal based on the subband components are fundamental concepts in signal processing. The partitioning of the input signal into several frequency bands is done by the so-called analysis filter bank and reconstruction by the synthesis filter bank. The subband components of the input signal are usually decimated to reduce the amount of computational load in applications where the subband components need to be processed. In maximally decimated filter banks, each subband component is represented with the minimum number of samples per unit time. Such maximally decimated filter banks are of particular interest in frequency-domain coders. In fact, the motivation for studying maximally decimated filter banks largely stems from frequency-domain speech coding because in such coders properties of aural perception can be exploited to achieve higher speech quality at lower bit rates.

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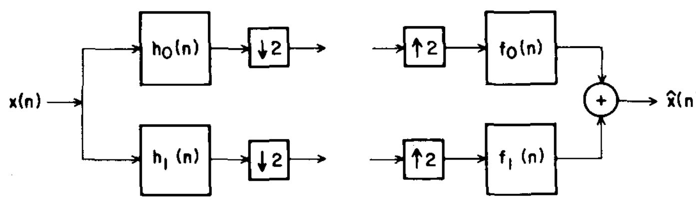

Fig. 1. The quadrature mirror filter bank.

The quadrature mirror filter (QMF) pair (Fig. 1) has been the basis of most filter banks used in frequency-domain speech coders since its development by Croisier *et al.* [35] and later by Esteban and Galand [21]. The analysis section of the QMF pair splits the input signal into two Subband signals having equal bandwidth and decimates each subband signal by a factor of two. The synthesis section interpolates the two subband signals by a factor of two and recombines them through another filter pair to produce the output. The impulse responses of the analysis filters are represented by $h_0(n)$ and $h_1(n)$ and their transfer functions by $H_0(z)$ and $H_1(z)$. The impulse responses of the synthesis filters are represented by $f_0(n)$ and $f_1(n)$ and their transfer functions by $F_0(z)$ and $F_1(z)$. The input signal is denoted by $x(n)$ with z-transform $X(z)$ and the reconstructed signal by $\hat{x}(n)$ with z-transform $\hat{X}(z)$. The QMF pair has the desirable property that when the analysis and synthesis filter banks are connected together, the input signal can be reconstructed at the output with arbitrarily small error. This is because 1) the analysis and synthesis filters are chosen so as to completely cancel the aliasing components caused by the decimation process, and 2) the filter frequency responses are designed to overlap and add in such a way that the overall frequency response approximates a delay at all frequencies.

An important class of filter banks is the uniform filter bank where the input signal is split into equal bands. Maximally decimated uniform filter banks with more than two bands may be implemented using tree structures in which the input signal is successively divided into two equal bands at each stage of the tree using the analysis section of the QMF pair and the output signal is reconstructed by successively recombining the subband components at each stage of the tree using the synthesis section of the QMF pair [1], [10], [21]–[26]. A disadvantage of this approach is that the number of uniform bands, say r, is restricted to be a power of two. Uniform maximally decimated parallel QMF banks (Fig. 2), which do not have this

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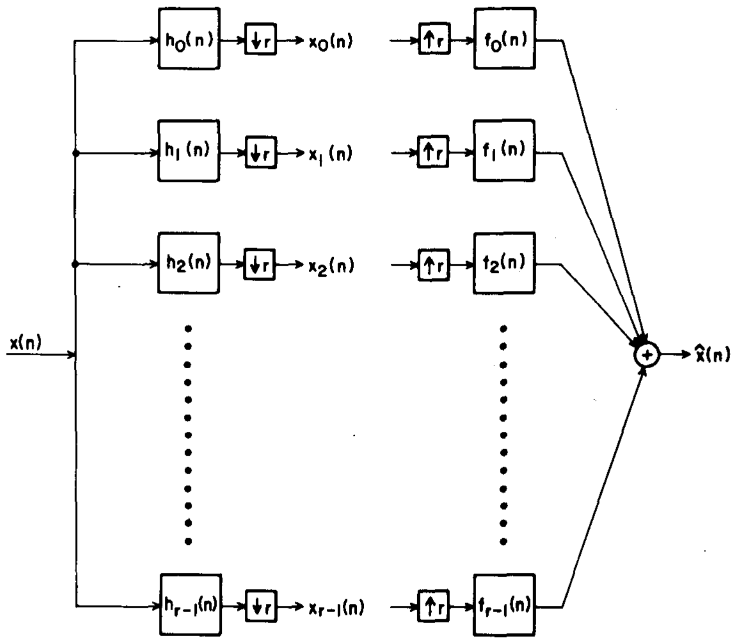

Fig. 2. r-band parallel QMF bank.

restriction, are implemented by passing the signal through r analysis filters and decimating each filter output by a factor of r. The signal is reconstructed by interpolating each subband signal by a factor of r and recombining them through r synthesis filters [10], [27]. The impulse responses of the r analysis filters are represented by $h_0(n), h_1(n), \cdots, h_{r-1}(n)$, and their transfer functions by $H_0(z), H_1(z), \cdots, H_{r-1}(z)$. The impulse responses of the r synthesis filters are represented by $f_0(n), f_1(n), \cdots, f_{r-1}(n)$, and their transfer functions by $F_0(z), F_1(z), \cdots, F_{r-1}(z)$. A comparison between parallel QMF structures and tree structures can be found in [10], when r is a power of two. It is shown in [10] that the complexity of the parallel approach is comparable to the tree approach and in addition has several advantages, such as smaller group delay and smaller signal storage requirement, over the tree approach.

In this paper, we restrict ourselves to maximally decimated uniform parallel QMF banks. We also assume that the analysis filters of these uniform filter banks are all frequency-translated versions of a common baseband filter. Thus

$$H_l(e^{j\omega}) = H(e^{j(\omega - 2\pi l/r)})$$
$$h_l(n) = e^{j2\pi ln/r} h(n)$$

where $H(z)$ is the prototype low-pass filter and $h(n)$ its impulse response. If we define

$$W = e^{-j2\pi/r}$$

then

$$H_l(z) = H(zW^l)$$
$$h_l(n) = W^{-ln} h(n).$$

Such uniform filter banks, where the analysis filters are derived by frequency modulation of a common baseband filter, are referred to as uniform DFT filter banks [1, ch. 7]. The common baseband filter is assumed to be centered at $\omega = 0$. This corresponds to the "even-type" uniform channel stacking arrangement [1, ch. 7]. The objective of this paper is to develop the theory, implementation, and design of such filter banks. A unified framework is presented in this paper which allows various issues such as aliasing, amplitude and phase distortions, efficiency of implementation, stability of filters, etc., to be addressed based on a common ground.

The filter banks developed in this paper differ from the results of Galand and Nussbaumer [10], Rothweiler [37], and Chu [39] in a number of respects. We indicate here how aliasing can be exactly canceled with an arbitrary number of channels r and with no assumptions regarding the exact frequency shaping achieved by the filters (for example, no assumption is made that nonadjacent filters have completely nonoverlapping frequency responses). Galand and Nussbaumer have indicated tree structures for accomplishing this, and their results hold when r is a power of 2. The results in [37] and [39] hold for arbitrary r, under the assumption that nonadjacent filters in the analysis bank do not overlap.

The filters belonging to the filter banks presented in this paper are such that all filters are derived as in the above equation by complex modulation. Accordingly, the filtered signals $x_k(n)$ in Fig. 2 are complex even for real $x(n)$. Depending upon the actual application, this may or may not be an inconvenience. In any case, there are several important reasons for studying these filter banks. First, complete theoretical results are developable, addressing aliasing and other distortions in a unified and quantiative manner. Second, parallel r-channel QMF banks with filters having real impulse responses (so that $x_k(n)$ are real for real $x(n)$), and which are entirely free from aliasing, are readily developed from the results of this paper; in fact, all

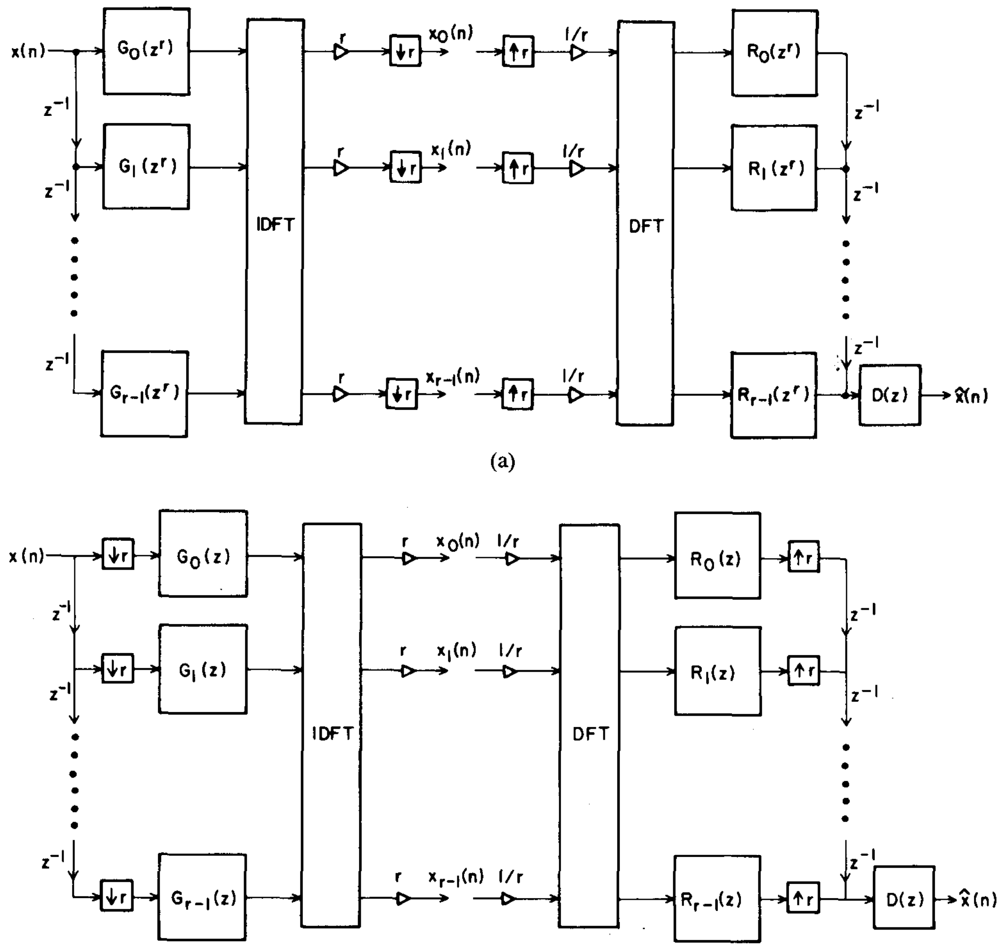

Fig. 3. (a) Structure for analysis and synthesis filter banks. (b) Alternative structure for analysis and synthesis filter banks.

the properties we study herein can be translated to the case of real-QMF banks, as will be reported elsewhere. Finally, there are several other contexts such as transmultiplexer design [3], [22]–[24], spectral analysis, and synthesis of speech [1, ch. 7], and so on, where the concept of uniform DFT filter banks finds applications.

The organization of the paper is as follows. In Section II, the QMF equations (i.e., the equations that must be satisfied if the input signal is to be reconstructed exactly by the cascade of analysis and synthesis banks) are derived. Closed-form expressions for the synthesis filters that are needed for exact reconstruction are also derived. In Section III, the concept of decimated filters is introduced and expressions for the synthesis filters needed for exact reconstruction are obtained in terms of the decimated filters. Structures for the analysis and synthesis filter banks are derived. The study of FIR filter banks (i.e., where all analysis and synthesis filters are FIR) is motivated and conditions that must be satisfied for exact reconstruction by the FIR filter bank are derived. In Section IV, we investigate situations for which the cascade of analysis and synthesis FIR filter banks has transmission zeros at some frequencies for any arbitrary linear-phase FIR prototype low-pass filter $H(z)$. Clearly, such situations are to be avoided while designing FIR filter banks. Properties of FIR filter banks are derived in Section V. In Section VI, a procedure for designing linear-phase FIR filter banks is described. Examples of the design algorithm are given in Section VII. In Section VIII, various aspects of IIR filter banks are discussed.

II. Derivation of the QMF Equations

In this section, we derive the QMF equations which are a set of linear equations that must be satisfied for exact reconstruction of the incoming signal in the absence of any processing of the decimated subband signals. Using these equations, we derive closed-form expressions for each synthesis filter in terms of analysis filters.

