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THEORY AND NUMERICAL ANALYSIS FOR EXACT DISTRIBUTIONS OF FUNCTIONALS OF A DIRICHLET PROCESS[1]

BY EUGENIO REGAZZINI,[2] ALESSANDRA GUGLIELMI AND GIULIA DI NUNNO

Università degli Studi di Pavia, CNR-IAMI (Milano) and Università degli Studi di Pavia

Dedicated to the memory of Luciano Daboni

The distribution of a mean or, more generally, of a vector of means of a Dirichlet process is considered. Some characterizing aspects of this paper are: (i) a review of a few basic results, providing new formulations free from many of the extra assumptions considered to date in the literature, and giving essentially new, simpler and more direct proofs; (ii) new numerical evaluations, with any prescribed error of approximation, of the distribution at issue; (iii) a new form for the law of a vector of means. Moreover, as applications of these results, we give: (iv) the sharpest condition sufficient for the distribution of a mean to be symmetric; (v) forms for the probability distribution of the variance of the Dirichlet random measure; (vi) some hints for determining the finite-dimensional distributions of a random function connected with Bayesian methods for queuing models.

1. Introduction. In Walker, Damien, Laud and Smith (1999), the authors and discussants offer a systematic survey of recent research in Bayesian nonparametric statistics. As in any concise exposition of a rather large topic, these authors have been forced to omit some aspects of the literature. In particular, they disregard the problem of finding the *exact distributions of functionals* (means, for example) of random probability measures. In our opinion, this problem deserves some attention since in statistical work many interesting properties of a family of distributions happen to be indicated by vectors of suitable means. The importance of this subject is also stressed in Diaconis and Kemperman (1996) and, more recently, in Conti (1999).

The last two decades have produced various contributions to the solution of the problem, and a short survey of these would be desirable to complete the picture

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[2]Also affiliated with CNR-IAMI, Milano, Italy.

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offered in Walker, Damien, Laud and Smith (1999). It is, instead, important to know the exact expression of the distribution concerned in practical applications, for this permits us to obtain numerical evaluations with *any prescribed error of approximation*. This is the distinguishing element that marks the difference between deterministic evaluation of the error and those evaluations based on the simulation of the underlying process. Interesting examples of the simulation-based approach can be found in Florens and Rolin (1994), Gelfand and Mukhopadhyay (1995), Muliere and Tardella (1998), Gelfand and Kottas (2002), Guglielmi, Holmes and Walker (2002) and Guglielmi and Tweedie (2001).

We provide here an elementary and self-contained theory that covers the case of means of a Dirichlet random measure by combining the construction suggested in Regazzini (1998) with a simple argument presented in Hannum, Hollander and Langberg (1981), via a suitable inversion formula for multidimensional characteristic functions. The paper has been written with the following objectives in mind:

1. A review of the main results concerning the exact distribution of a mean of a Dirichlet process, by providing new formulations free of redundant extra conditions and equipped with new and more direct proofs;
2. A suitable approximation technique for exact distributions;
3. A presentation of new results concerning the exact distribution of a vector of means of a Dirichlet random measure.

Accordingly, Section 2 of this paper introduces the notation and some elementary results used here. It provides a new proof of a condition for a mean of a Dirichlet process to be finite. It also presents the characteristic function of a mean of a gamma random measure. In Section 3 this characteristic function is combined with the Hannum, Hollander and Langberg device to give the distribution of a mean of a Dirichlet process, via the Gurland inversion formula for characteristic functions. As an application, a sufficient condition for the distribution to be symmetric is provided. This result was formulated by Hannum, Hollander and Langberg, under the extra condition that the first moment of α is finite. Additional formulas, reminiscent of the Liouville–Weyl fractional integral, are given in Section 4, together with a few illustrative examples. Section 5 presents multidimensional extensions of previous results, together with two applications: (a) an expression for the distribution of the variance, to be added to that of Cifarelli and Melilli (2000); and (b) the expression of a distribution connected with problems posed in Conti (1999). Numerical evaluations are considered in Section 6. The univariate probability distribution function is uniformly approximated to any given degree of precision by a distribution function that is easily computed. Finally, several examples are given to illustrate this approximation procedure.

2. Preliminary basic results.

2.1. *Notation.* A metric or topological space $\mathbb{S}$ will always be endowed with its Borel class $\mathscr{B}(\mathbb{S})$ generated by the topology in $\mathbb{S}$, unless a σ-field is otherwise specified. Denote the one-dimensional Euclidean space by $\mathbb{R}$, and the space of finite measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ by $\mathbb{M}$. Space $\mathbb{M}$ is endowed with the σ-field induced by all evaluation maps $\pi_B : m \mapsto m(B)$, $m \in \mathbb{M}$ and $B \in \mathscr{B}(\mathbb{R})$. Such a σ-field coincides with $\mathscr{B}(\mathbb{M})$ when $\mathbb{M}$ is equipped with the topology of weak convergence [see, e.g., Dawson (1993), Lemma 3.2.3, page 42]. Hence, the set of all probability measures on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$, $\mathbb{M}_1 := \{m \in \mathbb{M} : m(\mathbb{R}) = 1\}$, is an element of $\mathscr{B}(\mathbb{M})$.

Next consider a probability space $(\Omega, \mathscr{F}, P)$ and call *random finite measure* any measurable mapping ξ between Ω and $\mathbb{M}$. Given any measurable function ψ from $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ into itself, recall that $\xi(\psi) := \int \psi \, d\xi$ is the limit of $\xi(\psi_n)$ where $\psi_n := \sum_{k=1}^{\nu_n} x_k^{(n)} \mathbb{I}_{A_k^{(n)}}$ is, for every n, a simple (measurable) function such that $\psi_n \to \psi$ pointwise, $0 \le |\psi_1| \le |\psi_2| \le \cdots \le |\psi|$, and $\psi_n \to \psi$ uniformly on any set on which ψ is bounded. It is easily verified that $\xi(\psi_n)$ and $\xi(\psi)$ are measurable functions from $(\Omega, \mathscr{F})$ into $(\overline{\mathbb{R}}, \mathscr{B}(\overline{\mathbb{R}}))$. In particular, $I_\psi := \{m \in \mathbb{M} : m(|\psi|) < +\infty\}$ belongs to $\mathscr{B}(\mathbb{M})$ and $\{\xi(|\psi|) < +\infty\}$ is an element of $\mathscr{F}$.

If α is a finite measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ such that $a := \alpha(\mathbb{R}) > 0$, ξ_α will designate a *gamma random measure* with parameter α, that is, a random finite measure such that, for any partition $\{A_1, \ldots, A_n\}$ of $\mathbb{R}$ in $\mathscr{B}(\mathbb{R})$, the random variables $\xi_\alpha(A_1), \ldots, \xi_\alpha(A_n)$ are independent, and $\xi_\alpha(A_k)$ has a gamma distribution with shape parameter $\alpha(A_k)$ and scale parameter 1, for $k = 1, \ldots, n$. This distribution is considered degenerate at 0 if $\alpha(A_k) = 0$.

The process $\xi_{1\alpha} := \xi_\alpha(\mathbb{R})^{-1} \xi_\alpha \mathbb{I}_{\{\xi_\alpha(\mathbb{R}) > 0\}}$ is called *Dirichlet process* with parameter α. Clearly, $P\{\xi_{1\alpha} \in \mathbb{M}_1\} = 1$ and the random vector $(\xi_{1\alpha}(A_1), \ldots, \xi_{1\alpha}(A_{n-1}))$ has the $(n-1)$-dimensional Dirichlet distribution with parameter $(\alpha(A_1), \ldots, \alpha(A_n))$ whenever $\alpha(A_k) > 0$ for every k. Note that the above random vector and $\xi_\alpha(\mathbb{R})$ are stochastically independent. Compare, for example, Section 3.3 in Bilodeau and Brenner (1999).

The notations $Q(\cdot; \alpha)$ and $Q_1(\cdot; \alpha)$ are used here to designate the probability distributions of ξ_α and $\xi_{1\alpha}$, respectively.

2.2. *Finiteness of* $\xi_{1\alpha}(\psi)$. Before giving explicitly the distribution $\xi_{1\alpha}(\psi)$ must have, we consider the conditions under which this random variable is (essentially) finite. Feigin and Tweedie (1989) have proved that $\xi_{1\alpha}(\psi)$ is (essentially) finite if, and only if, $x \mapsto \log(1 + |\psi(x)|)$ belongs to $L^1(\alpha)$. Their proof uses the fact that the Dirichlet distribution, that is $Q_1(\cdot; \alpha)$, can be thought of as the invariant distribution of a specific measure-valued Markov chain. A different proof, based on a generalized Stieltjes transform of the distribution of $\xi_\alpha(\psi)$, is given in

Cifarelli and Regazzini (1996). This remarkable equivalence is derived, by means of simple and direct arguments, from the following elementary statement:

$$\phi(t;\psi_n;\alpha) = \exp\biggl(-\int_{\mathbb{R}} \log\bigl(1 - it\psi_n(x)\bigr)\alpha(dx)\biggr), \qquad t \in \mathbb{R}, \tag{1}$$

where $\phi(t;\psi_n;\alpha) := \int_{\mathbb{M}} e^{itm(\psi_n)} Q(dm;\alpha)$ and $(\psi_n)_{n\geq 1}$ is the same as in Section 2.1. Note that throughout this paper, $\log z$ indicates the principal determination of the logarithm of the complex number z; that is, $\log z = \log|z| + i \operatorname{Arg} z$, where $\operatorname{Arg} z$ is chosen in $(-\pi, \pi]$.

PROPOSITION 1. *The following conditions are equivalent*:

(a) $Q(I_\psi;\alpha) = 1$.
(b) $Q_1(I_\psi \cap \mathbb{M}_1;\alpha) = 1$.
(c) $\int_{\mathbb{R}} \log|1 + it\psi(x)|\alpha(dx) < +\infty$, $t \in \mathbb{R}$.

PROOF. (a) implies (b) and (b) implies (a) by definition of $\xi_{1\alpha}$. An indirect argument can be used to prove that (a) implies (c). Thus, assume that the pair (α, ψ) satisfies (a), and that $\int_{\mathbb{R}} \log|1 - it_0\psi|\, d\alpha = +\infty$ for some t_0. Then $\int_{\mathbb{R}} \log|1 - it\psi|\, d\alpha = +\infty$ for every $t \neq 0$. Now, for any sequence $(\psi_n)_{n\geq 1}$ as defined in the previous section, $m(\psi_n) \to m(\psi)$ for every m in I_ψ, and the dominated convergence theorem would yield $\phi(t;\psi_n;\alpha) \leftarrow \int_{I_\psi} \exp\{itm(\psi)\} \times Q(dm;\alpha) = E(\exp\{it\xi_\alpha(\psi)\}) =: \phi(t;\psi;\alpha)$. On the other hand, equation (1) and the monotone convergence theorem would give us a characteristic function which is zero at each $t \neq 0$, a contradiction.

Conversely, assume that (c) holds. In this case, the dominated convergence theorem can be applied to show that $\phi(t;|\psi_n|;\alpha) := E(\exp\{it\xi_\alpha(|\psi_n|)\})$ converges to $\exp\{-\int_{\mathbb{R}} \log(1 - it|\psi|)\, d\alpha\}$ as $n \to +\infty$, for every t, which is continuous at $t = 0$, so that $\xi_\alpha(|\psi_n|)$ converges in distribution to a real-valued random variable. To complete the proof it suffices to note that $\xi_\alpha(|\psi_n|) \to \xi_\alpha(|\psi|)$ pointwise on Ω. □

We assume throughout that the conditions of Proposition 1 hold.

2.3. *Characteristic function of* $\xi_\alpha(\psi)$. In the present treatment of the probability distribution of $\xi_{1\alpha}(\psi)$, the characteristic function of $\xi_\alpha(\psi)$ plays a leading role. Using a somewhat different approach, the same distribution was deduced from its Stieltjes transform of order a. See Cifarelli and Regazzini (1978, 1979a, b, 1990, 1993, 1996) and Guglielmi (1998).

PROPOSITION 2. *For the characteristic function* $\phi(\cdot;\psi;\alpha)$ *of* $\xi_\alpha(\psi)$ *we have*

$$\begin{aligned}\phi(t;\psi;\alpha) &= \int_{\mathbb{M}} \exp\{itm(\psi)\} Q(dm;\alpha)\\ &= \exp\biggl\{-\int_{\mathbb{R}} \log\bigl(1 - it\psi(x)\bigr)\alpha(dx)\biggr\}, \qquad t \in \mathbb{R}.\end{aligned}$$

Moreover, the integral identity

$$\int_{\mathbb{M}_1} \big(1 - itp(\psi)\big)^{-a} Q_1(dp;\alpha) = \exp\bigg\{-\int_{\mathbb{R}} \log\big(1 - it\psi(x)\big)\alpha(dx)\bigg\}, \qquad t \in \mathbb{R},$$

holds true.

PROOF. The former assertion can be proved by applying the same argumentation used in the final part of the proof of Proposition 1.

For the latter assertion, note that

$$\begin{aligned}\phi(t;\psi_n;\alpha) &= E\Bigg(\exp\Bigg\{it\sum_{k=1}^{\nu_n} x_k^{(n)} \xi_\alpha(A_k^{(n)})\Bigg\}\Bigg)\\ &= E\Bigg(\exp\Bigg\{it\xi_\alpha(\mathbb{R})\sum_{k=1}^{\nu_n} x_k^{(n)} \xi_{1\alpha}(A_k^{(n)})\Bigg\}\Bigg)\end{aligned}$$

in the notation of the previous section. Recall now that $(\xi_{1\alpha}(A_1^{(n)}), \dots, \xi_{1\alpha}(A_{\nu_n}^{(n)}))$ and $\xi_\alpha(\mathbb{R})$ are independent random elements, and $\xi_\alpha(\mathbb{R})$ has a gamma distribution with shape parameter a and scale parameter 1. Hence, by a well-known conditioning argument [cf., e.g., Kallenberg (1997), page 29],

$$\begin{aligned}\phi(t;\psi_n;\alpha) &= \int_{\mathbb{M}_1}\Bigg(\int_0^{+\infty} \exp\Bigg\{itx\sum_{k=1}^{\nu_n} x_k^{(n)} p(A_k^{(n)})\Bigg\}\frac{1}{\Gamma(a)} e^{-x} x^{a-1}\,dx\Bigg) Q_1(dp;\alpha)\\ &= \int_{\mathbb{M}_1} \big(1 - itp(\psi_n)\big)^{-a} Q_1(dp;\alpha)\end{aligned}$$

and the thesis follows easily from the dominated convergence theorem. □

REMARK 1. The second part of Proposition 2 is an extension of the Markov–Krein identity. This fact was first stressed by Diaconis and Kemperman (1996). The simple proof here above is obtained from an argument used in Regazzini (1998), and is close to the proof given more recently in Tsilevich, Vershik and Yor (2000). For a generalization of the Markov–Krein identity, see (8) in Regazzini (1998) and Lijoi and Regazzini (2001).

