Theory of Anisotropic Thin-Walled Beams

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Asymptotically correct, linear theory is presented for thin-walled prismatic beams made of generally anisotropic materials. Consistent use of small parameters that are intrinsic to the problem permits a natural description of all thin-walled beams within a common framework, regardless of whether cross-sectional geometry is open, closed, or strip-like. Four "classical" one-dimensional variables associated with extension, twist, and bending in two orthogonal directions are employed. Analytical formulas are obtained for the resulting 4×4 cross-sectional stiffness matrix (which, in general, is fully populated and includes all elastic couplings) as well as for the strain field. Prior to this work no analytical theories for beams with closed cross sections were able to consistently include shell bending strain measures. Corrections stemming from those measures are shown to be important for certain cases. Contrary to widespread belief, it is demonstrated that for such "classical" theories, a cross section is not rigid in its own plane. Vlasov's correction is shown to be unimportant for closed sections, while for open cross sections asymptotically correct formulas for this effect are provided. The latter result is an extension to a general contour of a result for I-beams previously published by the authors. [S0021-8936(00)03003-8]

1 Introduction

The following discussion is restricted to the theory of prismatic beams where the three-dimensional constitutive law and strain-displacement relationships can be considered linear. Any beam theory is associated with introduction of variables which depend only on the coordinate along the beam axis. For a general type of deformation at least four such one-dimensional variables have to be introduced: extensional, torsional, and two bending variables (corresponding to deformation along two orthogonal directions). The corresponding one-dimensional governing equations are uncoupled for isotropic beams with doubly symmetric cross sections and are given by Euler-Bernoulli theory for extension and bending and St. Venant theory for torsion. If one wishes to extend this theory to composite beams, the governing equations become coupled due to the appearance of off-diagonal terms in the cross-sectional stiffness matrix. This 4×4 stiffness matrix C_{ab} characterizes elastic properties of the beam. Then, the strain energy per unit length is expressed in terms of the four one-dimensional strain measures as

$$2F_{\text{classical}} = \alpha_a C_{ab} \alpha_b \quad \text{where} \quad \alpha^T = \{U_1', U_2'', U_3'', \theta'\}. \quad (1)$$

For thin-walled beams this problem was first posed in Reissner and Tsai [1]. However, the approach employed therein led to a complicated set of equations, especially in the case of closed cross sections. The solution of those equations was presented only for a special type of three-dimensional constitutive equations.

The introduction of the variational-asymptotic method in context of anisotropic beams Berdichevsky [2] allowed the treatment of this problem from a different perspective: beam theory can obtain three-dimensional elasticity without making any *ad hoc* assumptions using the small parameter $\frac{a}{l} \ll 1$, where a is a characteristic dimension of the cross section and l is a the wavelength of deformation along the beam reference line. For a general (but not thin-walled) cross section the problem is reduced to a system of two-dimensional equations on a cross section. A development of a numerical solution of this problem is presented in Cesnik and Hodges [3].

Applying the variational-asymptotic procedure to thin-walled cross sections where another small parameter exists, namely $\frac{h}{a} \ll 1$ (where h is a wall thickness), allows one to start with shell theory rather than three-dimensional elasticity. Rather than having to solve a two-dimensional problem over the cross-sectional plane, one instead solves a one-dimensional problem over the length of the thin walls. This dimensional reduction can be also conducted in another way: the asymptotic procedure with respect to $\frac{h}{a}$ can be applied directly to the two-dimensional cross-sectional problem that results when starting with three-dimensional elasticity. Both approaches lead to the same final results, but the latter procedure is more computationally involved.

The former procedure was used in Berdichevsky et al. [4] to obtain analytical solutions for closed sections. The resulting convenient cross-sectional stiffness formulas published in that paper are presently widely used in engineering community. Although shell bending strain measures were neglected in that paper, these for most practical purposes do not affect final stiffness results. However, as shown below, for certain material properties the deviation of their results from the asymptotically correct results might be significant.

Concerning the application of the variational-asymptotic method to beams with open cross sections, an I-beam was viewed as an assembly of strips in Volovoi et al. [5]. Asymptotically correct formulas were obtained therein which account for Vlasov's correction. Those results are generalized here for beams with arbitrary open contours.

2 Present Approach

Beams are considered thin walled if $h \ll a, R$ where R is a characteristic radius of curvature of the midsurface. No assumptions are made about the relative orders of a and R, and shell theory is employed. A curvilinear system of coordinates is introduced (see Fig. 1), with s and ξ being contour and through-the-thickness coordinates, respectively; $\mathbf{r} = x_i \mathbf{x}_i$ is a position vector of the shell midsurface, vectors are denoted with bold letters. The notation to be used is
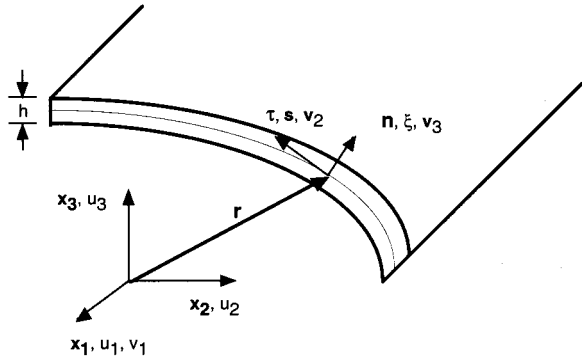
Fig. 1 Configuration and coordinate system

$$(\dot{\ }) \equiv \frac{d(\ )}{ds} \quad (\ )' \equiv \frac{d(\ )}{dx_1}$$

$$\boldsymbol{\tau} = \dot{\mathbf{r}} = \dot{x}_2 \mathbf{x}_2 + \dot{x}_3 \mathbf{x}_3$$