Let us denote the decimated subband signals as $x_0(n)$, $x_1(n),\cdots,x_{r-1}(n)$ with z-transforms $X_0(z), X_1(z),\cdots$, $X_{r-1}(z)$. Each decimated subband signal $x_k(n)$ is obtained by first passing the input signal $x(n)$ through the filter $H_k(z)$ and then decimating it by a factor of r. Thus, $X_k(z)$ can be expressed as (see [1, sec. 2.3.2])

$$X_k(z) = \frac{1}{r}\sum_{l=0}^{r-1} H_k(z^{1/r}W^{-l})X(z^{1/r}W^{-l}),$$
$$0 \leqslant k \leqslant r-1. \tag{1}$$

Each signal $x_k(n)$ is next passed through an interpolator as shown in Fig. 3. It is then passed through the corresponding synthesis filter $F_k(z)$ and the outputs of synthesis filters are added to produce $\hat{x}(n)$. Thus

$$\hat{X}(z) = \sum_{k=0}^{r-1} F_k(z) X_k(z^r). \tag{2}$$

Substituting for $X_k(z)$ from (1) in the above equation, we get

$$\begin{aligned}\hat{X}(z) &= \sum_{k=0}^{r-1} F_k(z) \frac{1}{r} \sum_{l=0}^{r-1} H_k(zW^{-l}) X(zW^{-l}) \\ &= \frac{1}{r} \sum_{l=0}^{r-1} X(zW^{-l}) \sum_{k=0}^{r-1} F_k(z) H_k(zW^{-l}). \end{aligned} \tag{3}$$

But the analysis filters $H_k(z)$ are assumed to be related to $H(z)$ by $H_k(z) = H(zW^k)$ and therefore

$$H_k(zW^{-l}) = H(zW^{k-l}) = H_{(k-l) \bmod r}(z) \tag{4}$$

whence

$$\hat{X}(z) = \frac{1}{r} \sum_{l=0}^{r-1} X(zW^{-l}) \sum_{k=0}^{r-1} F_k(z) H_{(k-l) \bmod r}(z). \tag{5}$$

If the reconstructed signal $\hat{x}(n)$ is to be an exact replica of the input signal, except for a delay of n_0 samples, then we must have

$$\hat{X}(z) = X(z) z^{-n_0} \tag{6}$$

and so the following conditions must hold:

$$\frac{1}{r} \sum_{k=0}^{r-1} F_k(z) H_{(k-l) \bmod r}(z) = z^{-n_0} \delta(l), \qquad 0 \leqslant l \leqslant r-1 \tag{7}$$

where $\delta(l)$ is the usual Kronecker delta defined as

$$\delta(l) = \begin{cases} 1 & \text{if } l = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Expressing (7) in matrix form,* we have the following *circulant* set of equations:

$$\boldsymbol{H}(z) \boldsymbol{f}(z) = \boldsymbol{c}(z) \tag{8}$$

where

$$\boldsymbol{H}(z) = \begin{pmatrix} H_0(z) & H_1(z) & \cdots & H_{r-1}(z) \\ H_{r-1}(z) & H_0(z) & \cdots & H_{r-2}(z) \\ \vdots & & & \vdots \\ H_1(z) & H_2(z) & \cdots & H_0(z) \end{pmatrix} \tag{8a}$$

$$\boldsymbol{f}(z) = [F_0(z) F_1(z) \cdots F_{r-1}(z)]^T \tag{8b}$$

$$\boldsymbol{c}(z) = [rz^{-n_0} \quad 0 \cdots 0]^T. \tag{8c}$$

The set of linear equations described in (7) or (8) are the required QMF equations, and they represent necessary and sufficient conditions for exact reconstruction (except for a delay) of the original signal. The aliasing cancellation matrix (AC matrix) formulation derived in [2] by Smith and Barnwell for an arbitrary analysis filter bank reduces to (7) or (8) if the analysis filters obey $H_k(z) = H(zW^k)$.

**Notations used in the paper*: Capital bold-faced letters are used to represent matrices and small bold-faced letters for vectors. The identity matrix is represented by $\boldsymbol{I}$. The conjugate transpose of a matrix $\boldsymbol{W}$ is denoted by $\boldsymbol{W}^\dagger$, whereas transpose is denoted by $\boldsymbol{W}^T$.

A general matrix formulation has also been derived in [41] by Ramstad and in [42] by Vetterli. Equations (7) or (8) are identical to (5-33) derived in [31] by Smith.

In order to obtain a closed-form expression for the synthesis filters in terms of the analysis filters, we exploit the circulant nature of the matrix $\boldsymbol{H}(z)$. Every circulant matrix, such as $\boldsymbol{H}(z)$, can be expressed as

$$\boldsymbol{H}(z) = \boldsymbol{W} \Lambda(z) \boldsymbol{W}^{-1} \tag{9}$$

where $\boldsymbol{W}$ is the r-point DFT matrix, i.e.,

$$\boldsymbol{W} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W & \cdots & W^{r-1} \\ \vdots & & & \vdots \\ 1 & W^{r-1} & \cdots & W^{(r-1)(r-1)} \end{bmatrix} \tag{9a}$$

and where

$$\Lambda(z) = \begin{bmatrix} \Lambda_0(z) & & & \\ & \Lambda_1(z) & & 0 \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & \Lambda_{r-1}(z) \end{bmatrix} \tag{9b}$$

$$\Lambda_i(z) = \sum_{k=0}^{r-1} H_k(z) W^{ik}, \qquad 0 \leqslant i \leqslant r-1. \tag{9c}$$

Recall that $\boldsymbol{W}/\sqrt{r}$ is unitary. Hence

$$\boldsymbol{W}^\dagger \boldsymbol{W} = \boldsymbol{W} \boldsymbol{W}^\dagger = r\boldsymbol{I} \tag{10a}$$

and

$$\boldsymbol{W}^{-1} = \boldsymbol{W}^\dagger / r. \tag{10b}$$

Substituting for $\boldsymbol{H}(z)$ from (9) in (8), we get

$$\boldsymbol{W} \Lambda(z) \boldsymbol{W}^{-1} \boldsymbol{f}(z) = \boldsymbol{c}(z) \tag{11a}$$

$$\boldsymbol{f}(z) = \boldsymbol{W} \Lambda^{-1}(z) \boldsymbol{W}^{-1} \boldsymbol{c}(z). \tag{11b}$$

In view of (8c), (9a), and (9b), eq. (11b) simplifies to

$$\boldsymbol{f}(z) = \boldsymbol{W} \begin{bmatrix} \Lambda_0^{-1}(z) \\ \vdots \\ \vdots \\ \vdots \\ \Lambda_{r-1}^{-1}(z) \end{bmatrix} z^{-n_0}. \tag{12}$$

Thus

$$F_i(z) = \sum_{k=0}^{r-1} \Lambda_k^{-1}(z) W^{ik} \cdot z^{-n_0}, \qquad 0 \leqslant i \leqslant r-1. \tag{13a}$$

In words, *the ith synthesis filter transfer function $F_i(z)$ in the synthesis bank is the r-point DFT of the reciprocals of the r-point DFT sequence of the analysis bank* $\{H_k(z)\}$. Substituting for $\Lambda_k(z)$ from (9c) in (13a), we get

$$F_i(z) = \sum_{k=0}^{r-1} \left\{ \frac{W^{ik}}{\sum_{l=0}^{r-1} H_l(z) W^{lk}} \right\} z^{-n_0}, \qquad 0 \leqslant i \leqslant r-1. \tag{13b}$$

We have thus derived a closed-form expression for the synthesis filters in terms of the analysis filters.

III. Analysis and Synthesis Filter Banks Using Decimated Filters

In this section, uniform DFT, parallel, QMF filter banks are studied using the concept of decimated filters. The lth decimated filter with transfer function $G_l(z)(0 \leqslant l \leqslant r-1)$, corresponding to $H(z)$, is defined as the filter whose impulse response is obtained by retaining every rth sample of $h(n)$, the impulse response of the prototype low-pass filter, starting from the lth sample. Thus

$$G_l(z) = \sum_{p=0}^{\infty} h(l+pr)z^{-p}, \qquad 0 \leqslant l \leqslant r-1. \quad (14a)$$

The transfer function $H(z)$ can therefore be written as

$$H(z) = \sum_{l=0}^{r-1} z^{-l}G_l(z^r). \quad (14b)$$

This representation has been used earlier by other authors; for example see Bellanger *et al.* in [3]. The notion of decimated filters provides additional insight into the implementation and design aspects of uniform DFT parallel QMF filter banks. We begin this section by deriving expressions for the analysis filters in terms of the decimated filters. Next, expressions for the synthesis filters in terms of the decimated filters are developed. Structures for the analysis and synthesis filter bank are derived. Finally, for the case when both the analysis and synthesis filters are restricted to be FIR filters, the condition for exact reconstruction of the original incoming signal is derived in terms of the decimated filters.

The transfer function $H_k(z)$ of the kth analysis filter can be expressed, using (14b) as

$$H_k(z) = H(zW^k) = \sum_{l=0}^{r-1} z^{-l}W^{-kl}G_l(z^r). \quad (15)$$

The above equation can also be expressed in matrix form as

$$\boldsymbol{h}(z) = \boldsymbol{W}^{\dagger}\boldsymbol{g}(z) = r\boldsymbol{W}^{-1}\boldsymbol{g}(z) \quad (16)$$

where

$$\boldsymbol{h}(z) = [H_0(z)H_1(z)\cdots H_{r-1}(z)]^T \quad (16a)$$

$$\boldsymbol{g}(z) = [G_0(z^r) \quad z^{-1}G_1(z^r)\cdots z^{-(r-1)}G_{r-1}(z^r)]^T. \quad (16b)$$

We have thus expressed the analysis filter bank in terms of the decimated filters of the prototype $H(z)$.

Equation (16) can also be written as

$$r\boldsymbol{g}(z) = \boldsymbol{W}\boldsymbol{h}(z) \quad (17)$$

or in terms of components

$$rz^{-l}G_l(z^r) = \sum_{k=0}^{r-1} H_k(z)W^{lk}. \quad (18)$$

But the right-hand side of (18) is just the lth eigenvalue $\Lambda_l(z)$ of the matrix $\boldsymbol{H}(z)$ (see (9c)). Thus

$$\Lambda_l(z) = rz^{-l}G_l(z^r), \qquad 0 \leqslant l \leqslant r-1. \quad (19)$$

The eigenvalues of the circulant matrix $\boldsymbol{H}(z)$ are thus very simply related to the decimated filters by the above expression. A consequence of this relationship is that, if any of the decimated filters has a transmission zero at some frequency, then the corresponding eigenvalue becomes zero causing $\boldsymbol{H}(z)$ to become singular at the same frequency. We elaborate further on the physical significance of these singularity issues in Section V.

Substituting (19) into (13a), we obtain

$$F_i(z) = \frac{1}{r}\sum_{k=0}^{r-1} \frac{W^{ik}}{z^{-k}G_k(z^r)} z^{-n_0}, \qquad 0 \leqslant i \leqslant r-1. \quad (20)$$

We have thus obtained an expression for the synthesis filters in terms of the decimated filters of the analysis prototype $H(z)$. In order to derive a similar equation as (16) for the synthesis filters, we proceed by defining

$$R_k(z) = \prod_{\substack{l=0 \\ l \neq k}}^{r-1} G_l(z), \qquad 0 \leqslant k \leqslant r-1 \quad (21)$$

and

$$D(z) = \frac{z^{-n_0}}{z^{-(r-1)}\prod_{l=0}^{r-1}G_l(z^r)}. \quad (22)$$

Then

$$F_i(z) = \frac{1}{r}D(z)\sum_{k=0}^{r-1} R_k(z^r)z^{-(r-1-k)}W^{ik}, \qquad 0 \leqslant i \leqslant r-1 \quad (23)$$

or in matrix form

$$\boldsymbol{f}(z) = \frac{1}{r}D(z)\boldsymbol{W}\boldsymbol{r}(z) \quad (24)$$

where

$$\boldsymbol{r}(z) = [z^{-(r-1)}R_0(z^r) \quad z^{-(r-2)}R_1(z^r)\cdots R_{r-1}(z^r)]^T. \quad (24a)$$

One can easily derive the structures for the analysis and synthesis filter banks using (16) and (24). The vector of r outputs of the analysis filters is given by $\boldsymbol{h}(z)X(z) = r\boldsymbol{W}^{-1}\boldsymbol{g}(z)X(z)$. If the vector of r inputs to the synthesis filters is $\boldsymbol{y}(z)$, the reconstructed signal is given by $\hat{X}(z) = \boldsymbol{f}^T(z)\boldsymbol{y}(z) = 1/rD(z)\boldsymbol{r}^T(z)\boldsymbol{W}\boldsymbol{y}(z)$, in view of the symmetry of $\boldsymbol{W}$, i.e., $\boldsymbol{W}^T = \boldsymbol{W}$. The analysis and synthesis filter banks can therefore be drawn as in Fig. 3(a). Without the IDFT and DFT, the filter banks are identical to the decimating and interpolating structures used by Constantinides and Valenzuela in transmultiplexing [4]. An alternative structure with the decimator moved to the left and interpolator moved to the right is shown in Fig. 3(b). This can be justified because the DFT, IDFT, and multiplication operations are all memoryless (see [1, sect. 3.1.2]). The analysis and synthesis filter banks are now seen to be identical to the polyphase structures discussed in [1], [3], and [5]. Identical filter banks for the case $r = 2$ have been given by Barnwell in [6] and [30].

In the structures of Fig. 3, we have a general means of perfectly reconstructing a signal $x(n)$ (within a fixed delay) after the signal has been split into r bands and each component decimated by r. The claim is that $\hat{x}(n)=x(n-n_0)$ with no aliasing error and no reconstruction error. Accordingly, the overall system of Fig. 3 is a linear shift-invariant system (even though the decimator and interpolator building blocks are time-varying) with transfer function z^{-n_0}.

The question that remains to be answered is, under what conditions is the synthesis filter bank realizable for a given analysis filter bank? For realizability, the components $R_k(z)$ and $D(z)$ are required to be causal and stable. Notice that if the prototype low-pass filter $H(z)$ is stable, then so are the decimated filters $G_l(z)$. As a result, $R_k(z)$ defined as in (21) is stable for all k. It only remains to concern ourselves with the stability of $D(z)$. In general, $D(z)$ as given by (22) is not stable. In fact, if $H(z)$ is a linear-phase FIR filter with a symmetric impulse response, it is easily shown (property 4, Section V) that $\prod_{l=0}^{r-1}G_l(z^r)$ is a linear-phase FIR filter. Hence, $D(z)$ is necessarily unstable in this case. This motivates us to find conditions under which $D(z)$ can be deleted in Fig. 3, without deteriorating beyond tolerance, the reconstructed signal.