3. The distribution of a mean of a Dirichlet process.

3.1. *Basic tools.* Write $F_1(\cdot;\psi;\alpha)$ for the distribution function of $\xi_{1\alpha}(\psi)$ and, for any $g:\mathbb{R}\to\mathbb{R}$, let $C(g)$ denote the set of the continuity points of g. Explicit forms of $F_1(\cdot;\psi;\alpha)$, based on its Stieltjes transform of order a, are given in the aforesaid papers by Cifarelli and Regazzini under the extra condition that $\alpha((-\infty,\nu]) = 0$ for some $\nu \in \mathbb{R}$. A similar expression for $F_1(\cdot;\psi;\alpha)$ is obtained here without this constraint, using a trick of Hannum, Hollander and Langberg (1981); that is,

$$F_1(\sigma;\psi;\alpha) := P\{\xi_{1\alpha}(\psi)\le\sigma\} = P\{\xi_\alpha(\psi-\sigma)\le 0\}, \qquad \sigma\in\mathbb{R}.$$

See also Tamura (1988) and Section 0 in Cifarelli and Regazzini (1990). Thus, writing $\operatorname{Im} z$ and $\operatorname{Re} z$ for the imaginary and real part of z, respectively, by a well-known inversion formula [cf. Gurland (1948)], we obtain

$$F_1(\sigma;\psi;\alpha) = \frac{1}{2} - \frac{1}{\pi}\lim_{\substack{\varepsilon\searrow 0\\ T\nearrow+\infty}}\int_\varepsilon^T \frac{1}{t}\operatorname{Im}\phi(t;\psi-\sigma;\alpha)\,dt \tag{2}$$

at each σ in $C(F_1)$.

Since the law of $\xi_{1\alpha}(\psi)$ coincides with the distribution of $\int x\xi_{1\alpha\circ\psi^{-1}}(dx)$ (i.e., the distribution of the mean of a Dirichlet process with parameter $\alpha\circ\psi^{-1}$), the technique of the proofs is the same for any ψ. To avoid cumbersome notation, we specialize to the case $\psi(x)\equiv x$, $x\in\mathbb{R}$. We now denote the distribution function of $\int x\xi_{1\alpha}(dx)$ by $F_1(\cdot;\alpha)$.

3.2. *Main result for the probability distribution of* $\xi_{1\alpha}(\psi)$. As a matter of fact, the principal value integral in (2) can be replaced by a Lebesgue integral. Throughout this paper a measurable function X on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ is said to be α-degenerate if there is a point $\bar{x}$ such that $\alpha\{X=\bar{x}\}=a$; that is, $\alpha\circ X^{-1}=a\delta_{\bar{x}}$.

PROPOSITION 3. *The equality*

$$F_1(\sigma;\psi;\alpha) = \frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty}\frac{1}{t}\operatorname{Im}\biggl\{\exp\biggl(-\int_{\mathbb{R}}\log\bigl[1+it\bigl(\sigma-\psi(x)\bigr)\bigr]\alpha(dx)\biggr)\biggr\}\,dt$$

holds true for every σ *in* $\mathbb{R}$ *when* ψ *is not* α-*degenerate, and for every* $\sigma\ne\bar{x}$ *when* $\alpha\circ\psi^{-1}=a\delta_{\bar{x}}$.

Expanding the imaginary part, we obtain

$$\begin{aligned} F_1(\sigma;\psi;\alpha) = \frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty}\frac{1}{t}\exp\biggl(-\frac{1}{2}\int_{\mathbb{R}}\log\bigl[1+t^2\bigl(\sigma-\psi(x)\bigr)^2\bigr]\alpha(dx)\biggr)& \\ \times\sin\biggl\{\int_{\mathbb{R}}\arctan\bigl[t\bigl(\sigma-\psi(x)\bigr)\bigr]\alpha(dx)\biggr\}\,dt&. \end{aligned} \tag{3}$$

It is easy to verify that $F_1(\cdot;\psi;\alpha)=\mathbb{I}_{[\bar{x},+\infty)}(\cdot)$ if $\alpha\circ\psi^{-1}=a\delta_{\bar{x}}$.

PROOF OF PROPOSITION 3. Note that $\rho(t) := t^{-1}\operatorname{Im}\phi(t;\psi-\sigma;\alpha)$ coincides with $t^{-1}\exp\{-\int_{\mathbb{R}}\log|1+it(\sigma-x)|\alpha(dx)\}\sin\{\int_{\mathbb{R}}\arctan[t(\sigma-x)]\alpha(dx)\}$ when $\psi(x)\equiv x$, and observe that, for any $b>0$, t in $(0,b)$,

$$|\rho(t)| \le \frac{1}{t}\int_{\mathbb{R}}|\arctan\{t(\sigma-x)\}|\alpha(dx) \le \frac{\pi}{2}\int_{\mathbb{R}}\frac{|\sigma-x|}{\sqrt{1+t^2(\sigma-x)^2}}\alpha(dx),$$

the second inequality being a consequence of Pfaff's transformation. Compare, for example, Andrews, Askey and Roy [(1999), page 69]. Moreover, when $t \geq b$,

$$\begin{aligned}|\rho(t)| &\leq t^{-1}\exp\bigl(-2^{-1}\alpha\{[\sigma-\eta,\sigma+\eta]^c\}\log[1+(t\eta)^2]\bigr)\\ &= t^{-1}\bigl(1+(t\eta)^2\bigr)^{-\alpha\{[\sigma-\eta,\sigma+\eta]^c\}/2}\end{aligned}$$

is clearly valid for any $\eta > 0$. Assume that α is not a degenerate measure. We can then associate a positive η for which $\alpha\{[\sigma-\eta,\sigma+\eta]^c\} > 0$ to each σ. Thus the limit in (2) can be replaced with a Lebesgue integral on $(0,+\infty)$, and the proposition is proved for any σ in $C(F_1)$, provided that α is not degenerate. For the extension to any real σ, see Remark 2 below. □

The following proposition provides an explicit formula for the density function of F_1. It represents a substantial improvement over an analogous result provided in Cifarelli and Regazzini (1990), where the densities were determined under the further hypothesis that $a > 1$. Note that this condition is satisfied by the parameter α of any posterior Dirichlet distribution, in the presence of a sequence of exchangeable observations.

PROPOSITION 4. *If ψ is not α-degenerate, then $F_1(\cdot;\psi;\alpha)$ is absolutely continuous with respect to the Lebesgue measure and, for any $\sigma \in \mathbb{R}$,*

$$\begin{aligned}&f_1(\sigma;\psi;\alpha)\\ &\quad= \frac{1}{\pi}\mathbb{I}_{C(A_\psi)}(\sigma)\int_0^{+\infty}\mathrm{Re}\biggl[\exp\biggl(-\int_{\mathbb{R}}\log[1+it(\sigma-\psi(x))]\alpha(dx)\biggr)\\ &\qquad\qquad\times\int_{\mathbb{R}-\{\sigma\}}\frac{\alpha(dx)}{1+it(\sigma-\psi(x))}\biggr]dt\end{aligned}$$

is a density function for $F_1(\cdot;\psi;\alpha)$.

We use the following notation: $A(x) := \alpha(-\infty,x]$, $A_\psi(x) := \alpha\circ\psi^{-1}(-\infty,x]$.

PROOF OF PROPOSITION 4. With $\psi(x)\equiv x$, it is enough to prove the above statement for any σ in $C(F_1)\cap C(A)$. Applying the Fubini–Tonelli theorem and the theorem of differentiation under the integral sign [see, e.g., Folland (1984), pages 64 and 54, respectively] yields the following equalities:

$$\begin{aligned}F_1(\sigma;\alpha) &= \frac{1}{2}-\frac{1}{\pi}\int_0^{+\infty}\frac{1}{t}\mathrm{Im}\biggl[\int_{\mathbb{M}}\exp\biggl(it\int_{\mathbb{R}}(x-\sigma)m(dx)\biggr)Q(dm;\alpha)\biggr]dt\\ &= \frac{1}{2}-\frac{1}{\pi}\int_0^{+\infty}\frac{1}{t}\mathrm{Im}\biggl[\int_{\mathbb{M}}\exp\biggl(it\int_{\mathbb{R}}xm(dx)\biggr)\\ &\qquad\times\biggl(\int_0^{\sigma}e^{-itym(\mathbb{R})}(-itm(\mathbb{R}))\,dy+1\biggr)Q(dm;\alpha)\biggr]dt\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\{F_1(0^+;\alpha) + F_1(0^-;\alpha)\} \\
&\quad - \frac{1}{\pi}\int_0^{+\infty}\frac{1}{t}\,\mathrm{Im}\biggl[\int_0^{\sigma}\int_{\mathbb{M}}\frac{\partial}{\partial y}\exp\Bigl(it\int_{\mathbb{R}}(x-y)m(dx)\Bigr)Q(dm;\alpha)\,dy\biggr]dt
\end{aligned}
$$

and, from Proposition 2,

$$
\begin{aligned}
F_1(\sigma;\alpha) &= \frac{1}{2}\{F_1(0^+;\alpha) + F_1(0^-;\alpha)\} \\
&\quad + \frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\biggl[\int_0^{\sigma}\exp\Bigl(-\int_{\mathbb{R}}\log[1-it(x-y)]\alpha(dx)\Bigr) \\
&\qquad\qquad \times \int_{\mathbb{R}}[1+it(y-u)]^{-1}\alpha(du)\,dy\biggr]dt.
\end{aligned}
$$

Thus, interchanging the integrals would give the conclusion. For this reason we verify the conditions of the Fubini–Tonelli theorem for $\sigma > 0$. Given any $b > 0$, for any t in $(0, b]$,

$$
\int_0^{\sigma}\Bigl|\exp\Bigl(-\int_{\mathbb{R}}\log[1-it(x-y)]\alpha(dx)\Bigr)\Bigr|\int_{\mathbb{R}}\bigl[1+t^2(y-u)^2\bigr]^{-1/2}\alpha(du)\,dy \le \sigma a.
$$

Moreover, since α is not degenerate, some $\eta > 0$ can be chosen so that $\gamma := \inf_{y\in(0,\sigma]}\alpha\{[y-\eta, y+\eta]^c\} > 0$ and, therefore, when $t \ge b$,

$$
\begin{aligned}
&\int_{\mathbb{R}}\int_0^{\sigma}\Bigl|\exp\Bigl(-\int_{\mathbb{R}}\log[1-it(x-y)]\alpha(dx)\Bigr)\Bigr|\bigl[1+t^2(y-u)^2\bigr]^{-1/2}dy\,\alpha(du) \\
&\quad \le \frac{1}{t(1+t^2\eta^2)^{\gamma/2}}\biggl\{2a\log t + \int_{\mathbb{R}}\log\Bigl[\Bigl(\sigma - u + \sqrt{b^{-2}+(\sigma-u)^2}\Bigr) \\
&\qquad\qquad \times \Bigl(u + \sqrt{b^{-2}+u^2}\Bigr)\Bigr]\alpha(du)\biggr\},
\end{aligned}
$$

that is, finite by Proposition 1. Therefore, by the Fubini–Tonelli theorem, $(t, y, u) \mapsto |\exp\{-\int_{\mathbb{R}}[1-it(x-y)]\alpha(dx)\}||1+it(y-u)|^{-1}$ is integrable on $(0,+\infty)\times[0,\sigma]\times\mathbb{R}$ with respect to $\lambda^2\otimes\alpha$, where λ^2 denotes the Lebesgue measure on $\mathbb{R}^2$. The integrability on $(0,+\infty)\times[\sigma,0]\times\mathbb{R}$, with $\sigma < 0$, can be proved by the same argument. Thus,

$$
\begin{aligned}
F_1(\sigma;\alpha) &= \int_{-\infty}^{\sigma}\frac{1}{\pi}\int_0^{+\infty}\mathrm{Re}\biggl[\exp\Bigl(-\int_{\mathbb{R}}\log[1-it(x-y)]\alpha(dx)\Bigr) \\
&\qquad\qquad \times \int_{\mathbb{R}}[1+it(y-u)]^{-1}\alpha(du)\biggr]dt\,dy.
\end{aligned}
$$

In particular, using the same upper bounds as in the preceding computation on $[b,+\infty)$, it can be shown that $\int_0^{+\infty}\mathrm{Re}(\exp\{-\int_{\mathbb{R}}\log[1-it(x-y)]\alpha(dx)\} \times\int_{\mathbb{R}}[1+it(y-u)]^{-1}\alpha(du))\,dt$ is finite for every $y\in C(A)$. □

REMARK 2. To prove Proposition 4 it suffices to assume that (3) is valid on $C(F_1)$. Since the thesis of Proposition 4 is the absolute continuity of F_1, it is clear that the proof of Proposition 3 is now complete.

A new formula for f_1 can be obtained, as a straightforward corollary of an expression of the posterior distribution of $\xi_{1\alpha}(\psi)$ given in Regazzini, Lijoi and Prünster (2003), at the end of Section 4.

PROPOSITION 5. *If ψ is not α-degenerate and there is some point x_0 such that $\alpha\{x_0\} \geq 1$, then*

$$f_1(\sigma;\psi;\alpha) = \frac{a-1}{\pi}\mathbb{I}_{C(A_\psi)}(\sigma) \times \int_0^{+\infty} \operatorname{Re}\exp\biggl(-\int_{\mathbb{R}} \log[1+it(\sigma-\psi(x))]\alpha(dx)\biggr)\,dt$$

is a density function for $F_1(\cdot;\psi;\alpha)$.

PROOF. In the expression of f_1 given in Proposition 4, change the variable t to $z = T^{-1} + \sigma - it^{-1}$, and argue as below in Section 4, just before (5). Then check that the limit (as $\varepsilon \searrow 0$ and $T \nearrow +\infty$) of the resulting integral on $-(\gamma_2 \cup \gamma_3)$ coincides with the expression given in Proposition 9(i). $\square$

Note that the conditions under which the above formula holds true are still satisfied by the parameter α of any posterior Dirichlet distribution, in the presence of a sequence of exchangeable observations.

