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THEORY OF AN INTERACTING STRING AND DUAL RESONANCE MODEL

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A B S T R A C T

In this paper we study the problem of an interacting string. In the case of an open string the interaction is introduced adding in the Lagrangian an additional term which describes the interaction of the string with an external "photon-like" field. The interaction is acting only at the ends of the string to keep the invariance of the Lagrangian under reparametrization. The equation of motion of the string is exactly solved in the case of a monochromatic external field. All the spectrum properties already known in the case of a free string are left unchanged by the interaction. In addition we show that the probability of emission of n "photons" off the string reproduces the dual amplitude of the conventional model with "photons" as external particles.

In the case of a closed string the interaction is obtained introducing in the free Lagrangian a four-dimensional metric tensor $g_{\mu\nu}(x)$ associated with a curved space as in the theory of general relativity. Also in this case the equation of motion of the string in an external "gravitational" field are exactly solved for a monochromatic "graviton" field. All the main features of the Shapiro-Virasoro model are then reproduced starting from this description of a closed string.

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1. INTRODUCTION

The dual resonance model (DRM) is a quite unconventional model [1]. Actually it is a set of rules for constructing scattering amplitudes among hadrons in accordance with the requirements of crossing symmetry, good asymptotic behaviour (e.g., Regge behaviour) and Lorentz covariance without explicit reference to any Lagrangian or equation of motion. Its structure has been a matter of study during the last few years. Several people have tried to grasp the basic principles which are the cornerstones of the dual physics The scheme which came out of this analysis, is very general and beautiful and exhibits several interesting physical features.

The properties of crossing and asymptotic behaviour are realized via an infinite set of one-particle states (resonances) lying on linear Regge trajectories scaled by one unit and all with the same universal slope. This built-in fundamental constant α' is responsible for many appealing and unconventional features of the DRM. The number of resonances appearing at a certain level is growing up exponentially with the mass [2]; their norm is positive definite for any value of the space-time dimension $D \leq 26$ [3]. For the critical value $D=26$ the model presents the highest degree of symmetry and the spectrum is constructed only with the transverse excitations [Refs. 3) and 4)].

The physical particles are characterized by certain vertex operators $\mathcal{V}_\alpha(z,\pi)$ which have definite transformation properties under the gauge operators L_n [5),6)].

These operators L_n play a double important role. On one hand they characterize [7] the physical states decoupling the states with non-positive norm. On the other hand, their commutation relations with the physical vertices give the nice properties of the scattering amplitudes [5].

In connection with the DRM and with the goal to have an understanding of duality in terms of basic principles people have attempted to connect the modes of vibration of a relativistic string with the oscillators which appear in the DRM [8]. In the early treatment of Nambu, Nielsen and Susskind the connection between the string and the space-time was somewhat obscure. This point has been clarified only recently writing down an action integral for the string in analogy with what is done for describing the motion of a point-like particle [9), 10)]. A new fundamental constant α' comes here into the game for dimensional reasons in the action integral. In this way the gauge group has a very simple physical interpretation; it is a consequence of the invariance of the action under any change of co-ordinates on the surface [9),10)]. In particular the choice of an "orthonormal" parametrization on the surface gives automatically the vanishing of the gauge operators L_n's. The arbitrariness of the parametrization of the world sheet has been completely exploited in Ref. 11) where the string variables σ and τ have been related to physical observable quantities. τ has been taken proportional to some kind of time variable in analogy with the case of a relativistic point-like particle and σ proportional to the fraction of the total energy included between one end of the string and the point σ. With this choice of gauge the only independent degrees of freedom are the transverse ones. After the quantization it has been shown that this theory is consistent with the theory of relativity only for $D = 26$ and in this case it reproduces all the features of spectrum of the DRM at the same value of D. Other attempts to reproduce the spectrum of the DRM have been also made using a field theoretical description [12)].

All the previous developments are based on a free string; they do not give any information on the interaction between strings. On the other hand, the DRM gives a detailed information not only about the spectrum of the hadrons, but also about the couplings among the various particles of the spectrum. If the string is not only an analog model of the DRM, one should be able to understand the interaction among the hadrons starting from the string picture. A first attempt along this direction is described in Ref. 10), where a term of interaction with an external field at the ends of the string has been explicitly introduced; however, it is still lacking a consistent picture for an interacting string.

A further step along this direction has been made by Gervais and Sakita [13)] who were able to write down the n point dual amplitude in terms of the string variables using the path integral formalism.

In this paper we start from a string interacting with an external field and we reproduce the main features of the DRM (both the conventional model with $\alpha_0 = 1$ and the Shapiro-Virasoro model with $\alpha_0 = 2$). All the main properties of the spectrum derived in the free case are left unchanged when we introduce a term describing the interaction of a string. In addition we can construct an n point scattering amplitude which reproduces the dual amplitudes and gives also a very intuitive interpretation of them.

In the case of an open string we add to the free Lagrangian $\mathcal{L}_0$ a term which describes the interaction of a string with an external "photon-like" (spin 1 and massless) field $A_\mu(x)$ at the ends. The total action will then be

$$S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \left\{ \mathcal{L}_0 + A_\mu(x)\, j^\mu \right\} \tag{1.1}$$

with

$$j^\mu = \dot{x}^\mu \left[g_0\, \delta(\sigma) + g_\pi\, \delta(\sigma - \pi) \right] \tag{1.2}$$

g_0 and g_π are the coupling constants at the ends $\sigma = 0, \pi$ of the string. $\mathcal{L}_0$ is the Lagrangian of the free string $[\hbar = c = \alpha' = 1]$.

The interaction term preserves the invariance of the action under a change of parametrization on the world sheet of the string. An "orthonormal" parametrization can always be chosen even in the case with the interaction [Eq. (4.15)]. Varying the action (1.1) one can get the equations of motion of a string in an external "electromagnetic" field. With an interaction acting only at the ends the equation of motion of the string is unchanged with respect to the free case [Eq. (4.17)]. The external field modifies only the boundary conditions at the ends of the string [Eqs. (4.18), (4.19)].

All these non-linear equations can be solved exactly in the case of a monochromatic external field $[A_\mu(x) = \varepsilon_\mu e^{ikx}]$ and the result is:

$$x_\mu(\tau,\sigma) = x_{0\mu}(\tau,\sigma) + \left[\varepsilon_\mu k_\nu - \varepsilon_\nu k_\mu\right] \left[R_+ \int^{\tau-\sigma} \dot{x}(\tau',0)\, e^{ik\cdot x(\tau',0)} d\tau' + R_- \int^{\tau+\sigma} \dot{x}^\nu(\tau',0)\, e^{ik\cdot x(\tau',0)} d\tau' \right] \tag{1.3}$$

where

$$R_\pm = -2\pi \frac{g_\pi + g_0 e^{\pm 2i(P\cdot K)\pi}}{2 \sin(2PK)\pi} \tag{1.4}$$

P is the momentum of the string.

The solution (1.3) is the sum of two terms. The term $x_0^\mu(\tau,\sigma)$ represents the most general solution of the free equation and can be expanded in terms of an infinite set of harmonic oscillators. The other term describes the interaction with the external field; it shows simple poles for the values $2Pk = n$ (integer number) which correspond to the frequencies of resonance of the string. The string in fact resonates when the frequency of the photon can excite one of its eigenfrequencies. The resonance may disappear if the coupling constants at $\sigma = 0$ and $\sigma = \pi$ are related in such a way that the numerator in (1.4) is vanishing. $R_{\pm}$ plays the role of a kind of signature factor allowing only even (odd) poles when $g_0 = g_{\pi}$ $[g_0 = -g_{\pi}]$. Of course, for general values of g_0 and g_{π} both the even and odd poles are allowed.

An iterative solution is given also in the case when the "photon" field is not monochromatic (see Appendix C).

The choice of the "orthonormal" gauge leaves still the possibility of making gauge transformations which do not change the action integral. We show that the generators L_n of these gauge transformations are constants of motion and satisfy the Virasoro algebra. In addition in presence of interaction they retain the same expression in terms of the free solution as in the non-interacting case. In the transverse gauge we can then show that the only independent modes of vibration are the transverse ones; the others can be expressed as a function of the transverse oscillators and the functional relation is unchanged with respect to the free case. This theory can be quantized using standard methods and it is of course relativistic invariant only for $D = 26$ as in the free case.

In Section 7 we solve the equations of motion of the string in the case of a monochromatic field varying with τ $[A_{\mu}(x) = \varepsilon_{\mu}(\tau) e^{ik(\tau)\cdot x}]$. This is done to allow emission of "photons" with arbitrary polarization and four-momentum.

We can then evaluate the probability of emission of n "photons" off the string and this probability reproduces the n point "photon" amplitude of the DRM.

All the previous considerations are based on the interaction of the string with the "strong photon". In the DRM we have a complete "democracy" among the various particles of the spectrum; it is the Virasoro group which plays the role of fully characterizing the physical states. We are then led to think that we can proceed with any excited states as we did in the case of the "photon". On the other hand, in the string approach the interaction Lagrangian must keep the invariance of the action under the reparametrization group. In the case of an interaction localized at the ends of the string the previous requirement leads to the commutation relation

(8.15) with the generators L_f of the group which leaves the action invariant.

This is the same relation satisfied by the vertices associated with the physical particles in DRM. We have then a one-to-one correspondence between the physical vertices in DRM and the interaction Lagrangians which keep the invariance under reparametrizations.

However, with the excited states we have an additional problem related to the fact that the translation from the classical to the quantum theory is not straightforward. In fact we have problems of reordering except for the case of the on mass shell "strong photon" [conditions (5.4)]. These problems will be discussed in Section 8 where we give the rules for constructing interaction Lagrangians associated with excited states.

As the open string is related to the conventional model with $\alpha_0 = 1$, the close string is connected with the Shapiro-Virasoro model (SVM) [14] where the "strong photon" is replaced by the "strong graviton". In this case it is quite natural to introduce the interaction in a geometrical way without adding an additional term in the Lagrangian as in the case of the open string. This is obtained introducing in the free Lagrangian a four-dimensional metric tensor $g_{\mu\nu}$ associated with a curved space as in the theory of general relativity.

In this case the equation of motion of a string in an external gravitational field is given by

$$\partial_+ \partial_- x^\mu + \Gamma^\mu_{\nu\rho}(x)\, \partial_+ x^\nu \partial_- x^\rho = 0 \tag{1.5}$$

where $\Gamma^\mu_{\nu\rho}(x)$ is the Christoffel symbol and $\partial_\pm$ are the derivatives with respect to the light-cone variables $\xi_\pm = \tau \pm \sigma$. (1.5) can be solved for a monochromatic "graviton" field using methods analogous to the case of the "strong photon". All the main features of the SVM can then be reproduced starting from our description of the closed string.

The paper is organized in a way that all the sections are very much self-contained. This should allow the reader to go directly over those sections that interest him more. For instance, the Lagrangian and Hamiltonian formalisms are independent from each other; the reader can easily skip one of them or even both of them if he is mainly interested in the solution of the equations of motion.

The paper is organized as follows.

In Section 2 we discuss from a very intuitive point of view the main features of the string giving also some simple arguments which make more clear all the further more formal developments.

Section 3 is devoted to the discussion of the Lagrangian formalism. Starting from the Lagrangian we derive the equation of motion of a string in an external "electromagnetic" field and we discuss the invariance under re-parametrizations.

In Section 4 we set up the Hamiltonian formalism. Using this formalism we rederive the equations of motion and we prove the consistency of our Lagrangian. The last part of this section is devoted to the study of the symmetry properties of the action and to the construction of the generators of symmetry transformations for our action integral. These generators are the gauge operators L_n which are constants of motion and satisfy the Virasoro algebra.

In Section 5 we discuss the solution of the non-linear equations of motion of a string in an external "electromagnetic" field and then we show that the transverse modes are the only independent modes of vibration of the string.

In Section 6 after the quantization procedure the three-point function is explicitly constructed.

In Section 7 we construct the probability amplitude for the emission of n "photons" and we give the rules to translate any quantity appearing in the string model into the corresponding expression of the DRM and vice versa.

Section 8 is devoted to the excited states. The closed string and its interaction with an external gravitational field is described in Section 9.

Section 10 is devoted to some conclusions and final remarks.

In Appendix A we give the relation between some four-dimensional quantities as the electromagnetic current and the energy momentum tensor and the correspondent two-dimensional quantities. Appendix B is devoted to the solution of the equation of motion of a string in an external electromagnetic field.

In Appendix C we give a perturbative solution when the external field is a superposition of two monochromatic photons. Finally, Appendix D is devoted to the Hamiltonian formalism in the case of a closed string.

2. ELEMENTARY CONSIDERATIONS ON THE STRING

Our primary task is to show that the dual string theory is more than a purely analog model for the mass spectrum of DRM. Indeed, we shall see that the dual string, even at the classical level, allows us to build up an intuitive picture of some important features of the dual couplings. One simple example, only, will be given here, a more systematic study will be found in the next Sections.

It has been recognized for a long time that the coupling of a strong photon to any physical state of DRM gives a universal gyromagnetic ratio [4),15)] $G = 2$, where G is defined by $\mu = (Gg/2mc)J$, μ is the dipole magnetic moment, m the mass of the state, J its angular momentum and g the coupling constant. The same result can be easily reproduced in the dual string by placing a charge g at one end of the string, as we shall see shortly [see also Ref. 16)]. In order not to obscure the elementary nature of our considerations, we limit ourselves for the moment to a very simple set of string motions pointed out in Ref. 11), made by rigid rotations of a straight string [11)] of length $2a$ with an angular velocity ω such that the ends move at the speed of light, i.e., $\omega a = c$. The co-ordinates of a point of this string in the c.m. frame are

$$\underset{\sim}{x}_T = r\, e^{i \frac{c}{a} t} \tag{2.1}$$

where $\underset{\sim}{x}_T = x_1 + ix_2$ is the two-dimensional vector in the rotation plane 1, 2, t is the time and $|r| \leq a$ is the distance of the point from the c.m. The transverse velocity is clearly

$$v_T = c\, \frac{r}{a} \tag{2.1'}$$

while the covariant momentum density Π_μ is defined as:

$$\Pi_\mu = \rho_0 c \frac{dx_\mu}{ds} = \rho_0 \gamma_T \frac{d}{dt} x_\mu \tag{2.2}$$

or

$$\Pi_T = \rho_0 v_T \gamma_T \quad , \quad \Pi_0 = \rho_0 c \gamma_T$$

where s/c is the proper time of each point of the string and t is the time in the c.m. frame, γ_T is the Lorentz factor $(1-(v_T^2/c^2))^{-\frac{1}{2}}$, and ρ_0 is a universal constant with the dimension (mass/length) which may be interpreted as the rest mass density of the string measured in a non-inertial frame which rotates with the string. The density of the mass ρ (r), measured in the c.m. clearly is

$$\rho(r) = \frac{1}{c} \Pi_0 \tag{2.3}$$

thus, we have, for the total mass,

$$m = 2\int_0^a \rho(r)\, dr = \pi \rho_0 a \tag{2.4}$$

The angular momentum along the rotation axis is

$$J = 2\int_0^a v_T r \rho(r)\, dr = 2c\rho_0 a^2 \int_0^1 dx \frac{x^2}{\sqrt{1-x^2}} = \frac{c}{2\rho_0 \pi} m^2 \tag{2.5}$$

Then the masses belong to a "linear Regge trajectory" with a slope

$$\alpha' = \left[2\pi\rho_0 \hbar c^3\right]^{-1} \tag{2.6}$$

If we put a charge g at the end r = a of the string, it generates a current $i = g(\omega/2\pi) = g(c/2\pi a)$. The dipole magnetic moment of this current is $\mu = (i/c)\mathcal{Q}$, where $\mathcal{Q}$ is the area of the surface bounded by the trajectory of the charge, then $\mathcal{Q} = \pi a^2$. So we have

$$\mu = \frac{g}{2\pi a} \pi a^2 = \frac{g}{mc} J \tag{2.7}$$

Thus, even in this case the gyromagnetic ratio is $G = 2$. It could also be shown that this result even holds for the most general motion of the string.

In view of this coincidence one could proceed to the higher multipole moments to check whether this coincidence continues to hold. More simply one can test whether the total coupling of a resonance of DRM to the strong photon coincides with the coupling of the corresponding string state with an external electromagnetic field.

The coupling of a vector potential A to the current generated by the charged end is $g(V/c)\cdot A$, where V is the velocity of the charge. The effect of this field is governed by a non-linear differential equation that we are able to solve explicitly when the field is a monochromatic wave, which is just the case considered in DRM. We shall see that in such a case this coupling coincides exactly with the photon vertex of DRM.

Before we proceed to describe the salient features of the motion of the string in such a field, it is convenient to introduce some standard notations of the string theory which allow us to write in a very simple form the general solution for the motion of a free string. First of all we replace the two quantities t and r which label the string co-ordinates by the two adimensional parameters τ and σ , $0 \leq \sigma \leq \pi$, that are, for the motion (2.1)

$$\tau = \frac{ct}{a} \qquad (2.8)$$

and

$$\sigma = \arccos \frac{r}{a} \qquad (2.9)$$

A more general definition will be found in the next Sections. With this choice of parameters the equation of motion of the dual string is simply

$$\frac{\partial^2 x_\mu}{\partial \sigma^2} - \frac{\partial x_\mu}{\partial \tau^2} = 0 \qquad (2.10)$$

The general solution represents a superposition of waves travelling along the two directions in the string with a constant velocity $v = d\sigma/d\tau = 1$, i.e.,

$$x_\mu(\tau,\sigma) = f_\mu(\tau+\sigma) + g_\mu(\tau-\sigma) \qquad (2.11)$$

The condition that no momentum flows across the ends requires, in the case of free string, $f_\mu(\xi) = g_\mu(\xi)$.

The equations (2.10), that describe the motion of any classical string, must be supplemented for a dual string by further constraints which follow from the requirement that the action be proportional to the area of the surface described by the string in the space-time (and hence invariant under reparametrization).

This constraint is $(\partial x/\partial\tau \pm \partial x/\partial\sigma)^2 = 0$, or $(\partial f(\xi)/\partial\xi)^2 = 0$. In view of the relation

$$\frac{1}{2}\frac{\partial}{\partial\tau}x_\mu(\tau,0) = f_\mu(\tau)$$

the above constraint means simply that the end points of the string must travel at the speed of light.

When the charged end interacts with an external electromagnetic field, some momentum is injected into the string, then the outgoing wave f_μ is no longer equal to the ingoing one g_μ, however, the Eq. (2.10) is unchanged. This means that the disturbance (phonons) produced by the field goes through the string with a velocity u which is constant and equal to 1 in the σ, τ plane *). After a "time" π the disturbance reaches the end at $\sigma = \pi$ where it is reflected and comes back at $\sigma = 0$ after a total time $\Delta\tau = 2\pi$.

This characteristic time, which in dimensional units is $\Delta t = 4\pi\alpha'\hbar E$, where E is the energy of the string, seems to play an important role in the theory of interacting string. Indeed we shall see that if the string interacts with the external field for a time $\Delta\tau \leq 2\pi$, the motion is linear in the field so that the disturbance produced at each time is not affected by the photons absorbed before.

A new phenomenon happens when the interacting time is $> 2\pi$. In such a case the field may interact at a time τ with the phonons produced at the time $\tau - 2\pi$, hence the response of the string is no longer linear in the field or, what is the same, in the coupling constant g.

It has to be noticed that the same action which gives the dynamics of strings with free ends can be used to describe a different mechanical system set up by closed strings. It is already known that such a system

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*) Actually in terms of the physical parameters r and t, the speed of sound along the string is a function of the transverse velocity v_T. For instance, for a small distrubance which propagates in the straight string subject to the rigid motion of Eq. (2.1), one has from (2.8), (2.9) and (2.11):

$$u = \left|\frac{dr}{dt}\right| = \left|\frac{dr}{d\sigma}\,\frac{d\sigma}{d\tau}\,\frac{d\tau}{dt}\right| = c\sqrt{1-\frac{r^2}{a^2}} = c\,\gamma_T^{-1}$$

One can show that the last expression holds also for the most general string [17].

reproduces the spectrum of the Shapiro-Virasoro model (SVM), and we shall see in Section 9 that it can also describe, in a curved space, the interaction with a "strong graviton". Here we want to show, to conclude this section, that the slope of the Regge trajectory of the closed string states is $\frac{1}{2}\alpha'$, when α' is the slope of the open string. Indeed a simple set of motions of the closed string is set up by two straight strings of the kind described in (2.1) which adhere to each other in order to form a unique string whose mass density ρ_c is twice that of the ordinary open string, that is $\rho_c = 2\rho$.

Now, Eq. (2.5) says that the angular momentum J_c of the closed string is twice that of the open string, whereas the squared mass is of course $m_c^2 = 4m^2$. It follows then from Eq. (2.5) that the Regge trajectory for the closed string is

$$\frac{J_c}{\hbar} = c^2\left(\frac{\alpha'}{2}\right) m_c^2 \tag{2.9}$$

We conclude then that the string picture of DRM allows us to connect the slope of the conventional model to that of SVM, which in DRM formulation are obviously unrelated. It is interesting to notice that the string value of the slope of the SVM is just the one needed to identify, in critical space-time dimensions, the spectrum of SVM with that of the Pomeron sector of the conventional model, as it has been shown recently by Olive and Scherk [18]. We defer to the last Section some further comments on the relationship between closed strings and the Pomeron contributions to the conventional model.