Consider the case when $H(z)$ is a FIR filter. Let

$$H(z)=\sum_{n=0}^{N-1}h(n)z^{-n}.$$

Furthermore, let

$$N-1=m_0+m_1 r, \qquad 0\leqslant m_0\leqslant r-1. \tag{25}$$

Then the decimated filters are given by

$$G_l(z)=\begin{cases}\sum_{p=0}^{m_1}h(l+pr)z^{-p}, & 0\leqslant l\leqslant m_0\\ \sum_{p=0}^{m_1-1}h(l+pr)z^{-p}, & m_0+1\leqslant l\leqslant r-1.\end{cases} \tag{26}$$

Since all the decimated filter transfer functions are polynomials in z^{-1}, the filters $R_l(z)(0\leqslant l\leqslant r-1)$, defined in (21), are also polynomials in z^{-1}. Thus, all the filters shown in the structures for analysis and synthesis filter banks can be realized as FIR filters with the exception of $D(z)$. If we exclude $D(z)$, then the overall transfer function of the cascade of the analysis and synthesis filter banks would simply be

$$T(z)=\frac{z^{-n_0}}{D(z)}=z^{-(r-1)}\prod_{l=0}^{r-1}G_l(z^r). \tag{27}$$

This is because when $D(z)$ is included, the overall transfer function is simply z^{-n_0}. Note that the effect of deleting $D(z)$ is to introduce an amplitude distortion of the form $1/|D(e^{j\omega})|$ and a phase distortion of the form $-\arg[D(e^{j\omega})]$. (The reconstructed signal however, *continues to be free* from aliasing.) The effects of these two distortions are minimized by approximating $T(z)$ as closely as possible with a pure delay operator.

The overall transfer function $T(z)$ has powers of z^{-1} starting from $r-1$ to N_T, where

$$\begin{aligned}N_T&=r-1+(m_0+1)m_1 r+(r-1-m_0)(m_1-1)r\\ &=(N-r)r+r-1.\end{aligned} \tag{28}$$

The power of z^{-1}, which lies midway, is $r-1+(N-r)\cdot r/2$. In Section IV, it is shown that in order to avoid *singularity situations*, $N-r$ must necessarily be even, and so its mid-value is guaranteed to be an integer. So one possible design criterion for minimizing the distortion is to approximate $T(z)$ by a delay of $r-1+(N-r)r/2$, i.e.,

$$T(z)=z^{-(r-1)}\prod_{l=0}^{r-1}G_l(z^r)=z^{-(r-1+r(N-r)/2)} \tag{29}$$

or equivalently

$$\prod_{l=0}^{r-1}G_l(z)=z^{-(N-r)/2}. \tag{30}$$

Thus, if we design the prototype transfer function $H(z)$ so that in addition to being a good low-pass filter it also satisfies (29) or (30) as closely as possible, then the reconstructed signal $\hat{x}(n)$ will be a good approximation of the original incoming signal $x(n)$.

For FIR analysis-synthesis filter banks, the filter $D(z)$ is excluded, and so the synthesis filters are now given by

$$F_i(z)=\frac{1}{r}\sum_{k=0}^{r-1}R_k(z^r)z^{-(r-1-k)}W^{ik}, \qquad 0\leqslant i\leqslant r-1 \tag{31}$$

instead of the expressions in (23). By substituting the new expressions for the synthesis filters in the left-hand side of the QMF equation (7), one can verify that

$$\frac{1}{r}\sum_{k=0}^{r-1}H_{(k-l)\bmod r}(z)F_k(z)=T(z)\delta(l), \qquad 0\leqslant l\leqslant r-1. \tag{32}$$

This confirms our earlier statement that the effect of deleting $D(z)$ is only to introduce an amplitude distortion and a phase distortion, and the reconstructed signal is still completely free from aliasing. Minimizing the effect of this distortion is equivalent to satisfying the first QMF equation as closely as possible, i.e.,

$$T(z)=\frac{1}{r}\sum_{k=0}^{r-1}H_k(z)F_k(z)=z^{-n_0}. \tag{33}$$

For the choice of $n_0=r-1+r\cdot(N-r)/2$, this is the same as the design criterion expressed by (29) and (30).

Comment on Exact Reconstruction: The term "exact reconstruction" means a situation where aliasing, amplitude, and phase distortions are completely eliminated, so that $\hat{x}(n)=x(n-n_0)$. With FIR filters in the analysis and synthesis stages, this is possible provided we make the following choices:

$$G_l(z^r)=z^{-k_l r} \quad R_l(z^r)=z^{-(p-k_l)r}, \qquad 0\leqslant l\leqslant r-1$$

where k_l and p are arbitrary nonnegative integers, with $p\geqslant k_l$. The overall delay n_0 is given by $n_0=r-1+pr$. With $G_l(z^r)$ restricted to be delays in the above manner, the frequency-shaping achievable by $H_0(z)$ is indeed very limited, i.e., $H_0(z)$ does not give a "good" low-pass response. Such pathological situations of "exact" reconstruction will not be considered further.

In general, with $H_0(z)$ required to accomplish better frequency shapings, it is possible to eliminate either amplitude distortion or phase distortion completely, but not both. Aliasing, of course, can always be eliminated by proper choice of $F_k(z)$ for given $H_k(z)$.

IV. Singularity Issues

In this section, we investigate the conditions under which the input signal cannot be reconstructed by the FIR filter bank for any arbitrary linear-phase low-pass filter $H(z)$. Clearly, these conditions are to be avoided.

The overall transfer function of the FIR filter bank is $T(z)$ and is given by (27), i.e.,

$$T(z) = z^{-(r-1)} \prod_{l=0}^{r-1} G_l(z^r).$$

The input signal is unconditionally nonreconstructable iff $G_l(e^{j\omega_0}) = 0$ for some l and some ω_0 unconditionally. In other words, one of the decimated filters $G_l(z)$ has a transmission zero at some frequency ω for any arbitrary linear-phase filter $H(z)$. Equivalently, the corresponding eigenvalue of $\boldsymbol{H}(z)$ (see (19)) has zeros at frequencies $(\omega_0 + 2\pi p)/r$ for $0 \leqslant p \leqslant r-1$ causing $\boldsymbol{H}(z)$ to become singular at these frequencies.

If $h(n)$, the impulse response of $H(z)$, is real, then the lth decimated filter will also have a transmission zero at $2\pi - \omega_0$ and, accordingly, the eigenvalue $\Lambda_l(z)$ will have zeros at frequencies $(2\pi - \omega_0 + 2\pi p)/r$ for $0 \leqslant p \leqslant r-1$ as well. Let us now investigate the condition under which this situation can arise.

In the range $0 \leqslant l \leqslant m_0$, the transfer function of the lth decimated filter $G_l(e^{j\omega})$ is given by

$$G_l(e^{j\omega}) = \sum_{k=0}^{m_1} h(l+kr) e^{-j\omega k}.$$

This vanishes unconditionally when one of the following cases arises [7, ch. 3].

i) The lth decimated filter is a linear-phase filter of odd order (even length) with a symmetric impulse response. For this case, $G_l(e^{j\omega})$ becomes necessarily zero at $\omega = \pi$.

ii) The lth decimated filter is a linear-phase filter of odd order with an antisymmetric impulse response. For this case, $G_l(e^{j\omega})$ becomes necessarily zero at $\omega = 0$.

iii) The lth decimated filter is a linear-phase filter of even order with an antisymmetric impulse response. For this case, $G_l(e^{j\omega})$ becomes necessarily zero at $\omega = 0$ and $\omega = \pi$.

For both case i) and ii), m_1 must be odd, and we must have

$$h(l+kr) = h(l+(m_1-k)r), \qquad 0 \leqslant k \leqslant \frac{m_1-1}{2}$$

for case i) and

$$h(l+kr) = -h(l+(m_1-k)r), \qquad 0 \leqslant k \leqslant \frac{m_1-1}{2}$$

for case ii). For case iii), m_1 must be even, and we must have

TABLE I
Conditions Under Which the $(m_0/2)$-nd Decimated Filter Becomes Singular

Parity of m_1	Parity of m_0	Nature of $h(n)$	Zeros of $G_{\frac{m_0}{2}}(e^{j\omega})$	Zeros of $\Lambda_{\frac{m_0}{2}}(e^{j\omega})$
Odd	Even	Symmetric	$\omega = \pi$	$\omega = \frac{\pi}{r}, \frac{3\pi}{r}, \ldots, \frac{(2r-1)\pi}{r}$
Odd	Even	Antisymmetric	$\omega = 0$	$\omega = 0, \frac{2\pi}{r}, \ldots, \frac{(2r-2)\pi}{r}$
Even	Even	Antisymmetric	$\omega = 0, \pi$	$\omega = 0, \frac{\pi}{r}, \frac{2\pi}{r} \ldots, \frac{(2r-1)\pi}{r}$

$$h(l+kr) = -h(l+(m_1-k)r), \qquad 0 \leqslant k \leqslant \frac{m_1}{2}.$$

Case i) conditions hold for an arbitrary linear-phase low-pass filter with a symmetric impulse response iff

$$N-1-l = l + m_1 r$$

or

$$2l = m_0.$$

In other words, if m_1 is odd and m_0 even, $G_{m_0/2}(e^{j\omega})$ vanishes at $\omega = \pi$ or equivalently $\Lambda_{m_0/2}(e^{j\omega})$ vanishes at all odd multiples of π/r for an arbitrary linear-phase filter $H(z)$ with a symmetric impulse response. Similarly, it can be shown that if m_1 is odd and m_0 even, $G_{m_0/2}(e^{j\omega})$ vanishes at $\omega = 0$ and, hence, $\Lambda_{m_0/2}(e^{j\omega}) = 0$ at all even multiples of π/r for an arbitrary linear-phase filter with an antisymmetric impulse response. If both m_1 and m_0 are even, it can be shown that $G_{m_0/2}(e^{j\omega})$ vanishes at $\omega = 0$ and $\omega = \pi$ and, hence, $\Lambda_{m_0/2}(e^{j\omega}) = 0$ at all multiples of π/r for an arbitrary linear-phase filter with an antisymmetric impulse response. These results are summarized in Table I.

In the range $m_0 + 1 \leqslant l \leqslant r-1$, the frequency response of the lth decimated filter is given by

$$G_l(e^{j\omega}) = \sum_{k=0}^{m_1-1} h(l+kr) e^{-j\omega k}.$$

As in the previous range, this will vanish for any $\{h(n)\}$ at either $\omega = \pi$ or $\omega = 0$ or both corresponding to one of the three cases. For the first two cases, m_1 must be even, and we must have

$$h(l+kr) = h(l+(m_1-1-k)r), \qquad 0 \leqslant k \leqslant \frac{m_1}{2} - 1$$

for case i), and

$$h(l+kr) = -h(l+(m_1-1-k)r), \qquad 0 \leqslant k \leqslant \frac{m_1}{2} - 1$$

for case ii). For case iii), m_1 must be odd, and we must have

$$h(l+kr) = -h(l+(m_1-1-k)r), \qquad 0 \leqslant k \leqslant \frac{m_1-1}{2}.$$

TABLE II
CONDITIONS UNDER WHICH $((m_0 + r)/2)$-ND DECIMATED FILTER BECOMES SINGULAR

Parity of m_1	Parity of $(m_0 + r)$	Nature of $h(n)$	Zero of $G_{\frac{m_0+r}{2}}(e^{j\omega})$	Zero of $\Lambda_{\frac{m_0+r}{2}}(e^{j\omega})$
Even	Even	Symmetric	$\omega = \pi$	$\omega = \frac{\pi}{r}, \frac{3\pi}{r}, \cdots, (2r-1)\frac{\pi}{r}$
Even	Even	Antisymmetric	$\omega = 0$	$\omega = 0, \frac{2\pi}{r}, \cdots, (2r-2)\frac{\pi}{r}$
Odd	Even	Antisymmetric	$\omega = 0, \pi$	$\omega = 0, \frac{\pi}{r}, \cdots, \frac{(2r-1)\pi}{r}$

Case i) conditions hold for arbitrary linear-phase $H(z)$ with a symmetric impulse response iff

$$N-1-l = l + (m_1 - 1)r$$

or

$$2l = m_0 + r.$$

Thus, for m_1 even and $(m_0 + r)$ even, $G_{(m_0+r)/2}(e^{j\omega})$ vanishes at $\omega = \pi$ or equivalently $\Lambda_{(m_0+r)/2}(e^{j\omega})$ vanishes at all odd multiples of π/r for an arbitrary linear-phase filter $H(z)$ with a symmetric impulse response. Similarly, it can be shown that if m_1 is even and $(m_0 + r)$ even, $G_{(m_0+r)/2}(e^{j\omega})$ vanishes at $\omega = 0$ or equivalently $\Lambda_{(m_0+r)/2}(e^{j\omega}) = 0$ at all even multiples of π/r for an arbitrary linear-phase filter $H(z)$ with an antisymmetric impulse response. If m_1 is odd and $(m_0 + r)$ is even, it can be shown that $G_{(m_0+r)/2}(e^{j\omega})$ vanishes at $\omega = 0$ and $\omega = \pi$ or equivalently $\Lambda_{(m_0+r)/2}(e^{j\omega}) = 0$ at all multiples of π/r for an arbitrary linear-phase filter $H(z)$ with an antisymmetric impulse response. These results are summarized in Table II.

To summarize, a singularity situation arises for a linear-phase FIR filter with a symmetric impulse response only when i) m_0 is even and m_1 is odd, and ii) $(m_0 + r)$ is even and m_1 is even. The combinations of r, m_0, m_1, which result in the above conditions, are listed in Table III. Such conditions are avoided by restricting ourselves to the following choices of r, m_0, m_1:

i) odd r, odd m_0, odd m_1
ii) odd r, even m_0, even m_1
iii) even r, odd m_0, odd m_1
iv) even r, odd m_0, even m_1.

But for choices i) and ii), N is odd and for choices iii) and iv), N is even. Thus, singularity can be avoided simply by choosing N and r to be both odd or both even. For the case $r = 2$, this result is well-known [1].