3.3. *Symmetry of the probability distribution of $\xi_{1\alpha}(\psi)$.* As a simple application of the previous formulas we show how the symmetry of the distribution function A_ψ produces the same property for F_1, improving on Corollary 2.6 in Hannum, Hollander and Langberg (1981). We prove this by using the change of variable $y = \sigma t$ in (3) to obtain the following expressions for $F_1(\sigma;\psi;\alpha)$:

$$\frac{1}{2} + \frac{1}{\pi}\int_0^{+\infty} y^{-1}\exp\biggl(-\frac{1}{2}\int_{\mathbb{R}} \log[1+y^2(1-\psi(x)/\sigma)^2]\alpha(dx)\biggr) \times \sin\biggl[\int_{\mathbb{R}} \arctan\bigl(y(1-\psi(x)/\sigma)\bigr)\alpha(dx)\biggr]\,dy \qquad \text{if } \sigma > 0,$$

$$\frac{1}{2} - \frac{1}{\pi}\int_0^{+\infty} y^{-1}\exp\biggl(-\frac{1}{2}\int_{\mathbb{R}} \log[1+y^2(1-\psi(x)/\sigma)^2]\alpha(dx)\biggr) \times \sin\biggl[\int_{\mathbb{R}} \arctan\bigl(y(1-\psi(x)/\sigma)\bigr)\alpha(dx)\biggr]\,dy \qquad \text{if } \sigma < 0$$

and by observing that $\int_0^{+\infty} y^{-1}(1+y^2)^{-a/2}\sin(a\arctan y)\,dy = \int_0^{\pi}(\sin z)^{-1}\times(\cos z)^{a-1}\sin(az)\,dz = \pi/2$ by 3.638 (3) in Gradshteyn and Ryzhik (1994). Then, setting

$$q(y;\sigma)=\frac{1}{\pi y}\biggl\{\exp\biggl(-\frac{1}{2}\int_{\mathbb{R}}\log(1+y^2)\alpha(dx)\biggr)\sin\biggl(\int_{\mathbb{R}}\arctan y\alpha(dx)\biggr)$$
$$-\exp\biggl(-\frac{1}{2}\int_{\mathbb{R}}\log\bigl[1+y^2\bigl(1-\psi(x)/\sigma\bigr)^2\bigr]\alpha(dx)\biggr)$$
$$\times\sin\biggl(\int_{\mathbb{R}}\arctan\bigl(y(1-\psi(x)/\sigma)\bigr)\alpha(dx)\biggr)\biggr\}$$

for any $\sigma\neq 0$, the relation

$$\int_0^{+\infty}q(y;\sigma)\,dy=\begin{cases}1-F_1(\sigma;\psi;\alpha), & \text{if } \sigma>0,\\ F_1(\sigma;\psi;\alpha), & \text{if } \sigma<0,\end{cases}\tag{4}$$

follows easily.

As usual, let $\alpha\circ\psi^{-1}$ denote the distribution of $\psi(X)$ when X has distribution α.

PROPOSITION 6. *If $\alpha\circ\psi^{-1}$ is symmetric, that is, $\alpha\{\psi(X)\le -y\}=\alpha\{\psi(X)\ge y\}$ for every y, then $\xi_{1\alpha}(\psi)$ is a symmetric random variable.*

PROOF. With $\psi(x)\equiv x$, the hypothesis becomes: α is symmetric. Then $\alpha(g(X))=\alpha(g(-X))$, for any real measurable function g. Using this in the expression of $q(y;\sigma)$ with $\sigma>0$, we see that $q(y;\sigma)=q(y;-\sigma)$ and, from (4),

$$1-F_1(\sigma;\alpha)=\int_0^{+\infty}q(y;\sigma)\,dy=\int_0^{+\infty}q(y;-\sigma)\,dy=F_1(-\sigma;\alpha).\qquad\square$$

4. Additional formulas for the distribution of a mean of a Dirichlet process. This section is a survey of expressions for F_1 and f_1 reminiscent of the Liouville–Weyl fractional integral $I_+^a g(x):=\Gamma(a)^{-1}\int_{-\infty}^x(x-\xi)^{a-1}g(\xi)\,d\xi$; see, for instance, Oldham and Spanier (1974). A natural question now is: Why make further efforts to obtain new forms for the distributions of means of a Dirichlet process? The answer is, essentially, that while forms provided in Section 3 are suitable for the numerical evaluation of the distributions under consideration, the formulas we deduce here can better express these distributions in terms of elementary or special functions. Cifarelli and Regazzini appear to be the first to have proved formulas of this type, under some extra assumptions. Compare, for example, Cifarelli and Regazzini (1990, 1993). The present treatment is based on weakened hypotheses, and profits by much simpler arguments: a suitable change of variable for the expressions of F_1 and f_1 in Propositions 3 and 4, respectively, and a discretization of A that provides *dominant* functions suitable for some particular applications of the Lebesgue dominated convergence theorem.

First, with the change of variable $z = T^{-1} + \sigma - it^{-1}$ we obtain

$$\int_{\varepsilon}^{T} \frac{1}{t} \exp\left(-\int_{\mathbb{R}} \log[1 - it(x-\sigma)]\alpha(dx)\right) dt$$
$$= \int_{\gamma_1} \left(\frac{1}{T} + \sigma - z\right)^{a-1} \exp\left(-\int_{\mathbb{R}} \log\left(\frac{1}{T} + x - z\right)\alpha(dx)\right) dz,$$

where γ_1 denotes the straight-line segment joining $(T^{-1} + \sigma - i\varepsilon^{-1})$ and $(T^{-1} + \sigma - iT^{-1})$. Moreover, Cauchy's theorem (on the integral of functions on a contour in $\mathbb{C}$) yields

$$\int_{\gamma_1} \left(\frac{1}{T} + \sigma - z\right)^{a-1} \exp\left(-\int_{\mathbb{R}} \log\left(\frac{1}{T} + x - z\right)\alpha(dx)\right) dz$$
$$= -\int_{\gamma_2 \cup \gamma_3} \left(\frac{1}{T} + \sigma - z\right)^{a-1} \exp\left(-\int_{\mathbb{R}} \log\left(\frac{1}{T} + x - z\right)\alpha(dx)\right) dz,$$

where γ_2 stands for the straight-line segment from $(T^{-1} + \sigma - iT^{-1})$ to $(2T^{-1} + \sigma - \varepsilon^{-1} - iT^{-1})$, and γ_3 is the arc of the circle $|z - T^{-1} - \sigma + iT^{-1}| = (\varepsilon^{-1} - T^{-1})$ from point $(2T^{-1} + \sigma - \varepsilon^{-1} - iT^{-1})$ to point $(T^{-1} + \sigma - i\varepsilon^{-1})$. Now, applying the change of variable $\xi = z - T^{-1} + iT^{-1}$ to the integrals on γ_2 and γ_3, and the dominated convergence theorem to the integral on γ_3, (2) yields

$$F_1(\sigma; \alpha) = -\frac{1}{\pi} \lim_{\substack{\varepsilon \searrow 0 \\ T \nearrow +\infty}} \int_{(1/T + \sigma - 1/\varepsilon, \sigma)} g(\xi, T)\, d\xi, \tag{5}$$

where $g(\xi, T) = |\sigma - \xi + \frac{i}{T}|^{a-1} \exp(-\int_{\mathbb{R}} \log|x - \xi + \frac{i}{T}|\alpha(dx)) \times \sin\{(a-1)\arctan(T(\sigma - \xi))^{-1} - \int_{(\xi,+\infty)} \arctan(T(\sigma-\xi))^{-1}\alpha(dx) - \int_{(-\infty,\xi)}[\pi - \arctan(T(\sigma-\xi))^{-1}]\alpha(dx) - \frac{\pi}{2}\alpha\{\xi\}\}$.

Then resort to the above-mentioned discretization to obtain the following result.

PROPOSITION 7. *The equality*

$$F_1(\sigma; \psi; \alpha) = -\frac{1}{\pi} \lim_{T \nearrow +\infty} \int_{-\infty}^{\sigma} \operatorname{Im}\Bigg\{(\sigma - \xi + i/T)^{a-1}$$
$$\times \exp\left(-\int_{\mathbb{R}} \log(\psi(x) - \xi + i/T)\alpha(dx)\right)\Bigg\} d\xi$$

holds true for every σ when ψ is not α-degenerate, and for any $\sigma \neq \bar{x}$ if $\alpha \circ \psi^{-1} = a\delta_{\bar{x}}$.

The proof of this statement, as well as the proofs of all the other propositions in this section, is omitted. They can all be found in Regazzini, Guglielmi and Di Nunno (2000).

A sharper result can be formulated when A_ψ has no jump of size greater than or equal to 1.

PROPOSITION 8. *If ψ is not α-degenerate, and $\alpha \circ \psi^{-1}$ has no jump of size greater than or equal to* 1, *then*

$$\begin{aligned} F_1(\sigma;\psi;\alpha) &= \frac{1}{\pi}\int_{-\infty}^{\sigma}(\sigma-\xi)^{a-1}\exp\biggl(-\int_{\mathbb{R}-\{\xi\}}\log|x-\xi|\alpha\circ\psi^{-1}(dx)\biggr) \\ &\qquad \times \sin\{\pi A_\psi(\xi)\}\,d\xi \\ &= I_+^a\biggl(\exp\biggl(-\int_{\mathbb{R}-\{\sigma\}}\log|x-\sigma|\alpha\circ\psi^{-1}(dx)\biggr) \\ &\qquad \times \sin\{\pi A_\psi(\sigma)\}\biggr), \qquad \sigma\in\mathbb{R}. \end{aligned}$$

As far as the density function is concerned, we have the following proposition.

PROPOSITION 9. *Suppose that ψ is not α-degenerate, and assume σ is in $C(A_\psi)$.*

(i) *If $a > 1$, then*

$$\begin{aligned} f_1(\sigma;\psi;\alpha) = -\frac{1}{\pi}\lim_{T\nearrow+\infty}\int_{-\infty}^{\sigma}\mathrm{Im}\biggl\{&(a-1)\bigl(\sigma-\xi+i/T\bigr)^{a-2} \\ &\times\exp\biggl(-\int_{\mathbb{R}}\log\bigl(\psi(x)-\xi+i/T\bigr)\alpha(dx)\biggr)\biggr\}\,d\xi. \end{aligned}$$

(ii) *If $a > 1$ and the saltus of A_ψ at each discontinuity point is smaller than* 1, *then*

$$\begin{aligned} f_1(\sigma;\psi;\alpha) &= \frac{a-1}{\pi}\int_{-\infty}^{\sigma}(\sigma-\xi)^{a-2}\exp\biggl(-\int_{\mathbb{R}-\{\xi\}}\log|x-\xi|\alpha\circ\psi^{-1}(dx)\biggr) \\ &\qquad \times \sin\{\pi A_\psi(\xi)\}\,d\xi. \end{aligned}$$

(iii) *If $a = 1$, then*

$$f_1(\sigma;\psi;\alpha) = \frac{1}{\pi}\exp\biggl(-\int_{\mathbb{R}-\{\sigma\}}\log|x-\sigma|\alpha\circ\psi^{-1}(dx)\biggr)\sin\{\pi A_\psi(\sigma)\}.$$

(iv) *If* $0 < a < 1$, *then*

$$f_1(\sigma;\psi;\alpha) = \frac{1-a}{\pi} \lim_{T\nearrow+\infty} \int_{-\infty}^{\sigma} \mathrm{Im}\Big\{(a-1)(\sigma-\xi+i/T)^{a-2} \times \Big[\exp\Big(-\int_{\mathbb{R}} \log(\psi(x)-\xi+i/T)\alpha(dx)\Big) - \exp\Big(-\int_{\mathbb{R}} \log(\psi(x)-\sigma+i/T)\alpha(dx)\Big)\Big]\Big\}\,d\xi.$$

Cifarelli and Regazzini (1990, 1993) have formulated representations (ii) and (iii) under the additional assumption that $A(\tau)=0$, for some $\tau > -\infty$.

Some examples illustrating the above results may be helpful.

EXAMPLES. (a) Let α be the uniform distribution on [0, 1]. Then f_1 can be deduced from Proposition 9(iii); that is,

$$f_1(\sigma;\alpha) = \frac{1}{\pi}\exp\Big(-\int_{(0,1)-\{\sigma\}} \log|x-\sigma|\,dx\Big)\sin(\pi\sigma) = \frac{e}{\pi}\frac{1}{\sigma^{\sigma}(1-\sigma)^{1-\sigma}}\sin(\pi\sigma), \qquad \sigma\in(0,1).$$

(b) Let α be the standard Gaussian distribution. According to Proposition 9(iii), $\int_{\mathbb{R}-\{\sigma\}} \log|x-\sigma|\alpha(dx)$ must be evaluated in order to obtain $f_1(\sigma;\alpha)$. Then, for any real σ,

$$\begin{aligned}\int_{\mathbb{R}-\{\sigma\}} \log|x-\sigma|\alpha(dx) &= e^{-\sigma^2/2}\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}}(\log|z|)e^{-z^2/2-\sigma z}\,dz \\ &= \frac{e^{-\sigma^2/2}}{\sqrt{2\pi}}\Big\{\int_0^{+\infty}(\log z)e^{-z^2/2+\sigma z}\,dz + \int_0^{+\infty}(\log z)e^{-z^2/2-\sigma z}\,dz\Big\} \\ &= \frac{e^{-\sigma^2/2}}{\sqrt{2\pi}}\frac{\partial}{\partial\nu}\Big\{\frac{\Gamma(\nu)}{2^{\nu/2}}\Psi\Big(\frac{\nu}{2},\frac{1}{2};\frac{\sigma^2}{2}\Big)\Big\}\Big|_{\nu=1},\end{aligned}$$

where Ψ is the Tricomi confluent hypergeometric function.