3. THE LAGRANGIAN FORMALISM

In this Section we use the Lagrangian formalism to derive the equations of motion of the string in presence of an external "electromagnetic field". General properties of the free and interaction Lagrangian densities are examined. We shall also deduce some identities appearing in the Lagrangian formalism which will manifest themselves as "constraints" between dynamical variables in the Hamiltonian formulation of the theory.

i) The interaction Lagrangian density and the equations of motion

In the free case the action integral is given by

$$S_{free} = \int_0^{\pi} d\sigma \int_{\tau_i}^{\tau_f} d\tau \, \mathcal{L}_{free} = -\frac{1}{2\pi\alpha'\hbar c^2} \int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \left[(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2 \right]^{1/2} \tag{3.1}$$

Here we are using the same notations as in Ref. 11). $x^\mu(\tau,\sigma)$ are the string space-time co-ordinates and the dot and prime denote differentiation with respect to τ and σ respectively, τ and σ are curvilinear co-ordinates on the surface swept out by the string during its motion. Sometimes, we shall also use the symbols ξ^0 and ξ^1 instead of τ and σ, and the notation $x^\mu{}_{,0}$ $x^\mu{}_{,1}$ for the derivatives of the x^μ, with respect to ξ^0 and ξ^1 respectively.

Following the suggestion of the previous section we now introduce an external electromagnetic field $A_\mu(x)$ and we write down the interaction Lagrangian density in the τ, σ space in the form

$$\mathcal{L}_{int} = \frac{1}{c} \rho(\sigma) \, \dot{x}^\mu A_\mu(x) \tag{3.2}$$

where $\rho(\sigma)$ is the "charge" density on the string and τ is interpreted as an internal time co-ordinate in the parameter space. The same form of $\mathcal{L}_{int}$ may also be recovered, according to Ref. 19), starting from the usual interaction Lagrangian density in the Minkowski space:

$$A_\mu(x) \; j^\mu(x) \tag{3.3}$$

and looking for the corresponding density in the parameter space (see Appendix A).

The action integral in presence of the interaction becomes:

$$S = \iint d^2\xi \left(\mathcal{L}_{free} + \mathcal{L}_{int.} \right) \tag{3.4}$$

In the following we will choose a unit system where $\alpha' = \hbar = c = 1$. We now recall the fundamental property of the free action integral to be invariant with respect to arbitrary reparametrization of the τ, σ variables on the two-dimensional world surface of the string. This is a consequence of the geometrical meaning of the integrand in (3.1) (area element on the string surface). This invariance is the analog in the string model of the gauge invariance under the Virasoro group in the dual models. In order to maintain such important invariance property in presence of interactions we are forced to give an intrinsic meaning to the Eq. (3.2). This can be done placing "charges" at the ends of the string, which are the only points unaffected by reparametrization.

Consequently, we choose [19)]

$$\rho(\sigma) = g_0\,\delta(\sigma) + g_\pi\,\delta(\sigma-\pi) \tag{3.5}$$

so that

$$S_{int} = \int_{\tau_i}^{\tau_f} d\tau \left\{ g_0\, \dot{x}^\mu(\tau,0)\, A_\mu(x(\tau,0)) + g_\pi\, \dot{x}^\mu(\tau,\pi)\, A_\mu(x(\tau,\pi)) \right\} \tag{3.6}$$

and the homogeneity in $\dot{x}$ insures the invariance of S_{int} under reparametrizations. The equations of motion are obtained in the standard way, imposing that the variation δS of the action (3.4) vanishes when the "fields" $x^\mu(\tau,\sigma)$ are arbitrarily varied, with the only condition

$$\delta x^\mu(\tau_i,\sigma) = \delta x^\mu(\tau_f,\sigma) = 0 \tag{3.7}$$

One gets:

$$\delta S = -\iint d^2\xi\; \partial_i \left(\frac{\partial \mathcal{L}_{free}}{\partial x^\mu_{,i}} \right) \delta x^\mu + \int d\tau \left(\frac{\partial \mathcal{L}_{free}}{\partial x'_\mu}\, \delta x^\mu \right) \Bigg|_{\sigma=0}^{\sigma=\pi} + \tag{3.8}$$

$$+ g_\pi \int d\tau \left[\frac{\partial\, \dot{x}_\mu A^\mu}{\partial x^\nu} - \frac{d}{d\tau} \frac{\partial\, \dot{x}_\mu A^\mu}{\partial \dot{x}^\nu} \right] \delta x^\nu \Bigg|_{\sigma=0} + g_0 \int d\tau \left[\frac{\partial\, \dot{x}_\mu A^\mu}{\partial x^\nu} - \frac{d}{d\tau} \frac{\partial\, \dot{x}_\mu A^\mu}{\partial \dot{x}^\nu} \right] \delta x^\nu \Bigg|_{\sigma=\pi} = 0$$

Hence the equations of motion are not modified with respect to the free case

$$\partial_i \left(\frac{\partial \mathcal{L}_{free}}{\partial x^\mu_{,i}} \right) = 0 \tag{3.9}$$

Instead, the boundary conditions are modified and are given by:

$$\frac{\partial \mathcal{L}_{free}}{\partial x'^{\mu}} + g_{\pi}\left(\frac{\partial\, \dot{x}_{\mu}A^{\mu}}{\partial x^{\nu}} - \frac{d}{d\tau}\, \frac{\partial\, \dot{x}_{\mu}A^{\mu}}{\partial \dot{x}^{\nu}} \right) = 0 \ , \ \sigma=\pi$$

$$-\frac{\partial \mathcal{L}}{\partial x'^{\mu}} + g_{0}\left(\frac{\partial\, \dot{x}_{\mu}A^{\mu}}{\partial x^{\nu}} - \frac{d}{d\tau}\, \frac{\partial\, \dot{x}_{\mu}A^{\mu}}{\partial \dot{x}^{\nu}} \right) = 0 \ , \ \sigma=0 \qquad (3.10)$$

The physical meaning of the new boundary conditions is clear. In the free case $(\partial \mathcal{L}_{free})/(\partial x'^{\mu}) = 0$ expresses the fact that no four-momentum flows across the boundaries of the string. The new boundary equations describe the exchange of momentum between the string and the external field.

ii) Invariance properties of the Lagrangian

We want now to stress these particular features of the string model that do not depend on the particular Lagrangian, and are consequences only of the invariance of the action integral with respect to a reparametrization group containing two arbitrary functions.

We begin by establishing the conditions that a Lagrangian must satisfy so that the action integral is invariant under such a group. We suppose for simplicity that the Lagrangian is function of the fields x^{μ} and their first derivatives only $x^{\mu}{}_{,i}$ and we exclude for the moment explicit dependence on the parameters ξ^{i}. These conditions are met by the free Lagrangian appearing in (3.1). The modifications to our considerations due to the σ dependence of the interaction term are trivial and postponed to the end of this subsection. We now perform an infinitesimal co-ordinate transformation:

$$\bar{\xi}^{i} = \xi^{i} + \varepsilon^{i}(\xi^{0}, \xi^{1}) \qquad (i = 0,1) \qquad (3.11)$$

with ε^{i} infinitesimal arbitrary functions of the ξ^{i} which do not change the points $\sigma = 0$ and $\sigma = \pi$ [that is $\varepsilon^{1}(\sigma = 0, \tau) = \varepsilon^{1}(\sigma = \pi, \tau) = 0$]. The fields x^{μ} will correspondingly undergo the variation:

$$\delta x^{\mu} = - x^{\mu}{}_{,j}\, \varepsilon^{j} \qquad (3.12)$$

Straightforward calculations give the change in the functional dependence of S as

$$\delta S = \iint d^2\xi \left[\frac{\partial \mathcal{L}}{\partial x^\mu} \delta x^\mu + \frac{\partial \mathcal{L}}{\partial x^\mu_{,i}} \delta x^\mu_{,i} + \partial_i \left(\varepsilon^i \mathcal{L} \right) \right] \tag{3.13}$$

or, using (3.9)

$$\delta S = \iint d^2\xi \left\{ \left[\left(\mathcal{L}\, \delta^i_j - \frac{\partial \mathcal{L}}{\partial x^\mu_{,i}} x^\mu_{,j} \right) \varepsilon^j \right]_{,i} + L_\mu x^\mu_{,j} \varepsilon^j \right\} \tag{3.14}$$

where L_μ is a shorthand notation for the left-hand side of the equations of motion.

If S has to be invariant under (3.11) the integrand in (3.14) has to vanish identically. Moreover, because of the appearance of the arbitrary functions ε^i, the coefficient of ε^i and $\varepsilon^j{}_{,i}$ must vanish separately. By imposing the vanishing of the coefficient of $\varepsilon^j{}_{,i}$ we get that a Lagrangian of a theory invariant under reparametrizations must satisfy the equations: *

$$\mathcal{L}\, \delta^i_j - \frac{\partial \mathcal{L}}{\partial x^\mu_{,i}} x^\mu_{,j} = 0 \tag{3.15}$$

or, explicitly

$$\mathcal{L} - \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \dot{x}^\mu = 0 \quad (3.15a) \qquad x'^\mu \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = 0 \quad (3.15b)$$

$$\dot{x}^\mu \frac{\partial \mathcal{L}}{\partial x'^\mu} = 0 \quad (3.15c) \qquad \mathcal{L} - \frac{\partial \mathcal{L}}{\partial x'^\mu} x'^\mu = 0 \quad (3.15d)$$

---

*) Note that the quantities

$$\left(\mathcal{L}\, \delta^i_j - \frac{\partial \mathcal{L}}{\partial x^\mu_{,i}} x^\mu_{,j} \right) \varepsilon^j$$

whose divergence appears in the integrand of (3.14) are the generators of the infinitesimal transformation which leave the Lagrangian invariant. The fact that they vanish identically is peculiar of a theory covariant with respect to transformations containing arbitrary functions [20]. As we shall see below the relations (3.15) give rise to the primary constraints of the Hamiltonian formalism.

The requirement that the coefficient of ε^j be zero, does not introduce any new relation as it stems from the following identity:

$$\left(\mathcal{L}\, \delta^i_j - \frac{\partial \mathcal{L}}{\partial x^\mu{}_{,i}}\, x^\mu{}_{,j} \right)_{,i} = - L_\mu\, x^\mu{}_{,j} \tag{3.16}$$

It is interesting to note that (3.16) together with (3.15) explicitly shows that the equations of motion are not algebraically independent, but satisfy the two conditions

$$L_\mu\, x^\mu{}_{,j} = 0 \qquad (j = 0, 1) \tag{3.17}$$

strictly analogous to the Bianchi identities of the gravitation theory and the $F^{\mu\nu}{}_{,\mu\nu} = 0$ identity of the electromagnetism *). The presence of such identities among the left-hand side of equations of motion is clearly related to the necessity of imposing auxiliary gauge (or co-ordinate) conditions in order to restrict the number of (formally) different solutions. In the string model we may impose two such conditions: a convenient choice being, for example [9),10)]

$$\dot{x}^2 + x'^2 = \dot{x}\cdot x' = 0 \tag{3.18}$$

By adjoining the gauge conditions to the equations of motion, one obtains an overdetermined system of differential equations that is solvable just in virtue of the aforementioned identities.

When we pass to the consideration of our actual total Lagrangian (3.4) some minor modifications of our previous consideration are needed, because the interaction term (3.2) does explicitly depend on σ. We already stressed, however, that the explicit dependence of the Lagrangian on σ can be compatible with the invariance under reparametrization if (only if) it appears in a form of a delta function at the boundaries [16)]. The Eq. (3.6) shows then explicitly that the homogeneity of the interaction term (3.2) in $\dot{x}$ gives us automatically an invariant action integral **).

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*) Note that since our transformation law (3.12) does not involve the derivatives of ε^j, the identities (3.17) are algebraic rather than differential, as it is in the case of gravitation and electromagnetism.

**) Conditions for the invariance of more general interaction Lagrangians may be found in Section 9.

As far as identities (3.17) are concerned they are obviously the same as in the free case because the equations of motion (3.9) are left unchanged, in the case of an interaction at the ends. For the deduction of the constraints in the Hamiltonian formalism, it is useful to rewrite the Eqs. (3.15) in terms of our total Lagrangian. By the substitution $\mathcal{L} \to \mathcal{L}_{tot} - \mathcal{L}_{int}$ one obtains:

$$\mathcal{L}_{tot} - \frac{\partial \mathcal{L}_{tot}}{\partial \dot{x}^\mu} \dot{x}^\mu = 0 \quad , \quad \frac{\partial \mathcal{L}_{tot}}{\partial \dot{x}^\mu} x'^\mu = A_\mu x'^\mu \rho(\sigma) \tag{3.19}$$

$$\frac{\partial \mathcal{L}_{tot}}{\partial x'^\mu} \dot{x}^\mu = 0 \quad , \quad \frac{\partial \mathcal{L}_{tot}}{\partial x'^\mu} x'^\mu - \mathcal{L}_{tot} = - A_\mu \dot{x}^\mu \rho(\sigma)$$

We conclude this subsection with the calculation of the stress energy tensor in the case of the free string. Let us consider the string co-ordinates x^μ as "fields"; the energy momentum tensor is given by

$$t^i_{\ j} = \delta^i_j \mathcal{L} - \frac{\partial \mathcal{L}}{\partial x^\mu_{,i}} x^\mu_{,j} \qquad (i,j = 0,1) \tag{3.20}$$

We find, however, that because of conditions (3.15)

$$t^i_{\ j} = 0 \tag{3.21}$$

identically.

Obviously, this stress tensor has nothing to do with the true energy momentum tensor of the string in the four-dimensional space *). As it is shown in Appendix A, it turns out to be

$$T^{\mu\nu}(y) = -\frac{1}{2\pi}\int d^2\xi \ \delta^{(4)}(y - x(\xi)) \frac{(\dot{x}\cdot x')[\dot{x}^\mu x'^\nu + x'^\mu \dot{x}^\nu] - \dot{x}^2 x'^\mu x'^\nu - x'^2 \dot{x}^\mu \dot{x}^\nu}{[(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2]^{1/2}} \tag{3.22}$$

where $y^\mu = x^\mu(\xi^0, \xi^1)$ are the parametric equations of the surface described by the string.

---

*) For the sake of simplicity we refer to a four-dimensional space in which the string moves; of course, our treatment works for any number (> 2) of dimensions of the Minkowski space.

iii) Deduction of the constraint equations in the Hamiltonian formalism

We have mentioned before some analogies between the string theory and those of gravitation and electromagnetism. In all these covariant theories we may always carry out a transformation getting new solutions of the differential equations satisfying the same initial conditions, but formally different at later times. That means that the equations cannot determine the solution uniquely from a set of properly chosen initial conditions; it must be impossible to solve the equations with respect to the second time derivatives. In our case the coefficients of the second derivatives are given by

$$\Lambda_{\mu\nu} = \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu} \tag{3.23}$$

We conclude that $\Lambda_{\mu\nu}$ must be a singular matrix. In fact we may ascertain this fact by differentiating with respect to $\dot{x}^\nu$ the general equations (3.15a) and (3.15b), obtaining:

$$\dot{x}^\mu \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = 0 \quad , \quad x'^\mu \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^\mu \partial \dot{x}^\nu} = 0 \tag{3.24}$$

[Note that because of the linearity in $\dot{x}^\mu$ of the interaction Lagrangian (3.2) the same equations (3.24) hold also for the total Lagrangian.]

From these equations we deduce that $\|\Lambda_{\mu\nu}\|$ is a singular matrix of rank two, possessing the two null eigenvectors $\dot{x}^\mu$ and x'^μ.

In the Hamiltonian formulation one defines

$$\pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \tag{3.25}$$

so that the Jacobian $\|(\partial\pi_\mu/\partial \dot{x}^\nu)\| \equiv \|\Lambda_{\mu\nu}\|$ is zero and we cannot solve for the velocities in terms of the momenta. The conclusion is that we must have two primary constraints in the Hamiltonian formalism, i.e., two relations between the momenta π^μ , the field co-ordinates x^μ and their σ-derivatives.

If we again limit ourselves to the general Lagrangians not explicitly dependent on ξ^0, ξ^1, Eqs. (3.15) give us the form of these two constraints.

Equation (3.15b) gives directly the constraint

$$\Pi_\mu x'^\mu = 0 \tag{3.26}$$

To find the second constraint we differentiate (3.15d) with respect to $\dot{x}^\nu$ and contract with π^ν obtaining:

$$x'^\mu \frac{\partial \pi^2}{\partial x'^\mu} = 2\,\pi^2 \tag{3.27}$$

so that

$$\pi^2 = f(x'^\mu, x^\mu) \tag{3.28}$$

with f homogeneous function of degree 2 in x'^μ. In the case of the free string we have no explicit dependence on x^μ in the Lagrangian or other four vectors which can be used; so that (3.28) should become:

$$\pi^2 + \alpha x'^2 = 0 \tag{3.29}$$

where α is a constant. This constraint is immediately verifiable to hold with $\alpha = [1/2\pi]^2$.

When the "strong-photon" field interaction is turned on the modified constraints equations may be found in the exactly same way by using Eqs. (3.19) instead of (3.15) or more simply observing that the conjugate momentum π^μ in presence of the interaction term (3.2) is modified with respect to the free case only by the constant term

$$\rho(\sigma)\, A^\mu \tag{3.30}$$

In both ways we find as modified constraints the equations

$$\left(\Pi_\mu - \rho(\sigma) A_\mu\right) x'^\mu = 0 \tag{3.31}$$

and

$$\left(\pi - \rho(\sigma) A_\mu\right)^2 + \left(\frac{x'}{2\pi}\right)^2 = 0 \tag{3.32}$$

respectively.

4. THE HAMILTONIAN METHOD

i) Equation of motion

In the previous section we developed the Lagrangian formalism to write down the equations of motion of the string and study the constraints imposed on the Lagrangian by the invariance under the reparametrization group. This formalism is very convenient to constrain a theory to be relativistic invariant. In fact one has to have a Lorentz invariant action integral and this will automatically insure that any development from this action will be in agreement with the theory of relativity.

In this Section we will consider the Hamiltonian method which, on the other hand, is very convenient when we want to quantize a classical system. In fact the transition from the classical to the quantum theory can be done very easily interpreting the dynamical variables as operators acting in a given Hilbert space and transforming the Poisson brackets of two dynamical variables into the commutator of the corresponding operators. Of course, one has to deal then with additional problems related to the non-commutativity of the quantities appearing in the quantum theory. This fact may spoil the Lorentz covariance of the theory which, therefore, must be checked again at the quantum level.

We start with the action integral:

$$S = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \; \mathcal{L}(x, \dot{x}, x')$$

$$\mathcal{L}(x,\dot{x},x') = \mathcal{L}_{free}(\dot{x},x') + \rho(\sigma)\, \dot{x}_\mu A^\mu(x) \tag{4.1}$$

which is invariant under any reparametrization which leaves unchanged the points $\sigma = 0, \pi$. $\rho(\sigma)$ is given in (3.5) and the field $A_\mu(x)$ is chosen of the form

$$A_\mu(x) = \varepsilon_\mu e^{ikx} \tag{4.1'}$$

with the additional conditions:

$$k^2 = \varepsilon^2 = k\cdot\varepsilon = 0 \tag{4.1''}$$

We discuss these conditions in the next Section.

As a consequence of this invariance the Hamiltonian density of the system is identically vanishing [see (3.19a)]:

$$\mathcal{H} = \dot{x} \cdot \Pi - \mathcal{L} = 0 \tag{4.2}$$

where Π_μ is the conjugate momentum

$$\Pi_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} \tag{4.3}$$

This is a choice that we can always make in a relativistic invariant theory where we do not want to have one particular time playing a special role; we want to have the possibility of various times τ which are all on the same footing *).

The vanishing of the Hamiltonian does not mean that we cannot describe the dynamical system using a Hamiltonian formalism, but it is connected with the symmetry properties of our Lagrangian. They manifest themselves in the Hamiltonian formalism in a set of constraints which involve the dynamical variables x^μ and π^μ [21].

---

*) We have the same features in the relativistic theory of a free point-like particle where the Lagrangian can be written as:

$$\mathcal{L} = -m \int_{\tau_i}^{\tau_f} \sqrt{-\dot{x}^2}\, d\tau \qquad ; \quad \dot{x}_\mu = \frac{dx_\mu}{d\tau}$$

where τ is the proper time. The Hamiltonian is then identically zero as a consequence of the homogeneity of first degree of $\mathcal{L}$:

$$H = \dot{x} \cdot p - \mathcal{L} = 0 \qquad ; \quad p_\mu = \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}$$

Therefore, the Hamiltonian can be taken to be proportional to the constraint

$$H = \lambda \left(p^2 + m^2 \right)$$

Starting from this Hamiltonian one can then evaluate the equations of motion.

In the case of the Lagrangian (4.1) we have an infinity *) of primary constraints given by the following equations deduced in Section 3 [Eqs. (3.31), (3.32)]

$$\left(\Pi - \varepsilon e^{ikx} \varrho(\sigma)\right) \cdot x' = 0 \tag{4.4}$$

$$\left(\Pi - \varepsilon e^{ikx} \varrho(\sigma)\right)^2 + \left(\frac{x'}{2\pi}\right)^2 = 0 \tag{4.5}$$

which can be recast in the more unified form

$$L^{(\pm)}(\tau,\sigma) = \frac{1}{4}\left(\Pi - \varepsilon e^{ikx} \varrho(\sigma) \pm \frac{x'}{2\pi}\right)^2 = 0 \tag{4.6}$$

Due to the existence of such constraints the Hamiltonian of our problem is not uniquely determined; we can always add to $\mathcal{H}$ any combination of the constraints without any change on the physics of our system. Therefore, we are free to write the Hamiltonian in the form

$$H = \frac{\pi}{2}\int_0^{\pi} d\sigma \left[f_+(\tau,\sigma)\left(\Pi - \varepsilon e^{ikx} \varrho(\sigma) + \frac{x'}{2\pi}\right)^2 + f_-(\tau,\sigma)\left(\Pi - \varepsilon e^{ikx} \varrho(\sigma) - \frac{x'}{2\pi}\right)^2 \right] \tag{4.7}$$

where $f_\pm(\tau, \sigma)$ are two arbitrary functions.