For a linear-phase FIR filter with an antisymmetric impulse response, a singularity situation arises only when m_0 or $(m_0 + r)$ is even. The combinations of r, m_0, m_1, which result in such a condition, are listed in Table IV. Such a condition can therefore be avoided by restricting ourselves to even r and odd m_0, or equivalently, even r and even N. We thus see that linear-phase FIR filters with an antisymmetric impulse response have limited use. Furthermore, such filters have a zero at $\omega = 0$ and therefore cannot be used if we want a low-pass filter centered at the zero frequency. For this reason, we assume throughout the remainder of this paper that all prototype linear-phase FIR filters have a symmetric impulse response.

TABLE III
CONDITIONS UNDER WHICH SINGULARITY OCCURS, WHEN $H_0(z)$ HAS SYMMETRIC IMPULSE RESPONSE

r	m_0	m_1	N
Odd	Even	Odd	Even
Even	Even	Odd	Odd
Odd	Odd	Even	Even
Even	Even	Even	Odd

TABLE IV
CONDITIONS UNDER WHICH SINGULARITY OCCURS, WHEN $H_0(z)$ HAS ANTISYMMETRIC IMPULSE RESPONSE

r	m_0	m_1	N
Odd	Even	Odd	Even
Odd	Even	Even	Odd
Even	Even	Odd	Odd
Even	Even	Even	Odd
Odd	Odd	Odd	Odd
Odd	Odd	Even	Even

V. PROPERTIES OF FIR FILTER BANKS

In this section, we derive several useful properties of FIR filter banks discussed in the previous section.

Property 1

For FIR filter banks, the synthesis filters, like the analysis filters, can be derived from a basic filter $F(z)$, which is the same as $F_0(z)$. The synthesis filters are related to $F(z)$ by the following equation:

$$F_i(z) = W^{-i}F(zW^i). \tag{34}$$

Proof: Using (31)

$$F_i(z) = \frac{1}{r}\sum_{k=0}^{r-1} R_k(z^r)z^{-(r-1-k)}W^{ik}$$

$$= W^{i(r-1)}\cdot\frac{1}{r}\sum_{k=0}^{r-1} R_k(z^rW^{ir})z^{-(r-1-k)}W^{-i(r-1-k)}$$

$$= W^{-i}F_0(zW^i) = W^{-i}F(zW^i).$$

Corollary: When $n_0 = r-1+(N-r)\cdot r/2$, the synthesis filters required for exact reconstruction of the original signal, except for a delay of n_0, and whose transfer functions are given by (23) also satisfy this property.

Proof: Consider the filter $D(z)$, defined by (22). For $n_0 = r-1+r(N-r)/2$, $D(z)$ has the following property:

$$D(zW^i) = \frac{z^{-[(N-r)/2\cdot r+r-1]}W^{-i((N-r)/2\cdot r+r-1)}}{z^{-(r-1)}\cdot W^{-i(r-1)}\prod_{l=0}^{r-1}G_l(z^rW^{ir})} = D(z).$$

Thus, if the synthesis filters are given by (23) rather than (31), then (34) continues to be true.

Property 2

The length of the basic synthesis filter $F(z)$ for FIR filter banks is

$$N_f = (N-r+2)r - N. \tag{35}$$

Proof:

$$F(z) = F_0(z) = \frac{1}{r}\sum_{i=0}^{r-1} R_i(z^r)z^{-(r-1-i)}$$

where

$$R_i(z^r) = \prod_{\substack{k=0\\k\neq i}}^{r-1} G_k(z^r).$$

The degree of $R_i(z^r)$ using (27) and (28) is simply obtained as follows:

degree of $R_i(z^r)$

$$= \text{degree of } \left\{\prod_{k=0}^{r-1} G_k(z^r)\right\} - \text{degree of } \{G_i(z^r)\}$$

$$= N_T - (r-1) - \text{degree of } \{G_i(z^r)\}$$

$$= \begin{cases} (N-r)r - m_1 r, & 0 \leqslant i \leqslant m_0 \\ (N-r)r-(m_1-1)r, & m_0+1 \leqslant i \leqslant r-1. \end{cases}$$

Clearly, degree of $F(z)$

$$= \max_{0\leqslant i\leqslant r-1}\{\text{degree of } z^{-(r-1-i)}R_i(z^r)\}$$

$$= \max\Big\{\max_{0\leqslant i\leqslant m_0}\{\text{degree of } z^{-(r-1-i)}\cdot R_i(z^r)\},$$

$$\max_{m_0+1\leqslant i\leqslant r-1}\{\text{degree of } z^{-(r-1-i)}\cdot R_i(z^r)\}\Big\}$$

$$= \max\{r(N-r) - m_1 r + r - 1, (N-r)r - m_1 r + r - 1 + (r - m_0 - 1)\}$$

$$= r(N-r) - m_1 r + r - 1 + r - m_0 - 1$$

$$= (N-r+2)r - N - 1.$$

Thus, the length of the filter $F(z)$ is $N_f = (N-r+2)r - N$.

Comments on the length of F(z): Note that the length of the FIR filter $F(z)$ is much higher than that of the analysis prototype filter $H_0(z)$ for large r. This is unlike the parallel QMF structures proposed elsewhere, such as in [37] and [39]. The reason for this increased length is that aliasing is guaranteed to be perfectly canceled, regardless of the exact nature and quality of $H_0(z)$.

For composite r (i.e., r not a prime), one can obtain a compromise between parallel and tree structures by decomposing r, say, as $r = r_1r_2$, and decomposing the analysis stage (and synthesis stage) into two subsections, in a manner analogous to the tree structures in [10]. Such decomposition may lead to reduced overall length of the synthesis filters. Notice that aliasing is guaranteed to be perfectly canceled even with this decomposition.

Property 3

If the basic analysis filter $H(z)$ is a linear-phase FIR filter, the decimated filters obey the mirror image property

$$G_l(z) = G_{m_0-l}(z^{-1})z^{-m_1}, \qquad 0 \leqslant l \leqslant m_0 \tag{36}$$

$$G_l(z) = G_{r+m_0-l}(z^{-1})z^{-(m_1-1)}, \qquad m_0+1 \leqslant l \leqslant r-1. \tag{37}$$

Proof: Because of the linear-phase property of $H(z)$

$$h(l+kr) = h(N-1-l-kr) = h((m_1-k)r+m_0-l).$$

Thus, (26) gives

$$G_l(z) = \sum_{k=0}^{m_1} h((m_1-k)r+m_0-l)z^{-k}$$

$$= z^{-m_1}\sum_{k=0}^{m_1} h(kr+m_0-l)z^k$$

$$= z^{-m_1}G_{m_0-l}(z^{-1})$$

which establishes (36). Equation (37) follows in a similar manner.

Property 4

If the basic analysis filter $H(z)$ is a linear-phase FIR filter, then the overall transfer function $T(z)$ has linear phase.

Proof: In view of (36) and (37), $T(z)$ given by (27) has the property that if z_k is a zero, then so is z_k^{-1}. Hence, $T(z)$ has linear phase.

Property 5

If the basic analysis filter $H(z)$ is a linear-phase FIR filter, the basic synthesis filter $F(z)$ also has linear phase.

Proof:

$$F(z)=\frac{1}{r}\sum_{i=0}^{r-1}z^{-(r-1-i)}R_i(z^r) \qquad \text{(using (31))}$$

$$F(z^{-1})=\frac{1}{r}\sum_{i=0}^{r-1}z^{+(r-1-i)}\prod_{\substack{k=0\\k\neq i}}^{r-1}G_k(z^{-r}) \qquad \text{(using (21))}$$

$$=\frac{1}{r}\sum_{i=0}^{m_0}z^{r-1-i}\prod_{\substack{k=0\\k\neq i}}^{m_0}\left\{G_{m_0-k}(z^r)z^{m_1 r}\right\}\cdot\prod_{k=m_0+1}^{r-1}\left\{G_{m_0+r-k}(z^r)\cdot z^{(m_1-1)r}\right\}$$

$$+\frac{1}{r}\sum_{i=m_0+1}^{r-1}z^{r-1-i}\prod_{k=0}^{m_0}\left\{G_{m_0-k}(z^r)z^{m_1 r}\right\}\cdot\prod_{\substack{k=m_0+1\\k\neq i}}^{r-1}\left\{G_{m_0+r-k}(z^r)\cdot z^{(m_1-1)r}\right\}$$

$$\text{(using (36) and (37))}$$

$$=\frac{1}{r}\sum_{i=0}^{m_0}z^{(r-1-i)}z^{(N-r)\cdot r}z^{-m_1 r}\prod_{\substack{k=0\\k\neq m_0-i}}^{r-1}\{G_k(z^r)\}$$

$$+\frac{1}{r}\sum_{i=m_0+1}^{r-1}z^{(r-1-i)}z^{(N-r)r}z^{-(m_1-1)r}\cdot\prod_{\substack{k=0\\k\neq m_0+r-i}}^{r-1}\{G_k(z^r)\}$$

$$=\frac{1}{r}\sum_{i=0}^{m_0}z^{r-1-(m_0-i)}z^{(N-r)r}z^{-m_1 r}\prod_{\substack{k=0\\k\neq i}}^{r-1}\{G_k(z^r)\}$$

$$+\frac{1}{r}\sum_{i=m_0+1}^{r-1}z^{r-1-(m_0+r-i)}z^{(N-r)r}z^{-(m_1-1)r}\cdot\prod_{\substack{k=0\\k\neq i}}^{r-1}\{G_k(z^r)\}$$

$$=\frac{1}{r}\sum_{i=0}^{r-1}z^{r-1-(m_0-i)}z^{(N-r)r}z^{-m_1 r}\prod_{\substack{k=0\\k\neq i}}^{r-1}\{G_k(z^r)\}$$

$$=z^{(N-r+2)r-N-1}\cdot\frac{1}{r}\sum_{i=0}^{r-1}z^{-(r-1-i)}\prod_{\substack{k=0\\k\neq i}}^{r-1}\{G_k(z^r)\}$$

$$=z^{N_f-1}\cdot F(z).$$

Thus, if z_k is a zero of $F(z)$, then so is z_k^{-1}. Hence, $F(z)$ has linear phase.

Property 6

If the impulse response $h(n)$ of the basic analysis filter $H(z)$ is real, then the impulse response $f(n)$ of the basic synthesis filter is also real.

Proof: This property is easily proved by simply observing that coefficients of the powers of z^{-1} in the decimated filters are real and so the coefficients of z^{-1} in the basic synthesis filter $F(z)$, given by (31), are also real.

VI. Design of Linear-Phase FIR Filter Banks

In this section, we describe a procedure for designing linear-phase FIR filter banks. Our procedure uses the same distortion measure as in Johnston's technique [8] as well as in a technique proposed by Jain and Crochiere [9]. This distortion measure is a weighted sum of the ripple energy E_r of the overall transfer function $T(z)$ and the stopband energy E_s of the basic analysis filter $H(z)$. Thus

$$E=E_r+\alpha E_s.$$

The analysis filter $H(z)$ is designed by minimizing E subject to the constraint that the total energy of $H(z)$ is unity, i.e., $\sum_{n=0}^{N-1}h^2(n)=1$. The same normalization constraint has been employed by Jain and Crochiere [9].

The design procedure is basically a gradient algorithm. Before we describe the algorithm, we obtain expressions for the normalization constraint, ripple energy E_r, and stopband energy E_s. The gradient computation is discussed in Section VI-D, and finally the gradient algorithm is described in Section VI-E.

A. Normalization Constraint

The normalization constraint is

$$\sum_{n=0}^{N-1}h^2(n)=1.$$

For linear-phase FIR filters, $h(n)=h(N-1-n)$. So let us define for even N

$$d(n)=\sqrt{2}\,h(n), \qquad 0\leqslant n\leqslant\frac{N}{2}-1 \tag{38a}$$

$$N_d=\frac{N}{2}-1 \tag{38b}$$

and for odd N

$$d(n)=\begin{cases}\sqrt{2}\,h(n), & 0\leqslant n\leqslant\dfrac{N-1}{2}-1\\ h(n), & n=\dfrac{N-1}{2}\end{cases} \tag{39a}$$

$$N_d=\frac{N-1}{2}. \tag{39b}$$

Then the normalization constraint is also given by

$$\sum_{n=0}^{N_d}d^2(n)=1$$

or

$$\boldsymbol{d}^T\boldsymbol{d}=1 \tag{40}$$

where

$$\boldsymbol{d}^T=[d(0)\quad d(1)\quad\cdots\quad d(N_d)]. \tag{40a}$$

B. Ripple Energy

The overall transfer function of the filter bank is $T(z)$, which is given by (see (32))

$$\begin{aligned} T(z) &= \frac{1}{r}\sum_{k=0}^{r-1} H_k(z)F_k(z) \\ &= \frac{1}{r}\sum_{k=0}^{r-1} H(zW^k)F(zW^k)W^{-k} \\ &= \frac{1}{r}\sum_{n=0}^{N+N_f-2}\left\{\sum_{i=0}^{N-1} h(i)f(n-i)\sum_{k=0}^{r-1} W^{-(n+1)k}\right\}z^{-n} \\ &= \sum_{k=0}^{N-r} z^{-(kr+r-1)}\left\{\sum_{i=0}^{N-1} h(i)f(kr+r-1-i)\right\}. \end{aligned}$$

Thus, the impulse response $t(n)$ of the overall transfer function $T(z)$ is given by

$$t(n) = \begin{cases} \sum_{i=0}^{N-1} h(i)f(kr+r-1-i), & \text{if } n = kr+r-1, \; 0 \leqslant k \leqslant N-r \\ 0, & \text{otherwise} \end{cases} \tag{41}$$

Note that $f(n)$ is assumed to be zero for n outside the range $[0, N_f-1]$ in the above expression. It follows from (41) that the overall impulse response $t(n)$ can be obtained by first convolving the prototype analysis filter impulse response $h(n)$ with the prototype synthesis filter impulse response $f(n)$, decimating the convolved sequence by a factor of r (starting from the $(r-1)$th sample), and finally interpolating the decimated sequence by a factor of r.