(c) If α is the Cauchy distribution, namely $\alpha(dx) = 1/(\pi(1+x^2))\,dx$, x in $\mathbb{R}$, then, by Proposition 9(iii),

$$f_1(\sigma;\alpha) = \frac{1}{\pi}\exp\Big\{-\int_{\mathbb{R}-\{\sigma\}} \frac{\log|x-\sigma|}{\pi(1+x^2)}\,dx\Big\}\sin\Big(\frac{\pi}{2}+\arctan\sigma\Big),$$

where $\sin(\frac{\pi}{2} + \arctan\sigma) = (1+\sigma^2)^{-1/2}$ and

$$\begin{aligned}\int_{\mathbb{R}-\{\sigma\}} \frac{\log|x-\sigma|}{\pi(1+x^2)}\,dx &= \int_0^{+\infty} \frac{\log\xi}{\pi(1+\sigma^2+\xi^2-2\sigma\xi)}\,d\xi \\ &\quad + \int_0^{+\infty} \frac{\log\xi}{\pi(1+\sigma^2+\xi^2+2\sigma\xi)}\,d\xi \\ &= \frac{1}{\pi}\left\{\arccos\frac{|\sigma|}{\sqrt{1+\sigma^2}}\log(1+\sigma^2)^{1/2} \right. \\ &\qquad \left. + \left(\pi - \arccos\frac{|\sigma|}{\sqrt{1+\sigma^2}}\right)\log(1+\sigma^2)^{1/2}\right\} \\ &= \frac{1}{2}\log(1+\sigma^2).\end{aligned}$$

Thus, as is well known, if α is Cauchy, f_1 is also Cauchy.

(d) Suppose $\alpha(dx) = \mu e^{-\mu x}\mathbb{I}_{(0,+\infty)}(x)\,dx$. Then the following holds true for every $\sigma > 0$:

$$\begin{aligned}f_1(\sigma;\alpha) &= \frac{1}{\pi}\exp\left(-\int_0^{+\infty}\log|x-\sigma|\mu e^{-\mu x}\,dx\right)\sin(\pi e^{-\mu\sigma}) \\ &= \frac{1}{\sigma\pi}\exp\left(e^{-\sigma\mu}\mathrm{Ei}(\sigma\mu)\right),\end{aligned}$$

where Ei stands for the exponential integral.

5. Distributional results for a vector of means of a Dirichlet process. In studying generalizations of the previous results to random vectors $(\xi_{1\alpha}(\psi_1), \dots, \xi_{1\alpha}(\psi_d))$, we shall assume that $\psi_1, \dots, \psi_d$ are measurable real-valued functions on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, satisfying the conditions of Proposition 1, that is, $\int \log|1 + it_k\psi_k|\,d\alpha < +\infty$ for every t_k in $\mathbb{R}$ and $k = 1, \dots, d$. Under these conditions, the characteristic function ϕ_d of $(\xi_\alpha(\psi_1), \dots, \xi_\alpha(\psi_d))$ is defined by $\phi_d(t;\psi;\alpha) = E(\exp\{i\langle t, \xi_\alpha(\psi)\rangle\})$ with $\psi := (\psi_1, \dots, \psi_d)$ and $\xi_\alpha(\psi) := (\xi_\alpha(\psi_1), \dots, \xi_\alpha(\psi_d))$, for any $t := (t_1, \dots, t_d) \in \mathbb{R}^d$. Since $\phi_d(\cdot;\psi;\alpha) \equiv \phi(1; \langle\cdot,\psi\rangle;\alpha)$, Proposition 2 can be employed to obtain an expression for ϕ_d.

PROPOSITION 10. *For any t in $\mathbb{R}^d$*,

$$\phi_d(t;\psi;\alpha) = \exp\left\{-\int_{\mathbb{R}}\log(1 - i\langle t,\psi\rangle)\,d\alpha\right\}.$$

Applying again the trick of Hannum, Hollander and Langberg, the distribution function $F_d(\cdot;\psi;\alpha)$ of $(\xi_{1\alpha}(\psi_1), \dots, \xi_{1\alpha}(\psi_d))$ obeys

$$F_d(\sigma;\psi;\alpha) = P\{\xi_\alpha(\psi_1 - \sigma_1) \le 0, \dots, \xi_\alpha(\psi_d - \sigma_d) \le 0\}$$

for any $\sigma := (\sigma_1, \dots, \sigma_d) \in \mathbb{R}^d$. Thus, the problem of finding F_d can be solved by inverting ϕ_d at the origin. We shall use the Gurland formula (d-dimensional version) to do so, obtaining

$$\begin{aligned}
&(-1)^{d+1} 2^d F_d(\sigma; \psi; \alpha) \\
&\quad = A_0 + \sum_{k=1}^{d} \frac{A_k}{(\pi i)^k} \\
&\qquad \times \sum_{1 \le j_1 < \cdots < j_k \le d} \mathrm{PV} \int \frac{\phi_k(t_1, \dots, t_k; \psi_{j_1} - \sigma_{j_1}, \dots, \psi_{j_k} - \sigma_{j_k}; \alpha)}{t_1 \cdots t_k} \, dt_1 \cdots dt_k
\end{aligned}$$

for every $\sigma \in C(F_d)$, where $A_0, A_1, \dots, A_d$ satisfy the system of equations $\{\sum_{k=0}^{d-r-1} \binom{d-r}{k} A_{r+k} = 1,\ A_d = -1$ for $r = 0, 1, \dots, d-1\}$, and PV $\int$ designates the *principal value integral*, that is, $\mathrm{PV}\int = \lim_{\varepsilon \searrow 0,\, T \nearrow +\infty} \int_{\varepsilon < |t| < T}$ with $\varepsilon = (\varepsilon_1, \dots, \varepsilon_d)$, $T = (T_1, \dots, T_d)$ and $|t| = (|t_1|, \dots, |t_d|)$.

5.1. *Representation of the distribution function and density function.* In the above inversion formula, PV $\int$ can be replaced by a Lebesgue integral, as in the one-dimensional case. The technique of the proof is the same in any dimension $d \ge 2$ but, to avoid cumbersome notation, we specialize to the case $d = 2$.

Let $g(t; \psi - \sigma; \alpha)$ be the function defined by

$$\begin{aligned}
&t_1 t_2 g(t; \psi - \sigma; \alpha) \\
&\quad := \mathrm{Re}\{\phi_2(t_1, t_2; \psi_1 - \sigma_1, \psi_2 - \sigma_2; \alpha) - \phi_2(t_1, -t_2; \psi_1 - \sigma_1, \psi_2 - \sigma_2; \alpha)\} \\
&\quad = \exp\left(-\int_{\mathbb{R}} \log|1 - i(\psi_1 - \sigma_1)t_1 - i(\psi_2 - \sigma_2)t_2| \, d\alpha\right) \\
&\qquad \times \cos\left\{\int_{\mathbb{R}} \arctan[(\psi_1 - \sigma_1)t_1 + (\psi_2 - \sigma_2)t_2] \, d\alpha\right\} \\
&\qquad - \exp\left(-\int_{\mathbb{R}} \log|1 - i(\psi_1 - \sigma_1)t_1 + i(\psi_2 - \sigma_2)t_2| \, d\alpha\right) \\
&\qquad \times \cos\left\{\int_{\mathbb{R}} \arctan[(\psi_1 - \sigma_1)t_1 - (\psi_2 - \sigma_2)t_2] \, d\alpha\right\}.
\end{aligned}$$

Throughout the section, a measurable vector (X_1, X_2) on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is said to be α-degenerate if there is a straight line r such that $\alpha\{x \in \mathbb{R} : (X_1(x), X_2(x)) \in r\} = a$.

PROPOSITION 11. *If* (ψ_1, ψ_2) *is not* α*-degenerate, then*

$$F_2(\sigma;\psi;\alpha) = \frac{1}{2}[F_1(\sigma_1;\psi_1;\alpha) + F_1(\sigma_2;\psi_2;\alpha)] - \frac{1}{4} - \frac{1}{2\pi^2}\int_{(0,+\infty)^2} g(t;\psi-\sigma;\alpha)\,dt$$

for any σ *in* $\mathbb{R}^2$.

The proof is given in the Appendix.

By analogy with the one-dimensional case, we can conjecture that:

(C) *Under the conditions of Proposition* 11, F_2 *is absolutely continuous* (*with respect to the Lebesgue measure* λ^2 *in* $\mathbb{R}^2$) *and there is* $a_0 > 0$ *such that*

$$f_2(\sigma;\psi;\alpha) = \frac{1}{2\pi^2}\int_{(0,+\infty)^2} \frac{1}{t_1 t_2}\,\mathrm{Re}\biggl\{\frac{\partial^2}{\partial\sigma_1\partial\sigma_2}\biggl(\exp\biggl(\int_{\mathbb{R}}\log[1 - it_1\hat{\psi}_1 + it_2\hat{\psi}_2]\,d\alpha\biggr) - \exp\biggl(\int_{\mathbb{R}}\log[1 - it_1\hat{\psi}_1 - it_2\hat{\psi}_2]\,d\alpha\biggr)\biggr)\biggr\}\,dt_1\,dt_2$$

is an expression of the density function of F_2, *for any* α *with* $\alpha(\mathbb{R}) > a_0$, λ^2*-almost every* σ.

Hence, in the following subsection, the soundness of any result involving the above expression of f_2 is subject to the validity of (C).

5.2. *Two applications of the multidimensional results.* Cifarelli and Melilli (2000) have recently dealt with the probability distribution of $V = V(\xi_{1\alpha}) := (\int x^2\,d\xi_{1\alpha} - [\int x\,d\xi_{1\alpha}]^2)$. While their results do reveal some interesting aspects of the structure of the law of V, they do not, it seems to us, provide a feasible algorithm for the numerical evaluation of the distribution of V. Moreover, the authors confine themselves to considering α parameters such that $\alpha((-\infty, \nu]) = 0$ for some $\nu \in \mathbb{R}$ and $a \in \mathbb{N}$. Finding the exact distribution of V as an almost direct application of Proposition 11 and conjecture (C) constitutes the subject of the first part of this subsection. Our approach—free of any redundant condition—produces formulas reminiscent of the expression of the distribution of $\int x\,d\xi_{1\alpha}$; consequently, we may reasonably expect they will provide computable approximations as will be illustrated in Section 6.

We start by specializing some of the previous propositions when $\psi_1(x) \equiv x$ and $\psi_2(x) \equiv x^2$, under the hypothesis that $\int \log(1 + x^2)\alpha(dx) < +\infty$. Proposition 11 and conjecture (C) provide us with the expressions of the distribution function H and of the density function h of $(\int x\, d\xi_{1\alpha}, \int x^2\, d\xi_{1\alpha})$:

$$H(\sigma_1,\sigma_2) = \frac{1}{2}\{H_1(\sigma_1) + H_2(\sigma_2)\} - \frac{1}{4} + \frac{1}{2\pi^2}\int_{(0,+\infty)^2} g^*(t;\sigma)\, dt,$$

$$h(\sigma_1,\sigma_2) = \frac{1}{2\pi^2}\int_{(0,+\infty)^2}\left[\frac{\partial^2 g^*}{\partial\sigma_1\partial\sigma_2}(t;\sigma)\right] dt, \qquad \lambda^2\text{-a.e.},$$

where H_1 (h_1) and H_2 (h_2) are the distribution functions (densities) of $\int x\, d\xi_{1\alpha}$ and $\int x^2\, d\xi_{1\alpha}$, respectively, and $g^*(t;\sigma) = (t_1t_2)^{-1}\,\mathrm{Re}[\exp\{-\int \log[1 - it_1(x-\sigma_1) + it_2(x^2-\sigma_2)]\alpha(dx)\} - \exp\{-\int \log[1 - it_1(x-\sigma_1) - it_2(x^2-\sigma_2)]\alpha(dx)\}]$, $t \in (0,+\infty)^2$ and $\sigma \in \mathbb{R}^2$.

It is now easy to formulate the probability distribution function H^* of $([\int x\, d\xi_{1\alpha}]^2, \int x^2\, d\xi_{1\alpha})$, that is,

$$\begin{aligned} H^*(w_1,w_2) &= H(\sqrt{w_1}, w_2) - H(-\sqrt{w_1}, w_2) \\ &= \frac{1}{2}\big[H_1(\sqrt{w_1}) - H_1(-\sqrt{w_1})\big] \\ &\quad + \frac{1}{2\pi^2}\int_{(0,+\infty)^2}\big\{g^*(t;\sqrt{w_1},w_2) - g^*(t;-\sqrt{w_1},w_2)\big\}\, dt \end{aligned}$$

for $w_1 \ge 0$ and $w_2 \in \mathbb{R}$. Analogously, the density function h^* of the same random vector can be expressed by

$$h^*(w_1,w_2) = \frac{1}{2\pi^2}\int_{(0,+\infty)^2}\frac{\partial^2\zeta}{\partial w_1\partial w_2}(t;w_1,w_2)\, dt$$

with $\zeta(t;w_1,w_2) := g^*(t;\sqrt{w_1},w_2) - g^*(t;-\sqrt{w_1},w_2)$, and

$$\gamma_V(v) := \int_0^{+\infty} h^*(w_1, w_1+v)\, dw_1, \qquad v \ge 0,$$

is a density for the probability distribution of V.

In the numerical evaluation of the probability distribution of V, the following relations might be of use:

$$s_n \nearrow P\{V \le v\}, \qquad S_n \searrow P\{V \le v\} \qquad \text{as } n \to +\infty,$$

where

$$s_n = \frac{1}{2} + \frac{1}{2\pi^2}\sum_{k\ge 0}\int_{(0,+\infty)^2}\left[\zeta\left(t;\frac{k+1}{2^n},\frac{k}{2^n}+v\right) - \zeta\left(t;\frac{k}{2^n},\frac{k}{2^n}+v\right)\right] dt,$$

$$S_n = \frac{1}{2} + \frac{1}{2\pi^2}\sum_{k\ge 0}\int_{(0,+\infty)^2}\left[\zeta\left(t;\frac{k+1}{2^n},\frac{k+1}{2^n}+v\right) - \zeta\left(t;\frac{k}{2^n},\frac{k+1}{2^n}+v\right)\right] dt.$$

Our second application indicates how to deal with the problem of finding the exact distribution of the random vector $W = (W(z_1),\dots,W(z_d))$ defined in

Conti (1999). In view of the definition of W, one can start from the law of $\widetilde{W} := ([1-\lambda\int x\,d\xi_{1\alpha}]\mathbb{I}\{0 < 1-\lambda\int x\,d\xi_{1\alpha} < 1\}, B(z_1), \dots, B(z_d))$ where λ is any fixed number in $(0, 1)$, $z_1, \dots, z_d$ belong to a suitable subinterval of $\mathbb{R}$, the parameter α has support included in $\mathbb{N}$ and $B(z_j) := \int z_j^x \xi_{1\alpha}(dx)$ with $j = 1, \dots, d$. For the sake of illustration, let $d = 1$.