We can assume then canonical Poisson brackets at equal τ :

$$\begin{aligned} &\left\{ x^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma') \right\} = g^{\mu\nu}\, \delta(\sigma - \sigma') \\ &\left\{ x^\mu(\tau,\sigma), x^\nu(\tau,\sigma') \right\} = \left\{ \Pi^\mu(\tau,\sigma), \Pi^\nu(\tau,\sigma') \right\} = 0 \end{aligned} \tag{4.8}$$

so that the equation of motion of a dynamical variable becomes in the Hamiltonian form

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*) Equations (4.4) and (4.5) give a constraint for any value of σ ; any combination of these equations at different σ (in particular σ- derivative) is still a primary constraint.

$$\frac{dO}{dt} = \{O, H\} + \frac{\partial O}{\partial t} \tag{4.9}$$

In particular, for x and Π we obtain in our case (no explicit dependence on τ)

$$\dot{x} = \{x, H\} \qquad ; \quad \dot{\Pi} = \{\Pi, H\} \tag{4.10}$$

Using the Hamiltonian (4.7) and the Poisson brackets (4.8), we get:

$$\frac{\dot{x}}{2\pi} = \frac{1}{2}\left[f_+(\tau,\sigma) + f_-(\tau,\sigma)\right]\left[\Pi - \varepsilon\rho(\sigma)e^{ikx}\right] + \frac{1}{2}\left[f_+(\tau,\sigma) - f_-(\tau,\sigma)\right]\frac{x'}{2\pi} \tag{4.11}$$

In order to see in a more direct way the analogy between our problem and the dual resonance model we choose the particular gauge where

$$\Pi = \frac{\dot{x}}{2\pi} + \varepsilon\rho(\sigma)e^{ikx} \tag{4.12}$$

Therefore, the functions $f_\pm(\tau, \sigma)$ are completely determined:

$$f_+(\tau,\sigma) = f_-(\tau,\sigma) = 1 \tag{4.13}$$

and the Hamiltonian becomes

$$H = \pi \int_0^\pi d\sigma \left[\left(\Pi - \varepsilon e^{ikx}\rho(\sigma)\right)^2 + \left(\frac{x'}{2\pi}\right)^2\right] \tag{4.14}$$

By substituting (4.12) in (4.6) it is easy to check that this particular gauge corresponds to the choice of an orthogonal parametrization such that

$$\frac{1}{16\pi^2}\left(\dot{x} \pm x'\right)^2 = 0 \tag{4.15}$$

As we will see in the following the Fourier components of the Eq. (4.15) say just that the Virasoro operators L_n are zero.

The equation of motion for Π gives in this particular gauge

$$\dot{\Pi}_\mu(\tau,\sigma) = \frac{x''_\mu(\tau,\sigma)}{2\pi} + i\, k_\mu \varepsilon_\nu \left(\Pi - \varepsilon\, \rho(\sigma)\, e^{ikx}\right)^\nu \rho(\sigma)\, e^{ikx} - \qquad (4.16)$$

$$- \frac{x'(\tau,0)}{2\pi}\,\delta(\sigma) + \frac{x'(\tau,\pi)}{2\pi}\,\delta(\sigma-\pi)$$

The use of (4.12) gives finally the equation of motion of the string

$$\ddot{x} - x'' = 0 \qquad (4.17)$$

with the boundary conditions at $\sigma = 0, \pi$:

$$i\, g_0 \left(\varepsilon^\mu k^\nu - k^\mu \varepsilon^\nu\right) \dot{x}_\nu\, e^{ikx} - \frac{x'^\mu}{2\pi} = 0 \quad \text{at} \quad \sigma = 0 \qquad (4.18)$$

$$i\, g_\pi \left(\varepsilon^\mu k^\nu - k^\mu \varepsilon^\nu\right) \dot{x}_\nu\, e^{ikx} + \frac{x'^\mu}{2\pi} = 0 \quad \text{at} \quad \sigma = \pi \qquad (4.19)$$

These equations describe the motion of the string in an external "electromagnetic" field and are, of course, the same as those obtained in the previous section by means of the least action principle.

ii) Consistency conditions

In the second part of this section we will check the consistency of our Lagrangian. In fact the Lagrangian describing a certain classical system cannot be completely arbitrary; it has to satisfy certain consistency conditions which insure the consistency of the corresponding equation of motion.

Going from the Lagrangian to the Hamiltonian formalism we get a number of constraints which connect the dynamical variables for any value of τ. On the other hand, the evolution of a dynamical variable is determined by the Eq. (4.9). Therefore, to have a consistent formalism we must require the validity of the constraints all along the motion of our system.

In our case, this leads to the requirement that

$$\left\{ L^{(\pm)}(\tau,\sigma)\ ,\ H \right\} = 0 \tag{4.20}$$

The use of the equation of motion and of the constraints (4.6) is allowed only after the computation of the Poisson brackets.

Using the canonical Poisson brackets (4.8) it is easy to evaluate the Poisson brackets between the $L^{(\pm)}(\tau,\sigma)$'s:

$$\left\{ L^{(\pm)}(\tau,\sigma)\ ,\ L^{(\pm)}(\tau,\sigma') \right\} = \pm \frac{1}{4\pi} \left(\Pi - \varepsilon \rho e^{ikx} \pm \frac{x'}{2\pi} \right)^{\mu}(\tau,\sigma) \left(\Pi - \varepsilon e^{ikx} \rho \pm \frac{x'}{2\pi} \right)(\tau,\sigma') \cdot \frac{d}{d\sigma} \delta(\sigma - \sigma') \tag{4.21}$$

$$\left\{ L^{(+)}(\tau,\sigma)\ ,\ L^{(-)}(\tau,\sigma') \right\} = \frac{i}{2}\, \delta(\sigma-\sigma') \left(k_\mu \varepsilon_\nu - k_\nu \varepsilon_\mu \right) \frac{x'^{\mu}}{2\pi} \Pi^{\nu} e^{ikx} \rho(\sigma) \tag{4.22}$$

The right-hand side of (4.22) is vanishing for any τ due to the boundary conditions (4.18), (4.19) and the conditions (4.1"). Then we get

$$\left\{ L^{(\pm)}(\tau,\sigma),\ H \right\} = \pm \frac{1}{4} \frac{d}{d\sigma} \left(\Pi - \varepsilon e^{ikx} \rho(\sigma) \pm \frac{x'}{2\pi} \right)^2 \tag{4.23}$$

The right-hand side of (4.23) is the derivative with respect to σ of the primary constraints and hence it is still a primary constraint. In such a way we have shown that the commutators of the primary constraints with the Hamiltonian vanish and do not generate any secondary constraint.

In conclusion the constraints present in our problem are consistent with the τ -evolution and if they are satisfied at a certain value of τ they continue to be valid for any value of τ .

iii) Symmetry properties

In the last part of this section we study the symmetry properties of the action (4.1) in the particular gauge when an orthonormal parametrization has been chosen. We will see that a number of symmetry properties of the DRM will be related to the invariance of our Lagrangian under reparametrization.

We have seen that the action is invariant under any change of the parameters

$$\delta \xi_\pm = \varepsilon_\pm f_\pm(\xi_+, \xi_-) \tag{4.24}$$

which leaves the points $\sigma = 0, \pi$ invariant. We use the light-cone variables $\xi_\pm = \tau \pm \sigma$ only for our convenience. $\varepsilon_\pm$ are small quantities and $f_\pm$ are arbitrary functions. If we choose, however, the orthonormal gauge and then we make any transformation of the type (4.24), in general the conditions

$$\frac{1}{16\pi^2}\left(\dot{x} \pm x'\right)^2 = 0 \tag{4.25}$$

are not valid any more. In fact, only a subset of the gauge transformations preserves the relation (4.25).

It is easy to check that the transformations which do not modify the relations (4.25), are of the particular type

$$\delta \xi_\pm = \varepsilon_\pm f_\pm(\xi_\pm) \tag{4.26}$$

This is strictly related to the fact that, when the equations of motion are satisfied, the quantities

$$\left(\dot{x} \pm x'\right)^2 = \left[2 \frac{\partial x}{\partial \xi_\pm}\right]^2 = 0 \tag{4.27}$$

are function only of $\xi_\pm$ respectively and are the generators of transformations on these variables separately.

What happens here is analogous to the case of the free quantum electrodynamics where the action is invariant, under gauge transformations:

$$A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \Lambda(x) \tag{4.28}$$

where $\Lambda(x)$ is an arbitrary function. However, if we choose a particular gauge like for instance the Lorentz gauge $\Lambda(x)$ is constrained to satisfy the d'Alembert equation:

$$\Box \Lambda(x) = 0 \tag{4.29}$$

The additional requirement that the transformations (4.26) leave invariant the points $\sigma = 0, \pi$ imposes some restrictions on the functions f_+ and f_-:

$$f_+(\xi) = f_-(\xi) = f(\xi) \quad ; \quad f_\pm(\xi+\pi) = f_\pm(\xi-\pi) \quad ; \quad \varepsilon_+ = \varepsilon_- = \varepsilon \tag{4.30}$$

In conclusion the most general transformation which preserves the gauges (4.27) and leaves invariant the points $\sigma = 0, \pi$ can be written as:

$$\delta \xi_\pm = \varepsilon f(\xi_\pm) \tag{4.31}$$

where $f(\xi)$ is periodic of period 2π.

The generators of the previous transformations can then be written in terms of the constraints:

$$L_f = \frac{\pi}{2} \int_0^\pi d\sigma \left[\left(\frac{\dot{x} + x'}{2\pi} \right)^2 f(\tau+\sigma) + \left(\frac{\dot{x} - x'}{2\pi} \right)^2 f(\tau-\sigma) \right] \tag{4.32}$$

Using the equations of motion and the periodicity of $f(\xi)$ [Eq. (4.30)] it is easy to show the following properties of L_f:

$$\frac{d}{d\tau} L_f = \left[\left(\frac{\dot{x} + x'}{2\pi} \right)^2 (\tau+\sigma)\, f(\tau+\sigma) - \left(\frac{\dot{x} - x'}{2\pi} \right)^2 (\tau-\sigma)\, f(\tau-\sigma) \right]\Bigg|_0^\pi = 0 \tag{4.33}$$

$$\left\{ L_f , L_g \right\} = L_{f \otimes g} \tag{4.34}$$

where

$$f(\rho) \otimes g(\rho) = f(\rho)\frac{d}{d\rho}g(\rho) - g(\rho)\frac{d}{d\rho}f(\rho) \tag{4.35}$$

being $f(\rho)$ and $g(\rho)$ two functions of period 2π. The Eq. (4.33) states that L_f is a constant of motion and it is in agreement with the equation of evolution

$$\frac{d}{d\tau} L_f = \{ L_f , H \} \tag{4.36}$$

as we have checked above.

The expression (4.34) tells us that the generators of the transformations which leave invariant our Lagrangian, form a closed algebra.

Starting from the generators (4.32) the transformation properties of any dynamical variable O under a transformation of the type (4.34) are given by the equation *)

$$\delta O = \varepsilon \{ L_f , O \} \tag{4.37}$$

In the particular case $O = x$ we have:

$$\delta x^\mu(\tau,\sigma) = \varepsilon \{ L_f , x^\mu(\tau,\sigma) \} = -\frac{\varepsilon}{2}\left[(\dot{x}+x')^\mu f(\tau+\sigma) + (\dot{x}-x')^\mu f(\tau-\sigma) \right] \tag{4.38}$$

which is in agreement with the expression (3.12) in the case of a transformation of the type (4.31). For $f(\tau) = 1$ one finds again the result (4.11) in the particular gauge where (4.12) and (4.14) are valid. At $\sigma = 0$ one gets:

$$\delta x^\mu(\tau,0) = -\varepsilon \, \dot{x}^\mu(\tau,0) \, f(\tau) \tag{4.39}$$

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*) We are evaluating the change $\bar{O}(\tau) - O(\tau)$ in a point with the same co-ordinate value τ (local variation).

$$\delta \dot{x}^\mu = - \varepsilon \frac{d}{d\tau} \left(\dot{x}^\mu(\tau,0)\, f(\tau) \right) \tag{4.40}$$

which are the same as the transformations of the operators $Q(e^{i\tau})$ and $P(e^{i\tau})$ in the DRM.

At $\sigma = \pi$ one gets similar transformation properties:

$$\delta x^\mu(\tau,\pi) = - \varepsilon\, \dot{x}^\mu(\tau,\pi)\, f(\tau+\pi) \tag{4.41}$$

$$\delta \dot{x}^\mu(\tau,\pi) = - \varepsilon \frac{d}{d\tau} \left[\dot{x}^\mu(\tau,\pi)\, f(\tau+\pi) \right] \tag{4.42}$$

where the periodicity of $f(\tau)$ has been used.

Starting from the expressions (4.39) and (4.40) it is easy to evaluate the transformation properties of the Lagrangian $\mathcal{L}_i$ which describes the interaction of the string with the external field:

$$\left\{ \varepsilon \cdot \dot{X}(\tau,0)\, e^{ikx(\tau,0)} , L_f \right\} = \frac{d}{d\tau} \left[\varepsilon \cdot \dot{x}(\tau,0)\, e^{ikx(\tau,0)} f(\tau) \right] \tag{4.43}$$

In the DRM these transformation properties characterize the vertices associated with the physical particles.

The transformation property (4.43) is not peculiar of the particular choice of $\mathcal{L}_i$; it follows in a quite general way from the invariance of $\mathcal{L}_i$ under reparametrizations.

A more general case of a Lagrangian depending on higher τ-derivatives will be discussed in Section 8.

5. SOLUTION OF THE EQUATIONS OF MOTION AND ITS DISCUSSION

i) Solution of the equation of motion

In the first part of this Section we solve the equations of motion of a string in an external electromagnetic field. They have been derived and discussed in the previous sections using both the Lagrangian and Hamiltonian formalisms:

$$\ddot{x} - x'' = 0 \tag{5.1}$$

$$\frac{x'^{\mu}}{2\pi} = i g_0 \left(\varepsilon^{\mu} k^{\nu} - \varepsilon^{\nu} k^{\mu}\right) \dot{x}_{\nu} e^{ikx} \qquad \text{at} \quad \sigma = 0 \tag{5.2}$$

$$\frac{x'^{\mu}}{2\pi} = - i g_{\pi} \left(\varepsilon^{\mu} k^{\nu} - \varepsilon^{\nu} k^{\mu}\right) \dot{x}_{\nu} e^{ikx} \qquad \text{at} \quad \sigma = \pi \tag{5.3}$$

with the requirement that the photon be physical and on-mass shell:

$$k^2 = k \cdot \varepsilon = \varepsilon^2 = 0 \tag{5.4}$$

We impose the condition $\varepsilon^2 = 0$ (photon circularly polarized) because in this case the Equations of motion can be exactly solved without the use of a perturbative approach.

The only difference with the free case is given by the boundary conditions which in presence of an external field give rise to an exchange of momentum between the field and the string. We remark that (5.2) and (5.3) are consistent with the constraints (4.25) at $\sigma = 0, \pi$ for the presence of the skew tensor $\varepsilon^{\mu} k^{\nu} - \varepsilon^{\nu} k^{\mu}$:

$$\dot{x}(\tau,0) \cdot x'(\tau,0) = \dot{x}(\tau,\pi) \cdot x'(\tau,\pi) = 0 \tag{5.5}$$

On the other hand, the conditions (5.4) impose that

$$x'^2(\tau,0) = x'^2(\tau,\pi) = 0 \tag{5.6}$$

The constraint equations (4.25) give then the relations:

$$\dot{x}^2(\tau,0) = \dot{x}^2(\tau,\pi) = 0 \tag{5.7}$$

The most general solution of (5.1) can be written in the form

$$x^{\mu}(\tau,\sigma) = F^{\mu}(\tau+\sigma) + G^{\mu}(\tau-\sigma) \tag{5.8}$$

We must now constraint this function to satisfy the boundary conditions. This in principle may be a very difficult task because the boundary conditions are not linear for the presence of the term e^{ikx}; therefore, a linear combination of two solutions is not in general again a solution. However, the conditions (5.4) simplify considerably the problem and enable us to solve the equations (5.2), (5.3) very easily.

In fact, as a consequence of (5.4) the quantities $k_{\mu} x^{\mu}$ and $\varepsilon_{\mu} x^{\mu}$ satisfy the boundary conditions of the free case which constrain the functions $F(\tau+\sigma)$ and $G(\tau-\sigma)$ along the directions k^{μ} and ε^{μ} as follows:

$$\binom{k}{\varepsilon}_{\mu} F^{\mu}(\tau) = \binom{k}{\varepsilon}_{\mu} G^{\mu}(\tau) \;\; ; \;\; \binom{k}{\varepsilon}_{\mu} \frac{d}{d\tau} F^{\mu}(\tau+\pi) = \binom{k}{\varepsilon}_{\mu} \frac{d}{d\tau} F^{\mu}(\tau-\pi) \tag{5.9}$$

so that

$$\binom{k}{\varepsilon}_{\mu} x^{\mu}(\tau,\sigma) = \binom{k}{\varepsilon}_{\mu} \left[F^{\mu}(\tau+\sigma) + F^{\mu}(\tau-\sigma) \right] \tag{5.10}$$

As a consequence of this simplification the right-hand side of the equations (5.2) and (5.3), being only function of the free solution, is a known function and so these conditions can be easily fulfilled.

We want to stress that the simplifications occurring here are based on the three important facts:

i) photon on-mass shell: $k^2 = 0$;

ii) physical external photon: $k \cdot \varepsilon = 0$;

iii) choice of a monochromatic external photon field with $\varepsilon^2 = 0$. In fact, if one of these conditions is not satisfied we are unable to find a solution of the non-linear boundary conditions. For the point i) this may be connected with the difficulties met in the extrapolation off-mass shell of the DRM.

The formal solution of Eqs. (5.2) and (5.3) is then easily obtained starting from an $x^{\mu}(\tau,\sigma)$ of the following form:

$$x^{\mu}(\tau,\sigma) = \left[F^{\mu}(\tau+\sigma)+F^{\mu}(\tau-\sigma)\right] + A\left[\varepsilon^{\mu}k^{\nu}-\varepsilon^{\nu}k^{\mu}\right]\left[h_{\nu}(\tau+\pi-\sigma)+h_{\nu}(\tau-\pi+\sigma)\right]+ \\ + B\left[\varepsilon^{\mu}k^{\nu}-\varepsilon^{\nu}k^{\mu}\right]\left[g_{\nu}(\tau+\sigma)+g_{\nu}(\tau-\sigma)\right] \tag{5.11}$$

The term $\left[F^{\mu}(\tau+\sigma)+F^{\mu}(\tau-\sigma)\right]$ represents the most general solution of the free case. The other two terms take into account the presence of the external field. We have put two terms in the solution (5.11) in order to separate completely the effect of the boundary condition at $\sigma=0$ from that of the boundary condition at $\sigma=\pi$. In fact, the term with the function $h\,[g]$ does not give any contribution to the boundary condition at $\sigma=\pi$ $[\sigma=0]$. In addition the form (5.11) insures that the components of $x^{\mu}(\tau,\sigma)$ along k^{μ} and ε^{μ} are free.

The actual determination of the functions g and h is given in Appendix B. Here we write down directly the solution

$$\dot{x}^{\mu}(\tau,\sigma) = \dot{x}^{\mu}_{o}(\tau,\sigma) + \left[\varepsilon^{\mu}k^{\nu}-\varepsilon^{\nu}k^{\mu}\right]\left[R_{+}\dot{x}_{o\nu}(\tau-\sigma;0)e^{ikx(\tau-\sigma;0)} + R_{-}\dot{x}_{o\nu}(\tau+\sigma;0)e^{ikx(\tau+\sigma;0)}\right] \tag{5.12}$$

where

$$R_{\pm} = \frac{2\pi\left(g_{\pi} + g_{0}\, e^{\pm 2i(P.k)\pi}\right)}{2\sin(2Pk)\pi} \tag{5.13}$$

and P_{μ} is the total momentum of the string.

Integrating (5.12) with respect to τ we determine $x^{\mu}(\tau,\sigma)$ within a constant which can be included in the free part of the solution:

$$x^{\mu}(\tau,\sigma) = x^{\mu}_{0}(\tau,\sigma) + \left(\varepsilon^{\mu}k^{\nu}-\varepsilon^{\nu}k^{\mu}\right)\left[R_{+}\int^{\tau-\sigma}\dot{x}_{\nu}(\tau',0)e^{ikx(\tau',0)}d\tau' + R_{-}\int^{\tau+\sigma}\dot{x}_{\nu}(\tau',0)e^{ikx(\tau',0)}d\tau'\right] \tag{5.14}$$

A particularly symmetric way of writing down $x^{\mu}(\tau,\sigma)$ is the following

$$x^{\mu}(\tau,\sigma) = x^{\mu}_{0}(\tau,\sigma) + \left(\varepsilon^{\mu}k^{\nu}-\varepsilon^{\nu}k^{\mu}\right)\left[S_{+}\int_{\tau-\sigma-\pi}^{\tau-\sigma+\pi}\dot{x}_{\nu}(\tau',\pi)\,e^{ikx(\tau',\pi)}d\tau' + \right. \\ \left. + S_{-}\int_{\tau+\sigma-\pi}^{\tau+\sigma+\pi}\dot{x}_{\nu}(\tau',\pi)\,e^{ikx(\tau',\pi)}d\tau'\right] \tag{5.15}$$

where

$$S_{\pm} = \frac{g_{\pi} + g_0 e^{\pm 2i(P.k)\pi}}{4i \sin^2 (2P.k)\pi} \tag{5.16}$$

$x_0^{\mu}(\tau, \sigma)$ satisfies the boundary condition of the free case and has the form

$$x_0^{\mu}(\tau,\sigma) = F^{\mu}(\tau+\sigma) + F^{\mu}(\tau-\sigma) \tag{5.17}$$

where the derivative $\dot{F}^{\mu}(\tau)$ is periodic of period 2π.