Ideally, we want $T(z)$ to obey the design condition in (29), (30), or (33) which are all equivalent to

$$t(n) = \begin{cases} 1, & \text{if } n = \frac{N-r}{2}\cdot r + r - 1 \\ 0, & \text{otherwise} \end{cases}.$$

So, the ripple energy E_r of $T(z)$ is simply

$$E_r = \sum_{\substack{k=0 \\ k \neq (N-r)/2}}^{N-r} t^2(kr+r-1). \tag{42}$$

Substituting from (41), we get

$$E_r = \sum_{i=0}^{N-1}\sum_{j=0}^{N-1} h(i)h(j) \sum_{\substack{k=0 \\ k \neq (N-r)/2}}^{N-r} f(kr+r-1-i) \cdot f(kr+r-1-j).$$

For linear-phase FIR filters, the ripple energy can also be expressed in terms of $d(i)$'s defined by (38a) and (39a). Thus

$$E_r = \sum_{i=0}^{N_d}\sum_{j=0}^{N_d} d(i)d(j)\beta_{ij} \sum_{\substack{k=0 \\ k \neq (N-r)/2}}^{N-r} \cdot\left[f(kr+r-1-i) + f\left(kr+r-1-\overline{N-1-i}\right)\right] \cdot\left[f(kr+r-1-j) + f(kr+r-1-(N-1-j))\right] \tag{43}$$

where

$$\beta_{ij} = \frac{1}{2}, \qquad 0 \leqslant i \leqslant N_d, 0 \leqslant j \leqslant N_d \tag{43a}$$

for even N and

$$\beta_{ij} = \begin{cases} \frac{1}{2} & 0 \leqslant i \leqslant N_d - 1, \quad 0 \leqslant j \leqslant N_d - 1 \\ \frac{1}{2\sqrt{2}} & i = N_d, \quad 0 \leqslant j \leqslant N_d - 1 \\ \frac{1}{2\sqrt{2}} & 0 \leqslant i \leqslant N_d - 1, \quad j = N_d \\ \frac{1}{4} & i = N_d, \quad j = N_d \end{cases} \tag{43b}$$

for odd N. In matrix form, we can express (43) as

$$E_r = \boldsymbol{d}^T \boldsymbol{A} \boldsymbol{d} \tag{44}$$

where

$$\boldsymbol{A} = [a_{ij}]_{(N_d+1)\times(N_d+1)} \tag{44a}$$

$$a_{ij} = \beta_{ij} \sum_{\substack{k=0 \\ k \neq (N-r)/2}}^{N-r} [f(kr+r-1-i) + f(kr+r-1-(N-1-i))] \cdot [f(kr+r-1-j) + f(kr+r-1-(N-1-j))]. \tag{44b}$$

Note that the matrix $\boldsymbol{A}$ is symmetric and nonnegative definite.

C. Stopband Energy

The stopband energy of $H(z)$ is given by

$$E_s = \frac{1}{\pi}\int_{\omega_s}^{\pi} |H(e^{j\omega})|^2 \, d\omega \tag{45}$$

where ω_s is the stopband edge and lies between π/r and $2\pi/r$. Equation (45) can be rewritten as

$$E_s = \sum_{i=0}^{N-1}\sum_{k=0}^{N-1} h(i)h(k)\frac{1}{\pi}\int_{\omega_s}^{\pi} e^{j\omega(i-k)}\, d\omega.$$

For linear-phase FIR filters, E_s can be expressed in terms of the vector $\boldsymbol{d}$. With some manipulation, we can show that

$$E_s = \boldsymbol{d}^T \boldsymbol{V} \boldsymbol{d} \tag{46}$$

where

$$\boldsymbol{V} = [v_{ij}]_{(N_d+1)\times(N_d+1)} \tag{46a}$$

and

$$v_{ij} = \begin{cases} 2\beta_{ij}\left[-\dfrac{\sin\omega_s(i-j)}{\pi(i-j)} - \dfrac{\sin\omega_s(N-1-i-j)}{\pi(N-1-i-j)}\right] & \text{if } i \neq j \text{ and } i+j \neq N-1 \\ 2\beta_{ij}\left[\dfrac{\pi-\omega_s}{\pi} - \dfrac{\sin\omega_s(N-1-i-j)}{\pi(N-1-i-j)}\right] & \text{if } i = j \text{ and } i+j \neq N-1 \\ 2\beta_{ij}\left[\dfrac{-\sin\omega_s(i-j)}{\pi(i-j)} + \dfrac{\pi-\omega_s}{\pi}\right] & \text{if } i \neq j \text{ and } i+j = N-1 \\ 2\beta_{ij}\dfrac{2(\pi-\omega_s)}{\pi} & \text{if } i = j \text{ and } i+j = N-1 \end{cases} \quad (46b)$$

The matrix V, like A, is symmetric and nonnegative definite. In addition, V is nonsingular and, hence, positive definite unless $\omega_s = \pi$.

D. Gradient Computation

The error measure E is given by

$$E = E_r + \alpha E_s = \boldsymbol{d}^T A \boldsymbol{d} + \alpha \boldsymbol{d}^t V \boldsymbol{d}.$$

We would like to find the gradient of E with respect to the vector $\boldsymbol{d}$. Note that the elements of the matrix V are constants which do not depend on the elements of vector $\boldsymbol{d}$. But the elements of matrix A do depend on the elements of $\boldsymbol{d}$.

The gradient $\boldsymbol{g}$ is given by

$$\boldsymbol{g} = \left[\frac{\partial E}{\partial d(0)} \quad \frac{\partial E}{\partial d(1)} \cdots \frac{\partial E}{\partial d(N_d)}\right]^T \quad (47)$$

and can be computed using

$$\boldsymbol{g} = 2A\boldsymbol{d} + 2\alpha V\boldsymbol{d} + \boldsymbol{t} \quad (48)$$

where

$$\boldsymbol{t} = \left[\boldsymbol{d}^T A'_0 \boldsymbol{d} \quad \boldsymbol{d}^T A'_1 \boldsymbol{d} \cdots \boldsymbol{d}^T A'_{N_d} \boldsymbol{d}\right]^T; \quad (48a)$$

$$A'_l = \left[\frac{\partial a_{ij}}{\partial d(l)}\right]_{(N_d+1)\times(N_d+1)}. \quad (48b)$$

The partial derivative of the elements a_{ij} of the matrix A with respect to $d(l)$ is given by

We now describe a procedure for computing the derivatives of $f(n)$ with respect to $d(l)$. Let

$$l = l_0 + l_1 r, \qquad 0 \leqslant l_0 \leqslant r-1.$$

Then, for a general FIR filter (not necessarily linear phase)

$$\frac{\partial F(z)}{\partial h(l)} = \frac{\partial F_0(z)}{\partial h(l)} = \frac{\partial}{\partial h(l)}\left[\frac{1}{r}\sum_{i=0}^{r-1} z^{-(r-1-i)} \prod_{\substack{k=0\\k\neq i}}^{r-1} G_k(z^r)\right]$$

(see eq. (31))

$$= \frac{1}{r}\sum_{\substack{i=0\\i\neq l_0}}^{r-1} z^{-(r-1-i)} z^{-l_1 r} \prod_{\substack{k=0\\k\neq i\\k\neq l_0}}^{r-1} G_k(z^r), \qquad 0 \leqslant l \leqslant N-1. \quad (50)$$

Now, $N-1-l = m_0 - l_0 + (m_1 - l_1)r$. So, for $m_0 - l_0 \geqslant 0$, we have

$$\frac{\partial F(z)}{\partial h(N-1-l)} = \frac{1}{r}\sum_{\substack{i=0\\i\neq m_0-l_0}}^{r-1} z^{-(r-1-i)} z^{-(m_1-l_1)r} \cdot \prod_{\substack{k=0\\k\neq i\\k\neq m_0-l_0}}^{r-1} G_k(z^r). \quad (51a)$$

For $m_0 - l_0 < 0$, we have

$$\frac{\partial F(z)}{\partial h(N-1-l)} = \frac{1}{r}\sum_{\substack{i=0\\i\neq r+m_0-l_0}}^{r-1} z^{-(r-1-i)} z^{-(m_1-l_1-1)r} \cdot \prod_{\substack{k=0\\k\neq i\\k\neq m_0-l_0+r}}^{r-1} G_k(z^r). \quad (51b)$$

$$\frac{\partial a_{ij}}{\partial d(l)} = \beta_{ij} \sum_{\substack{k=0\\k\neq(N-r)/2}}^{N-r} \left\{\left[\frac{\partial f(kr+r-1-i)}{\partial d(l)} + \frac{\partial f(kr+r-1-(N-1-i))}{\partial d(l)}\right] \cdot \left[f(kr+r-1-j) + f(kr+r-1-(N-1-j))\right] + \left[f(kr+r-1-i) + f(kr+r-1-(N-1-i))\right] \cdot \left[\frac{\partial f(kr+r-1-j)}{\partial d(l)} + \frac{\partial f(kr+r-1-(N-1-j))}{\partial d(l)}\right]\right\},$$

$$0 \leqslant i \leqslant N_d, \quad 0 \leqslant j \leqslant N_d, \quad 0 \leqslant l \leqslant N_d. \quad (49)$$

For linear-phase filters, we have the additional relation $h(l)=h(N-1-l)$. Consequently

$$\frac{\partial F(z)}{\partial d(l)}=\begin{cases}\frac{1}{\sqrt{2}}\left[\frac{\partial F(z)}{\partial h(l)}+\frac{\partial F(z)}{\partial h(N-1-l)}\right] & 0\leqslant l\leqslant N_d-1(\text{odd } N)\\ & 0\leqslant l\leqslant N_d(\text{even } N)\\ \frac{\partial F(z)}{\partial h(l)} & l=N_d(\text{odd } N)\end{cases}\tag{52}$$

The partial derivatives $\partial f(n)/\partial d(l)$ are simply obtained as coefficients of z^{-n} in $\partial F(z)/\partial d(l)$.

E. Gradient Algorithm

For a given d, the error measure is

$$E(d)=d^TAd+\alpha d^TVd$$

and also the vector d is of unit length, i.e.,

$$d^Td=\|d\|^2=1.\tag{53}$$

Now if d is changed to $d+\Delta d$ such that $d+\Delta d$ is also of unit length, then

$$E(d+\Delta d)-E(d)\approx\Delta d^Tg$$

where Δd is assumed to be sufficiently "small" so that we can ignore second-order terms. Let us choose

$$\Delta d=-\Gamma g+\nu g_\perp\tag{54}$$

where $g_\perp$ is a vector orthogonal to g, and Γ is the stepsize, a positive number which controls the change in error and ν is a number chosen so as to ensure that $d+\Delta d$ has unit length. A vector $g_\perp$ such that

$$g_\perp^Tg=0\tag{55}$$

is generated using

$$g_\perp=(g^Tg)d-\mu g\tag{56}$$

where

$$\mu=g^Td.\tag{57}$$

One can also verify easily the following relations using (53), (55), (56), and (57):

$$g_\perp^Td=g^Tg-\mu^2\tag{58a}$$

$$g_\perp^Tg_\perp=g^Tg\left[g^Tg-\mu^2\right].\tag{58b}$$

From (58b), it follows that

$$g^Tg\geqslant\mu^2.\tag{58c}$$

We now evaluate the constant ν that is needed to ensure that $d+\Delta d$ is of unit length, i.e.,

$$(d+\Delta d)^T(d+\Delta d)=1.$$

Substituting for Δd from (54) in the above expression and using (53), we get

$$\nu^2g_\perp^Tg_\perp+2\nu g_\perp^Td+\Gamma^2g^Tg-2\Gamma g^Td=0.$$

Using (57), (58a), and (58b), we get

$$g^Tg\left[g^Tg-\mu^2\right]\nu^2+2\nu\left[g^Tg-\mu^2\right]+\Gamma^2g^Tg-2\Gamma\mu=0.\tag{59}$$

Solving for ν, we obtain

$$\nu=\frac{1}{g^Tg}\cdot\left\{-1\pm\sqrt{\frac{\left(\mu+\sqrt{g^Tg}-\Gamma g^Tg\right)\left(-\mu+\sqrt{g^Tg}+\Gamma g^Tg\right)}{(g^Tg-\mu^2)}}\right\}.$$

A smaller ν and, hence, a smaller $\|\Delta d\|$ is obtained with the (+)ve sign. So this is preferred. Thus

$$\nu=\frac{1}{g^Tg}\cdot\left\{-1+\sqrt{\frac{\left(\mu+\sqrt{g^Tg}-\Gamma g^Tg\right)\left(-\mu+\sqrt{g^Tg}+\Gamma g^Tg\right)}{(g^Tg-\mu^2)}}\right\}.\tag{60}$$

For ν to be real, we must have

$$\frac{\mu-\sqrt{g^Tg}}{g^Tg}\leqslant\Gamma\leqslant\frac{\mu+\sqrt{g^Tg}}{g^Tg}.$$

But Γ is a positive number and $\mu-\sqrt{g^Tg}\leqslant 0$ from (58c). So the range of permissible values for Γ is $[0,\Gamma_{max}]$, where Γ_{max} is given by

$$\Gamma_{max}=\left(\mu+\sqrt{g^Tg}\right)/g^Tg.\tag{61}$$

To summarize, a decrease in the error E can be obtained by changing every unit length vector d to another unit length vector $d+\Delta d$, where Δd is as in (54). In this equation, g is the gradient of the error E evaluated at d, $g_\perp$ is the vector orthogonal to the gradient g and is given by (56), Γ is the stepsize that must lie in the range $[0,\Gamma_{max}]$, and ν is a constant that ensures that $d+\Delta d$ is of unit length and is given by (60). Thus, by modifying the vector d every iteration, the error E can be reduced to within acceptable limits.