The distribution function G of $\widetilde{W}$, at $(\sigma_1, \sigma_2) \in [0, +\infty] \times \mathbb{R}$, is given then by

$$
\begin{aligned}
G(\sigma_1, \sigma_2)&\\
&= P\biggl\{\sum_{k\ge 1} k\xi_{1\alpha}\{k\} \ge \frac{1-\sigma_1}{\lambda}, \sum_{k\ge 1} z_1^k \xi_{1\alpha}\{k\} \le \sigma_2\biggr\}\\
&= \frac{1}{4} + \frac{1}{2\pi}\int_0^{+\infty} t^{-1}\,\mathrm{Im}\biggl[\exp\biggl(-\sum_{k\ge 1}\log\biggl[1 - it\biggl(k - \frac{1-\sigma_1}{\lambda}\biggr)\biggr]\alpha\{k\}\biggr)\\
&\qquad\qquad - \exp\biggl(-\sum_{k\ge 1}\log[1 - it(z_1^k - \sigma_2)]\alpha\{k\}\biggr)\biggr]\,dt\\
&\quad + \frac{1}{2\pi^2}\int_{(0,+\infty)^2} (t_1 t_2)^{-1}\,\mathrm{Re}\biggl[\exp\biggl(-\sum_{k\ge 1}\log\biggl[1 - it_1\biggl(k - \frac{1-\sigma_1}{\lambda}\biggr)\\
&\qquad\qquad - it_2(z_1^k - \sigma_2)\biggr]\alpha\{k\}\biggr)\\
&\qquad\qquad - \exp\biggl(-\sum_{k\ge 1}\log\biggl[1 - it_1\biggl(k - \frac{1-\sigma_1}{\lambda}\biggr)\\
&\qquad\qquad + it_2(z_1^k - \sigma_2)\biggr]\alpha\{k\}\biggr)\biggr]\,dt_1\,dt_2.
\end{aligned}
$$

At this stage, standard reasoning yields the distribution of $W(z_1) := [1 - \tilde{\lambda} - z_1 + \tilde{\lambda}B(z_1)]^{-1}(1-\rho)(1-z_1)\mathbb{I}_{0<\rho<1}$ where $\tilde{\lambda}$ is a random variable with a Beta distribution, $\tilde{\lambda}$ and $\xi_{1\alpha}$ are stochastically independent, and $\rho := \tilde{\lambda}\int x\,d\xi_{1\alpha}$.

Multidimensional laws with $d \ge 2$ lack the relative simplicity of the case where $d = 1$ and require further study.

6. Numerical analysis for the one-dimensional case. As we have already said in Section 1, we are interested in providing a procedure for the numerical evaluation of $F_1(\cdot; \psi; \alpha)$ that meets any prescribed error of approximation. By approximation error we mean here the distance, in the uniform metric, between the approximating distribution and the exact one. This objective could also be reached with a direct numerical evaluation of (2). This is exactly what Tamura (1988) has done, referring, in turn, to Davies (1973). Davies sketches an algorithm that requires further conditions, for example, the finiteness of the expectation of the distribution to be evaluated. Moreover, Davies' bounds for the error are not easily expressed only in terms of the characteristic function. Hughett (1998) has extended the work of Davies by obtaining error bounds in the L_p-norm (for any $1 \le p \le +\infty$), based on conditions on the decay of the distribution function to be determined.

Our idea has been to take advantage of (3) for an accurate approximation of the distribution at issue, that does not require adding any of the extra conditions considered by these authors.

Here, we consider only the case $\psi(x) \equiv x$ $(x \in \mathbb{R})$, since, as we have already remarked, extension to more general ψ can be dealt with by suitably changing the parameter measure. To evaluate this distribution function, we first produce a computable approximation of it. Thus, given $\varepsilon > 0$, we look for a finite measure α_n supported by a finite set $\{x_0^{(n)}, x_1^{(n)}, \dots, x_{k_n}^{(n)}\}$, for which $x_j^{(n)} < x_{j+1}^{(n)}$, $\forall\, j$, and such that the approximation error $\sup_{\sigma\in\mathbb{R}} |F_1(\sigma;\alpha) - F_1(\sigma;\alpha_n)|$ is less than ε. We can then evaluate $F_1(\cdot;\alpha_n)$, it being an easy task to evaluate $F_1(\cdot;\alpha)$ numerically when α has finite support; see (3).

6.1. *Approximation of* F_1. The procedure for the approximation is split into a number of steps. Step 0 is technical, and provides some useful inequalities. Steps 1 and 2 analyze the approximation of F_1 outside a compact set (the tails of F_1), and inside the compact set, respectively. The reader is referred to Section 3 for basic results and notation. In particular, see Section 3.3 for the definition of q.

STEP 0. *Let* $\rho \in (0,1)$, $\eta \in (0,a)$; *moreover, take* $M_1 < 0$ *and* $M_2 > 0$ *such that* $A((1-\rho)M_2) > a - \eta$ *and* $A((1-\rho)M_1) < \eta$. *Then, for* $b > 0$,

$$\left|\int_b^{+\infty} q(y;\sigma)\,dy\right| \le I(b,1,a/2) + I\big(b,\rho,(a-\eta)/2\big)$$

for any $\sigma \in (-\infty, M_1) \cup [M_2, +\infty)$ *with*

$$I(u,v,z) := \frac{1}{\pi}\int_u^{+\infty} \frac{1}{y(1+(yv)^2)^z}\,dy, \qquad u,v,z > 0.$$

Indeed,

$$\begin{aligned}\left|\int_b^{+\infty} q(y;\sigma)\,dy\right| &\le \int_b^{+\infty} \frac{1}{\pi y(1+y^2)^{a/2}}\,dy \\ &\quad + \int_b^{+\infty} \frac{1}{\pi y}\exp\left\{-\frac{1}{2}\int_{\{x\,:\,|1-x/\sigma|>\rho\}} \log[1+(y\rho)^2]\alpha(dx)\right\}dy\end{aligned}$$

$$\leq I(b,1,a/2) + I\left(b,\rho,\inf_{\sigma\in[M_1,M_2]^c}\alpha\{x:|1-x/\sigma|>\rho\}/2\right)$$

$$\leq I(b,1,a/2) + \begin{cases} I\big(b,\rho,A(M_2(1-\rho))\big), & \text{if } \sigma \geq M_2,\\ I\big(b,\rho,a-A(M_1(1-\rho))\big), & \text{if } \sigma < M_1.\end{cases}$$

The following proposition controls the approximation in the tails.

STEP 1. *Given* $\varepsilon > 0$, *there exist* $M_1 = M_1(\varepsilon)$ *and* $M_2 = M_2(\varepsilon)$ *such that* $M_1 < 0 < M_2$ *and*

$$\sup_{\sigma\in[M_1,M_2)^c}|F_1(\sigma;\alpha) - F_1(\sigma;\tilde{\alpha})| < \varepsilon, \tag{6}$$

for any finite measure $\tilde{\alpha}$ *with support contained in* $[M_1, M_2)$.

Values for M_i $(i = 1, 2)$ are specified in the proof, given in the Appendix.

Let us now evaluate $|F_1(\sigma;\alpha) - F_1(\sigma;\alpha_n)|$ when $\sigma \in [M_1, M_2)$, for some measure α_n with the proprieties indicated at the beginning of the present section.

STEP 2. *Given any* $\varepsilon > 0$, *we can determine the constants* M_1, M_2 *in Step* 1 *and a finite measure* α_n *with support* $\{x_0^{(n)}, x_1^{(n)}, \dots, x_{k_n}^{(n)}\} \subset [M_1, M_2)$ *in such a way that*

$$\sup_{\sigma\in[M_1,M_2)}|F_1(\sigma;\alpha) - F_1(\sigma;\alpha_n)| < \varepsilon. \tag{7}$$

Explicit estimates of M_i $(i = 1, 2)$ and an expression for α_n are given in the Appendix.

Obviously, when the support of α is bounded, Steps 0 and 1 can be omitted.

6.2. *Some illustrative examples.* To illustrate the approximation procedure described in Section 6.1, we consider the case where α is Gaussian, $a = 1$, and $\varepsilon = 0.005$. We start from the analysis of the error produced by the approximation of the tails of the distribution, recalling that, by Step 1, there exist $M_1 < 0 < M_2$ such that (6) holds for any measure α_n with finite support contained in $[M_1, M_2]$. Since the measure α is symmetric (with respect to the origin), we explicitly compute the approximation error connected with the right tail only. According to Step 0, with $\rho = 0.9$, $\eta = 0.00001$, if $M_1 = -150$ and $M_2 = 150$, then $A((1-\rho)M_2) > a - \eta$, $A((1-\rho)M_1) < \eta$. Moreover, assuming $b = 10^3$, we obtain $I(b,1,a/2) + I(b,\rho,(a-\eta)/2) \leq 0.000672$. As shown in the Appendix [see (9) and (10)], $\Delta_1(b;\sigma)$ is bounded by the sum of four functions of σ, which vanish as $\sigma \to +\infty$. Therefore $\sup_{\sigma\geq M_2}\Delta_1(b;\sigma)$ is bounded by that sum, evaluated at $\sigma = M_2$. To this end, it is convenient to resort to a numerical integration method. We employ standard numerical integration procedures on $\mathbb{R}^2$ in the IMSL/Fortran library, which give $\sup_{\sigma \ge M_2} \Delta_1(b;\sigma) \le 0.001693$, in our case. Similarly, we compute the sum of the integral functions, evaluated at $\sigma = M_2$, that bounds $\sup_{\sigma \ge M_2} \Delta_2(b;\sigma)$ [see (11) in the Appendix], yielding $\sup_{\sigma \ge M_2} \Delta_2(b;\sigma) \le 0.001706$. Summing up, $\sup_{\sigma \in [M_1, M_2)^c} |F_1(\sigma;\alpha) - F_1(\sigma;\alpha_n)| \le 0.004071 < \varepsilon$.

We now compute the approximation error in $[M_1, M_2)$ and determine explicitly α_n (see the Appendix for the notation). In this case, measure α_n has support $\{x_1^{(n)}, \dots, x_{n-1}^{(n)}, x_n^{(n)}\}$, with $x_j^{(n)} := A^{-1}(j/n)$, $j = 1, \dots, n-1$, $x_n^{(n)} := M_2$ and $\alpha_n(\{x_j^{(n)}\}) = 1/n$, $j = 1, \dots, n$. Moreover, $\omega_n = 1/n$. We find that $3I(c, h, \frac{Q_h}{2}) + I(c, h, \frac{Q_h}{2} - \omega_n) \le 0.000880$, when $h = 0.2$, $c = 3.5 \cdot 10^4$, and n is an integer between 9000 and 10000. Furthermore, setting $l = 0.1$, $n = 9315$, we obtain $J_1(c, M_1, M_2) + \omega_n J_2(c, M_1, M_2) \le 0.004120$, so that (7) gives $\sup_{\sigma \in [M_1, M_2)} |F_1(\sigma;\alpha) - F_1(\sigma;\alpha_n)| \le 0.005$. Obviously, since there exist many b's, c's, h's and l's satisfying the hypotheses in Steps 0–2, we have taken here those values which seem to minimize the bounds.

The scheme illustrated here for a Gaussian distribution maintains its validity for any finite measure on $\mathbb{R}$.

Table 1 presents the approximation error, together with the number of points in the support of the approximating α_n for $\alpha(\cdot) := a\alpha_0(\cdot)$, some values of a, and some common probability measures α_0. The figures illustrate the shape of the density functions associated with the approximating distribution functions, α_n, taken into account in Table 1.

Figure 1 provides us with a direct qualitative comparison of the shapes of the densities corresponding to the different values of a considered, when α is proportional to the Gaussian (i), the Beta (ii), and the Gamma distribution (iii). These pictures show that, as we expected, the concentration increases with a. Figure 2(i) exhibits both the exact density and that associated with α_n when α is

TABLE 1

Approximation error (first row) and number of points in the support of α_n (second row), corresponding to each α_0

	a		
$\boldsymbol{\alpha_0}$	**1**	**10**	**100**
Gauss(0, 1)	0.005	0.005	0.010
	9315	69170	702400
Beta(1, 4)	0.005	0.005	0.010
	5817	21120	487800
Gamma(0.5, 1)	0.005	0.005	0.010
	9256	48350	494400
Cauchy(0, 1)	0.005	0.005	0.010
	23155	179530	1486500

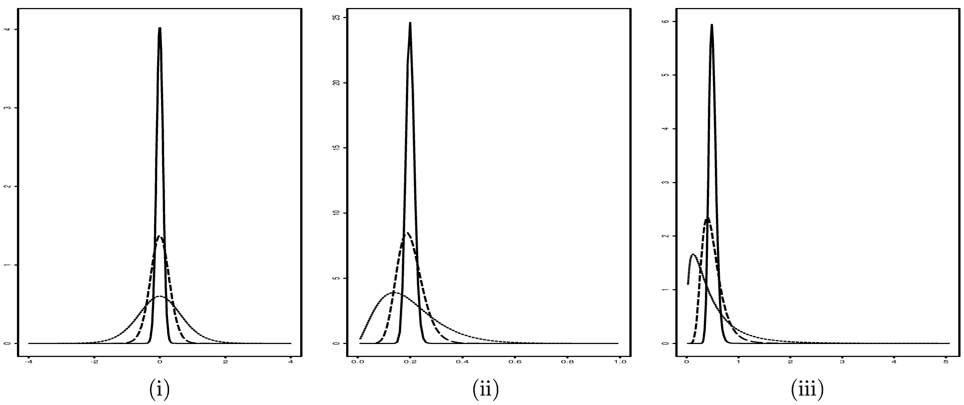

FIG. 1. *Densities corresponding to the approximating parameter* α_n *for* α *proportional to*: (i) Gauss(0, 1), (ii) Beta(1, 4), (iii) Gamma(0.5, 1). *In each figure the dotted line corresponds to* $a = 1$, *the dashed line to* $a = 10$ *and the solid line to* $a = 100$.

the standard Cauchy distribution; in this case, the exact and the approximating distribution functions are shown in Figure 2(ii). In both graphs the two functions are indistinguishable.