As in the free case $x_0^{\mu}(\tau, \sigma)$ can be written as a superposition of oscillators:

$$x_0^{\mu}(\tau,\sigma) = q_0^{\mu} + \sqrt{2}\left[p_0^{\mu}\tau + i \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{\alpha_n^{\mu}}{n} \cos n\sigma \, e^{-in\tau} \right] \tag{5.18}$$

where p_0^{μ} is connected to the momentum P^{μ} of the free string by the relation $\sqrt{2}P^{\mu} = p_0^{\mu}$. For those components of $x^{\mu}(\tau, \sigma)$ which behave as free the momentum of the interacting string coincides with that of the free string; that is why we have used the relation $2P\cdot k = \sqrt{2}p_0\cdot k$.

The expression (5.14) describes the motion of a string interacting at the ends with an external, monochromatic electromagnetic field.

It shows simple poles for integer values of $(2Pk)$:

$$2P.k = n \qquad n \text{ is integer} \tag{5.19}$$

These poles have a very transparent physical meaning. If the frequency of the photon is such to satisfy the relation (5.19), then it can excite an eigen-frequency of the string and we have the case of a resonance.

If, however, the signature factor (5.13) vanishes:

$$g_{\pi} + g_0 (-1)^n = 0 \tag{5.20}$$

then the momentum pumped in one end of the string is taken back out to the field from the other end; so the resonance cannot be formed.

These resonances correspond to the resonances appearing in the DRM and the term of interaction at $\sigma = \pi$ corresponds to the twisted vertex as we will see later on; that is why we have called signature factor the expression (5.13).

ii) Poisson brackets and gauge operators

In the second part of this section we discuss the Poisson brackets between our dynamical variables and we construct explicitly the gauge operators I_n. In the gauge defined by the constraints (4.15) the Poisson brackets at equal time between $x^\mu(\tau,\sigma)$ and $\dot{x}^\mu(\tau,\sigma)$ are given by:

$$\left\{ x^\mu(\tau,\sigma) , x^\nu(\tau,\sigma') \right\} = \left\{ \dot{x}^\mu(\tau,\sigma) , \dot{x}^\nu(\tau,\sigma') \right\} = 0$$
$$\left\{ x^\mu(\tau,\sigma) , \dot{x}^\nu(\tau,\sigma') \right\} = 2\pi g^{\mu\nu} \delta(\sigma - \sigma') \tag{5.21}$$

Starting from the decomposition of $\dot{x}^\mu(\tau,\sigma)$ in terms of harmonic oscillators and using the canonical Poisson brackets between $\dot{x}(\tau,\sigma)$ and $\dot{x}(\tau,\sigma')$, we get the following Poisson brackets:

$$\left\{ \alpha_{n,\mu} , \alpha_{m,\nu} \right\} = -in\, g_{\mu\nu}\, \delta_{n+m;0} \tag{5.22}$$

as in the free case.

Using the equation of motion (5.1) it is then easy to derive the Poisson brackets between the quantities in (5.21) at different τ ; the result at $\sigma = 0$ is the same as in DRM for the commutators between $Q^\mu(e^{-i\tau})$ and $P^\mu(e^{-i\tau})$.

Let us prove the following theorem.

If the quantities $x^\mu(\tau,\sigma)$ and $\dot{x}^\nu(\tau,\sigma)$ satisfy the Poisson brackets (5.21) at equal time and the equation of motion (5.1) is valid, then the Poisson brackets for any value of σ and τ are given by:

$$\left\{ x^\mu(\tau,\sigma) , x^\nu(\tau',\sigma') \right\} = 2 \sum_{n=-\infty}^{\infty} \frac{\cos n\sigma \cos n\sigma'}{n} \sin n(\tau'-\tau) \tag{5.23}$$

We can write the left-hand side of (5.23) in the following form

$$\left\{x^{\mu}(\tau,\sigma)\,,x^{\nu}(\tau',\sigma')\right\} = \sum_{n=0}^{\infty}\frac{(\tau'-\tau)^{n}}{n!}\left\{x^{\mu}(\tau,\sigma),\frac{d^{n}}{d\tau^{n}}x^{\nu}(\tau,\sigma')\right\} \qquad (5.24)$$

Using the equation of motion (5.1) and the Poisson brackets at equal τ we get

$$\left\{x^{\mu}(\tau,\sigma),\,x^{\nu}(\tau',\sigma')\right\} = 2\pi g^{\mu\nu}\sum_{n=0}^{\infty}\frac{(\tau'-\tau)^{2n+1}}{(2n+1)!}\frac{d^{2n}}{d\sigma'^{2n}}\,\delta(\sigma-\sigma') \qquad (5.25)$$

The series can be summed up and it is equal to the right-hand side of (5.23).

At $\sigma = \sigma' = 0$ one gets:

$$\left\{x^{\mu}(\tau,0)\,,x^{\nu}(\tau',0)\right\} = 2\pi\, g^{\mu\nu}\,\varepsilon(\tau'-\tau) \qquad (5.26)$$

which is the same result than one has in the DRM for the commutator between two Q's.

In the last part of this section we evaluate the gauge operators L_n. Starting from the solution of the equation of motion we get:

$$\frac{1}{2}\left(\dot{x}\pm x'\right)^{\mu} = \left[\frac{\dot{x}_0^{\mu}\pm x_0'^{\mu}}{2} + R_{\mp}\left(\varepsilon^{\mu}k^{\nu}-\varepsilon^{\nu}k^{\mu}\right)\dot{x}_{0\nu}(\tau\pm\sigma;0)\,e^{ikx(\tau\pm\sigma;0)}\right] \qquad (5.27)$$

where

$$\frac{\dot{x}_0^{\mu}\pm x_0'^{\mu}}{2} = \dot{F}^{\mu}(\tau\pm\sigma) \qquad (5.28)$$

Squaring (5.27) we got:

$$\frac{1}{4}\left(\dot{x}\pm x'\right)^{2} = \frac{1}{4}\left(\dot{x}_0\pm x_0'\right)^{2} = \dot{F}(\tau\pm\sigma)^{2} \qquad (5.29)$$

after the use of the boundary conditions (5.2) and (5.3) and of the following identity:

$$\left(\dot{x}\pm x'\right)(\tau,\sigma) = \left(\dot{x}\pm x'\right)(\tau\pm\sigma\,;0) \qquad (5.30)$$

Equation (5.29) is very important because it shows that, also when the interaction is turned on, the constraints have the same expression in terms of the oscillators α_n as in the free case.

The gauge operators L_f can then be obtained from expression (4.32):

$$L_f = \frac{1}{2\pi}\int_0^{\pi} d\sigma \left[\dot{F}^2(\tau+\sigma)\, f(\tau+\sigma) + \dot{F}^2(\tau-\sigma)\, f(\tau-\sigma) \right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma\, \dot{F}^2(\tau+\sigma) f(\tau+\sigma) \qquad (5.31)$$

L_f is a constant of motion; it does not depend on τ. We get the same expression for L_f if instead we integrate over τ:

$$L_f = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\sigma\, \dot{F}^2(\tau+\sigma)\, f(\tau+\sigma) = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\tau\, \dot{F}^2(\tau+\sigma)\, f(\tau+\sigma) \qquad (5.32)$$

The resulting expression does not depend of course on σ ; so we can evaluate the integral fixing $\sigma = 0$ and using the properties (5.5) and (5.6):

$$L_f = \frac{1}{8\pi}\int_{-\pi}^{\pi} d\tau\, f(\tau)\, \dot{x}^2(\tau,0) \qquad (5.33)$$

which is the expression that one gets in the DRM provided that the identification $x(\tau,0) \leftrightarrow P(e^{-i\tau})$ is made. The choice of the complete set of functions

$$f(\tau) = e^{in\tau} \qquad (5.34)$$

gives then the familiar operators L_n

$$L_n = \frac{1}{8\pi}\int_{-\pi}^{\pi} d\tau\, \dot{x}^2(\tau,0)\, e^{in\tau} \qquad (5.35)$$

In terms of the harmonic oscillators we get the well-known expression:

$$L_{-n} = \frac{1}{2}\sum_{\ell=-\infty}^{\infty} \alpha_{-\ell}\, \alpha_{\ell+n} \qquad (5.36)$$

iii) Independent degrees of freedom

In the last part of this section we compute the number of independent degrees of freedom of a string in an external electromagnetic field. Our treatment will follow strictly the one used in Ref. 11) for the free case; in fact, our situation is quite analogous to that of the free theory because we have shown that the interaction does not spoil the gauges.

As we have already seen, the constraint conditions (4.25) do not eliminate completely any arbitrarity of gauge; we can still make transformations of the type (4.31) without changing the physical content of our equations. If, however, we want to select the independent degrees of freedom of the string we cannot leave any freedom of gauge.

Following the procedure of Ref. 11) we identify τ with some time co-ordinate:

$$n\cdot x = 2(n.P)\,\tau + n\cdot q_0 \tag{5.37}$$

where $n^2 \leq 0$ and the quantity $2(n\cdot P)$ is a constant of motion.

In our case the momentum of the string is not a constant of motion because there is a flow of momentum from the field to the string; so, in general, we cannot identify the quantity P appearing in (5.37) with the momentum of the string. However, if we choose for the vector n^μ the following quantity

$$\sqrt{2}\, n^\mu = \frac{k^\mu}{k_0} \tag{5.38}$$

then the component of the momentum of the string along the direction n^μ is a constant of motion. In this case the quantity P^μ appearing in the Eq. (5.37) is the total momentum of the string. The choice of a light-like vector n^μ is very natural in our approach; it is convenient to show very clearly the analogy existing between the string and the DRM.

The derivative with respect to τ of (5.37) gives:

$$n\cdot \dot{x} = 2\,(P\cdot n) \tag{5.39}$$

which is constant along σ.

One can define then the following expression [11]

$$2(p\cdot n)\sigma = \int_0^\sigma d\sigma' \, (n\cdot\dot{x}) \tag{5.40}$$

which is independent of τ ; it is a constant of motion.

Equation (5.40) gives to σ a very transparent physical meaning; it is proportional to the momentum of the string along n^μ included between the boundary $\sigma = 0$ and the point σ considered.

Choosing for the vector n^μ the following form:

$$n^\mu = \frac{1}{\sqrt{2}}\left(1, \underset{\sim}{0}, -1\right) \quad ; \quad n_- = 1 \, , \quad n_+ = \underset{\sim}{n} = 0 \tag{5.41}$$

where

$$n_\pm = \frac{1}{\sqrt{2}}\left(n_0 \pm n_3\right) \tag{5.42}$$

the condition (5.37) implies that

$$\alpha_{n,+} = 0 \qquad \text{for any} \quad n \neq 0 \tag{5.43}$$

Using this equation in the expression for the L_n's one gets the relation between the $\alpha_{n,-}$'s and the transverse oscillators:

$$\alpha_{n,-} = \frac{1}{p_{0+}} \, \mathcal{L}_n \tag{5.44}$$

which is the same as that found in the case of a free string. In terms of the transverse harmonic oscillators $\mathcal{L}_n$ is given by:

$$\mathcal{L}_n = \frac{1}{2}\sum_{\ell=-\infty}^{\infty} \alpha_{\perp;-\ell} \; \alpha_{\perp;n+\ell} \tag{5.45}$$

In conclusion the independent degrees of freedom are given by the transverse oscillators. The longitudinal and scalar modes, which must be there to insure Lorentz covariance, can be expressed as a function of the transverse oscillators.

6. QUANTIZATION AND RELATION WITH THE DUAL RESONANCE MODEL

In this Section we proceed to quantize the string interacting with an external electromagnetic field.

As in iii) of Section 5 we will follow strictly the treatment given for the free case in Ref. 11). The rule for quantizing a classical system is a very simple one when we have its description in the Hamiltonian formalism. We have to make in fact the dynamical variable x and π into operators acting in a linear space and the Poisson brackets into commutators:

$$i\,\{\text{Poisson brackets}\} \longrightarrow [\text{commutators}] \tag{6.1}$$

However, in any problem with constraints [Eq. (4.15)] connecting together the dynamical variables we can proceed in two different and equivalent ways.

The first one consists in quantizing only the independent variables; the others will be then expressed in the same linear space as a function of the independent ones. This is the non-covariant procedure of quantization and it is very convenient to study the norm of the linear space; a coherent quantum theory requires the positiveness of the norm of the physical states. The second procedure of quantization consists instead in quantizing all the degrees of freedom and then in restricting the physical subspace by means of certain supplementary conditions. This is the covariant quantization and it shows manifestly the right properties of covariance under the Lorentz group.

We have introduced in Section 3 a covariant Hamiltonian formalism; so we proceed now to the covariant quantization. The equal time Poisson brackets become equal time commutators:

$$\left[x^{\mu}(\tau,\sigma), x^{\nu}(\tau,\sigma') \right] = \left[\pi^{\mu}(\tau,\sigma), \pi^{\nu}(\tau,\sigma') \right] = 0 \tag{6.2}$$

$$\left[x^{\mu}(\tau,\sigma), \pi^{\nu}(\tau,\sigma') \right] = i\, g^{\mu\nu}\, \delta(\sigma-\sigma') \tag{6.3}$$

The linear space where the operators act is built up by an infinite set of harmonic oscillators $\alpha_{n,\mu}$ acting on a vacuum state $|0\rangle$ and such that:

$$\left[\alpha_{m,\mu} , \alpha_{m,\nu} \right] = n g_{\mu\nu} \delta_{n+m;0} \tag{6.4}$$

This type of quantization shows manifestly the Lorentz covariance of the theory; however, the Lorentz metric tensor $g_{\mu\nu}$ does not make the linear space, defined by the commutation relations (6.4), have a positive definite norm. It is the gauge group that eliminates this contradiction between the quantum theory and the theory of relativity. This is obtained assuming that the space of the physical states is a subspace of the entire linear space; the vectors $|\Psi_1>$ and $|\Psi_2>$ belonging to the physical subspace must satisfy the following conditions:

$$\langle \Psi_1 | L_n | \Psi_2 \rangle = 0 \qquad n \neq 0 \tag{6.5}$$

$$\langle \Psi_1 | L_0 | \Psi_2 \rangle = \alpha_0 \langle \Psi_1 | \Psi_2 \rangle \tag{6.6}$$

The conditions (6.5) are the quantum transposition of the constraint equations (4.27) which cannot be imposed as operatorial identities. In the case of L_0 going to quantum theory we have some arbitrariness on the ordering of the operators. The choice (6.6) is discussed in detail in Ref. 11); α_0 is the intercept of the Regge trajectory.

The operators L_n are given by:

$$L_n = \frac{1}{8\pi} \int_{-\pi}^{\pi} d\tau \, e^{-in\tau} \; : \dot{x}^2(\tau, 0) : \tag{6.7}$$

and satisfy the algebra:

$$\left[L_n , L_m \right] = (n-m) L_{n+m} + \frac{D}{12} \delta_{n+m;0} \, n(n^2-1) \tag{6.8}$$

The physical states are then characterized by the subsidiary conditions:

$$L_n | \Psi \rangle = 0 \qquad n > 0 \tag{6.9}$$

$$L_0 | \Psi \rangle = \alpha_0 | \Psi \rangle \tag{6.10}$$

which insure the validity of (6.6) and (6.7) in the physical subspace.

This procedure of quantization is consistent with the theory of relativity only for the special values

$$\alpha_0 = 1 \qquad , \quad D = 26 \tag{6.11}$$

of the Regge intercept α_0 and of the space-time dimension D.

The requirement (4.27) does not fix unequivocally the gauge; we can in fact still make transformations of the type (4.31) without changing the gauge conditions (4.27).

In the classical theory the choice (5.37) eliminates this gauge arbitrariness; this condition is translated in the quantum theory as a relation valid in the subspace of the physical states:

$$\langle \Psi_1 | n \cdot x | \Psi_2 \rangle = \langle \Psi_1 | \left[n \cdot q_0 + 2(n \cdot P) \tau \right] | \Psi_2 \rangle \tag{6.12}$$

(6.12) is obviously valid if the physical states satisfy the subsidiary conditions:

$$n \cdot \alpha_m | \Psi \rangle = \alpha_{m,+} | \Psi \rangle = 0 \tag{6.13}$$

In conclusion the physical states are characterized by Eqs. (6.9), (6.10) and (6.13). A complete set of solutions of these equations is given by the transverse states as it has been shown in Ref. 3). They can be written in terms of the transverse operators $A_{n;i}$ defined in Ref. 4), and they have a positive definite norm.

In the last part of this section we evaluate the probability of emission of a photon from the string; the resulting expression is the same as in the DRM in agreement with the intuitive considerations of Section 2.

Starting from expression (4.14) of the total Hamiltonian, we can evaluate the interaction Hamiltonian:

$$H = H_0 + H_I \tag{6.14}$$

where

$$H_0 = \pi \int_0^{\pi} d\sigma \left[\Pi^2 + \left(\frac{x'}{2\pi}\right)^2 \right] \tag{6.15}$$

$$H_I = - \left[g_0\, \varepsilon \cdot \dot{x}(\tau,0)\, e^{ikx(\tau,0)} + g_\pi\, \varepsilon \cdot \dot{x}(\tau,\pi)\, e^{ikx(\tau,\pi)} \right] \tag{6.16}$$

In the case of our system H_I is related to the interaction Lagrangian L_I by the usual relation:

$$H_I = - L_I \tag{6.17}$$

If we take for simplicity $g_\pi = 0$ the probability of emission of one photon of momentum k^μ and polarization ε^μ is given by:

$$\langle \alpha |\ \varepsilon \cdot \dot{x}(\tau,0)\, e^{ikx(\tau,0)}\ | \beta \rangle \tag{6.18}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are the initial and final state of the string satisfying the conditions (6.9) and (6.10).

The components of $x^\mu(\tau,0)$ along ε^μ and k^μ satisfy the free equation of motion; so the expression we get is the same as in the DRM [6]. In conclusion, not only the magnetic moment, as we have seen in Section 2, but also the higher multipoles evaluated in the DRM are reproduced by a string type of description. The probability of emission of n photons will be described in the next section.

7. <u>INTERACTING STRING WITH A VARIABLE EXTERNAL FIELD. A STRING PICTURE OF n POINT FUNCTIONS OF DRM</u>

In this Section we wish to study the motion of the string interacting with an "electromagnetic" field for a limited time. In order to simulate at the classical level the emission or the absorption of photons with different momenta and polarizations we take ε_μ and k_μ as arbitrary functions of τ, that is

$$A_\mu(x,\tau) = \phi(\tau)\, \varepsilon_\mu(\tau)\, e^{ik(\tau)\cdot x} \tag{7.1}$$

where $\phi(\tau)$ is an arbitrary function different from 0 during the interacting time $\Delta\tau$. We shall see that in this case the string behaves in two different ways according to the value $\Delta\tau \leq 2\pi$ or $\Delta\tau > 2\pi$. For the sake of simplicity we take in the section $g_\pi = 0$, so that only one end of the string interacts with the field (7.1). The boundary conditions become

$$\frac{1}{2\pi} x'_\mu + g_0 \dot{x}_\nu \partial_\mu A^\nu(x,\tau) - g_0 \frac{d}{d\tau} A_\mu(x,\tau) = 0 \qquad \text{at } \sigma = 0$$

$$x'_\mu = 0 \qquad \text{at } \sigma = \pi \tag{7.2}$$

The equations of motion inside the string are unchanged, of course. From the conditions $k(\tau)^2 = \varepsilon(\tau)\cdot k(\tau) = \varepsilon(\tau)^2 = 0$, we have, as before

$$\varepsilon(\tau)\cdot x'(\tau,0) = k(\tau)\cdot x'(\tau,0) = 0 \tag{7.3}$$

thus $\varepsilon(\tau)\cdot x$ and $k(\tau)\cdot x$ coincide with the same components of the free string.