The exact choice of Γ controls the change in error at every iteration and, hence, the rate of convergence. For a large change in error at every iteration, a large value for Γ would be necessary. But a small Γ is necessary so that $\|\Delta d\|$ is small enough for the second-order effects to be negligible. At every iteration, it is necessary to verify that the chosen value for the stepsize, say Γ_0, is less than Γ_{max}. If it is not so, then the stepsize is reset to Γ_{max} for that iteration. In other words, the stepsize is chosen accord-

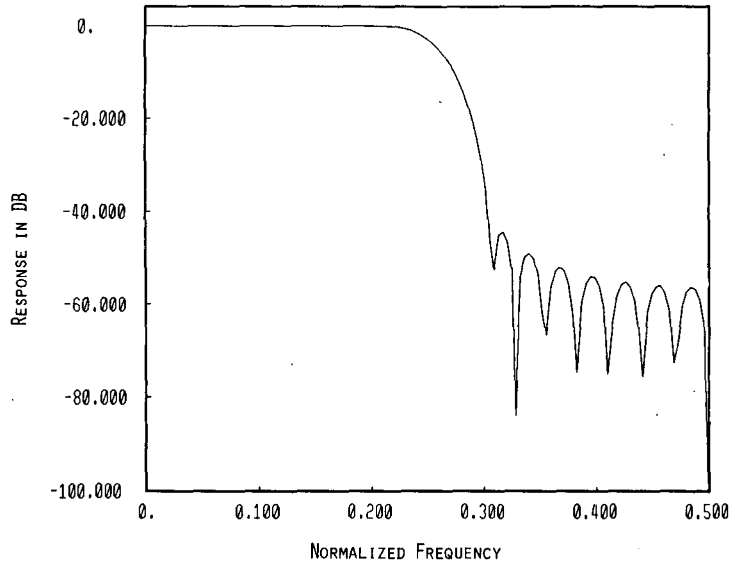

Fig. 4. Magnitude plot of prototype analysis filter $H(e^{j\omega})$ in Example 1.

ing to

$$\Gamma = \min(\Gamma_0, \Gamma_{\max}).$$

The initial value of $h(n)$ used in the gradient algorithm is the following:

$$h(n) = \begin{cases} 0, & 0 \leqslant n \leqslant \dfrac{N-r}{2} - 1 \\ \dfrac{1}{\sqrt{r}}, & \dfrac{N-r}{2} \leqslant n \leqslant \dfrac{N+r}{2} - 1. \\ 0, & \dfrac{N+r}{2} \leqslant n \leqslant N-1 \end{cases}$$

This corresponds to the case when the ripple energy is exactly zero. This is easily verified by noting that the corresponding decimated filters are given by

$$G_l(z) = \begin{cases} \dfrac{1}{\sqrt{r}} z^{-m_1/2}, & 0 \leqslant l \leqslant \dfrac{m_0 + r - 1}{2} \\ \dfrac{1}{\sqrt{r}} z^{-(m_1/2 - 1)}, & \dfrac{m_0 + r + 1}{2} \leqslant l \leqslant r - 1 \end{cases}$$

where m_1 is even and

$$G_l(z) = \begin{cases} \dfrac{1}{\sqrt{r}} z^{-(m_1+1)/2}, & 0 \leqslant l \leqslant \dfrac{m_0 - 1}{2} \\ \dfrac{1}{\sqrt{r}} z^{-(m_1-1)/2}, & \dfrac{m_0 + 1}{2} \leqslant l \leqslant r - 1 \end{cases}$$

when m_1 is odd. Thus, the overall transfer function $T(z)$, which is given by (27), i.e.,

$$T(z) = z^{-(r-1)} \prod_{l=0}^{r-1} G_l(z^r)$$

simply reduces to

$$T(z) = z^{-(r-1)} z^{-(N-r)r/2} \left(\frac{1}{r}\right)^{r/2}.$$

Hence, the ripple energy, defined in (42), is identically zero. Note that for this initial value of $h(n)$, the FIR filterbank satisfies (29) and (30) exactly.

VII. Examples

A Fortran program has been written which implements the design procedure described in the previous section. In this section, we present two examples:

i) a two-band filter bank with filter length being 32, i.e., $r = 2$, $N = 32$,

ii) a three-band filter bank with filter length being 49, i.e., $r = 3$, $N = 49$.

In both the examples, the stepsize Γ_0 was chosen to be 0.6 and the value of α was chosen to be 1.0. The value of ω_s in example i) was chosen to be $1.2\pi/r = 0.6\pi$ and $1.25\pi/r = 0.4167\pi$ in example ii). For both examples, magnitude plots of the prototype analysis low-pass filter transfer function $H(e^{j\omega})$ and the overall transfer function $T(e^{j\omega})$ are shown in Figs. 4–7. The filter coefficients $h(n)$ are also listed. The following two significant quantities are computed: i) the attenuation AL (in decibels) at the first sidelobe in the stopband of $|H(e^{j\omega})|$, and ii) the ripple ϵ (in decibels) which is defined as

$$\epsilon = \frac{1}{2}\left[\max_{\omega} |20 \log_{10} |T(e^{j\omega})|| - \min_{\omega} |20 \log_{10} |T(e^{j\omega})|| \right].$$

The values of AL and ϵ are provided in both the examples.

Example 1

For $r = 2$, $N = 32$, $\alpha = 1.0$, $\Gamma_0 = 0.6$, and $\omega_s = 0.6\pi$, the following coefficients were obtained at the end of 65 iterations (final values of $E_r = 0.1227320D-06$, $E_s = 0.6595251D-05$, and $E = 0.6717983D-05$):

$$h(0) = h(31) = 1.5811828831575D-03$$
$$h(1) = h(30) = -2.8662438710491D-03$$
$$h(2) = h(29) = -2.3423183894046D-03$$

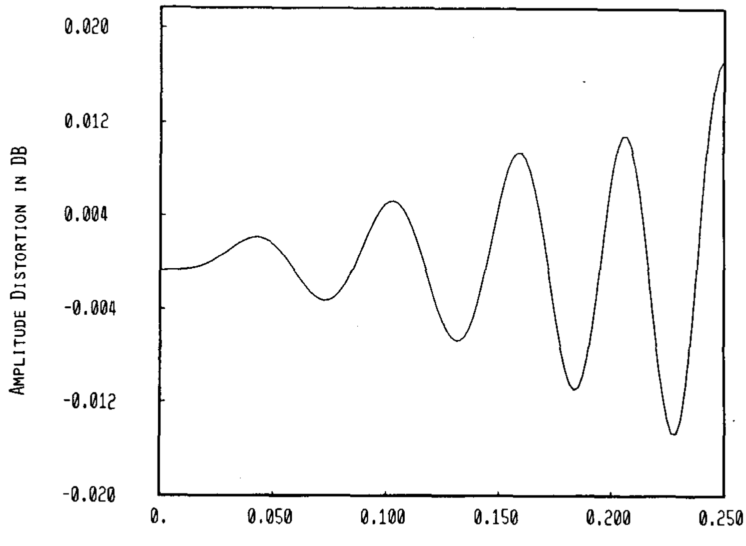

Fig. 5. Magnitude plot of overall transfer function $T(e^{j\omega})$ in Example 1.

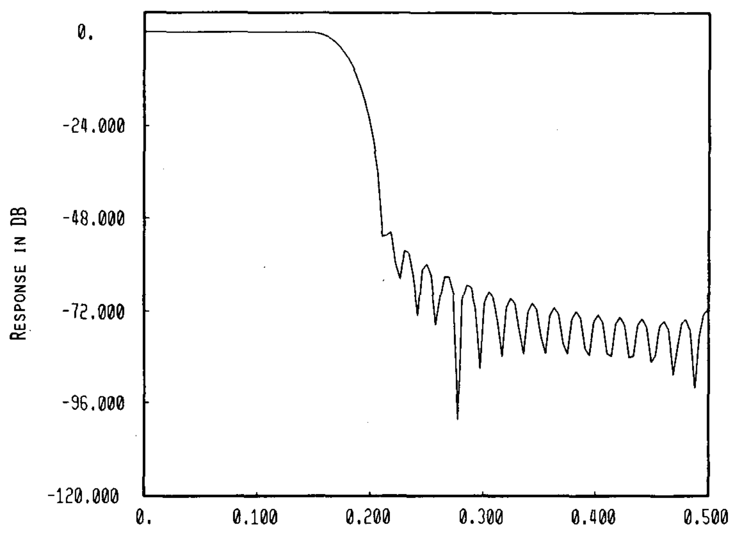

Fig. 6. Magnitude plot of prototype analysis filter $H(e^{j\omega})$ in Example 2.

$$h(3)=h(28)=7.7145269593883D-03$$
$$h(4)=h(27)=2.0430829579269D-03$$
$$h(5)=h(26)=-1.5749126593385D-02$$
$$h(6)=h(25)=6.0951717378978D-04$$
$$h(7)=h(24)=2.7997964556474D-02$$
$$h(8)=h(23)=-7.6685900872732D-03$$
$$h(9)=h(22)=-4.6334719445657D-02$$
$$h(10)=h(21)=2.3106797125640D-02$$
$$h(11)=h(20)=7.6067797614420D-02$$
$$h(12)=h(19)=-5.8423342570091D-02$$
$$h(13)=h(18)=-0.14099868001000$$
$$h(14)=h(17)=0.18421046025819$$
$$h(15)=h(16)=0.65812951736014.$$

The corresponding magnitude plots of the filter transfer function $H(e^{j\omega})$ and the overall transfer function $T(e^{j\omega})$ are shown in Figs. 4 and 5, respectively. The significant quantities AL and ϵ are 44.40 dB and 0.01596 dB, respectively. A plot of the synthesis prototype response $|F_0(e^{j\omega})|$ reveals that it has essentially the same low-pass nature of $H_0(z)$. Even though this can be understood based on an intuitive argument, no theoretical proof is attempted in this paper.

In the Jain-Crochiere design of a 32-tap QMF pair, the corresponding values for AL and ϵ are 44 dB and 0.015 dB. In [8], Johnston has listed two designs for a 32-tap QMF pair. The best set of values for AL and ϵ are 51 dB and 0.009 dB (this corresponds to the design with $\alpha=2.0$ and normalized transition bandwidth of 0.0625). Thus, the design for $r=2$ is comparable to both the Jain-Crochiere as well as Johnston's design of the QMF pair.

Example 2

For $r=3$, $N=49$, $\alpha=1.0$, $\Gamma_0=0.6$, and $\omega_s=0.4617\pi$, the following coefficients were obtained at the end of 350

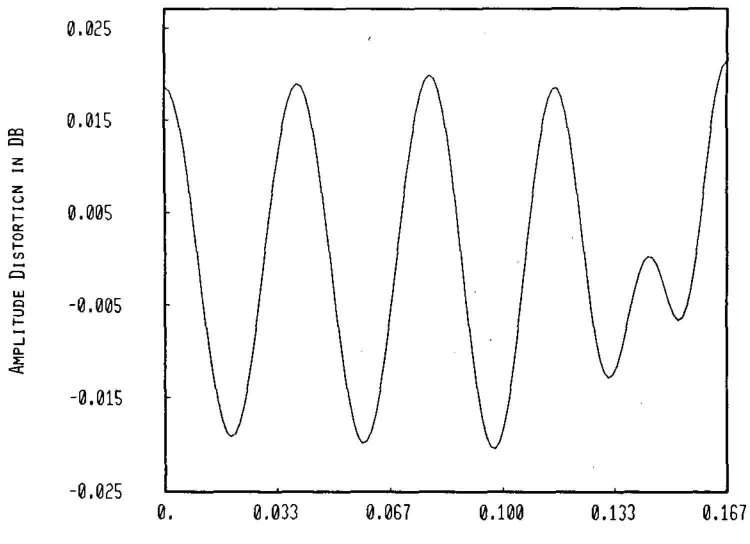

Fig. 7. Magnitude plot of overall transfer function $T(e^{j\omega})$ in Example 2.

iterations (final values of $E_r = 0.7415754D-07$, $E_S = 0.1145003D-05$, and $E = 0.1219241D-05$):

$$h(0) = h(48) = 0.9820884067901D-04$$
$$h(1) = h(47) = 1.3766090647386D-03$$
$$h(2) = h(46) = -4.7910880128979D-04$$
$$h(3) = h(45) = -3.1677291312554D-03$$
$$h(4) = h(44) = -2.4998799910234D-03$$
$$h(5) = h(43) = 2.8320063716334D-03$$
$$h(6) = h(42) = 7.1666739484746D-03$$
$$h(7) = h(41) = 2.9142318531781D-03$$
$$h(8) = h(40) = -8.2263531068544D-03$$
$$h(9) = h(39) = -1.3017340160472D-02$$
$$h(10) = h(38) = -1.0208555873822D-03$$
$$h(11) = h(37) = 1.8269886783613D-02$$
$$h(12) = h(36) = 2.0281750227610D-02$$
$$h(13) = h(35) = -5.7424052549831D-03$$
$$h(14) = h(34) = -3.5221072475563D-02$$
$$h(15) = h(33) = -2.8014927285150D-02$$
$$h(16) = h(32) = 2.2044792953158D-02$$
$$h(17) = h(31) = 6.4050915116369D-02$$
$$h(18) = h(30) = 3.4936657726858D-02$$
$$h(19) = h(29) = -6.0910885840952D-02$$
$$h(20) = h(28) = -0.12484310053556$$
$$h(21) = h(27) = -3.9743735579837D-02$$
$$h(22) = h(26) = 0.20915858960999$$
$$h(23) = h(25) = 0.48913886425113$$
$$h(24) = 0.61185356418355.$$

The corresponding magnitude plots of the filter transfer function $H(e^{j\omega})$ and the overall transfer function $T(e^{j\omega})$ are shown in Figs. 6 and 7, respectively. The significant quantities AL and ϵ are 51.53 dB and 0.02091 dB, respectively. A plot of the synthesis prototype response $|F_0(e^{j\omega})|$ reveals that it has essentially the same low-pass nature of $H_0(z)$. Even though this can be understood based on an intuitive argument, no theoretical proof is attempted in this paper.