We now take the case of $\alpha = a \cdot \text{Poisson}(\theta)$, with $\theta = 2$, $a = 1, 10$. We define α_n by $\alpha_n(\{k\}) = \alpha(\{k\})$, with $k = 0, \dots, [M_2] - 1$, and $\alpha_n(\{[M_2]\}) = a - A([M_2] - 1)$, where $M_2 = 300.000001$ for $a = 1$ and $M_2 = 1500.000001$ for $a = 10$. In this way, we achieve an error in (7) of less than 0.0075. It is interesting to see that Figure 3(i) confirms the existence of singularities for $f_1(\cdot; \psi; \alpha)$ when

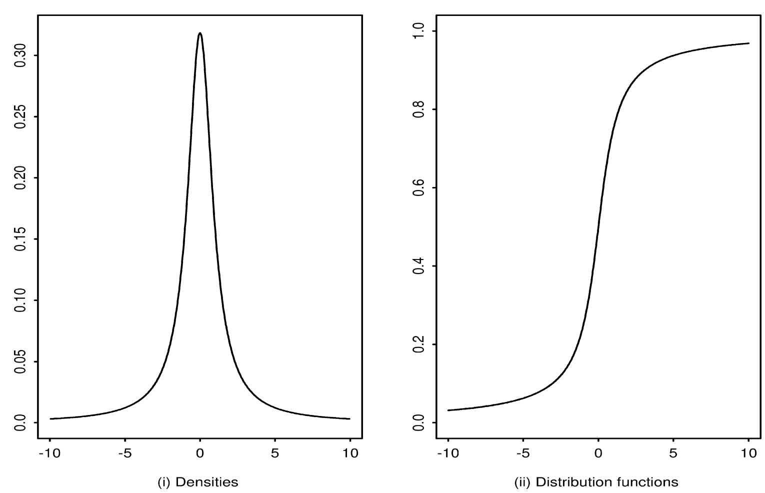

FIG. 2. *Densities and distribution functions* (*approximated and exact*) *corresponding to* α_n *when* α *is* Cauchy(0, 1).

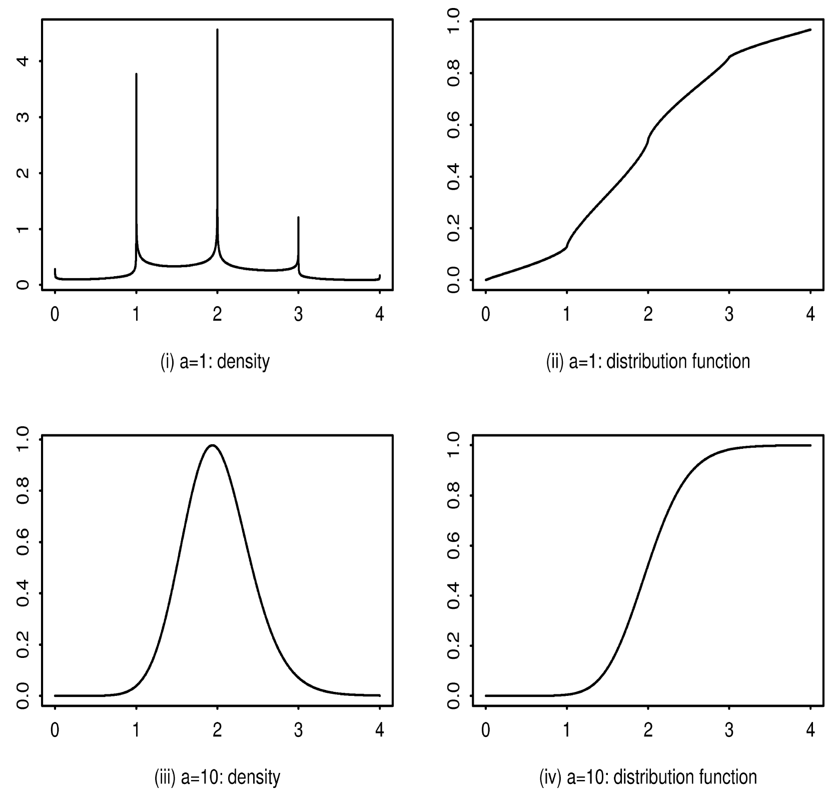

FIG. 3. *Densities and distribution functions corresponding to* α_n *when* α *is* $a \cdot \text{Poisson}(2)$, $a = 1, 10$.

$a = 1$; these singularities disappear when $a = 10$. Note that we have used the expression of $f_1(\cdot;\psi;\alpha_n)$ in Proposition 5 to plot Figure 3(iii).

Finally, we consider an example in which α is decomposed into an absolutely continuous component and a discrete one: $\alpha = 1/2\,\text{Beta}(1/9, 1) + \delta_{x_1} + \delta_{x_2}$, with $x_1 = 0.05$, $x_2 = 0.1$; compare Tamura (1988). This is the same as considering the posterior distribution of the mean of a Dirichlet process, with a parameter proportional to $1/2\,\text{Beta}(1/9, 1)$ given (x_1, x_2), when x_1, x_2 are conditionally independent and identically distributed observations. According to Step 2, with $M_1 = 0$, $M_2 = 1$, we choose $c = 10^6$, $h = l = 0.01$ and $n = 4144$ to achieve an error in (7) of less than 0.005. Hence the support of the approximating α_n, as defined in the Appendix, is $\{x_1^{(n)}, \dots, x_{1485}^{(n)},\ x_{1486}^{(n)} = x_1,\ x_{1487}^{(n)}, \dots, x_{1605}^{(n)},\ x_{1606}^{(n)} = x_2,\ x_{1607}^{(n)}, \dots, x_{2074}^{(n)}\}$, where $x_j^{(n)} = [(j/n)/a]^9$, $j = 1, \dots, 1485$, $[((j + 4143)/n - 1)/a]^9$, $j = 1487, \dots, 1605$, $[((j + 8286)/n - 2)/a]^9$, $j = 1607, \dots, 2074$, and $\alpha_n(\{x_j^{(n)}\}) = 1/n$ for all $j \neq 1486, 1606$, and $\alpha_n(\{x_1\}) = \alpha_n(\{x_2\}) = 1$. Figure 4 shows the density associated with α_n.

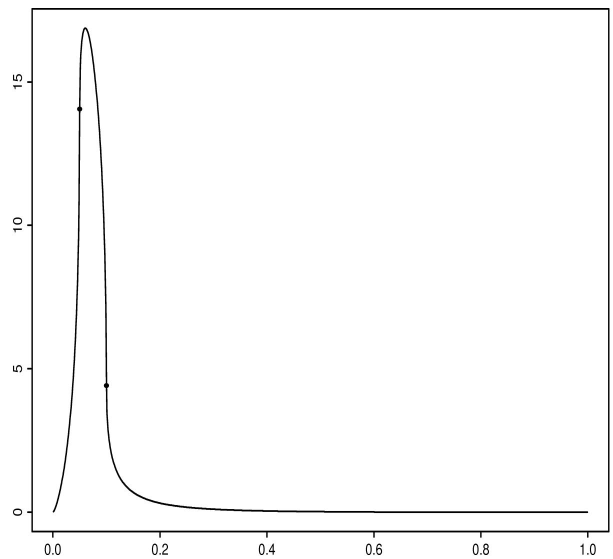

FIG. 4. *Density corresponding to α_n for $\alpha = 1/2\,\mathrm{Beta}(1/9,1) + \delta_{x_1} + \delta_{x_2}$. The points with abscissae x_1 and x_2 are marked.*

APPENDIX

A. Proof of Proposition 11. For $d = 2$, the Gurland formula reduces to

$$F_2(\sigma;\psi;\alpha) = \frac{1}{2}[F_1(\sigma_1;\psi_1;\alpha) + F_1(\sigma_2;\psi_2;\alpha)] - \frac{1}{4} - \frac{1}{2\pi^2} \lim_{\substack{\varepsilon\searrow 0\\ T\nearrow+\infty}} \int_{\varepsilon<t<T} g(t;\psi-\sigma;\alpha)\,dt$$

provided that $\sigma \in C(F_2)$. It is enough to prove that $|g|$ is integrable on $(0,+\infty)^2$ with respect to the Lebesgue measure λ^2 on $\mathbb{R}^2$. For brevity we adopt the following notation: $\hat{\psi}_i = \psi_i - \sigma_i$ $(i = 1,2)$; $\varphi_\pm = (\hat{\psi}_1 t_1 \pm \hat{\psi}_2 t_2)$; $\gamma_\pm = \arctan\varphi_+ \pm \arctan\varphi_-$; $\beta_i = \arctan(2t_i|\hat{\psi}_i||1-\varphi_+\cdot\varphi_-|^{-1})$; $Q_1(\eta_1,\eta_2) = \{\hat{\psi}_1 > \eta_1, \hat{\psi}_2 > \eta_2\}$, $Q_2(\eta_1,\eta_2) = \{\hat{\psi}_1 \le \eta_1, \hat{\psi}_2 > \eta_2\}$, $Q_3(\eta_1,\eta_2) = \{\hat{\psi}_1 \le \eta_1, \hat{\psi}_2 \le \eta_2\}$, $Q_4(\eta_1,\eta_2) = \{\hat{\psi}_1 > \eta_1, \hat{\psi}_2 \le \eta_2\}$, for any $\eta_i > 0$ $(i = 1,2)$; $l(x) = (2x_1x_2)^{-1}\log\{1 + 4x_1x_2[1 + (x_1 - x_2)^2]^{-1}\}$; $v(t_1,t_2) = 4|\hat{\psi}_1\hat{\psi}_2|t_1t_2\{1 + (|\hat{\psi}_1|t_1 - |\hat{\psi}_2|t_2)^2\}^{-1}$; the relation $f \lesssim g$ between positive functions means that $f \le Cg$ for some constant $C < +\infty$. We break the argument into several steps, and assume that $\sigma \in C(F_2)$.

STEP 1. $|g|\mathbb{I}_{A_1}$ is integrable with $A_1 := (\delta,+\infty)\times(\varepsilon,+\infty)$ for any pair (δ,ε) of strictly positive numbers. Indeed, $|g|\mathbb{I}_{A_1} \le (t_1t_2)^{-1}\mathbb{I}_{A_1} \times \{\exp[-\frac{1}{2}\int\log(1+\varphi_+^2)\,d\alpha] + \exp[-\frac{1}{2}\int\log(1+\varphi_-^2)\,d\alpha]\}$ and, since (ψ_1,ψ_2) is not

α-degenerate, there exist $a', b' > 0$ such that $\inf_{c\in\mathbb{R}^2} \alpha\{|c_1\hat{\psi}_1 + c_2\hat{\psi}_2| > a'\} = b'$. Hence, $|g|\mathbb{I}_{A_1} \le 2(t_1t_2)^{-1}\mathbb{I}_{A_1}\exp[-\log(1 + (a't_1t_2)^2)^{-b'/2}] \lesssim (t_1t_2)^{-b'-1}\mathbb{I}_{A_1}$, which is integrable with respect to λ^2.

STEP 2. $|g|\mathbb{I}_{A_2}$ is integrable with $A_2 := (0,\delta)\times(0,\varepsilon)$. To prove this, represent $g\mathbb{I}_{A_2}$ as the sum of two functions: $(t_1t_2)^{-1}\mathbb{I}_{A_2}$ $\{\exp[-\int\log|1 - i\varphi_+|\,d\alpha] - \exp[-\int\log|1 - i\varphi_-|\,d\alpha]\}$ $\cos[\int\arctan\varphi_+\,d\alpha] + (t_1t_2)^{-1}\mathbb{I}_{A_2}$ $\exp[-\int\log|1 - i\varphi_-|\,d\alpha]$ $\{\cos[\int\arctan\varphi_+\,d\alpha] - \cos[\int\arctan\varphi_-\,d\alpha]\}$. Then, apply elementary formulas for the difference of cos's and arctan's, and employ the well-known inequality $|\sin x| \le |x|$, to show that the latter addend in the above representation of $g\mathbb{I}_{A_2}$ is bounded by $(2t_1t_2)^{-1}\int|\gamma_+|\,d\alpha\int|\gamma_-|\,d\alpha\;\mathbb{I}_{A_2} \lesssim (t_1t_2)^{-1}\mathbb{I}_{A_2}\int_{\mathbb{R}^2}\{\beta_1(x)\mathbb{I}_{\{\varphi_+\cdot\varphi_-\neq 1\}}(x) + \pi\mathbb{I}_{\{\varphi_+\cdot\varphi_- > 1\}}(x)\}$ $\{\beta_2(y)\mathbb{I}_{\{\varphi_+\cdot\varphi_-\neq -1\}}(y) + \pi\mathbb{I}_{\{\varphi_+\cdot\varphi_- < -1\}}(y)\}\,\alpha(dx)\,\alpha(dy)$ that (since $\arctan x \lesssim x(1+x^2)^{-1/2}$) is dominated by

$$\begin{aligned}
&\mathbb{I}_{A_2}\int|\hat{\psi}_1(x)\hat{\psi}_2(y)|\{[1 + 2t_1^2\hat{\psi}_1^2(x)][1 + 2t_2^2\hat{\psi}_2^2(y)]\}^{-1/2}\\
&\qquad\times\mathbb{I}_{\{\varphi_+(x)\varphi_-(x)\neq 1\}\cap\{\varphi_+(y)\varphi_-(y)\neq -1\}}\alpha(dx)\alpha(dy)\\
&\quad + \mathbb{I}_{A_2}\int|\hat{\psi}_1(x)|\{t_2^2[1 + 2t_1^2\hat{\psi}_1^2(x)]\}^{-1/2}\mathbb{I}_{\{t_2|\hat{\psi}_2(y)|>1\}}\alpha(dx)\alpha(dy)\\
&\quad + \mathbb{I}_{A_2}\int|\hat{\psi}_2(y)|\{t_1^2[1 + 2t_2^2\hat{\psi}_2^2(y)]\}^{-1/2}\mathbb{I}_{\{t_1|\hat{\psi}_1(x)|>1\}}\alpha(dx)\alpha(dy)\\
&\quad + \mathbb{I}_{A_2}\int(t_1t_2)^{-1}\mathbb{I}_{\{|\hat{\psi}_1(x)|>1/t_1\}\cap\{|\hat{\psi}_2(y)|>1/t_2\}}\alpha(dx)\alpha(dy)
\end{aligned}$$

in the sense of $\lesssim$. In view of the conditions of Proposition 1, the classical Tonelli–Fubini theorem (with measure $\lambda^2\otimes\alpha\otimes\alpha$) can be applied to each of these addends to show that they are integrable with respect to λ^2. Coming to the former addend in the representation given at the beginning of this step, it is easy to verify that this addend is bounded by $(t_1t_2)^{-1}\mathbb{I}_{A_2}\int\log\{1 + 4t_1t_2|\hat{\psi}_1\hat{\psi}_2| \times (1+\varphi_-^2)^{-1}\}\,d\alpha$. Thus, through the Tonelli–Fubini theorem, it suffices to prove the finiteness of $\int\{\int(t_1t_2)^{-1}\mathbb{I}_{A_2}\log[1 + 4t_1t_2(1+\varphi_-^2)^{-1}]\,dt\}\,d\alpha$, which is tantamount to $\sum_{j=1}^4\int_{Q_j}(\int_{(0,\delta|\hat{\psi}_1(x)|)\times(0,\varepsilon|\hat{\psi}_2(y)|)} l(y)\,dy)\alpha(dx) < +\infty$ with $\eta_1 = k_1/\delta$, $\eta_2 = k_2/\varepsilon$. It is an easy task to show that $\int_{Q_3} < +\infty$. Thus, to deduce the same conclusion for $\int_{Q_4}$, it is enough to analyze $\int_{Q_4}[\int_{k_1}^{\delta|\hat{\psi}_1|}\int_0^{k_2} l(y)\,dy]\,d\alpha \lesssim \int_{Q_4}[\int_{k_1}^{\delta|\hat{\psi}_1|}\int_0^{k_2}(y_1y_2)^{-1}\;(\int_0^{M(y_2)}(1+\xi)^{-1}\,d\xi)\,dy]\,d\alpha$ with $M(x) := 4x\sqrt{1+x^2} \times\{1 + [\sqrt{1+x^2} - x]^2\}^{-1}$. To complete the argument, observe that the right-hand side can be bounded by $\int_{Q_4}\log(\delta|\hat{\psi}_1|/k_1)\,d\alpha < +\infty$. The same argument is used to verify that $\int_{Q_2}$ is finite. Finally, consider $\int_{Q_1}$ and note that, due to the previous intermediate statements, it will be finite if and only if

$\int_{Q_1}[\int_{k_1}^{\delta|\hat{\psi}_1|}\int_{k_2}^{\varepsilon|\hat{\psi}_2|} l(y)\,dy]\,d\alpha < +\infty$ holds. Employ the inequality $\log(1+|x|) \le |x|$ to show that the last iterated integral is bounded by $\int_{Q_1}[\int_{k_1}^{\delta|\hat{\psi}_1|}\frac{1}{v}|\arctan\frac{\varepsilon|\hat{\psi}_2|v-v^2}{v} - \arctan\frac{k_2 v - v^2}{v}|\,dv]\,d\alpha \lesssim \int_{Q_1}\log(\delta|\hat{\psi}_1|/k_1)\,d\alpha < +\infty$.