According to the value of the interacting time $\Delta\tau$ we have two kinds of solutions:

a) Assume $\Delta\tau \leq 2\pi$ or more precisely

$$A_\mu(x,\tau) = 0 \qquad \text{for} \quad \tau < 0 \;, \; \tau \geq 2\pi \tag{7.4}$$

In this case the general solution of Eq. (7.2) is

$$x_\mu(\tau,\sigma) = x_{0\mu}(\tau,\sigma) + \alpha_\mu(\tau-\sigma) + \alpha_\mu(\tau+\sigma-2\pi) \tag{7.5}$$

where $x_{0\mu}(\tau,\sigma)$ are the co-ordinates of the free string before the interaction, and

$$\alpha_\mu(\xi) = 2\pi g_0 A_\mu[\xi, x_0(\xi,0)] - 2\pi g_0 \int_0^\xi \dot{x}_{0\nu}(\xi',0)\left\{\partial_\mu A^\nu[\xi', x_0(\xi',0)]\right\} d\xi' \qquad \text{for } \xi < 2\pi \tag{7.6a}$$

$$\alpha_\mu(\xi + 2\pi) = \alpha_\mu(\xi) \qquad \text{for} \quad \xi \geq 2\pi \tag{7.6b}$$

$$\alpha_\mu(\xi) = 0 \qquad \text{for} \quad \xi < 0 \tag{7.6c}$$

Equation (7.5) has a transparent physical meaning: the external field produces a disturbance α_μ at the end $\sigma = 0$ which propagates along the string until it reaches the end $\sigma = \pi$ at the time $\tau = \pi$, where it generates a reflected wave. Equation (7.6b) says that the disturbance continues to propagate indefinitely between the two ends. Then, when $\tau > 2\pi$, the string becomes again a free string $y_0(\tau, \sigma)$, where

$$y_0(\tau,\sigma) = x_0(\tau,\sigma) + \alpha_\mu(\tau-\sigma) + \alpha_\mu(\tau+\sigma) \tag{7.7}$$

Thus, owing to the external field, the string undergoes the transition from the initial configuration $x_{0\mu}$ to the final one $y_{0\mu}$. It is worthwhile to note that the solution (7.5) is linear in the field A_μ (and in the coupling constant g_0); thus, there is no correlation among the photons absorbed or emitted at different times. We remind that $\Delta\tau = 2\pi$ is the characteristic time of the string, i.e., the time needed by a disturbance to go from one end to the other and come back (see Section 2).

b) Assume that the interacting time $\Delta\tau$ be greater than 2π, or more precisely

$$A_\mu(x,\tau) = 0 \quad \text{for} \quad \tau < 0 \;, \tau \geq 4\pi$$

Then the general solution is

$$x_\mu(\tau,\sigma) = x_{0\mu}(\tau,\sigma) + \alpha_\mu(\tau-\sigma) + \alpha_\mu(\tau+\sigma-2\pi) + \beta_\mu(\tau-\sigma) + \beta_\mu(\tau+\sigma-2\pi) \tag{7.8}$$

where α_μ fulfils the conditions (7.6a), (7.6b) and (7.6c), while

$$\beta_\mu(\xi) = 2\pi g_0 A_\mu[\xi, y_0(\xi,0)] - \int_{2\pi}^{\xi} 2\pi g_0 \dot{y}_{0\nu}(\xi',0)\{\partial_\mu A^\nu[\xi', y_0(\xi',0)]\} d\xi' \tag{7.9a}$$
$$\text{for } 4\pi > \xi \geq 2\pi$$

$$\beta_\mu(\xi + 2\pi) = \beta_\mu(\xi) \qquad \text{for } \xi \geq 4\pi \tag{7.9b}$$

$$\beta_\mu(\xi) = 0 \qquad \xi < 2\pi \tag{7.9c}$$

$y_0(\tau, \sigma)$ is defined in (7.7).

We see that in this case the general solution (7.8) is no longer linear in the field A_μ and in the coupling constant g_0; indeed β_μ is a non-linear function of y_0 which in turn depends on the field A_μ. The reason for such a non-linear behaviour is simply that the excitations produced by the field during the time $0 \leq \tau < 2\pi$, after a reflection at the end $\sigma = \pi$, come back and start to modify the motion of the charged end at $\tau \geq 2\pi$. Thus, the interaction of the field at $\tau \geq 2\pi$ depends on the photons absorbed or emitted before.

The presence of these two different behaviours is a direct consequence of the extended nature of the string. Indeed for a point-like object interacting with an electromagnetic field only the process b) may occur, while the linear behaviour of the case a) is a peculiar feature of the string.

The quantization of the interacting string in a potential (7.1) can be carried out in exactly the same way of the string interacting with a time-independent potential treated in Section 6. Even in this case the L_n operators have the same form of the free case and the commutators of the harmonic oscillator operators are the standard ones.

Let us consider a process in which a string state $|\alpha>$ emits at the time τ_1 a photon $\varepsilon_1 k_1$, at the time τ_2 a photon $\varepsilon_2 k_2 \ldots$ at the time τ_n the photon $\varepsilon_n k_n$, and finally jumps in the state $|\beta>$. The amplitude for such a process is

$$\langle\alpha|\, T\left(H_I(\tau_1 ; \varepsilon_1, k_1)\, H_I(\tau_2 ; \varepsilon_2, k_2) \cdots H_I(\tau_n ; \varepsilon_n, k_n)\right) |\beta\rangle \tag{7.10}$$

T indicates the time ordered product, $H_I(\tau ; \varepsilon, k)$ is the interaction Hamiltonian defined in (6.18), i.e,

$$H_I(\tau, \varepsilon, k) = g_0\ \varepsilon\cdot\dot{x}(\tau, 0)\, e^{ikx(\tau,0)} \tag{7.11}$$

The total amplitude for the emission of n "photons" can then be obtained integrating over the variables τ_i:

$$A(\alpha, \beta ; \varepsilon_i k_i) = \sum_{\text{perm.}} \int_{-\infty}^{0} \prod_{i=2}^{n} d\tau_i\, \vartheta(\tau_{i+1} - \tau_i) \langle\alpha|\, H_I(\tau_1) \cdots H_I(\tau_{n-1})\, H_I(\tau_n = 0)|\beta\rangle \tag{7.12}$$

τ_n has been fixed at zero as a consequence of the invariance of (7.10) under translation in τ. The integral over $d\tau_i$ is done along the negative real axis. The interaction Hamiltonian actually depends on $e^{-i\tau}$; so the integrals in (7.12) are not well defined. To make them convergent we can switch adiabatically the interaction multipying $H_I(\tau)$ by a factor which kills the interaction when $|\tau| \to \infty$:

$$H_I(\tau) \rightarrow H_I(\tau)\, e^{-\varepsilon|\tau|} \tag{7.13}$$

After the insertion of (7.13) into (7.12) and the evaluation of the integrals the (n+2) - point dual amplitude is recovered in the limit $\varepsilon \to 0$. It is well known that all the relevant quantities of DRM can be constructed out of the two fundamental operators $Q_\mu(z)$ and P_μ defined in Ref. 1). Thus, in order to have a dictionary to translate the DRM operators in the string language and vice versa we need only to express $Q_\mu(z)$ and $P_\mu(z)$ in terms of string variables. From direct inspection of Eq. (7.12) we can conclude at once that the correspondence principle between DRM and the string picture is

$$\begin{aligned} z &\longleftrightarrow e^{-i\tau} \\ \sqrt{2}\, Q_\mu(z) &\longleftrightarrow x_\mu(\tau,0) \\ -i\sqrt{2}\, P_\mu(z) &\longleftrightarrow \dot{x}_\mu(\tau,0) \end{aligned} \tag{7.14}$$

where z is the integration variable of the Koba-Nielsen circle, and $x_\mu(\tau,0)$ are the co-ordinates of the ends of the free string [indeed in Eq. (7.12) only the free components of the string co-ordinates are present.] Thus, in the string picture the Lie algebra generated by $Q_\mu(z)$ and $P_\mu(z)$, which is the starting point of DRM, results to be a direct consequence of the canonical commutators of the string co-ordinates defined in Eqs. (6.2), (6.3) which are in turn required solely by the local causality.

Having derived the n point dual amplitude for the emission of n "photons" and having established a correspondence principle between the string variables and those of the DRM, we want now to make some considerations on a possible and suggestive interpretation of the n point dual amplitude for any kind of excited state.

It is well known that the n point dual amplitude for the scattering of the states $\alpha_1, \ldots, \alpha_n$ is given by:

$$A_n = \sum_{\text{perm.}} B_n \tag{7.15}$$

where

$$B_n \sim \int_0^{2\pi} \prod_{i=1}^{n} d\tau_i \, \theta(\tau_i - \tau_{i+1}) < 0| \prod_{i=1}^{n} V_{\alpha_i}(\tau_i) |0> \tag{7.16}$$

$V_{\alpha_i}(\tau_i)$ is the vertex associated with the state $|\alpha_i>$. From the correspondence principle (7.14) the Koba-Nielsen variable τ is related to the time. The interesting feature of (7.16) is that the integration range is only between 0 and 2π and not infinite as in the case of (7.12). In this case the n point amplitude is written in a form where the external particles are treated in a completely symmetric way.

Expression (7.16) may suggest to interpret the n point dual scattering amplitude as an amplitude describing scattering processes where the intermediate resonant states live for less than the characteristic time $\tau = 2\pi$. This point will be discussed somemore in the conclusions.

8. INTERACTION WITH EXCITED STATES

i) General discussion

The success obtained so far in solving exactly the equations of motion of the string interacting with the strong photon field makes it natural to ask whether it is possible to treat in a similar way the interaction with the higher excited states of the DRM.

Of course the "photon" case is very peculiar because the simplicity of the treatment at the classical level joins up with an easy transition to quantum mechanics, and is related to this fact that the photon vertices are also the basic ingredients for the construction of the physical states of DRM.

For the other excited states the situation is quite different. The dual method suggests that for a state at the level N we take an interaction Lagrangian of the form (8.1) localized at the ends and depending on $x(\tau)$ and on its τ derivatives up to the Nth order. However, such an interaction raises immediately several problems. Already at the classical level the equations of motion (or more precisely the boundary conditions) are essentially non-linear so that an exact solution seems very difficult to obtain. Furthermore, the quantization procedure also gives serious troubles since the canonical commutation relations look to be inconsistent. On the other hand, the dual model suggests also that the interaction Lagrangian should be written in terms of the free fields. This circumstance was actually realized in the photon case, but is not consistently realized in the general case. We may then imagine two possibilities. Either we should start from a different interaction Lagrangian, which, however, will become equivalent to (8.1) when we take the equations of motion into account; or the dual amplitudes correspond to lowest order perturbation theory and the exact solution for the interacting string cannot be obtained in a closed form.

Here, we leave this question open to future investigations and we try to answer to some preliminary problems which are related to the transition between the DRM description in terms of vertex operators and the string description in terms of interaction Lagrangians. In fact, it is not a trivial task to see how the dual vertices, which are written in the normal ordered form in a quite non-transparent way, can also be written as functions of x and of its τ derivatives in a way which guarantees from the beginning the right transformation properties under the Virasoro gauges. This form has an obvious classical limit and can be assumed as the interaction Lagrangian for the string. However, since the string description corresponds to the DRM only for $D = 26$, we may expect that the dual vertices corresponding to states whose norm depends on D and is critical at $D = 26$, like the longitudinal Brower [3] states, will not have such a classical limit.

Actually, we shall proceed in the opposite direction. We shall start from a classical Lagrangian and we shall find the general conditions under which the action is invariant under change of parametrization [subsection ii)]. Then we consider the transition to quantum mechanics, assuming for $x(\tau, \sigma)$ the free operator expansion, and we analyze the possible sources of singularities coming from the reordering of the operators, requiring that the interaction Lagrangian becomes a regular operator [subsection iii)]. This has to be regarded as a heuristic procedure, rather than a rigorous one; nevertheless we believe it contains some interesting results. For example, it gives a natural explanation to the origin of the light-like vectors appearing in the physical states of DRM and which in the approach of Ref. 4) correspond to the "photon" momenta. Finally, in subsection iv) we work out as an example the interaction Lagrangians for the states at the mass level $N = 2$ and we will find that they correspond to the vertices of the transverse states of DRM which are coupled at $D = 26$.

ii) Interaction Lagrangian for the classical string

In analogy to the photon case we consider the interaction with excited state fields localized at the ends of the string and described by the interaction Lagrangian

$$\mathcal{L}_i = L\left(x, x^{(r)}\right) \rho(\sigma) \tag{8.1}$$

where we used the notations

$$x^{(r)}(\tau,\sigma) = \left(\frac{d}{d\tau}\right)^r x(\tau,\sigma) \tag{8.2}$$

$$\rho(\sigma) = g_0\, \delta(\sigma) + g_\pi\, \delta(\sigma-\pi) \tag{8.3}$$

and L is a function of $x(\tau,\sigma)$ and of its τ derivatives up to a given order.

We are interested in finding the conditions under which the action is invariant under a general parameter transformation which leaves the lines $\sigma = 0$ and $\sigma = \pi$ invariant. The total variation of the action is given by

$$\delta S = \int_{\tau_i}^{\tau_f} d\tau \int_0^\pi d\sigma\; \delta\mathcal{L} + \int_0^\pi d\sigma \left[\mathcal{L}\, \delta\tau\right]_{\tau_i}^{\tau_f} \tag{8.4}$$

where $\delta\mathcal{L}$ is the local variation of the Lagrangian function. Now the free Lagrangian $\mathcal{L}_0$ already contributes an invariant piece, while for $\mathcal{L}_i$ we immediately obtain the conditions

$$\delta L\left(x, x^{(r)}\right) = -\frac{d}{d\tau}\left[L\left(x, x^{(r)}\right)\delta\tau\right] \quad , \sigma = 0, \pi \tag{8.5}$$

which generalize Eq. (4.43) for the photon case. Taking now for each end point $(\sigma = 0, \pi)$ [22)]

$$\delta\tau = \varepsilon f(\tau) \tag{8.6}$$

we have

$$\delta x = - \varepsilon f(\tau) \dot{x} \tag{8.7}$$

$$\delta x^{(r)} = - \varepsilon \left(\frac{d}{d\tau}\right)^r \left(f(\tau) \dot{x}\right) \tag{8.7'}$$

from which

$$\delta L = \sum_{r=0}^{\infty} \frac{\partial L}{\partial x_\mu^{(r)}} \delta x_\mu^{(r)} = - \varepsilon \sum_{r=0}^{\infty} \frac{\partial L}{\partial x_\mu^{(r)}} \sum_{k=0}^{r} \binom{r}{k} f^{(k)}(\tau) x_\mu^{(r+1-k)} =$$

$$= - \varepsilon \sum_{k=0}^{\infty} f^{(k)}(\tau) \sum_{r=k}^{\infty} \binom{r}{k} \frac{\partial L}{\partial x_\mu^{(r)}} x_\mu^{(r+1-k)} \tag{8.8}$$

Equation (8.5) then becomes

$$\sum_{k=0}^{\infty} f^{(k)}(\tau) \sum_{r=k}^{\infty} \binom{r}{k} \frac{\partial L}{\partial x_\mu^{(r)}} x_\mu^{(r+1-k)} = f(\tau) \frac{dL}{d\tau} + \dot{f}(\tau) L \tag{8.9}$$

Since $f(\tau)$ is arbitrary, the coefficients of each $f^{(k)}(\tau)$ must be equal on the two sides of (8.9). The coefficients of $f(\tau)$ are identically equal, while for the others we obtain, at the points $\sigma = 0, \pi$, the conditions

$$\sum_{r=1}^{\infty} r \frac{\partial L}{\partial x_\mu^{(r)}} x_\mu^{(r)} = L \tag{8.10}$$

$$\sum_{r=k}^{\infty} \binom{r}{k} \frac{\partial L}{\partial x_\mu^{(r)}} x_\mu^{(r+1-k)} = 0 \quad , (k = 2, 3, \ldots) \tag{8.11}$$

Equation (8.10) says that L must be homogeneous of first degree in $d/d\tau$. In particular, if L is homogeneous in each $x^{(r)}$ of degree n_r we must have

$$\sum_{r=1}^{\infty} r\, n_r = 1 \tag{8.12}$$

Furthermore, if L depends at most on the Nth derivative of x, by taking $k=N$, $N-1$, etc., equations (8.11) give:

$$\frac{\partial L}{\partial x_\mu^{(N)}}\,\dot{x}_\mu = 0 \qquad (N \geq 2) \tag{8.13}$$

$$\frac{\partial L}{\partial x_\mu^{(N-1)}}\,\dot{x}_\mu + N\,\frac{\partial L}{\partial x_\mu^{(N)}}\,\ddot{x}_\mu = 0 \quad , \quad (N \geq 3) \text{ etc.} \tag{8.13'}$$

iii) Interaction Lagrangian in the operator form

We now consider the transition to quantum mechanics and we assume that the interaction Lagrangian (8.1) becomes a function of the free position operator (5.18) and of its derivatives, as discussed before.

If L_f is the generator of the transformation (8.7), such that in agreement with (4.38) we have

$$i\left[L_f, x^\mu(\tau,\sigma_0)\right] = f(\tau,\sigma_0)\,\dot{x}^\mu(\tau,\sigma_0) \quad , \quad (\sigma_0 = 0,\pi) \tag{8.14}$$

where $f(\tau, \sigma_0) = f(\tau+\sigma_0)$, Eq. (8.5) becomes

$$i\left[L_f, L(x, x^{(r)})\right] = \frac{d}{d\tau}\left[f(\tau)\, L(x, x^{(r)})\right] \tag{8.15}$$

where for simplicity we neglect the dependence on σ_0. Equation (8.15) coincides with the well-known transformation property of the vertex operators of DRM so we expect that the interaction operators L may correspond to the dual vertices. On the other hand (8.15) is automatically satisfied for any $f(\tau)$, provided that L obeys the conditions (8.10) and (8.11). Therefore, the present approach is also interesting from the point of view of the conventional DRM.

To describe the interaction with an excited state carrying momentum p, with $p^2 = 1-N$, we shall take a Lagrangian of the form

$$L(x, x^{(r)}) = G(x^{(r)})\, e^{ipx} \tag{8.16}$$

However e^{ipx} is singular for $p^2 \neq 0$ and $G(x^{(r)})$ may also be singular as it will contain products of operators taken at the same point (τ, σ_0). Therefore, the expression (8.16) needs to be defined by a suitable limiting procedure.

Specifically, we shall take the positive frequency (or destruction) part and the negative frequency (or creation) part of any $x^{(r)}$ operator at values of τ slightly displaced from the real axis, and precisely $\tau \to \tau - i(\eta/2)$, with $\eta > 0$ for the positive frequency parts and $\tau \to \tau + i(\eta/2)$ for the negative frequency parts. For example, we take

$$x(\tau,\sigma_0) = x^0(\tau,\sigma_0) + x^+\left(\tau - i\frac{\eta}{2},\sigma_0\right) + x^-\left(\tau + i\frac{\eta}{2},\sigma_0\right)$$

In this way any expression for (8.16) which is not intrinsically singular (see discussion below), is regular for $\eta \neq 0$ and can be easily brought to the normal ordered form. Then we can see whether this expression is singular or not in the limit $\eta \to 0$.

This procedure is not enough to find definite expressions for L, nevertheless it is a very useful method to analyze the possible singularities. As an example we calculate the commutator of $x(\tau,\sigma_0)$ and $\dot{x}(\tau',\sigma_0)$ and we find, calling $e^{-\eta} = \varepsilon$:

$$\left[x^\mu(\tau,\sigma_0), \dot{x}^\nu(\tau',\sigma_0)\right] = 2ig^{\mu\nu}\sum_{n=-\infty}^{\infty}\varepsilon^{|n|}e^{in(\tau'-\tau)} =$$

$$= 2ig^{\mu\nu}\frac{1-\varepsilon^2}{(1-\varepsilon)^2+4\varepsilon\sin^2\frac{\tau'-\tau}{2}} \xrightarrow[\varepsilon\to 1]{} 4\pi i g^{\mu\nu}\delta\left(2\sin\frac{\tau'-\tau}{2}\right) = \qquad (8.17)$$

$$= 4\pi i g^{\mu\nu}\delta(\tau'-\tau) \quad , \quad |\tau'-\tau| < 2\pi$$

In general, we find

$$\left[x_\mu^{(r)+}\left(\tau - i\frac{\eta}{2},\sigma_0\right), x_\nu^{(s)-}\left(\tau + i\frac{\eta}{2},\sigma_0\right)\right] = 2g_{\mu\nu}i^{s-r}C_{r+s}\left(e^{-\eta+i(\tau'-\tau)}\right) \qquad (8.18)$$

$$(r+s \geq 1)$$

where

$$C_r(\rho) = \left(\rho\frac{d}{d\rho}\right)^{r-1}\frac{\rho}{1-\rho} \qquad (8.18')$$

All c-number singularities coming from normal ordering can be expressed in terms of C_r functions. In the limit $\eta \to 0$ they have a leading singularity $C_r(e^{-\eta}) \sim (r-1)!\,\eta^{-r}$. We now may ask what kind of functions can be taken for G in (8.16). We shall consider the case where G is a sum of monomial terms satisfying the homogeneity condition (8.12), where n_r are not necessarily positive integers. For example, terms like $\sqrt{\dot{x}^2}$, $\sqrt{p\cdot\ddot{x}}$, $\dot{x}\cdot\ddot{x}/\dot{x}^2$ are acceptable from the classical point of view, however, they become meaningless as operators.

To examine this problem in a specific example we consider the function $(k\cdot\dot{x})^\alpha$. By a unitary transformation e^{ipq_0} this can be changed into $(1+k\cdot\dot{x})^\alpha$ which can be expanded in power series according to

$$\left(1 + k\dot{x}\right)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} \left(k\cdot\dot{x}\right)^n \tag{8.19}$$

and each term can now be normal ordered. This can be easily done using the relation

$$e^{\lambda(k\cdot\dot{x})} = e^{\lambda k(\sqrt{2}p_0 + \dot{x}^-)}\, e^{\lambda(k\cdot\dot{x}^+)}\, e^{\lambda^2 k^2 C_2} \tag{8.20}$$

and expanding both sides in power of λ. For the even and odd powers of $k\cdot\dot{x}$ we get

$$\begin{aligned}(k\cdot\dot{x})^{2n} = (2n-1)!!\,(2k^2C_2)^n + n\,(2n-1)!!\,(2k^2C_2)^{n-1} :(k\cdot\dot{x})^2: + \dots + \\ + :(k\cdot\dot{x})^{2n}:\end{aligned} \tag{8.21}$$

$$(k\cdot\dot{x})^{2n+1} = (2n+1)!!\,(2k^2C_2)^n\, k\cdot\dot{x} + \dots + :(k\cdot\dot{x})^{2n+1}: \tag{8.21'}$$

where C_2 stands for $C_2(e^{-\eta})$. Inserting these expressions into (8.19) we see that all the series for the c-number part and for the coefficients of the regular operators $k.\dot{x}$, $:(k.\dot{x})^2:$, etc., are always divergent for any $\eta > 0$, unless $k^2 = 0$.