VIII. General r-Band IIR QMF Banks

Based on the derivations of Sections II–V, it is a simple task to extend the solution to the QMF problem for the IIR case. In applications where phase distortion (but not amplitude distortion) can be tolerated, it is more appropriate to use IIR rather than FIR filters, as shown in this section. As pointed out in Section III, Bellanger *et al.* [3] have shown how an arbitrary digital filter transfer function $H(z)$ can be written in a form suitable for polyphase implementation as in (14b), i.e.,

$$H(z) = \sum_{k=0}^{r-1} z^{-k} G_k(z^r) \tag{14b}$$

where $G_k(z)$ is the transfer function of the kth decimated filter. Bellanger *et al.* have shown in [3] that when $H(z)$ is an IIR filter, the $G_k(z)$'s all have the same denominator and furthermore that the $G_k(z)$'s closely resemble allpass functions when $H(z)$ is a low-pass prototype with a passband width of about π/r. In this section, we employ the same polyphase structure along with an IDFT for the "analysis section" of the QMF bank. Without assuming the $G_k(z)$'s to be allpass or even approximately allpass, we show how to obtain a "synthesis bank" which can be employed along with the analysis bank so that the reconstructed signal $\hat{X}(z)$ is given by

$$\hat{X}(z) = A(z)X(z) \tag{62}$$

where $A(z)$ is a stable allpass function. The above equation essentially means that the aliasing error terms have been completely cancelled and the reconstructed signal only suffers from a phase distortion. For the case of $r = 2$, Barnwell [6] has shown how an IIR QMF bank can be constructed such that the reconstructed signal $\hat{X}(z)$ is related to $X(z)$ as in (62). The main purpose of this section is to generalize these results.

Let us assume that the given low-pass prototype $H(z)$ has been expressed as in (14b). Let us assume that the

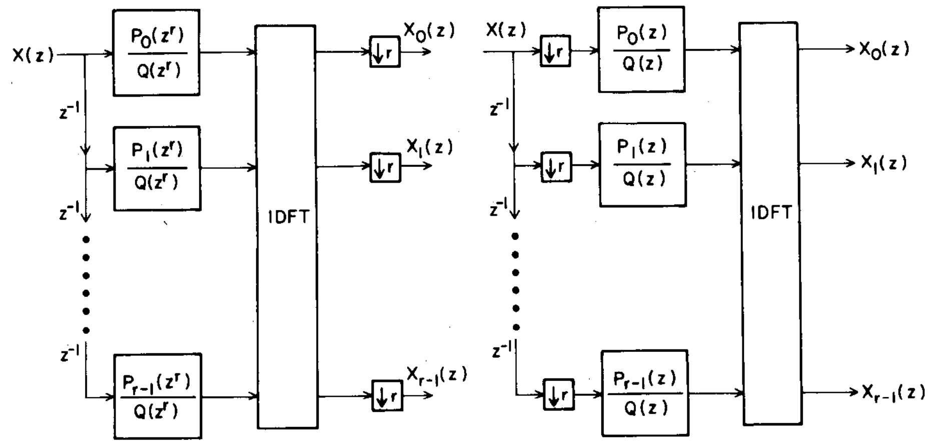

Fig. 8. The IIR filter bank.

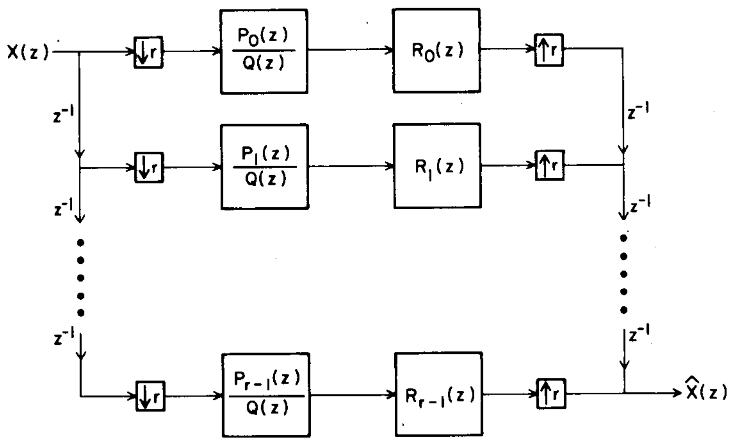

Fig. 9. Equivalent IIR filter bank used for theoretical study.

transfer function of the kth decimated filter $G_k(z)$ is given by

$$G_k(z) = P_k(z)/Q(z), \qquad 0 \leqslant k \leqslant r-1. \tag{63}$$

Notice that the denominator polynomial $Q(z)$ is the same for all k. The polynomials $P_k(z)$ and $Q(z)$ can be computed following Bellanger's approach in [3] if the poles of $H(z)$ are known. (Refer to the Appendix in this context, where we indicate how $P_k(z)$ and $Q(z)$ can be obtained without the explicit knowledge of the poles of $H(z)$.)

Both analysis and synthesis sections of the IIR filter-bank are shown in Fig. 8, where the quantities $R_0(z), \cdots, R_{r-1}(z)$ are to be determined such that (62) holds. Fig. 8 has been obtained here in analogy with the derivations and conclusions of Section III. Note that the DFT and IDFT blocks have been coalesced for analysis purposes, as shown in Fig. 9.

The overall structure of Fig. 9 is time invariant provided the quantities $P_k(z)R_k(z)/Q(z)$ are identical for all k. Under this condition, with

$$S(z) \triangleq P_k(z)R_k(z)/Q(z) \tag{64}$$

we can easily verify that

$$\hat{X}(z) = z^{-(r-1)}S(z^r)X(z). \tag{65}$$

Thus, the overall transfer function is $z^{-(r-1)}S(z^r)$. We therefore need only to choose $R_k(z)$ such that the right-hand side of (64) is a stable allpass function, independent of k. Under this condition, $\hat{X}(z)$ given by (65) is completely free from aliasing terms and amplitude distortion and only suffers a phase distortion.

Clearly, the choice $R_k(z) = Q(z)/P_k(z)$ accomplishes the desired goal, with $S(z)$ in (65) becoming unity. However, the synthesis bank is then not guaranteed to be stable because the polynomials $P_k(z)$ do not necessarily have all zeros strictly inside the unit circle. This problem can be handled by decomposing each $P_k(z)$ as

$$P_k(z) = P_{\min,k}(z)P_{\max,k}(z) \tag{66}$$

where $P_{\min,k}(z)$ is a minimum-phase polynomial which has all its zeros strictly inside the unit circle and $P_{\max,k}(z)$ is a maximum-phase polynomial which has all its zeros strictly outside the unit circle. We have implicitly assumed that $P_k(z)$ has no zeros on the unit circle. Indeed, if $P_k(z)$ had a zero at $z_0 = e^{j\omega_0}$ (and, hence, at $e^{-j\omega_0}$ if $h(n)$ is real), then the kth branch in Fig. 9 would have zeros at frequencies $(\omega_0 + 2\pi p)/r$ for $0 \leqslant p \leqslant r-1$ (and also at $(2\pi - \omega_0 + 2\pi p)/r$ for $0 \leqslant p \leqslant r-1$ if $h(n)$ is real). As discussed in Section IV, such a singularity situation would imply that $X(e^{j\omega})$ cannot be recovered by the filter bank at these frequencies. Thus, our assumption that $P_k(z)$ has no roots on the unit circle is entirely consistent with the conditions that would naturally be imposed by the requirements of reconstructability of the incoming signal.

Now define the set of transfer functions

$$T_k(z) = \frac{Q(z)}{P_{\min,k}(z)\hat{P}_{\max,k}(z)} \quad (67)$$

where $\hat{P}_{\max,k}(z)$ is the mirror image of $P_{\max,k}(z)$, i.e.,

$$\hat{P}_{\max,k}(z) = z^{-n_k}P_{\max,k}(z^{-1}). \quad (68)$$

In the above equation, n_k is the order of $P_{\max,k}(z)$. Clearly, $\hat{P}_{\max,k}(z)$ has minimum phase and, hence, $T_k(z)$ is stable. If $R_k(z)$ were taken to be equal to $T_k(z)$, then the kth branch in Fig. 9 would have transfer function $P_{\max,k}(z)/\hat{P}_{\max,k}(z)$. In order to make the branch transfer functions independent of k, it is therefore only necessary to set

$$R_k(z) = T_k(z)\prod_{\substack{l=0\\ l\neq k}}^{r-1}\frac{P_{\max,l}(z)}{\hat{P}_{\max,l}(z)}. \quad (69)$$

This choice leads to

$$S(z) = \frac{P_k(z)R_k(z)}{Q(z)} = \prod_{l=0}^{r-1}\frac{P_{\max,l}(z)}{\hat{P}_{\max,l}(z)} \quad (70)$$

which is clearly a stable allpass function, independent of k. Thus, we finally have (62), where

$$A(z) = z^{-(r-1)}S(z^r) = \text{a stable allpass function}. \quad (71)$$

Clearly, the complexity of the synthesis bank depends on the orders of $Q(z)$ and the factors $P_{\max,l}(z)$. If all the polynomials $P_k(z)$ happen to have minimum phase, then $R_k(z) = Q(z)/P_k(z)$ for all k, thus leading to the simplest possible synthesis bank. In this case, the cascade of the analysis and synthesis bank would be an identity system (except for a delay of $r-1$), leading to a perfect-reconstruction system. This observation has also been made by Smith [3, ch. 5].

Finally, note that closed-form expressions for the transfer functions $F_k(z)$ of the synthesis bank are once again given by (31) with $R_k(z)$ now given by (69). We wish to emphasize that no assumptions have been made about the nature of $H(z)$ except that none of the decimated filters $G_k(z)$ have transmission zeros on the unit circle. In the following subsection, we consider a specific instance of the above general setup.

A. A Special Case

If $H(z)$ is a low-pass transfer function with passband width nearly equal to π/r, then the decimated filter transfer functions $G_k(z)$ closely resemble allpass functions [3]. It might, in fact, turn out that $G_k(z)$ are exactly allpass functions. For the case of $r = 2$, this phenomenon has been known for some time. The interested reader is referred to Constantinides [4], Liu *et al.* [12], and Fettweis *et al.* [13], [14] for related results. If $G_k(z)$ is an allpass function, then, with N denoting the degree of $Q(z)$

$$P_k(z) = z^{-N}Q(z^{-1}), \qquad 0 \leqslant k \leqslant r-1 \quad (72)$$

if there are no cancellations in the ratio $P_k(z)/Q(z)$. Equation (72) implies that

$$G_k(z) = \frac{z^{-N}Q(z^{-1})}{Q(z)}, \qquad 0 \leqslant k \leqslant r-1. \quad (73)$$

Substituting in (14b), we have

$$H(z) = z^{-Nr}\frac{Q(z^{-r})}{Q(z^r)}[1+z^{-1}+\cdots+z^{-(r-1)}]. \quad (74)$$

But this is a trivial situation because the amplitude shaping provided by $H(z)$ is then entirely due to the FIR filter whose transfer function is $1+z^{-1}+\cdots+z^{-r+1}$. Accordingly, we rule out the possibility that $P_k(z)$ and $Q(z)$ are relatively prime. Assuming therefore that there *are* cancellations, the decimated filter transfer functions can be expressed in their minimal forms as

$$G_k(z) = \frac{\hat{\alpha}_k(z)}{\alpha_k(z)}, \qquad 0 \leqslant k \leqslant r-1 \quad (75)$$

where $\hat{\alpha}_k(z)$ is the mirror image of $\alpha_k(z)$. If we choose $R_k(z)$ to be the following stable allpass function:

$$R_k(z) = \prod_{\substack{l=0\\ l\neq k}}^{r-1} G_l(z) = \prod_{\substack{l=0\\ l\neq k}}^{r-1}\frac{\hat{\alpha}_l(z)}{\alpha_l(z)}, \qquad 0 \leqslant k \leqslant r-1 \quad (76)$$

then we have

$$S(z) = G_k(z)R_k(z) = \prod_{l=0}^{r-1} G_l(z) = \prod_{l=0}^{r-1}\frac{\hat{\alpha}_l(z)}{\alpha_l(z)} \quad (77)$$

which is independent of k. The overall transfer function $A(z)$ is again given by (71). Note that, in this special case, every transfer function building block, i.e., $G_k(z)$'s and $R_k(z)$'s, in the IIR filter bank has an allpass nature.

B. Relation to Low-Sensitivity IIR Filters Satisfying Allpass-Decomposition Property

Until this point, we have discussed only multirate filter structures in this paper. Let us now go back to single-rate digital filtering and review certain well-known properties satisfied by certain IIR digital filter transfer functions. These properties pertain to a concept called double-complemtarity [13], [14], [15], and lead to very efficient IIR digital filter implementations [16]. We wish to revisit these single-rate implementations in the context of multirate QMF banks and place in evidence their extreme suitability for multirate applications.