STEP 3. $|g|\mathbb{I}_{A_3}$ is integrable with $A_3 := (0,\delta)\times(\varepsilon,+\infty)$. To prove this, we start from the representation of g given at the beginning of Step 2, and treat the "trigonometric" part as in Step 2, while treating the "exponential" part as in Step 1. Hence, $(t_1 t_2)^{-1}\mathbb{I}_{A_3}\exp\{-\int\log|1-i\varphi_-|\,d\alpha\}\ \{\cos[\int\arctan\varphi_+\,d\alpha] - \cos[\int\arctan\varphi_-\,d\alpha]\} \lesssim (t_1 t_2)^{-1}\mathbb{I}_{A_3}\ \exp\{-\int\log|1-i\varphi_-|\,d\alpha\}\ \int|\arctan\varphi_+ + \arctan\varphi_-|\,d\alpha$, where $\exp\{-\int\log|1-i\varphi_-|\,d\alpha\} \le \exp\{-\frac{1}{2}\int\mathbb{I}_{\{|\varphi_-|>a' t_2\}}\times \log(a' t_2)^2\,d\alpha\} \le t_2^{-b'}$. Thus, the addend at issue is at most $\int_{A_3}(t_2^{-1-b'}|\hat{\psi}_1|(1+2t_1^2\hat{\psi}_1^2)^{-1/2} + t_2^{-1-b'}t_1^{-1}\ \alpha\{t_1|\hat{\psi}_1| > (1+\varepsilon^2\hat{\psi}_2^2)^{-1/2}\})$ in the sense of $\lesssim$. Using Proposition 1, we see that this function is integrable. Moreover, $(t_1 t_2)^{-1}\mathbb{I}_{A_3}\times [\exp\{-\int\log|1+i\varphi_+|\,d\alpha\} - \exp\{-\int\log|1-i\varphi_-|\,d\alpha\}]\ \cos[\int\arctan\varphi_+\,d\alpha] \lesssim (t_1 t_2)^{-1}\mathbb{I}_{A_3}\max(\exp\{-\int\log|1+i\varphi_+|\,d\alpha\}, \exp\{-\int\log|1-i\varphi_-|\,d\alpha\})\int|\log|1-i\varphi_+| - \log|1-i\varphi_-||\,d\alpha \lesssim t_1^{-1}t_2^{-1-b'}\mathbb{I}_{A_3}\int[\int_0^{v(t_1,t_2)}(1+\xi)^{-1}\,d\xi]\,d\alpha$, which is integrable, due to the conditions of Proposition 1.

STEP 4. An entirely analogous procedure establishes the integrability of $|g|\mathbb{I}_{A_4}$ with $A_4 := (\delta,+\infty)\times(0,\varepsilon)$.

Thus, the proposition is proved for any $\sigma \in C(F_2)$. To prove that it holds everywhere, it suffices to guarantee that $\sigma \mapsto \int_{(0,+\infty)^2} g(t;\psi-\sigma;\alpha)\,dt$ is continuous. To this end, note that: (a) the dominating functions in Steps 1–4 are bounded (from above) by integrable functions which do not depend on σ, whenever σ varies in any bounded rectangle R, and (b) $\sigma \mapsto g(t;\psi-\sigma;\alpha)$ is continuous. Thus the desired conclusion follows from a well-known corollary of the dominated convergence theorem. See, for example, Folland [(1984), page 54]. □

B. Proof of Step 1, Section 6.1. Take M_1 and M_2 and $b>1$, according to Step 0, in such a way that

$$I(b,1,a/2) + I\big(b,\rho,(a-\eta)/2\big) < \frac{\varepsilon}{3} \tag{8}$$

holds true. Then consider

$$\left|\int_0^b q(y;\sigma)\,dy\right| \le \Delta_1(b,\sigma) + \Delta_2(b,\sigma)$$

with

$$\Delta_1(b,\sigma) := \frac{1}{\pi}\int_0^b \frac{1}{y(1+y^2)^{a/2}}\bigg|\sin(a\arctan y) - \sin\biggl(\int_{\mathbb{R}} \arctan(y(1-x/\sigma))\alpha(dx)\biggr)\bigg|\,dy$$

and

$$\Delta_2(b,\sigma) := \frac{1}{\pi}\int_0^b \frac{1}{y(1+y^2)^{a/2}}\bigg|\sin\biggl(\int_{\mathbb{R}} \arctan\bigl(y(1-x/\sigma)\bigr)\alpha(dx)\biggr)\bigg| \times \bigg|1-\exp\biggl\{-\frac{1}{2}\int_{\mathbb{R}}\bigl[\log\bigl(1+y^2(1-x/\sigma)^2\bigr) - \log(1+y^2)\bigr]\alpha(dx)\biggr\}\bigg|\,dy.$$

Thus, for any $\sigma \geq M_2 > 0$, we have

$$\begin{aligned}
\Delta_1(b,\sigma) &\leq \frac{1}{\pi}\int_0^b \frac{1}{y(1+y^2)^{a/2}}\bigg|a\arctan y - \int_{\mathbb{R}} \arctan\bigl(y(1-x/\sigma)\bigr)\alpha(dx)\bigg|\,dy \\
&\leq \frac{1}{\pi}\int_{(-\infty,0]}\alpha(dx)\int_0^b \frac{1}{y(1+y^2)^{a/2}} \times \arctan\frac{|yx/\sigma|}{1+y^2(1-x/\sigma)}\,dy \qquad := \Delta_{11}(b,\sigma) \\
&\quad + \frac{1}{\pi}\int_{(0,\sigma(1+1/b^2)]}\alpha(dx)\int_0^b \frac{1}{y(1+y^2)^{a/2}} \times \arctan\frac{yx/\sigma}{1+y^2(1-x/\sigma)}\,dy \qquad := \Delta_{12}(b,\sigma) \\
&\quad + \frac{1}{\pi}\int_{(\sigma(1+1/b^2),+\infty)}\alpha(dx)\int_0^{\sqrt{\sigma/(x-\sigma)}} \frac{1}{y(1+y^2)^{a/2}} \times \arctan\frac{yx/\sigma}{1+y^2(1-x/\sigma)}\,dy \qquad := \Delta_{13}(b,\sigma) \\
&\quad + \frac{3}{2}\int_{(\sigma(1+1/b^2),+\infty)}\alpha(dx)\int_{\sqrt{\sigma/(x-\sigma)}}^b \frac{1}{y(1+y^2)^{a/2}}\,dy \qquad := \Delta_{14}(b,\sigma).
\end{aligned} \tag{9}$$

In particular, observe that Δ_{11} decreases to 0 as $\sigma \nearrow +\infty$, and

$$\Delta_{12} \le \frac{1}{\pi}\int_{(0,+\infty)} \alpha(dx) \int_0^b \frac{1}{y(1+y^2)^{a/2}} \times \arctan \frac{yx/\sigma}{1+y^2(1-x/\sigma)}\, dy \downarrow 0 \quad \text{as } \sigma \nearrow +\infty, \tag{10a}$$

$$\Delta_{13} \le \frac{1}{\pi}\int_0^b \frac{1}{y(1+y^2)^{a/2}} \int_{(\sigma(1+1/b^2),+\infty)} \arctan \frac{yx/\sigma}{1+y^2(1-x/\sigma)} \times \alpha(dx)\, dy \downarrow 0 \qquad \text{as } \sigma \nearrow +\infty, \tag{10b}$$

$$\begin{aligned}\Delta_{14} &\le \frac{3}{2}\int_{(\sigma(1+1/b^2),2\sigma]} \alpha(dx) \int_{\sqrt{\sigma/(x-\sigma)}}^b y^{-1}\, dy \\ &\quad + \frac{3}{2}\int_{(2\sigma,+\infty)} \alpha(dx) \int_{\sqrt{\sigma/(x-\sigma)}}^1 y^{-1}\, dy \\ &\quad + \frac{3}{2}\int_{(2\sigma,+\infty)} \alpha(dx) \int_1^b \frac{1}{y(1+y^2)^{a/2}}\, dy \\ &\le \frac{3}{2}\int_{(2\sigma,+\infty)} \log\sqrt{\frac{x-\sigma}{\sigma}}\alpha(dx) \\ &\quad + \frac{3}{2}\int_{(\sigma(1+1/b^2),+\infty)} \log\left(b\sqrt{\frac{x-\sigma}{\sigma}}\right)\alpha(dx) \\ &\quad + \frac{3}{2}\bigl(a - A(2\sigma)\bigr)\int_1^b \frac{dy}{y(1+y^2)^{a/2}} \downarrow 0 \qquad \text{as } \sigma \nearrow +\infty.\end{aligned} \tag{10c}$$

Moreover, for any $0 < y < b$, if $B_\sigma := (-\infty, 0] \cup (2\sigma, +\infty)$, then

$$\begin{aligned}&\int_{\mathbb{R}} \left|\log \frac{1+y^2(1-x/\sigma)^2}{1+y^2}\right| \alpha(dx) \\ &\quad = \int_{B_\sigma} \log \frac{1+y^2(1-x/\sigma)^2}{1+y^2}\alpha(dx) + \int_{B_\sigma^c} \log \frac{1+y^2}{1+y^2(1-x/\sigma)^2}\alpha(dx) \\ &\quad \le \int_{B_\sigma} \log \frac{1+b^2(1-x/\sigma)^2}{1+b^2}\alpha(dx) + \bigl(a - A(2\sigma)\bigr)\log(1+b^2) \\ &\qquad + \int_{(0,\sigma]} \log \frac{1+b^2}{1+b^2(1-x/\sigma)^2}\alpha(dx),\end{aligned}$$

where the sum of the first two terms on the right-hand side decreases to 0 as $\sigma \nearrow +\infty$, while, for $\sigma \geq M_2 > 0$,

$$\int_{(0,\sigma]} \log \frac{1+b^2}{1+b^2(1-x/\sigma)^2}\alpha(dx) \leq \int_{(0,M_2]} \log \frac{1+b^2}{1+b^2(1-x/M_2)^2}\alpha(dx) + \bigl(a - A(M_2)\bigr)\log(1+b^2).$$

Hence, for any y in $(0,b)$ and $\sigma \geq M_2$, there exists $K_1 = K_1(M_2) > 0$ such that

$$\exp\biggl\{\frac{1}{2}\int_{\mathbb{R}}\biggl|\log \frac{1+y^2(1-x/\sigma)^2}{1+y^2}\biggr|\alpha(dx)\biggr\} \leq K_1,$$

and, therefore, since $|1-e^{-v}| \leq |v|e^{|v|}$, we have

$$\begin{aligned}
\Delta_2(b,\sigma) \leq \frac{K_1}{2\pi}\int_0^b & \frac{1}{y(1+y^2)^{a/2}} \\
& \times \biggl[\int_{B_\sigma} \log \frac{1+y^2(1-x/\sigma)^2}{1+y^2}\alpha(dx) + \bigl(a - A(2\sigma)\bigr)\log(1+y^2)\biggr] \\
& \times \biggl[\int_{(-\infty,0]} \arctan(y(1-x/\sigma))\alpha(dx)\biggr]dy \\
& + \frac{K_1}{2\pi}\bigl(a - A(0)\bigr)\int_0^b \frac{\arctan y}{y(1+y^2)^{a/2}} \\
& \qquad \times \biggl[\int_{B_\sigma} \log \frac{1+y^2(1-x/\sigma)^2}{1+y^2}\alpha(dx) \\
& \qquad\qquad + \bigl(a - A(2\sigma)\bigr)\log(1+y^2)\biggr]dy \\
& + \frac{K_1}{2\pi}\int_0^b \frac{1}{y(1+y^2)^{a/2}}\biggl[\int_{B_\sigma} \log \frac{1+y^2(1-x/\sigma)^2}{1+y^2}\alpha(dx) \\
& \qquad\qquad + \bigl(a - A(2\sigma)\bigr)\log(1+y^2)\biggr] \\
& \times \biggl[\int_{(\sigma,+\infty)} \arctan\bigl(y(x/\sigma - 1)\bigr)\alpha(dx)\biggr]dy \qquad (11) \\
& + \frac{K_1}{2\pi}\int_0^b \frac{1}{y(1+y^2)^{a/2}}\biggl[\int_{(0,M_2]} \log \frac{1+y^2}{1+y^2(1-x/M_2)^2}\alpha(dx) \\
& \qquad\qquad + \bigl(a - A(M_2)\bigr)\log(1+y^2)\biggr]
\end{aligned}$$

$$
\begin{aligned}
&\times \left[\int_{(-\infty,0]} \arctan\bigl(y(1-x/\sigma)\bigr)\alpha(dx)\right] dy \\
&+ \frac{K_1}{2\pi}\bigl(a - A(0)\bigr) \int_0^b \frac{\arctan y}{y(1+y^2)^{a/2}} \\
&\qquad \times \left[\int_{(0,M_2]} \log \frac{1+y^2}{1+y^2(1-x/M_2)^2}\alpha(dx) \right. \\
&\qquad\qquad \left. + \bigl(a - A(M_2)\bigr)\log(1+y^2)\right] dy \\
&+ \frac{K_1}{2\pi}\int_0^b \frac{1}{y(1+y^2)^{a/2}} \left[\int_{(0,M_2]} \log \frac{1+y^2}{1+y^2(1-x/M_2)^2}\alpha(dx) \right. \\
&\qquad\qquad \left. + \bigl(a - A(M_2)\bigr)\log(1+y^2)\right] \\
&\times \left[\int_{(\sigma,+\infty)} \arctan\bigl(y(x/\sigma - 1)\bigr)\alpha(dx)\right] dy.
\end{aligned}
$$

The right-hand side of this inequality decreases to 0 as $M_2 \nearrow +\infty$, so that, for any $\varepsilon_1 > 0$, Δ is less than ε_1 for any σ greater than or equal to some $M_2 = M_2(\varepsilon_1)$.