The above result can be generalized and it is easy to realize that any non-integer function of $k\cdot x^{(r)}$ can be a regular operator only if $k^2 = 0$. On the contrary any expression of the form $(x^{(r)}.x^{(s)})^\alpha$, for α not positive integer, is never a well definite operator.

Coming back to $G(x^{(r)})$, it is easy to see that in order to satisfy all the conditions (8.11) for each term of G n_r must be positive integers for $r > 1$. If this were not the case, consider for a given $r > 1$ the term with the lowest value of n_r, which from the above discussion will be of the form

$$\left(k \cdot x^{(r)}\right)^{n_r} F\left(x^{(s)}\right)$$

This will contribute to (8.11) a term

$$\left(k \cdot x^{(r)}\right)^{n_r - 1} \left(k \cdot x^{(r+1-h)}\right) F\left(x^{(s)}\right) \quad , \; r \geq h \geq 2$$

which cannot be identically zero and cannot be cancelled by other terms. Thus n_2, n_3, etc., must be positive integers. No restriction may arise from (8.11) on n_1, while from (8.12) n_1 in general will be negative.

Next, we consider the singular operator e^{ipx}. From our limiting procedure we easily get

$$e^{ipx} = (1-\varepsilon)^{p^2} V(p) \tag{8.22}$$

where $\varepsilon = e^{-\eta}$ and

$$V(p) = :e^{ipx}: = e^{ipx^-} e^{ipx^0} e^{ipx^+} \tag{8.23}$$

Since $V(p)$ is regular, we may rewrite (8.22) in the form

$$V(p) = \lim_{\varepsilon \to 1} (1-\varepsilon)^{-p^2} e^{ipx} \tag{8.22'}$$

and we may think of expressing the factor $(1-\varepsilon)^{-p^2}$ in terms of powers of $\dot{x}$, $\ddot{x}$, etc., altogether of degree p^2 in $d/d\tau$.

To solve the ambiguity we require for both sides of (8.22') the same commutation relations with the gauge operators. Since we have

$$i\left[L_f, V(p)\right] = \left[f(\tau)\frac{d}{d\tau} + p^2 \dot{f}(\tau)\right] V(p) \tag{8.24}$$

$$i\left[L_f, e^{ipx}\right] = f(\tau)\frac{d}{d\tau} e^{ipx} \tag{8.25}$$

$$i\left[L_f\,,\,x^{(r)}\right] = \left(\frac{d}{d\tau}\right)^r \left(f(\tau)\,\dot{x}\right) \tag{8.26}$$

(8.24) suggests that $V(p)$ is a function of x and $\dot{x}$, and precisely of degree p^2 in $\dot{x}$. Then we consider

$$\begin{aligned}\tilde{V}(p) &= (k\cdot\dot{x})^{p^2} e^{ipx} = (1-\varepsilon)^{p^2} (k\cdot\dot{x})^{p^2} : e^{ipx} : = \\ &= e^{ipx^-} e^{ipx^0} \left[(1-\varepsilon)(k\cdot\dot{x}) + 2p\cdot k\right]^{p^2} e^{ipx^+}\end{aligned} \tag{8.27}$$

For the case of tachyon ($p^2 = 1$) the right-hand side of (8.27) has a well definite limit when $\varepsilon \to 1$ and is proportional to $V(p)$ provided only that $2p\cdot k \neq 0$, independent of k^2. For the higher excited states ($p^2 < 0$) the right-hand side of (8.27) is a regular operator only if $k^2 = 0$. In this case the limit $\varepsilon \to 1$ is again proportional to $V(p)$ for $2p\cdot k \neq 0$. Therefore, we may take in general

$$V(p) = \left(\frac{k\cdot\dot{x}}{2p\cdot k}\right)^{p^2} e^{ipx} \quad ; \quad k^2 = 0 \quad , \quad 2pk \neq 0 \tag{8.28}$$

The right-hand side is defined as the limit $\varepsilon \to 1$ of the normal ordered operator, and such a limit is actually independent of the vector k. We observe that for $k^2 = 0$, $(k.\dot{x})^{p^2}$ has always a regular inverse, so that (8.28) leads precisely to (8.22) for the highest singularity of e^{ipx}. If we require this to hold even for the tachyon, then the vertex operator for the tachyon is also given by (8.28).

The interaction Lagrangian (8.16) can be rewritten in the form

$$L(x, x^{(r)}) = F(x^{(r)})\; V(p) \tag{8.29}$$

where F and V are now regular operators. From (8.10), (8.11) and (8.28), F must obey the following conditions for the parameter invariance of the action:

$$\sum_{r=1}^{\infty} r \frac{\partial F}{\partial x_\mu^{(r)}} x_\mu^{(r)} = N F \tag{8.30}$$

$$\sum_{r=k}^{\infty} \binom{r}{k} \frac{\partial F}{\partial x_\mu^{(r)}} x_\mu^{(r+1-k)} = 0 \quad , \quad (k \geq 2) \tag{8.31}$$

These relations correspond to the transformation property of F under the gauge operators

$$i\left[L_f , F(x^{(r)})\right] = f(\tau) \dot{F}(x^{(r)}) + N \dot{f}(\tau) F(x^{(r)}) \tag{8.32}$$

which follows directly from (8.15) and (8.24).

If $F(x^{(r)})$ is a sum of monomial terms of degree n_r in $x^{(r)}$, then (8.30) requires for each term

$$\sum_{r=1}^{\infty} r\, n_r = N \tag{8.33}$$

We may assume that all n_r are non-negative integers, so that the solutions of (8.33) correspond to the partitions of N and we have a correspondence between F functions and excited states of DRM.

Finally, we have to require that the product FV in (8.29) is a regular operator. Since F by itself is supposed to be regular, singularities may only arise when the negative frequency part of V passes across F. We have

$$F\left(x^{(r)}\right) V(p) = e^{ipx^-} e^{ipx^0} \tilde{F}\left(x^{(r)}\right) e^{ipx^+} \tag{8.34}$$

where

$$\tilde{F}\left(x^{(r)}\right) = e^{-ip\left(x^- + x^0\right)} F\left(x^{(r)}\right) e^{ip\left(x^- + x^0\right)} =$$

$$= F\left[x^{(r)} + 2p\, i^{1-r}\left(C_r(\varepsilon) + \delta_{r,1}\right)\right] \tag{8.34'}$$

where for each single $x^{(r)}$ we made use of (8.18). Thus F must be such that $\tilde{F}$ is regular in the limit $\varepsilon \to 1$. If we had started from the product VF we would have obtained:

$$V(p)\ F(x^{(r)}) = e^{ipx^-} e^{ipx^0}\ \tilde{\tilde{F}}(x^{(r)})\, e^{ipx^+} \tag{8.35}$$

where

$$\tilde{\tilde{F}}\left(x^{(r)}\right) = e^{ipx^+}\ F(x^{(r)})\, e^{-ipx^+} = F\left[x^{(r)} + 2p\, i^{r+1} C_r(\varepsilon)\right] \tag{8.35'}$$

Since from the definition (8.18') we have

$$C_r\left(\frac{1}{\varsigma}\right) = (-1)^r \left[C_r(\varsigma) + \delta_{r,1}\right] \tag{8.36}$$

in the limit $\varepsilon \to 1$ $\tilde{F} = \tilde{\tilde{F}}$ and therefore F and V commute.

In conclusion, we started from an interaction Lagrangian of the form (8.29), with $V(p)$ given by (8.28), which is formally a function of x and of its τ derivatives. Such a Lagrangian leads to an action which is invariant under reparametrization provided that F obeys the conditions (8.30) and (8.31). After normal ordering the Lagrangian operator takes the form

$$L(x, x^{(r)}) = e^{ipx^-} e^{ipx^0}\ \tilde{F}\left(x^{(r)}\right) e^{ipx^+} \tag{8.37}$$

where

$$\tilde{F}\left(x^{(r)}\right) = \lim_{\varepsilon \to 1} F\left[x^{(r)} + 2p\, i^{r+1} C_r(\varepsilon)\right] \tag{8.38}$$

This limit must exist, and this insures that the operator on the right-hand side of (8.37) will satisfy the transformation property (8.15). We then expect that these operators correspond to dual vertices for excited states and this will be shown by a simple example in the following.

iv) Application to the states with $N = 2$

As an application of the preceding results, here we want to discuss the interaction with the excited states of the level $N = 2$. In this case the function F of (8.29) depends at most on $\ddot{x}$ and (8.31) gives the only condition

$$\frac{\partial F(\dot{x}, \ddot{x})}{\partial \ddot{x}^\mu}\, \dot{x}^\mu = 0 \tag{8.39}$$

Remembering (8.33), we will have two kinds of F functions, according to the partitions of $N = 2$: F_1 quadratic in $\dot{x}$ and F_2 linear in $\ddot{x}$:

$$F_1 = a_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu \qquad ; \; a_\mu{}^\mu = 0 \tag{8.40}$$

$$F_2 = \varepsilon \cdot \ddot{x} - \varepsilon \cdot \dot{x} \frac{k \cdot \ddot{x}}{k \cdot \dot{x}} + R(\dot{x}) \quad , \quad k^2 = \varepsilon \cdot k = 0 \tag{8.41}$$

In (8.40) $a_{\mu\nu}$ is an arbitrary symmetric tensor, with the condition of being traceless in order that F_1 be non-singular. In (8.41) ε and k are two arbitrary vectors, subject to the conditions $k^2 = \varepsilon \cdot k = 0$ for the second term to be regular. We also leave the freedom of an additional term R, quadratic in $\dot{x}$, to be chosen in such a way that $\widetilde{F}_2$ be regular.

From (8.38) we have

$$\widetilde{F}_1 = \lim_{\varepsilon \to 1} a_{\mu\nu} \left(\dot{x}^\mu - 2p^\mu C_1(\varepsilon)\right)\left(\dot{x}^\nu - 2p^\nu C_1(\varepsilon)\right) \tag{8.42}$$

The limit exists if $a_{\mu\nu}$ obeys the further condition

$$a_{\mu\nu}\, p^\nu = 0 \tag{8.43}$$

and in this case $\widetilde{F}_1 = F_1$. We can then write the interaction Lagrangian in the two equivalent forms

$$\begin{aligned} L_1(x, \dot{x}) &= F_1\, V(p) = a_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \, \frac{2p \cdot k}{k \cdot \dot{x}}\, e^{ipx} = \\ &= e^{ipx^-} e^{ipx^0} a_{\mu\nu}\, \dot{x}^\mu \dot{x}^\nu e^{ipx^+} \end{aligned} \tag{8.44}$$

We observe that if we require that the G function in (8.16) be a regular operator, we also get the condition

$$a_{\mu\nu}\, k^\nu = 0 \tag{8.45}$$

which corresponds to the vanishing of the gauge operators [3] K_n on the physical states of DRM for critical dimensions.

Concerning F_2 we get from (8.38)

$$\tilde{F}_2 - \tilde{R} = \varepsilon\cdot\ddot{x} - 2i\,\varepsilon\cdot p\,C_2(p) - \left(\varepsilon\cdot\dot{x} - 2p\cdot\varepsilon\,C_1(p)\right)\cdot\left(k\cdot\ddot{x} - 2i\,k\cdot p\,C_2(p)\right)\cdot$$

$$\cdot\left(k\cdot\dot{x} - 2\,k\cdot p\,C_1(p)\right)^{-1} = \varepsilon\cdot\ddot{x} - \frac{1}{1-\rho}\Big[2\rho\,\varepsilon\cdot p\left(i\,k\cdot\dot{x} - (1-\rho)\,k\cdot\ddot{x}\right) +$$

$$+ \varepsilon\cdot\dot{x}\left((1-\rho)^2\,k\cdot\ddot{x} - 2i\rho\,k\cdot p\right)\Big]\Big[(1-\rho)(k\cdot\dot{x}) - 2\rho\,k\cdot p\Big]^{-1} \tag{8.46}$$

We see that the double pole at $\rho = 1$ cancels, but remains a single pole with the residue

$$i\left(\varepsilon\cdot p\,\frac{k\cdot\dot{x}}{k\cdot p} - \varepsilon\cdot\dot{x}\right) \tag{8.47}$$

This pole can be cancelled by taking in (8.41)

$$R(\dot{x}) = i\,\frac{k\cdot\dot{x}}{2\,k\cdot p}\left(\varepsilon\cdot p\,\frac{k\cdot\dot{x}}{k\cdot p} - \varepsilon\cdot\dot{x}\right) \tag{8.48}$$

This gives in fact

$$\tilde{R} = i\left(\frac{k\cdot\dot{x}}{2k\cdot p} - \frac{\rho}{1-\rho}\right)\left(\frac{\varepsilon\cdot p}{k\cdot p}\,k\cdot\dot{x} - \varepsilon\cdot\dot{x}\right) \tag{8.49}$$

which summed to (8.46) gives in the limit $\rho \to 1$

$$\tilde{F}_2 = \varepsilon\cdot\ddot{x} - \frac{i}{k\cdot p}(\varepsilon\cdot\dot{x})(k\cdot\dot{x}) - i\,\varepsilon\cdot\dot{x} - \frac{\varepsilon\cdot p}{k\cdot p}\Big[k\cdot\ddot{x} - \frac{i}{k\cdot p}(k\cdot\dot{x})^2 - i\,k\cdot\dot{x}\Big] \tag{8.50}$$

Finally, for the interaction Lagrangian we have the two equivalent forms

$$L_2(x,\dot{x},\ddot{x}) = \left(\varepsilon\cdot\ddot{x} - \varepsilon\cdot\dot{x}\,\frac{k\cdot\ddot{x}}{k\cdot\dot{x}} + i\,\varepsilon\cdot p\,\frac{(k\cdot\dot{x})^2}{2(k\cdot p)^2} - \frac{i(\varepsilon\cdot\dot{x})(k\cdot\dot{x})}{2\,k\cdot p}\right)\cdot \tag{8.51}$$

$$\cdot\,\frac{2k\cdot p}{k\cdot\dot{x}}\,e^{ipx} = e^{ipx^-}\,e^{ipx^0}\,\tilde{F}_2\,e^{ipx^+}$$

where we have chosen the same k for F_2 and V to have a regular G.

It is easy to see that the normal ordered forms for L_1 and L_2 correspond to the vertex operators [4), 6)] for the transverse states of DRM. To find out directly the excited states corresponding to these vertices we may use the relation [5)]

$$\langle \phi_{1,2} | = \langle 0 | \frac{1}{2\pi} \int_0^{2\pi} e^{i\tau} L_{1,2}\, d\tau \tag{8.52}$$

and we find, in the old standard notations

$$\langle \phi_1 | = 2 \langle -p, 0 | a_{\mu\nu} a_1^{\mu} a_1^{\nu} \tag{8.53}$$

$$\langle \phi_2 | = 2 \langle -p, 0 | \left[\varepsilon\cdot a_2 - \frac{i}{k\cdot p}(\varepsilon\cdot a_1)(k\cdot a_1) - \frac{\varepsilon\cdot p}{k\cdot p}\left(k\cdot a_2 - \frac{i}{k\cdot p}(k\, a_1)^2\right)\right] \tag{8.54}$$

It is easy to verify that these states satisfy the gauge conditions.

$$\langle \phi_{1,2} | L_{-n} = \langle \phi_{1,2} | K_{-n} = 0 \tag{8.55}$$

they are orthogonal and their norm is

$$\langle \phi_1 | \phi_1 \rangle = 8 \, \mathrm{Tr} \, | a_{\mu\nu} |^2 \tag{8.56}$$

$$\langle \phi_2 | \phi_2 \rangle = 4 \varepsilon^2 \tag{8.57}$$

Finally, we may count the independent states $\langle \phi_1 |$ by counting the independent components of $a_{\mu\nu}$, with the conditions (8.40), (8.43), (8.45) and they are

$$\frac{1}{2} D(D+1) - D - (D-1) - 1 = \frac{1}{2}(D-2)(D-1) - 1$$

that is the same as the transverse states $\langle 0 | A_{1i} A_{1j}$. In fact for these states the tensor $a_{\mu\nu}$ is realized as follows: for $i \neq j$ we take

$$a_{\mu\nu} = u_\mu^{(i)} u_\nu^{(j)} \quad , \text{with} \quad u_\mu^{(i)} = \varepsilon_\mu^{(i)} + 2\left(\varepsilon^{(i)}\cdot p\right) k_\mu$$

where $\varepsilon^{(i)}$ is the unit vector along the i axis and $2k\cdot p = -1$. On the other hand for $i = j$ we may replace $<0|A_{1i}^2$ by some traceless combination since $<0|\sum_i A_{1i}^2$ is a null state. Similarly for the states $<\phi_2|$ ε can be taken transverse, due to the invariance $\varepsilon \to \varepsilon + \lambda k$. Then we can see that $<\phi_2|$ corresponds to the transverse state $<0|\underset{\sim}{A}_2\cdot\underset{\sim}{\varepsilon}$.

In conclusion our states (8.52) span the whole physical space of the level $N = 2$ for $D = 26$. As it was to be expected the Brower longitudinal states and the spin zero state are not found in our approach, and this means that the vertex operators for these states cannot be written as functions of x, $\dot{x}$ and $\ddot{x}$.

9. CLOSED STRING

Our previous considerations are based on open strings interacting at their ends with an external field; in this section we shall study the interaction of a closed string with an external field.

The connection with dual models is given by the Shapiro-Virasoro model (SVM) whose spectrum is reproduced by the quantization of a closed string, as well as by the Pomeron sector of the conventional dual model which is factorized, at the critical dimension, in terms of the same states of the Shapiro-Virasoro model.

In a closed string we cannot attach a charge anywhere without violating the invariance under reparametrizations; therefore, any interaction with an external field should involve the whole world sheet of the string. This corresponds to the fact that in the SVM the Koba-Nielsen variables are integrated in the whole complex plane. Moreover, in the SVM the fundamental zero mass particle is a "strong graviton" (spin 2) instead of the "strong photon" of the conventional DRM; this suggests that the interaction should be put in a geometrical way.

The approach we follow consists in taking the same Lagrangian as in the free case except that the metric tensor $g_{\mu\nu}$, which appears in all scalar products, is now taken to be a function of the co-ordinates. It is then the curvature of the four-dimensional space-time which provides the interaction of a string with an external gravitational field (as in the theory of general relativity). The action is then:

$$\mathcal{S} = \int_{\tau_i}^{\tau_f} d\tau \int_0^{\pi} d\sigma \; \mathcal{L}(x, \dot{x}, x') \tag{9.1}$$

where

$$\mathcal{L}(x,\dot{x},x') = -\frac{1}{4\pi\alpha'\hbar c^2}\left\{\left[\dot{x}^\mu x'^\nu g_{\mu\nu}(x) - \left(\dot{x}^\mu \dot{x}^\nu g_{\mu\nu}(x)\right)\left(x'^\mu x'^\nu g_{\mu\nu}(x)\right)\right]\right\}^{1/2} \tag{9.2}$$

The Lagrangian is still homogeneous of first degree both in $\dot{x}$ and x' and the action is therefore invariant under reparametrizations. We have taken σ varying in the interval $(0,2\pi)$ for the symmetry between σ and τ in the case of a closed string.

The equation of motion of a string follows from the variation of the action (9.1):

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} + \frac{d}{d\sigma}\frac{\partial \mathcal{L}}{\partial x'^\mu} - \frac{\partial \mathcal{L}}{\partial x^\mu} = 0 \tag{9.3}$$

The boundary conditions are replaced by the requirement that the string be close for any τ :

$$x^\mu(\tau,\sigma) = x^\mu(\tau, \sigma + 2\pi) \tag{9.4}$$

The Lagrangian (9.2) can be linearized as in the case of the open string choosing the "orthonormal" parametrization:

$$\partial_\pm x^\mu \partial_\pm x^\nu g_{\mu\nu}(x) = 0 \tag{9.5}$$

where we have introduced the "light-cone" parameters:

$$\xi_\pm = \tau \pm \sigma \tag{9.6}$$

and

$$\partial_\pm = \frac{\partial}{\partial \xi_\pm} \tag{9.7}$$

With such a choice of the parameters the Lagrangian (9.2) becomes:

$$\mathcal{L}\left[\partial_+ x, \partial_- x, x\right] = \partial_+ x^\mu \partial_- x^\nu g_{\mu\nu}(x) \tag{9.8}$$

and the equation of motion (9.3) reduces to:

$$2g_{\mu\nu}(x)\partial_+\partial_- x^\nu + \partial_- x^\nu \partial_+ x^\rho \frac{\partial g_{\mu\nu}}{\partial x^\rho} + \partial_+ x^\rho \partial_- x^\nu \frac{\partial g_{\mu\rho}}{\partial x^\nu} - \partial_+ x^\rho \partial_- x^\nu \frac{\partial g_{\rho\nu}}{\partial x^\mu} = 0 \tag{9.9}$$

Introducing the contravariant metric tensor $g^{\mu\nu}(x)$ and the Christoffel symbol

$$\Gamma^\mu_{\nu\rho}(x) = \frac{1}{2} g^{\mu\sigma}(x)\left[\frac{\partial g_{\sigma\nu}}{\partial x^\rho} + \frac{\partial g_{\sigma\rho}}{\partial x^\nu} - \frac{\partial g_{\rho\nu}}{\partial x^\sigma}\right] \tag{9.10}$$

the equation of motion can be written in a more compact way:

$$\partial_+ \partial_- x^\mu + \partial_+ x^\nu \partial_- x^\rho \, \Gamma^\mu_{\nu\rho}(x) = 0 \tag{9.11}$$

Equation (9.11) describes the surface of minimal area in the four-dimensional space with metric tensor $g_{\mu\nu}(x)$, and it strongly resembles the equation of the geodetics. It has to be supplemented by the two conditions (9.5), which specify the choice of the parameters, and by the periodicity equation (9.4) which, in terms of the variables $\xi_\pm$, reads:

$$x^\mu(\xi_+, \xi_-) = x^\mu(\xi_+ + 2\pi, \xi_- - 2\pi) \tag{9.12}$$

We shall now proceed as follows: first we shall treat the free case, namely when the space-time is flat and all the Christoffel symbols vanish; after we shall solve the equation of motion for a string in a space-time curved by a monochromatic external gravitational field.

i) Free case

The metric tensor is

$$g_{\mu\nu}(x) = \eta_{\mu\nu} \tag{9.13}$$

where

$$\eta_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu \quad \text{and} \quad \eta_{00} = -\eta_{11} = -\eta_{22} = -\eta_{33} = 1 \tag{9.14}$$

The Christoffel symbols vanish and Eq. (9.11) becomes

$$\partial_+ \partial_- x^\mu = 0 \tag{9.15}$$

The most general solution of (9.15) is given by:

$$x^\mu(\xi_+, \xi_-) = x_+^\mu(\xi_+) + x_-^\mu(\xi_-) \tag{9.16}$$

where $x_\pm(\xi_\pm)$ are arbitrary functions.