Two digital filter transfer functions $H_0(z)$ and $H_1(z)$ with the same common denominator polynomial are said to be "power complementary" [15] if

$$|H_0(e^{j\omega})|^2 + |H_1(e^{j\omega})|^2 = 1, \qquad \text{for all } \omega. \quad (78)$$

The pair $(H_0(z), H_1(z))$ is called a power-complementary pair. There exist certain power complementary pairs that can be decomposed into sums and differences of allpass functions [16]. This property is referred to as the allpass

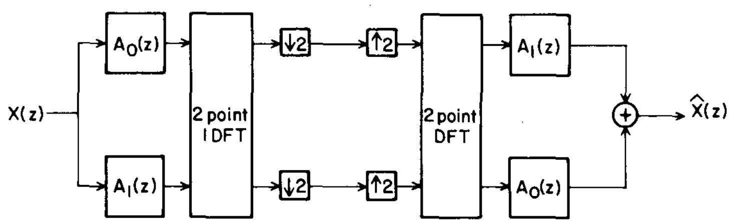

Fig. 10. Examples of an IIR filter bank with allpass building blocks. Here, $\hat{X}(z) = A_0(z)\cdot A_1(z)\cdot X(z)$.

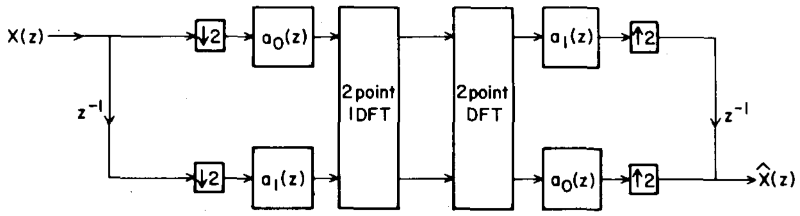

Fig. 11. Polyphase implementation of Fig. 10.

decomposition property. Thus

$$H_0(z) = \frac{1}{2}\left[A_0(z) + A_1(z)\right] \tag{79}$$

$$H_1(z) = \frac{1}{2}\left[A_0(z) - A_1(z)\right]. \tag{80}$$

Equivalently

$$A_0(z) = H_0(z) + H_1(z) \tag{81}$$

$$A_1(z) = H_0(z) - H_1(z). \tag{82}$$

Typical examples of such complementary pairs are odd-order digital Butterworth, Chebyshev, and Elliptic lowpass/high-pass pairs. (Note that if $H_0(z)$ is lowpass, then $H_1(z)$ is highpass in view of (78).)

An advantage of the decomposition of (79) and (80) is that it leads to dramatically efficient filter structures requiring very few multipliers. Moreover, if $A_0(z)$ and $A_1(z)$ are implemented in a "structurally lossless" manner [17], the resulting implementation is "structurally passive," giving rise to very low passband sensitivities [18]. The details of these issues are beyond the scope of this section and the reader is referred to [4], [12]–[18].

The main purpose of the subsection here is to indicate the suitability of these efficient structures for multirate applications. To be specific, let us consider the case of the QMF bank for $r = 2$. Let us choose the analysis filters $H_0(z)$, $H_1(z)$ in the usual manner [9], i.e.,

$$H_1(z) = H_0(-z). \tag{83}$$

In addition, let $H_0(z)$ and $H_1(z)$ be implementable in the forms shown in (79) and (80). Let the synthesis filters $F_0(z)$ and $F_1(z)$ be chosen in the usual manner [9], i.e.,

$$F_0(z) = H_0(z) \tag{84a}$$

$$F_1(z) = -H_1(z). \tag{84b}$$

Then there is perfect cancellation of the aliasing terms, and the reconstructed signal is

$$\hat{X}(z) = \left[H_0^2(z) - H_1^2(z)\right]X(z). \tag{85}$$

In view of the allpass decomposition of (79) and (80), the above equation leads to

$$\hat{X}(z) = A_0(z)A_1(z)X(z). \tag{86}$$

Thus, the overall transfer function is a stable allpass function and therefore the signal has been perfectly reconstructed to within a phase distortion. Fig. 10 shows the overall implementation. A similar reconstruction property is satisfied by certain suitably designed wave digital filter circuits, as shown by Fettweis [14].

In order to derive a polyphase implementation of the circuit in Fig. 10, first note that if $H_0(z)$ and $H_1(z)$ are related by (83), then

$$\begin{aligned} A_0(z) &= H_0(z) + H_1(z) \\ &= H_0(z) + H_0(-z) \\ &= a_0(z^2) \end{aligned} \tag{87}$$

$$\begin{aligned} A_1(z) &= H_0(z) - H_1(z) \\ &= H_0(z) - H_0(-z) \\ &= z^{-1}a_1(z^2). \end{aligned} \tag{88}$$

The polyphase implementation of the structure of Fig. 10 can now be obtained directly and is shown in Fig. 11.

Since $H_0(z)$ and $H_1(z)$ satisfy (78) and (83) simultaneously, they exhibit a certain symmetry around $\pi/2$. Fig. 12 shows a typical sketch of $|H_0(e^{j\omega})|^2$ and $|H_1(e^{j\omega})|^2$ for an equiripple case. Such transfer functions can be looked upon as logical extensions of half-band FIR transfer functions [7], [19]. These transfer functions, because of the symmetry around $\pi/2$, are also called symmetrical functions [4]. Note that if δ_1 and δ_2 represent the peak-ripples in the conventional sense (see Fig. 13), then these symmetrical filters have the design requirement

$$1-(1-2\delta_1)^2 = \delta_2^2$$

which should be taken into account while designing $H_0(z)$.

We now generalize the concepts mentioned so far in this subsection to the r-band case. The allpass decomposition

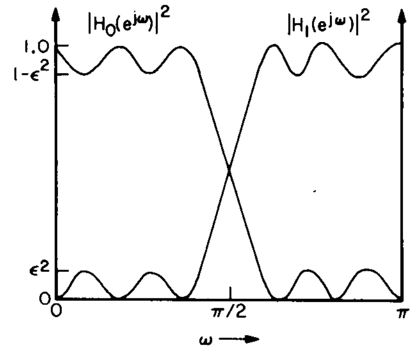

Fig. 12. Typical sketches of $|H_0^2|$ and $|H_1^2|$.

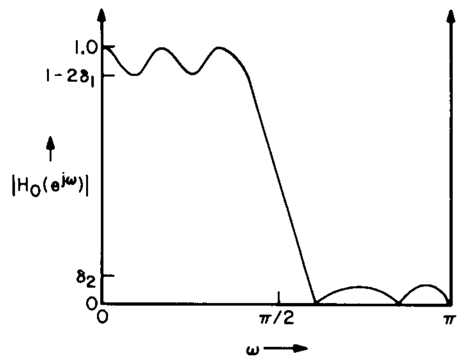

Fig. 13. Illustrating the design requirements of symmetrical filters.

property can be extended to r filters $H_0(z),\cdots,H_{r-1}(z)$ by defining a vector of allpass components $\boldsymbol{a}(z)$ such that

$$\boldsymbol{h}(z)=\frac{1}{r}\cdot \boldsymbol{W}^{\dagger}\cdot \boldsymbol{a}(z) \tag{89}$$

where

$$\boldsymbol{h}(z)=\left[H_0(z)\quad H_1(z)\cdots H_{r-1}(z)\right]^T$$

and

$$\boldsymbol{a}(z)=\left[A_0(z)\quad A_1(z)\cdots A_{r-1}(z)\right]^T \tag{89a}$$

$$|A_k(e^{j\omega})|=1 \qquad \text{for all } \omega, 0\leqslant k\leqslant r-1. \tag{89b}$$

and $\boldsymbol{W}$ is the usual r-point DFT matrix. Equations (79) and (80) are clearly special cases of (89). From (89), it follows that

$$\boldsymbol{h}^{\dagger}(e^{j\omega})\boldsymbol{h}(e^{j\omega})=\frac{1}{r}\boldsymbol{a}^{\dagger}(e^{j\omega})\boldsymbol{a}(e^{j\omega})=1 \tag{90}$$

since the components of $\boldsymbol{a}(e^{j\omega})$ all have unit magnitude. In other words, the set of r filter transfer functions $H_0(z), H_1(z),\cdots, H_{r-1}(z)$ satisfy the property

$$|H_0(e^{j\omega})|^2+|H_1(e^{j\omega})|^2+\cdots+|H_{r-1}(e^{j\omega})|^2=1. \tag{91}$$

Equation (78) is clearly a special case of the above equation. Note that if each allpass function is implemented in a structurally lossless manner, then each $H_k(z)$ is structurally passive because of (91) and, hence, has low passband sensitivity [17], [18].

Given a set of frequency domain specifications, one design issue is to compute the coefficients of $\boldsymbol{h}(z)$, or equivalently $\boldsymbol{a}(z)$, satisfying these specifications. This design issue is not within the scope of this subsection, although some preliminary work can be found in [15]. Another related question worth exploring is, how to precisely characterize the class of frequency responses $H_k(e^{j\omega})$ if they are to satisfy (89).

For uniform filter banks, i.e., where

$$H_k(z)=H_0(zW^k), \qquad 0\leqslant k\leqslant r-1$$

the vector $\boldsymbol{h}(z)$ can be related to the decimated filter transfer functions $G_l(z^r)$ by (16), i.e.,

$$\boldsymbol{h}(z)=\boldsymbol{W}^{\dagger}\boldsymbol{g}(z)$$

where $\boldsymbol{g}(z)$ is as defined in (16b), which is

$$\boldsymbol{g}(z)=\left[G_0(z^r)\quad z^{-1}G_1(z^r)\cdots z^{-(r-1)}G_{r-1}(z^r)\right]^T.$$

Thus, the uniform filter bank can satisfy the allpass decomposition property of (89) iff

$$A_k(z)=rz^{-k}G_k(z^r), \qquad 0\leqslant k\leqslant r-1. \tag{92}$$

In other words, the uniform filter bank can satisfy the allpass decomposition property iff the decimated filter transfer functions are allpass functions. But this is the same situation as the special case described in Section VIII-A. Thus, the design problem concerning $H(z)$ can be restated as follows: Given the prototype filter order N, how should $A_k(z)$ in (92) be designed so that the low-pass prototype $H(z)$ satisfies a given set of design requirements such as stopband energy level, transition bandwidth, etc. This requires further investigation.

IX. Summary

A unified theory of uniform DFT parallel QMF banks has been presented in this paper. Various aspects of FIR, as well as IIR, uniform DFT parallel QMF banks were addressed. A design procedure for FIR filter banks along with examples was presented. Improvements in design procedures for FIR filter banks, as well as design procedures for IIR filter banks, are currently being investigated. Extensions of the theory to GDFT (generalized DFT), as well as r-channel QMF banks having filters with real impulse responses are currently under study.

Appendix

Let $H(z)$ be an Nth-order IIR transfer function

$$H(z)=\frac{a_0+a_1z^{-1}+\cdots+a_Nz^{-N}}{1+b_1z^{-1}+\cdots+b_Nz^{-N}}. \tag{A1}$$

Let us assume, as is generally the case, that the poles of $H(z)$ are distinct. Then $H(z)$ can be expressed, using partial fraction expansion, as

$$H(z)=k_0+\sum_{l=1}^{N}\frac{k_l}{1-p_lz^{-1}} \tag{A2}$$

where the residues k_l and poles p_l may be complex. The causal impulse response corresponding to $H(z)$ is given by

$$h(n)=k_0\delta(n)+\sum_{l=1}^{N}k_lp_l^nu(n) \tag{A3}$$

where $\delta(n)$ and $u(n)$ denote the unit impulse and unit step functions, respectively. Clearly then, the inverse transformation of $G_k(z)$, defined by (14a), is given by

$$g_k(n) = h(rn+k) = k_0\delta(rn+k) + \sum_{l=1}^{N} k_l p_l^k p_l^{rn} u(rn+k), \quad n \geq 0, 0 \leq k \leq r-1. \tag{A4}$$

Hence

$$G_k(z) = k_0\delta(k) + \sum_{l=1}^{N} \frac{k_l p_l^k}{1 - p_l^r z^{-1}}. \tag{A5}$$

In other words, $Q(z)$ in (63) is given by

$$Q(z) = \prod_{l=1}^{N} (1 - p_l^r z^{-1}) \tag{A6}$$

and is independent of k.

In practice, it is not necessary to compute the poles p_l in order to obtain $G_k(z)$ for a given $H(z)$. The common denominator $Q(z)$ of the $G_k(z)$'s, given by

$$Q(z) = 1 + q_1 z^{-1} + \cdots + q_N z^{-N} \tag{A7}$$

can be found by simply solving the set of N linear equations [20]

$$\begin{bmatrix} g_0(N) & \cdots & g_0(2) & g_0(1) \\ g_0(N+1) & \cdots & g_0(3) & g_0(2) \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ g_0(2N-1) & \cdots & g_0(N+1) & g_0(N) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ \vdots \\ q_N \end{bmatrix} = - \begin{bmatrix} g_0(N+1) \\ g_0(N+2) \\ \vdots \\ \vdots \\ g_0(2N) \end{bmatrix}. \tag{A8}$$

The numerator coefficients, i.e., coefficients of $P_k(z)$ in (63), given by

$$P_k(z) = p_{k,0} + p_{k,1} z^{-1} + \cdots + p_{k,N} z^{-N} \tag{A9}$$

can be computed as

$$\begin{aligned} p_{k,0} &= g_k(0) \\ p_{k,1} &= g_k(1) + q_1 g_k(0) \\ &\vdots \\ p_{k,N} &= g_k(N) + q_1 g_k(N-1) + \cdots + q_N g_k(0). \end{aligned} \tag{A10}$$

Thus, given the rational transfer function $H(z)$, all the rational representations of the decimated filters $G_k(z^r)$ can easily be found.

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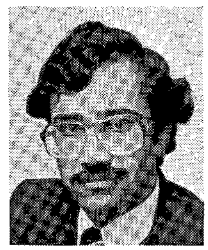

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