Similar results hold for $\sigma < M_1 < 0$.

Summing up, for any $\varepsilon > 0$, we choose $M_1 = M_1(\varepsilon) < 0 < M_2(\varepsilon) = M_2$ such that

$$
\Delta_1(b,\sigma) < \frac{\varepsilon}{3}, \qquad \Delta_2(b,\sigma) < \frac{\varepsilon}{3}, \qquad \sigma \in [M_1, M_2)^c,
$$

and this, combined with (4) and (8), yields Step 1. □

C. Proof of Step 2, Section 6.1. From (3) it follows that

$$
|F_1(\sigma;\alpha) - F_1(\sigma;\alpha_n)| \le \int_0^{+\infty} r(t;\sigma)\,dt,
$$

where $r(t;\sigma)$ denotes

$$
\begin{aligned}
&\frac{1}{\pi t}\left\{\exp\left(-\frac{1}{2}\int_{\mathbb{R}} \log[1+t^2(\sigma-x)^2]\alpha(dx)\right)\right. \\
&\quad \times \left|\sin\left(\int_{\mathbb{R}} \arctan(t(\sigma-x))\alpha(dx)\right)\right. \\
&\qquad \left. - \sin\left(\int_{\mathbb{R}} \arctan(t(\sigma-x))\alpha_n(dx)\right)\right| \\
&\quad + \left|\sin\left(\int_{\mathbb{R}} \arctan(t(\sigma-x))\alpha_n(dx)\right)\right|
\end{aligned}
$$

$$\times \left|\exp\left(-\frac{1}{2}\int_{\mathbb{R}} \log[1+t^2(\sigma-x)^2]\alpha(dx)\right) - \exp\left(-\frac{1}{2}\int_{\mathbb{R}} \log[1+t^2(\sigma-x)^2]\alpha_n(dx)\right)\right|\Bigg\}$$

for any t and σ. Now, if $c > 0$ and $\sigma \in [M_1, M_2)$, then

$$\begin{aligned}\int_c^{+\infty} r(t;\sigma)\,dt &\le \frac{2}{\pi}\int_c^{+\infty}\frac{1}{t}\exp\left(-\frac{1}{2}\int_{\mathbb{R}}\log[1+t^2(\sigma-x)^2]\alpha(dx)\right)dt \\ &\quad + \frac{1}{\pi}\int_c^{+\infty}\frac{1}{t}\left[\exp\left(-\frac{1}{2}\int_{\mathbb{R}}\log[1+t^2(\sigma-x)^2]\alpha(dx)\right)\right. \\ &\qquad \left. + \exp\left(-\frac{1}{2}\int_{\mathbb{R}}\log[1+t^2(\sigma-x)^2]\alpha_n(dx)\right)\right]dt \\ &\le 3I(c,h,Q_h/2) + I(c,h,Q_h^{(n)}/2) \\ &\le 3I(c,h,Q_h/2) + I(c,h,Q_h/2-\omega_n),\end{aligned}$$

where A_n denotes the distribution function of α_n, $Q_h := \inf_{\sigma\in[M_1,M_2)}\{A(\sigma-h)+a-A(\sigma+h)\}$, $Q_h^{(n)} := \inf_{\sigma\in[M_1,M_2)}\{A_n(\sigma-h)+a-A_n(\sigma+h)\}$, $h>0$, $\omega_n := \sup_{\sigma\in[M_1,M_2)}|A_n(\sigma)-A(\sigma)|$; the last inequality follows, since $Q_h^{(n)} \ge Q_h - 2\omega_n$, for all h and n. Therefore, for any $\varepsilon > 0$,

$$\int_c^{+\infty} r(t;\sigma)\,dt \le \frac{\varepsilon}{3} \tag{12}$$

for some $c > 1$. On the other hand,

$$\begin{aligned}&\int_0^c r(t;\sigma)\,dt \\ &\quad\le \frac{1}{\pi}\int_0^c \frac{1}{t}\exp\left(-\frac{1}{2}\int_{\mathbb{R}}\log[1+t^2(\sigma-x)^2]\alpha(dx)\right) \\ &\qquad\times\left|\int_{\mathbb{R}}\arctan(t(\sigma-x))\alpha(dx) - \int_{\mathbb{R}}\arctan(t(\sigma-x))\alpha_n(dx)\right|dt \\ &\qquad + \frac{1}{\pi}\int_0^c\frac{1}{t}\left|\sin\left\{\int_{\mathbb{R}}\arctan(t(\sigma-x))\alpha_n(dx)\right\}\right| \\ &\qquad\quad\times\exp\left(-\frac{1}{2}\int_{\mathbb{R}}\log[1+t^2(\sigma-x)^2]\alpha(dx)\right) \\ &\qquad\quad\times\left|1-\exp\left\{\frac{1}{2}\int_{\mathbb{R}}\log[1+t^2(\sigma-x)^2](\alpha_n-\alpha)(dx)\right\}\right|dt\end{aligned}$$

holds true. Moreover, observe that, integrating by parts, we obtain

$$\left|\int_{\mathbb{R}} \arctan(t(\sigma - x))\alpha(dx) - \int_{\mathbb{R}} \arctan(t(\sigma - x))\alpha_n(dx)\right| \\ \leq \int_{\mathbb{R}} \frac{t|A_n(x) - A(x)|}{1 + t^2(\sigma - x)^2}\, dx$$

and, for $0 < t < c$, $\sigma \in [M_1, M_2)$,

$$\left|\frac{1}{2}\int_{\mathbb{R}} \log[1 + t^2(\sigma - x)^2](\alpha_n - \alpha)(dx)\right| \\ \leq \int_{\mathbb{R}} |A_n(x) - A(x)| \frac{t^2|\sigma - x|}{1 + t^2(\sigma - x)^2}\, dx \\ \leq H_1(c, M_1) + H_2(c, M_2) + \frac{\omega_n}{2} K_3(c, M_1, M_2) := K_4,$$

where $H_1(c, M_1) := \frac{1}{2}\int_{(-\infty, M_1)} \log(1 + c^2(M_1 - x)^2)\alpha(dx) \downarrow 0$ as $M_1 \to -\infty$, $H_2(c, M_2) := \frac{1}{2}\int_{(M_2, +\infty)} \log(1 + c^2(M_2 - x)^2)\alpha(dx) \downarrow 0$ as $M_2 \to +\infty$, and

$$K_3(c, M_1, M_2) := \sup_{\sigma \in [M_1, M_2)} \log\bigl[(1 + c^2(\sigma - M_1)^2)(1 + c^2(\sigma - M_2)^2)\bigr] \\ = \begin{cases} 2\log\left[1 + c^2\left(\dfrac{M_2 - M_1}{2}\right)^2\right], & \text{if } c > \dfrac{2\sqrt{2}}{M_2 - M_1}, \\ \log\bigl[1 + c^2(M_2 - M_1)^2\bigr], & \text{if } c \leq \dfrac{2\sqrt{2}}{M_2 - M_1}. \end{cases}$$

Therefore, for $\sigma \in [M_1, M_2) \cap C(F_1)$, if $\zeta := 2\sqrt{2}/(M_2 - M_1)$, $c > \max(1, \zeta)$, $l > 0$, we can write

$$\int_0^c r(t;\sigma)\, dt \\ \leq \frac{1}{\pi}\int_{\mathbb{R}} \frac{|A_n(x) - A(x)|}{|\sigma - x|} \arctan(c|\sigma - x|)\, dx \\ + \frac{e^{K_4}}{\pi}\int_0^{\zeta} \frac{1}{t}\left(\int_{\mathbb{R}} \arctan(t(\sigma - x))\alpha_n(dx)\right) \\ \times \left|\frac{1}{2}\int_{\mathbb{R}} \log[1 + t^2(\sigma - x)^2](\alpha_n - \alpha)(dx)\right| dt \\ + \frac{e^{K_4}}{\pi}\int_{\zeta}^{c} \frac{1}{t}\exp\left(-\frac{1}{2}\int_{\mathbb{R}} \log[1 + t^2(\sigma - x)^2]\alpha(dx)\right) \\ \times \left|\frac{1}{2}\int_{\mathbb{R}} \log[1 + t^2(\sigma - x)^2](\alpha_n - \alpha)(dx)\right| dt$$

$$
\begin{aligned}
&\le \biggl\{\frac{c}{2}\int_{-\infty}^{M_1}\frac{A(x)}{\sqrt{1+c^2(\sigma-x)^2}}\,dx+\frac{c}{2}\int_{M_2}^{+\infty}\frac{1-A(x)}{\sqrt{1+c^2(\sigma-x)^2}}\,dx\\
&\qquad+\frac{e^{K_4}}{2}\int_0^{\zeta}\biggl[\int_{M_1}^{M_2}\biggl(\frac{M_2-x}{\sqrt{1+t^2(M_2-x)^2}}+\frac{x-M_1}{\sqrt{1+t^2(M_1-x)^2}}\biggr)\alpha_n(dx)\biggr]\\
&\qquad\qquad\times\biggl[\frac{1}{2}\int_{(-\infty,M_1)}\log[1+t^2(M_1-x)^2]\alpha(dx)\\
&\qquad\qquad\qquad+\frac{1}{2}\int_{(M_2,+\infty)}\log[1+t^2(M_2-x)^2]\alpha(dx)\biggr]\,dt\biggr\}\\
&\quad:= J_1(c,M_1,M_2)\downarrow 0 \qquad \text{as } M_1\searrow-\infty,\ M_2\nearrow+\infty\\
&\quad+\omega_n\biggl\{K_5(c,M_1,M_2)\\
&\qquad+\frac{e^{K_4}}{4}\int_0^{\zeta}\log\bigl[1+t^2(M_2-M_1)^2)\bigr]\\
&\qquad\qquad\times\biggl[\int_{M_1}^{M_2}\biggl(\frac{M_2-x}{\sqrt{1+t^2(M_2-x)^2}}\\
&\qquad\qquad\qquad+\frac{x-M_1}{\sqrt{1+t^2(M_1-x)^2}}\biggr)\alpha_n(dx)\biggr]\,dt\\
&\qquad+\frac{e^{K_4}}{\pi}\int_{\zeta}^{c}\frac{1}{t(1+l^2t^2)^{Q_l/2}}\log\biggl(1+t^2\biggl(\frac{M_2-M_1}{2}\biggr)^2\biggr)\,dt\biggr\},\\
&\quad:= \omega_n J_2(c,M_1,M_2),
\end{aligned}
$$

where

$$
\begin{aligned}
K_5(c,M_1,M_2) &:= \frac{1}{2}\sup_{\sigma\in[M_1,M_2)}\log\frac{c(M_2-\sigma)+\sqrt{1+c^2(M_2-\sigma)^2}}{c(M_1-\sigma)+\sqrt{1+c^2(M_1-\sigma)^2}}\\
&= \log\Bigl(c(M_2-M_1)/2+\sqrt{1+c^2((M_2-M_1)/2)^2}\Bigr).
\end{aligned}
$$

Therefore, for any $\varepsilon>0$ there exist M_1 and M_2 such that $J_1(c,M_1,M_2)<\frac{\varepsilon}{3}$. Furthermore, we can choose α_n, with finite support included in $[M_1,M_2]$, for which

$$
\omega_n J_2(c,M_1,M_2)<\frac{\varepsilon}{3}.
$$

To this end, for any n in $\mathbb{N}$, set

$$
I_{k,n} := [M_1,M_2)\cap\biggl\{\frac{k}{n}\le A<\frac{k+1}{n}\biggr\}, \qquad k\in\mathbb{N}_0,
$$

$$
\{k_1^{(n)},\dots,k_{\nu_n}^{(n)}\} := \{k\in\mathbb{N}_0 : I_{k,n}\ne\varnothing\}
$$

and introduce the function A_n defined by $A_n(x) = 0$ for $x < M_1$, $A_n(x) = \frac{k_i^{(n)}}{n}$ for x in $I_{k_i^{(n)},n}$ $(i = 1, \dots, \nu_n)$ and $A(x) = a$ for $x \geq M_2$. It is clear that A_n is a distribution function, and the corresponding measure has the desired proprieties.

Summing up,

$$\sup_{\sigma \in [M_1, M_2)} |F_1(\sigma; \alpha) - F_1(\sigma; \alpha_n)|$$
$$\leq 3I\left(c, h, \frac{Q_h}{2}\right) + I\left(c, h, \frac{Q_h}{2} - \omega_n\right) + J_1(c, M_1, M_2) + \omega_n J_2(c, M_1, M_2),$$

and the arguments above, combined with (12), show that Step 2 holds for some c, h, M_1, M_2 and n as specified above. □

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E. REGAZZINI
G. DI NUNNO
DIPARTIMENTO DI MATEMATICA "F. CASORATI"
UNIVERSITÀ DEGLI STUDI DI PAVIA
VIA FERRATA, 1
27100 PAVIA
ITALY
E-MAIL: eugenio@dimat.unipv.it
giulia@dimat.unipv.it

A. GUGLIELMI
ISTITUTO PER LE APPLICAZIONI
DELLA MATEMATICA E DELL'INFORMATICA
CONSIGLIO NAZIONALE DELLE RICERCHE
VIA AMPERE, 56
20131 MILANO
ITALY
E-MAIL: alessan@iami.mi.cnr.it