Taking the derivatives of Eq. (9.12) with respect to $\xi_\pm$ and using for $x^\mu(\xi_+, \xi_-)$ the expression (9.16), we get:

$$\partial_\pm x_\pm^\mu(\xi_\pm) = \partial_\pm x_\pm^\mu(\xi_\pm + 2\pi) \tag{9.17}$$

This relation implies that $x_\pm^\mu(\xi_\pm)$ are periodic functions except at most for a linear term, namely:

$$x_\pm^\mu(\xi_\pm) = P^\mu \xi_\pm + \chi_\pm^\mu(\xi_\pm) \tag{9.18}$$

where

$$\chi_\pm^\mu(\xi_\pm) = \chi_\pm^\mu(\xi_\pm + 2\pi) \tag{9.19}$$

The coefficient P^μ of the linear term in (9.18) is the same in x_+ and x_- in order to fulfill Eq. (9.12). As in the case of the open string P^μ has the meaning of total momentum of the string.

Our solution has still to satisfy the supplementary conditions (9.5) which, in a flat space-time and with x^μ given by Eq. (9.16), become:

$$\partial_\pm x_\pm^\mu(\xi_\pm)\, \partial_\pm x_\pm^\nu(\xi_\pm)\, \eta_{\mu\nu} = 0 \tag{9.20}$$

The connection with the SVM is now clear [22]; when the theory is quantized the coefficients of the Fourier expansion of $\chi_{\pm\mu}(\xi_\pm)$ give rise to the two sets of harmonic oscillators, the zero mode P^μ has the same eigenvalue in the two sets and the Fourier analysis of Eqs. (9.20) gives two sets of gauge operators.

ii) Interaction

Let us suppose now that the space-time is curved by the presence of an external gravitational field. As for the "electromagnetic" field in the case of the open string, we assume that the external field is monochromatic massless and with definite helicity. We then take a metric tensor of the following form:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \varepsilon_\mu \varepsilon_\nu e^{i k_\rho x^\rho} \tag{9.21}$$

$$g^{\mu\nu}(x) = \eta^{\mu\nu} - \varepsilon^\mu \varepsilon^\nu e^{i k_\rho x^\rho} \tag{9.22}$$

with the conditions:

$$\varepsilon_\mu \varepsilon^\mu = k_\mu \varepsilon^\mu = k_\mu k^\mu = 0 \tag{9.23}$$

which entail that the space-time is "flat" along the directions of k^μ and ε^μ, namely,

$$\varepsilon_\mu g^{\mu\nu} = \varepsilon_\mu \eta^{\mu\nu} \tag{9.24}$$

$$k_\mu g^{\mu\nu} = k_\mu \eta^{\mu\nu} \tag{9.25}$$

No ambiguity can therefore arise in raising or lowering indices of k and ε.

We can now evaluate the Christoffel symbol with the metric tensor given by Eqs. (9.21), (9.22) and (9.23) and write down the equations of motion:

$$\partial_+ \partial_- x^\mu + e^{ik\cdot x} \gamma^\mu_{\nu\rho} \partial_+ x^\nu \partial_- x^\rho = 0 \tag{9.26}$$

where

$$\gamma^{\mu}_{\nu\rho} = \frac{i}{2}\left[\varepsilon^{\mu}\varepsilon_{\nu}k_{\rho} + \varepsilon^{\mu}\varepsilon_{\rho}k_{\nu} - \varepsilon_{\nu}\varepsilon_{\rho}k^{\mu}\right] \tag{9.27}$$

The set of equations (9.26) must be supplemented with the periodicity conditions (9.12) and with the subsidiary relations (9.5), which now take the form:

$$\eta_{\mu\nu}\partial_{\pm}x^{\mu}\partial_{\pm}x^{\nu} + (\varepsilon_{\mu}\partial_{\pm}x^{\mu})(\varepsilon_{\nu}\partial_{\pm}x^{\nu})e^{ikx} = 0 \tag{9.28}$$

In order to solve Eq. (9.26) let us first notice that the equations of motion in the direction k_{μ} and ε_{μ} are the same as in the free case, namely:

$$\varepsilon_{\mu}\partial_{+}\partial_{-}x^{\mu} = 0 = k_{\mu}\partial_{+}\partial_{-}x^{\mu} \tag{9.29}$$

and their solution is given by

$$k_{\mu}x^{\mu} = k_{\mu}x^{\mu}_{0} \tag{9.30}$$

$$\varepsilon_{\mu}x^{\mu} = \varepsilon_{\mu}x^{\mu}_{0} \tag{9.31}$$

where x_0^{μ} is the solution given in Eqs. (9.15)-(9.19) for the free case.

Since only the components of x along ε and k are involved in the interaction term of Eq. (9.26), we may treat that term as a known term after having replaced x^{μ} by x_0^{μ} according to Eqs. (9.30) and (9.31). The equations of motion become then

$$\partial_{+}\partial_{-}x^{\mu} + e^{ikx_0}\gamma^{\mu}_{\nu\rho}\,\partial_{+}x^{\nu}_{0}\,\partial_{-}x^{\rho}_{0} = 0 \tag{9.32}$$

Integrating this equation with respect to ξ_{+} and ξ_{-} we get the general integral:

$$x^{\mu}(\xi_{+},\xi_{-}) = \varphi^{\mu}_{+}(\xi_{+}) + \varphi^{\mu}_{-}(\xi_{-}) + \gamma^{\mu}_{\nu\rho}F^{\nu}_{+}(\xi_{+},a_{+})F^{\rho}_{-}(\xi_{-},a_{-}) \tag{9.33}$$

where $\varphi^{\mu}_{\pm}(\xi_{\pm})$ are arbitrary functions and $F^{\nu}_{\pm}(\xi_{\pm},a_{\pm})$ is given by:

$$F^{\nu}_{\pm}(\xi_{\pm},a_{\pm}) = \int_{a_{\pm}}^{\xi_{\pm}} d\xi\;\dot{x}^{\nu}_{\pm 0}(\xi)\,e^{ikx_{\pm 0}(\xi)} \tag{9.34}$$

with

$$\bar{x}^{\nu}_{\pm o}(\xi) = \frac{d}{d\xi} \quad x^{\nu}_{\pm o}(\xi) \tag{9.35}$$

In order to impose the periodicity condition (9.12) the two following relations are useful:

$$F^{\nu}_{\pm}(\xi,a) = \frac{F^{\nu}_{\pm}(\xi \pm 2\pi, \xi) - F^{\nu}_{\pm}(a \pm 2\pi, a)}{e^{\pm 2\pi i k\cdot P} - 1} \tag{9.36}$$

$$F^{\nu}_{\pm}(\xi \pm 4\pi, \xi \pm 2\pi) = e^{\pm 2i\pi k\cdot P} F^{\nu}_{\pm}(\xi \pm 2\pi, \xi) \tag{9.37}$$

Equation (9.37) is a straightforward consequence of Eq. (9.36), which in turn is obtained integrating both sides of

$$\frac{d}{d\xi}\int_{\xi}^{\xi \pm 2\pi} \bar{x}^{\mu}_{\pm o}(\xi)\, e^{ik\bar{x}_{\pm o}(\xi)} d\xi = \left(e^{\pm 2i\pi k\cdot P} - 1\right)\bar{x}^{\mu}_{\pm o}(\xi)\, e^{ikx_{\pm o}(\xi)} \tag{9.38}$$

In deriving Eq. (9.38) use has been made of the periodicity properties of $\varepsilon \cdot x_{(\pm)o}$ and $k\cdot x_{(\pm)o}$; the factor $e^{\pm 2i\pi kP}$ comes just from the non-periodic linear term $k\cdot P\,\xi$ contained in $k\cdot x_{(\pm)o}(\xi)$.

Putting Eq. (9.36) into (9.33) we can rewrite the general integral of Eq. (9.32) in the following form

$$x^{\mu}(\xi_+,\xi_-) = \left[\Psi_+(\xi_+) + \Psi_-(\xi_-)\right] + \frac{1}{4\sin^2\pi k\cdot P}\,\gamma^{\mu}_{\nu\rho}\, F^{\nu}_{+}(\xi_+ + 2\pi, \xi_+) F^{\rho}_{-}(\xi_- - 2\pi, \xi_-) \tag{9.39}$$

where $\Psi_{\pm}(\xi_{\pm})$ are two functions, different from $\varphi_{\pm}(\xi_{\pm})$, but still arbitrary.

Using Eq. (9.37) it is quite easy to check that the second term in the right-hand side of (9.39) fulfills by itself the periodicity condition (9.12); we have then to impose that condition separately to $[\Psi_+ + \Psi_-]$. Therefore, $\Psi_+ + \Psi_-$ represents the free part of x^{μ}, as it satisfies both the free equation of motion and the periodicity condition, whereas the other term is due to the interaction.

Therefore, we may write:

$$x^{\mu}(\xi_+,\xi_-) = x_0^{\mu}(\xi_+,\xi_-) + x_{int.}^{\mu}(\xi_+,\xi_-) \tag{9.40}$$

where

$$x_{int.}^{\mu}(\xi_+,\xi_-) = \frac{1}{4\sin^2\pi k\cdot P}\,\gamma^{\mu}_{\nu\rho}\,F_+^{\nu}(\xi_+ + 2\pi,\xi_+)\,F_-^{\rho}(\xi_- - 2\pi,\xi_-) \tag{9.41}$$

This solution has many features in common with Eq. (5.15) which describes an open string interacting with an external "photon-like" field; we shall therefore limit ourselves to some essential remarks.

The string resonates whenever $P\cdot k$ is an integer, whereas in the case of an open string there is a resonance at any integer value of $2P\cdot k$. This is related to the fact that we have used the same slope as in the case of the open string; we know instead that in this case the slope of the Regge trajectory is a half of the α' (slope of the open string) which enters in the Lagrangian. If we take the actual slope of the Regge trajectory to be 1 then one gets $2(\alpha'/2)Pk = 2Pk =$ integer., as in the open string.

When we approach a pole $[P\cdot k = \text{integer}]$ our solution (9.41) diverges like $[\sin\pi kP]^{-2}$; its residue contains the cyclic integrals, which are characteristic of the construction of the transverse states.

Since x^{μ}_{int} is a function of x^{μ}_{0} the interacting string has still the same degrees of freedom as the free string, and it can be quantized in terms of the same set of harmonic oscillators. However, if we want that the spectrum of the physical states be left unchanged, it is necessary that the constraint equations, written in terms of x^{μ}_{0}, are still the same as in the free case. That is actually the case; in fact if one takes the left-hand side of Eq. (9.28) and calculates it with x^{μ} given by Eqs. (9.40) and (9.41) one finds the identity:

$$\eta_{\mu\nu}\partial_+ x^{\mu}\partial_- x^{\nu} + (\varepsilon_{\mu}\partial_+ x^{\mu})(\varepsilon_{\nu}\partial_- x^{\nu})e^{ikx} = \eta_{\mu\nu}\partial_+ x_0^{\mu}\partial_- x_0^{\nu} = 0 \tag{9.42}$$

when the right-hand side has just the form of the gauge operators in the free theory.

We shall not go through the details of the calculation, which are straightforward; rather we point out that the requirement that the gauge operators are left invariant in form by the interaction is very stringent and that it could hardly be fulfilled with a different approach. For instance, if one starts from the Lagrangian

$$\mathcal{L} = -\frac{1}{4\pi}\left[(\dot{x}\cdot x') - \dot{x}^2 x'^2\right]^{1/2} + \varepsilon\cdot\dot{x}\,\varepsilon\cdot x'\, e^{ik\cdot x} \tag{9.43}$$

namely a free Lagrangian plus an ad hoc interaction which resembles the "graviton" vertex, one would get in the "orthonormal" parametrization the same equations of motion (9.26), the same solution (9.40), (9.41), but different gauges; namely

$$\eta_{\mu\nu}\,\partial_+ x^\mu \partial_- x^\nu = 0 \tag{9.44}$$

and that would not lead, with x^μ given by Eqs. (9.40), (9.41) to

$$\eta_{\mu\nu}\,\partial_+ x_0^\mu\ \partial_- x_0^\nu = 0 \tag{9.45}$$

We will develop the Hamiltonian formalism for the closed string in Appendix D; it goes in the same way as in the case of an open string.

We want to discuss now some points about possible extensions of our approach.

i) With the choice of the metric tensor (9.21), (9.22), (9.23), the interaction term in the linearized Lagrangian just reproduces the vertex of the strong graviton in the SVM, that is:

$$\mathcal{L}_{int} = \varepsilon_\mu \partial_+ x^\mu\ \varepsilon_\nu \partial_- x^\nu\ e^{ikx} = \varepsilon\cdot\bar{X}_+(\xi_+)e^{ikx_+(\xi_+)}\cdot\varepsilon\cdot\bar{X}_-(\xi_-)e^{ikx_-(\xi_-)} \tag{9.46}$$

where the fact that the solutions along the directions k and ε are still the free ones has been used.

One could try by using the same techniques of Section 8 to reproduce the excited vertices; we shall only discuss here the most simple extension which concerns the other states of the SVM with zero mass [24]. They are:

$$\left(A_R^{+(1)} B_L^{+(1)} \pm A_L^{+(1)} B_R^{+(1)}\right) |0\rangle \tag{9.47}$$

where A^+ and B^+ are the transverse operators built up with the two sets of operators of the SVM, the subscripts R and L mean "right" and "left" handed polarization. The antisymmetric state is decoupled in the SVM because of the complete symmetry of the model; but it is not in the Pomeron factorization.

The interaction Lagrangians which correspond to the states (9.47) are:

$$\left(\varepsilon_\mu \bar{\varepsilon}_\nu \pm \bar{\varepsilon}_\mu \varepsilon_\nu\right) \partial_+ x_+^\mu \; \partial_- x_-^\nu \, e^{ikx} \tag{9.48}$$

where

$$\bar{\varepsilon}^2 = \varepsilon^2 = 0 \qquad \text{and} \quad \varepsilon \cdot \bar{\varepsilon} = 1 \tag{9.49}$$

The Lagrangian with the + sign would result from a metric tensor whose determinant varies from point to point, namely from a metric in which the modulus of a four-vector is not conserved for parallel transport; the other one with the - sign would result from a metric tensor with an antisymmetric part.

Unfortunately, the nice feature that $\varepsilon_\mu x^\mu$ is the same as in the free case does not hold in this case and the non-linear character of the equations of motion now requires for the solution an iterative process involving an infinite number of steps. This fact puts the "strong graviton" on a different ground not only with respect to the massive states of the SVM, but also with respect to the massless states present in that model.

ii) The second problem is whether the gravitational interaction may apply to an open string as well as to a closed one. In principle there is no difficulty in solving this problem and it would be interesting to see if, in this way, one can have some information on the couplings between an open string and the strong graviton which is a particular state of a closed string.

CONCLUSION AND FINAL REMARKS

We have developed a consistent relativistic theory of a string whose free ends interact with an external massless spin one field. We have shown that the equations of motion of the string are exactly solvable at least for special choices of the external field, and that this theory, when properly quantized, reproduces the interaction of the "strong photon" with the physical states of DRM. We have also seen the problems which arise when we try to extend this theory to more general massive external fields. Finally, we have studied the system of a closed string interacting with an external gravitational field; this system, once again consistently quantized, gives us exactly the coupling of the strong graviton to the physical states of the Shapiro-Virasoro model.

This is enough to show, we think, that the relativistic string theory is more than an analog model for the spectrum of DRM; indeed, it can be used to obtain informations on the couplings of DRM. We can then use the simple, intuitive picture given by the string to get more insight on the nature of dual phenomena. We wish to conclude with the following few remarks that can be regarded as naive, illustrative examples of the implications of the string view in dual models.

i) Our form of interaction is not fully satisfactory because we treat the interacting strings in an asymmetric way; indeed we simulate the effect of one of the three interacting strings by means of an external field. However, such a form enables us to guess that the interaction of open strings will take place at the ends, so that a two-particle resonating channel is represented in the space-time by two strings which glue together at one end in order to form a unique final string. As a consequence, if we draw in the σ, τ plane the trajectories of the ends of four strings which undergo the process $12 \rightarrow 34$ with resonances in the s (12) and t (23) channels, we obtain a planar quark diagram. More generally, it is not difficult to convince oneself that our way of attaching the strings leads to the graphical rules of planar duality, in the sense that the set of trajectories of the ends of n interacting strings form a planar diagram once we have ordered the external strings in such a way that contiguous strings give rise to resonating channels. Conversely, for every dual diagram we can reproduce the topological structure of the surface described in the space-time by the interacting strings. For instance, the non-planar, orientable single loop, which is currently associated with the Pomeron contribution to the four-particle amplitude, represents, in the string picture, a couple of strings which glue together at the two ends so as to form a closed string;

then this closed string breaks into two pieces which are the two final open strings. Thus we are led to conclude that in the string picture the Pomeron, like the Shapiro-Virasoro states, is associated with a closed string in full agreement with the general belief [25] that the Shapiro-Virasoro model provides a scattering theory for the Pomeron sector of the conventional DRM.

ii) We do not have yet a complete theory of interacting strings, thus, strictly speaking, we cannot say anything on the typical dynamical quantities like, for instance, the widths of the resonances. Nevertheless, we may conjecture that the lifetime of a string state is of the order of its characteristic time $\Delta t = 2\pi\alpha' E$, i.e., the time in which a disturbance goes through the string from one end to the other. Suppose indeed that a string absorbs a photon at one end, then it jumps into more excited state; this state cannot decay into the initial one until this absorbed excitation reaches the other end, in fact we have seen previously that the string can only radiate at the ends. It has to be pointed out that this value for the lifetime of the resonances is in good agreement with that found in DRM by Green and Veneziano [26] who predict that the average lifetime is greater than $\pi\alpha' E$, provided that the dual resonances are narrow.

If we push further our imagination and forget the rigour, we can relate this conjecture with our interpretation of n point functions given in Section 7. Indeed, we have suggested that these functions are amplitudes for processes which occur during the characteristic time of the interacting strings *); a peculiar feature of these processes is that any intermediate resonance propagates for a time less than its own characteristic time; hence, according to our conjecture, it should behave like a stable state. The fact that the widths of the resonances contributing to the n point functions are actually zero seems to support our conjecture.

iii) Clearly, the notion of characteristic time depends crucially on the fundamental length $\sqrt{\alpha'}$ which is present in the string theory. If we let $\alpha' \to 0$, the characteristic time, as well as the length of the string, goes to zero, thus one can argue that the n point function becomes, in the string picture, the amplitude for n particles interacting at the same point. This seems to support a recent conjecture of Veneziano [27], namely

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*) Actually, the characteristic time for a system of interacting strings is a function of their momenta $p_1, p_2, \ldots p_n$ and of the times $t_1, t_2, \ldots, t_n$ at which they interact.

that the n point functions of DRM may correspond, when the limit $\alpha' \to 0$ is properly carried out, to the interaction term ϕ^n of a non-polynomial Lagrangian for a field ϕ [28].

After the completion of this work the paper "String interacting picture of dual resonance model" by S. Mandelstam, which deals with the problem of the interaction among strings using the path integral formalism, was brought to our attention.

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APPENDIX A

In this Appendix we want to clarify the correspondence existing between physical quantities defined in the four-dimensional space and the corresponding two-dimensional quantities in the internal $\xi^0 \xi^1$ space.

Following an argument of Nambu [19], we begin to show that starting from the four-dimensional density

$$L_{int} = j_\mu(y) A^\mu(y) \tag{A.1}$$

we may obtain Eq. (3.2) as the correct interaction density in the parameter space.

Let $J^i(\xi^i)$ be the current in the internal space and assume that it is conserved. The four-dimensional current

$$j^\mu(y) = \iint d^2\xi\, \delta^{(4)}(x(\xi) - y)\, J^i x^\mu_{,i} = \iint d^2\xi\, \delta^{(4)}(x-y)(J^i x^\mu)_{,i} \tag{A.2}$$

is conserved if we choose approximately the density $J^i(\xi^i)$. In fact

$$j^\mu_{,\mu} = -\iint d^2\xi\, \partial_\mu \delta^{(4)}(x(\xi)-y) J^i x^\mu_{,i} = -\iint d^2\xi \left(J^i \delta^{(4)}(x-y)\right)_{,i} \tag{A.3}$$

The last integral is zero if

$$J^1(\tau, 0) = J^1(\tau, \pi) = 0 \quad \text{and} \quad x^0(\xi^0 = \pm\infty, \xi^1) = \pm\infty \tag{A.4}$$

so that

$$J^0(\xi^0 = \pm\infty, \xi^1) = 0 \tag{A.5}$$

There are many solutions of Eqs. (A.4), (A.5). We want, however, the invariance under reparametrization as has been explained in Section 3, so we choose:

$$J^1(\xi^i) = 0 \quad , \; J^0(\xi^i) = g_0\,\delta(\sigma) + g_\pi\,\delta(\sigma - \pi) = \rho(\xi^1) \tag{A.6}$$

With this choice the internal Lagrangian density corresponding to (A.1) is obtained:

$$\mathcal{L}_{int} = J^i(\xi_i)\, x^\mu_{,i}(\xi^i)\, A_\mu(x(\xi)) = \rho(\sigma)\frac{dx^\mu}{d\tau} A_\mu \tag{A.7}$$

so that the form (3.2) is recovered.

Next we proceed to calculate the stress energy tensor of the string in the four-dimensional space. Let us introduce general co-ordinates in the integral (3.1); so we can write:

$$S = -\frac{1}{2\pi}\iint d\sigma\, d\tau \left[\left(g_{\mu\nu} \dot{x}^{\mu} x'^{\nu} \right)^2 - \left(g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} \right)\left(g_{\mu\nu} x'^{\mu} x'^{\nu} \right) \right]^{1/2} \tag{A.8}$$

where $g_{\mu\nu}$ is the metric tensor of the Minkowski space in curvilinear co-ordinates. We define then $T^{\mu\nu}(y)$ through the standard formula:

$$\delta S = \frac{1}{2}\int d^4y\, \sqrt{-g}\; T^{\mu\nu}(y)\, \delta g_{\mu\nu} \tag{A.9}$$

where g is the determinant of $g_{\mu\nu}$ and δS is the variation of the action when the metric tensor is varied of $\delta g_{\mu\nu}$. Varying $g_{\mu\nu}$ in (A.8) and introducing an appropriate delta function we obtain

$$T^{\mu\nu}(y) = -\frac{1}{2\pi}\int d^2\xi\, \delta^{(4)}\left(y - x(\xi)\right) \frac{(\dot{x}\cdot x')\left[x'^{\mu} x'^{\nu} + \dot{x}^{\mu}\dot{x}^{\nu}\right] - \dot{x}^2 x'^{\mu} x'^{\nu} - x'^2 \dot{x}^{\mu}\dot{x}^{\nu}}{\left[(\dot{x}\cdot x')^2 - \dot{x}^2 x'^2\right]^{1/2}} \tag{A.10}$$

where again the Minkowski metric has been used. We may write (A.10) more coinciscly in the following way:

$$T^{\mu\nu}(y) = -\frac{1}{2\pi}\int d^2\xi\, \delta^{(4)}\left(y - x(\xi)\right) \gamma^{ij}(\xi)\, x^{\mu}_{,i}\, x^{\nu}_{,j} \tag{A.11}$$

where γ^{ij} $(i,j=0,1)$ are the contravariant components of the two-dimensional metric tensor on the surface:

$$\gamma_{ij} = x^{\mu}_{,i}\, x^{\nu}_{,j}\, \eta_{\mu\nu} \tag{A.12}$$

In the same way as we did for the current we may define $\gamma^{ij}(\xi)$ as the stress energy momentum tensor in the parameter space which correspond to the stress tensor in the four-dimensional space, through definition (A.11). This definition is corroborated by the following observation: let us work directly on the two-dimensional surface and write the action integral (3.1) in the form

$$\mathcal{S} = \int d^2\xi\, \sqrt{-\gamma} \tag{A.13}$$

where γ is the determinant of the γ_{ij}. We may define a two-dimensional stress tensor on the surface through the definition

$$\delta S = \int d^2\xi \, \sqrt{-\gamma} \, \tau^{ij} \, \delta\gamma_{ij} \tag{A.14}$$

Straightforward calculations give

$$\tau^{ij} = \gamma^{ij} \tag{A.15}$$

i.e., the same quantities which enter in the expression (A.11).

APPENDIX B

In this Appendix we derive the solution (5.12) starting from the expression (5.11) and imposing the boundary conditions (5.2) and (5.3).

This gives rise to the following equations for g and h:

$$\frac{1}{2\pi} A\, T^{\mu\nu}\left[\dot{h}_\nu(\tau+\pi) - \dot{h}_\nu(\tau-\pi)\right] = -i g_0\, T^{\mu\nu}\, \dot{x}_\nu(\tau,0)\, e^{ikx(\tau,0)} \tag{B.1}$$

$$\frac{1}{2\pi} B\, T^{\mu\nu}\left[\dot{g}_\nu(\tau+\pi) - \dot{g}_\nu(\tau-\pi)\right] = -i g_\pi\, T^{\mu\nu}\, \dot{x}_\nu(\tau,\pi)\, e^{ikx(\tau,\pi)} \tag{B.2}$$

where

$$T^{\mu\nu} = \varepsilon^\mu k^\nu - \varepsilon^\nu k^\mu \tag{B.3}$$

Then we get

$$\frac{1}{2\pi} A = g_0 \qquad\qquad \frac{1}{2\pi} B = g_\pi \tag{B.4}$$

and the following equations for the discontinuities:

$$T^{\mu\nu}\left[\dot{h}_\nu(\tau+\pi) - \dot{h}_\nu(\tau-\pi)\right] = -i\, T^{\mu\nu}\, \dot{x}_\nu(\tau,0)\, e^{ikx(\tau,0)} \tag{B.5}$$

$$T^{\mu\nu}\left[\dot{g}(\tau+\pi) - \dot{g}(\tau-\pi)\right] = -i\, T^{\mu\nu}\, \dot{x}_\nu(\tau,\pi)\, e^{ikx(\tau,\pi)} \tag{B.6}$$

An explicit solution of the equations (B.5) and (B.6) is given by:

$$T^{\mu\nu}\, \dot{h}_\nu(\tau) = -T^{\mu\nu}\, \dot{x}_\nu(\tau,\pi)\, \frac{e^{ikx(\tau,\pi)}}{2\sin(2P\cdot k)\pi} \tag{B.7}$$

$$T^{\mu\nu}\, \dot{g}_\nu(\tau) = -T^{\mu\nu}\, \dot{x}_\nu(\tau,0)\, \frac{e^{ikx(\tau,0)}}{2\sin(2P\cdot k)\pi} \tag{B.8}$$

where P is the four-momentum of the string.

In order to show that the previous functions satisfy the equations (B.5) and (B.6) we must prove the following relations:

$$k\cdot x(\tau\pm\pi,\pi) = k\cdot x(\tau,0) \pm 2P\cdot k\,\pi \tag{B.9}$$

$$T^{\mu\nu}\dot{x}_\nu(\tau\pm\pi,\pi) = T^{\mu\nu}\dot{x}_\nu(\tau,0) \tag{B.10}$$

The first relation follows from the fact that the component of x^μ along k^μ is behaving as in the free case; so we can use the expression for $(k\cdot x)$ given in terms of the oscillators (5.18). On the other hand, the external field does not carry momentum along k^μ and therefore we can identify $P\cdot k$ with the momentum of the string along the direction k^μ. The relation (B.10) is then a consequence of the fact that along the direction ε^μ we can carry the same analysis as along k^μ.

Using the relations (B.9) and (B.10) it is easy to prove that (B.7) satisfies the equation (B.5):

$$T^{\mu\nu}\left[\dot{\lambda}_\nu(\tau+\pi)-\dot{\lambda}_\nu(\tau-\pi)\right] = -\frac{T^{\mu\nu}\dot{x}_\nu(\tau,0)}{2\sin(2P\cdot k)\pi}\left[e^{ikx(\tau+\pi,\pi)}-e^{ikx(\tau-\pi,\pi)}\right]$$
$$= -T^{\mu\nu}\dot{x}_\nu(\tau,0)\,e^{ikx(\tau,0)} \tag{B.11}$$

The proof that (B.8) satisfies Eq. (B.6) can be then carried out in the same way. The insertion of (B.7) and (B.8) in (5.11) gives the solution of the equation of motion:

$$\dot{x}_\mu(\tau,\sigma) = \dot{F}_\mu(\tau+\sigma)+\dot{F}_\mu(\tau-\sigma)\quad(\varepsilon_\mu k_\nu-\varepsilon_\nu k_\mu)\left[R_+\dot{x}^\nu(\tau-\sigma;0)e^{ikx(\tau-\sigma;0)}+R_-\dot{x}^\nu(\tau+\sigma;0)\cdot e^{ikx(\tau+\sigma;0)}\right] \tag{B.12}$$

where

$$R_\pm = \frac{-2\pi\left(g_\pi+g_0\,e^{\pm2i(P\cdot k)\pi}\right)}{2\sin(2Pk)\pi} \tag{B.13}$$

APPENDIX C

We want to show here what happens when one studies the interaction of the string with two external "photon-like" fields with polarization and momentum respectively ε, k and ε', k'. We write the Lagrangian in the form

$$\mathcal{L} = \mathcal{L}_0 + g\left[\varepsilon\cdot\dot{x}\, e^{ikx} + \varepsilon'\cdot\dot{x}\, e^{ik'x}\right]\delta(\sigma) \qquad (C.1)$$

where, for the sake of simplicity we just consider the case of one charge at $\sigma = 0$.

The equations of motion of the string are the same as in the case of one monochromatic field, whereas the boundary conditions at $\sigma = 0$ become

$$\frac{x'_\mu}{2\pi}(\tau,0) = +ig\,\dot{x}^\nu(\tau,0)\left[f_{\mu\nu}\, e^{ikx(\tau,0)} + f'_{\mu\nu}\, e^{ik'x(\tau,0)}\right] \qquad (C.2)$$

with

$$-f_{\mu\nu} = \varepsilon_\mu k_\nu - \varepsilon_\nu k_\mu \quad , -f'_{\mu\nu} = \varepsilon'_\mu k'_\nu - \varepsilon'_\nu k'_\mu$$

For $\sigma = \pi$ they are of course $x'_\mu = 0$.

In this case we do not have any component of x^μ which satisfies the boundary conditions of the free case, so it is not possible to apply the procedure of Section 5 to find the solution. However, we can look for a perturbative solution in power series of $\tilde{g} = 2\pi g$

$$x^\mu = x^\mu_{(0)} + \tilde{g}\, x^\mu_{(1)} + \tilde{g}^2 x^\mu_{(2)} + \cdots \qquad (C.3)$$

For the first few orders in the coupling constant $\tilde{g}$ the boundary conditions (C.2) read

$$x'_{(0);\mu} = 0 \qquad (C.4)$$

$$x'_{(1),\mu} = -i\,\dot{x}^\nu_{(0)}\left[f_{\mu\nu}\, e^{ikx_{(0)}} + f'_{\mu\nu}\, e^{ik'x_{(0)}}\right] \qquad (C.5)$$

$$\begin{aligned} x'_{(2)\mu} = & -i\,\dot{x}^\nu_{(1)}\left[f_{\mu\nu}\, e^{ikx_{(0)}} + f'_{\mu\nu}\, e^{ik'x_{(0)}}\right] + \\ & + \dot{x}^\nu_{(0)}\left[f_{\mu\nu}(k\cdot x_{(1)})\, e^{ikx_{(0)}} + f'_{\mu\nu}(k'\cdot x_{(1)})\, e^{ik'x_{(0)}}\right] \end{aligned} \qquad (C.6)$$

$x_{(o)}$ is the free solution, $x_{(1)}$ is an obvious generalization of Eq. (5.14):

$$x^{\mu}_{(1)} = f^{\mu\nu}\left[A_-(k)\int^{\tau+\sigma} + A_+(k)\int^{\tau-\sigma}\right] d\xi\, \dot{x}_{(0)\nu}(\xi,0)\, e^{ikx_{(0)}(\xi,0)} + \\ + f'^{\mu\nu}\left[A_-(k')\int^{\tau+\sigma} + A_+(k')\int^{\tau-\sigma}\right] d\xi\, \dot{x}_{(0)\nu}(\xi,0)\, e^{ik'x_{(0)}(\xi,0)} \tag{C.7}$$

with

$$A_{\pm}(k) = \frac{e^{\pm 2iP\cdot k\pi}}{2\sin(2Pk)\pi} \tag{C.8}$$

It is easy to see that the second order term of the solution is

$$x_{(2)\mu}(\tau,\sigma) = \frac{\cos 2Pk'\pi}{\sin 2Pk'\pi}\, f_{\mu\nu}\Big\{ f'^{\nu\rho}\left[A_-(k+k')\int^{\tau+\sigma} + A_+(k+k')\cdot\int^{\tau-\sigma}\right]\cdot \\ \cdot\, \dot{x}_{\rho(0)}(\xi,0)\, e^{i(k+k')x_{(0)}(\xi,0)} d\xi + i\, k_\rho f'^{\rho\lambda}\left[A_-(k+k')\int^{\tau+\sigma} + \right. \\ \left. + A_-(k+k')\int^{\tau-\sigma}\right] d\xi\, \dot{x}^{\nu}_{(0)}(\xi,0)\, e^{ikx(\xi,0)} \int^{\xi} d\eta\, \dot{x}_{(0)\rho}(\eta,0)\, e^{ik'x_0(\eta,0)} \\ + (\varepsilon,k) \longleftrightarrow (\varepsilon',k') \Big\} \tag{C.9}$$

As it is clear from this expression, a peculiar feature of $x_{(2)}$ is that it shows single poles *) when $2P\cdot k$, $2P\cdot k'$ or $2P\cdot(k-k')$ are integer numbers. These poles correspond to the resonance structure of the dual amplitude, apart from the lack of poles in the crossed channels: the last ones seem to arise in fact from a purely quantistic cause and are essentially due to the "commutation rules" of the dual vertices [29].

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*) We remark that in (C.9) we preferred to use indefinite integrals, yet we might have used definite integrals extended to symmetrical intervals of 2π. However, in this case it would be more cumbersome to see that the poles are simple.

APPENDIX D

In this Appendix we sketch the Hamiltonian formalism in the case of a closed string embedded in a curved space-time; since the procedure is quite similar to that followed for the open string, we shall just summarize the results.

We start from the Lagrangian (9.2)

$$\mathcal{L} = -\frac{1}{4\pi}\left\{\left[\dot{x}^{\mu} x'^{\nu} g_{\mu\nu}(x)\right]^2 - \left[\dot{x}^{\mu}\dot{x}^{\nu} g_{\mu\nu}(x)\right]\left[x'^{\mu}x'^{\nu} g_{\mu\nu}(x)\right]\right\}^{1/2} \qquad (D.1)$$

and we choose as independent dynamical variables x^{μ}, and Π_{μ} defined by

$$\Pi_{\mu} = \frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}} = \frac{1}{(4\pi)^2 \mathcal{L}}\, g_{\mu\nu}\left[\left(\dot{x}^{\mu} x'^{\rho} g_{\mu\rho}\right) x'^{\nu} - \left(x'^{\mu} x'^{\rho} g_{\mu\rho}\right)\dot{x}^{\nu}\right] \qquad (D.2)$$

As a consequence of the reparametrization invariance of the action, the Hamiltonian density

$$\mathcal{H} = \dot{x}^{\mu}\Pi_{\mu} - \mathcal{L}$$

is identically vanishing. As in the case of the open string, we have constraints between the dynamical variables x^{μ} and Π_{μ}. They are

$$\begin{aligned} \mathcal{L}_1 &\equiv \frac{x'^{\mu}}{4\pi}\Pi_{\mu} = 0 \\ \mathcal{L}_2 &\equiv \frac{x'^{\mu}}{4\pi}\frac{x'^{\nu}}{4\pi} g_{\mu\nu}(x) + \Pi_{\mu}\Pi_{\nu}\, g^{\mu\nu}(x) = 0 \end{aligned} \qquad (D.3)$$

So, the Hamiltonian will be a combination of the constraints :

$$H = 4\pi \int_0^{2\pi} d\sigma \left[f_1(\sigma,\tau)\frac{x'^{\mu}}{4\pi}\Pi_{\mu} + f_2(\sigma,\tau)\left(\frac{x'^{\mu}}{4\pi}\frac{x'^{\nu}}{4\pi} g_{\mu\nu}(x) + \Pi_{\mu}\Pi_{\nu}\, g^{\mu\nu}(x)\right)\right] \qquad (D.4)$$

where f_1 and f_2 are two arbitrary functions. We want now to check the consistency of our formalism. Therefore, we must show that

$$\left\{ \mathcal{L}_{1,2}\, , H \right\} \approx 0 \qquad (D.5)$$

once that the equations of motion and the constraints themselves are taken into account. If we assume canonical Poisson brackets at equal τ

$$\left\{ x^{\mu}(\sigma,\tau), \Pi_{\nu}(\sigma',\tau) \right\} = \delta^{\mu}_{\ \nu}\ \delta(\sigma-\sigma')$$
$$\left\{ x^{\mu}(\sigma,\tau), x^{\nu}(\sigma',\tau) \right\} = \left\{ \Pi_{\mu}(\sigma,\tau), \Pi_{\nu}(\sigma',\tau) \right\} = 0 \qquad \text{(D.6)}$$

We may easily evaluate the Poisson brackets between $\mathcal{L}_1$ and $\mathcal{L}_2$:

$$\left\{ \mathcal{L}_1(\sigma,\tau), \mathcal{L}_1(\sigma',\tau) \right\} = \frac{1}{4} \left\{ \mathcal{L}_2(\sigma,\tau), \mathcal{L}_2(\sigma',\tau) \right\} =$$
$$= \frac{1}{4\pi} \left(\Pi_{\mu}(\sigma) \frac{x'^{\mu}}{4\pi}(\sigma') + \Pi_{\mu}(\sigma') \frac{x'^{\mu}}{4\pi}(\sigma) \right) \frac{\partial}{\partial\sigma} \delta(\sigma-\sigma') \qquad \text{(D.7)}$$

$$\left\{ \mathcal{L}_1(\sigma,\tau), \mathcal{L}_2(\sigma',\tau) \right\} = \frac{1}{2\pi} \Big[\Pi_{\mu}(\sigma) \Pi_{\nu}(\sigma') g^{\mu\nu}[x(\sigma')] +$$
$$\frac{x'^{\mu}}{4\pi}(\sigma) \frac{x'^{\nu}}{4\pi}(\sigma') g_{\mu\nu}[x(\sigma')] \Big] \frac{\partial}{\partial\sigma} \delta(\sigma-\sigma') - \frac{1}{4\pi} \Big[\Pi_{\mu}(\sigma) \Pi_{\nu}(\sigma) \frac{\partial}{\partial\sigma} g^{\mu\nu} + \frac{x'^{\mu}}{4\pi}(\sigma) \frac{x'^{\nu}}{4\pi}(\sigma) \frac{\partial}{\partial\sigma} g_{\mu\nu} \Big] \delta(\sigma-\sigma') \qquad \text{(D.8)}$$

the right-hand side of Eqs. (D.7) and (D.8) are again the primary constraints (D.3) or their σ derivatives; this result assures that the constraints $\mathcal{L}_1$ and $\mathcal{L}_2$ are first-class, then the formalism is consistent for any choice of the functions f_1 and f_2 (that is of the parametrization). We want to find now the equation of motion for x^{μ} and Π_{μ} ; it follows from the equations (D.4) and (D.6)

$$\frac{\dot{x}^{\mu}}{4\pi} = f_1(\sigma,\tau) \frac{x'^{\mu}}{4\pi} + 2 f_2(\sigma,\tau) \Pi_{\nu} g^{\mu\nu} \qquad \text{(D.9)}$$

If we choose the particular gauge where

$$\frac{\dot{x}^{\mu}}{4\pi} = \Pi_{\nu}\, g^{\mu\nu}(x) \quad , \qquad \text{(D.10)}$$

we completely fix the functions f_1 and f_2

$$f_1 = 0 \quad ; \quad f_2 = \frac{1}{2} \quad ; \qquad \text{(D.11)}$$

by substituting (D.10) in Eq. (D.3) we see that this choice corresponds to define a parametrization such that

$$\left(\dot{x}^{\mu}\dot{x}^{\nu}+x'^{\mu}x'^{\nu}\pm 2\,\dot{x}^{\mu}x'^{\nu}\right)g_{\mu\nu}(x)=0 \tag{D.12}$$

In this particular gauge the equation of motion for Π_{μ} is

$$\dot{\Pi}_{\rho}=-\frac{1}{8\pi}\,x'^{\mu}x'^{\nu}\frac{\partial}{\partial x^{\rho}}g_{\mu\nu}(x)-2\pi\,\Pi_{\mu}\Pi_{\nu}\frac{\partial}{\partial x^{\rho}}g^{\mu\nu}(x)+$$
$$+\frac{\partial}{\partial\sigma}\left(g_{\rho\nu}[x(\sigma)]\frac{x'^{\nu}(\sigma)}{4\pi}\right)-g_{\rho\nu}[x(\sigma')]\frac{x'^{\nu}(\sigma')}{4\pi}\,\delta(\sigma-\sigma')\Big|_{\sigma'=0}^{\sigma'=2\pi} \tag{D.13}$$

The last term in the right-hand side of this equation vanishes because of the periodicity condition (9.4). By eliminating Π_{μ} from Eqs. (D.10) and (D.13) we finally obtain the equation of motion for the closed string

$$\ddot{x}^{\rho}-x''^{\rho}+\left(\dot{x}^{\mu}\dot{x}^{\nu}-x'^{\mu}x'^{\nu}\right)\Gamma^{\rho}_{\nu\mu}=0 \tag{D.14}$$

which is identical to Eq. (9.11).

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29) We thank H.B. Nielsen for a very useful discussion on this point.