$$\mathbf{n} = \boldsymbol{\tau} \times x_1 = \dot{x}_3 \mathbf{x}_2 - \dot{x}_2 \mathbf{x}_3$$

$$r_\tau = \boldsymbol{\tau} \cdot r = \dot{x}_2 x_2 + x_3 \dot{x}_3 \qquad (2)$$

$$r_n = \mathbf{n} \cdot \mathbf{r} = x_2 \dot{x}_3 - x_3 \dot{x}_2$$

$$R = \dot{x}_2 / \ddot{x}_3 = -\dot{x}_3 / \ddot{x}_2.$$

Curvilinear displacements v_i are expressed in terms of Cartesian displacements u_i as

$$v_1 = u_1$$

$$v_2 = u_2 \dot{x}_2 + u_3 \dot{x}_3 \qquad (3)$$

$$v_3 = u_2 \dot{x}_3 - u_3 \dot{x}_2$$

Shell strain measures are taken from the works of Koiter [6] and Sanders [7], which for cylindrical shells yields

$$\gamma_{11} = v_{1,1} \qquad \rho_{11} = v_{3,11}$$

$$2\gamma_{12} = v_{1,2} + v_{2,1} \quad \rho_{12} = v_{3,12} + \frac{1}{4R}(v_{1,2} - 3v_{2,1}) \qquad (4)$$

$$\gamma_{22} = v_{2,2} + \frac{v_3}{R} \quad \rho_{22} = v_{3,22} - \left(\frac{v_2}{R}\right)_{,2}.$$

Here $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are the extensional (membrane) and bending strain measures, respectively. Then, the strain energy density of the shell has the form

$$2\mathrm{E}_\mathrm{shell} = h E_e^{\alpha\beta\gamma\delta} \gamma_{\alpha\beta}\gamma_{\gamma\delta} + h^3 E_b^{\alpha\beta\gamma\delta}\rho_{\alpha\beta}\rho_{\gamma\delta} + 2h^2 E_{eb}^{\alpha\beta\gamma\delta}\gamma_{\alpha\beta}\rho_{\gamma\delta} \tag{5}$$

where Greek indices vary from 1 to 2; $E_e^{\alpha\beta\gamma\delta}$ and $E_b^{\alpha\beta\gamma\delta}$ are two-dimensional material constants corresponding to membrane and bending deformation, respectively, and $E_{eb}^{\alpha\beta\gamma\delta}$ corresponds to coupling between these two types of deformation. These two-dimensional material constants are obtained from the reduced three-dimensional material constants $D^{\alpha\beta\gamma\delta}$ by use of the relations

$$\{E_e^{\alpha\beta\gamma\delta}, E_{eb}^{\alpha\beta\gamma\delta}, E_b^{\alpha\beta\gamma\delta}\} = \frac{1}{h}\int_{-h/2}^{h/2} D^{\alpha\beta\gamma\delta}\left\{1, \frac{\xi}{h}, \left(\frac{\xi}{h}\right)^2\right\} d\xi. \tag{6}$$

These constants are, in turn, obtained from the regular three-dimensional constants as

$$D^{\alpha\beta\gamma\delta} = E^{\alpha\beta\gamma\delta} - \frac{E^{\alpha\beta33}E^{\gamma\delta33}}{E^{3333}} - H_{\mu\lambda}G^{\alpha\beta\mu}G^{\gamma\delta\lambda}$$

where $H_{\mu\lambda}^{-1} = E^{\mu3\lambda3} - \dfrac{E^{\mu333}E^{\lambda333}}{E^{3333}}$

$$G^{\alpha\beta\mu} = E^{\alpha\beta\mu3} - \frac{E^{\alpha\beta33}E^{\mu333}}{E^{3333}}. \tag{7}$$

For the following derivation it is convenient to rewrite Eq. (5) as

$$2\mathrm{E}_\mathrm{shell} = \psi_1 Q_{ij}\psi_j + 2\phi_i S_{ij}\psi_j + \phi_i P_{ij}\phi_j \tag{8}$$

where $\psi^T \equiv \{\gamma_{11}, h\rho_{11}, h\rho_{12}\}$, and $\phi^T \equiv \{\gamma_{12}, \gamma_{22}, h\rho_{22}\}$; $i,j = 1\ldots3$ and 3×3 matrices Q_{ij}, S_{ij}, and P_{ij} are corresponding combinations of $E_e^{\alpha\beta\gamma\delta}$, $E_{eb}^{\alpha\beta\gamma\delta}$, and $E_b^{\alpha\beta\gamma\delta}$.

In the derivation below it is the axial coordinate x_1 that is distinct from the other two, so it is now convenient for Greek indices to vary from 2 to 3. The variational-asymptotic method Berdichevsky [2,8] is used in what follows. While we avoid a detailed discussion of this method, sufficient information is provided here to facilitate understanding of the derivation. We are using the term ''asymptotically correct'' concerning an approximate solution to denote its agreement with the expansion of the exact solution to a specified order in terms of a specific small parameter. It is clear that any theory which is not asymptotically correct will certainly fail to achieve the accuracy of one which is.

Setting up the Problem. Since only statics is considered, only the strain energy and work of external forces are present in the total functional. External forces are considered slowly varying so that our minimization is not affected by those forces. This leads to minimization of the strain energy density given in Eq. (5) with the strains given by Eqs. (4). Next, this functional is represented in terms of a series with respect to small parameters. A recursive procedure is invoked when perturbation of the previous approximation is used to obtain the following approximation. From this point of view ''classical'' approximation corresponds to the first (main) nonvanishing terms in that series.

In our case there are two small parameters: $\frac{a}{l}$ and $\frac{h}{a}$. These parameters are considered independent: for a given order of terms with respect to $\frac{a}{l}$ we sort out the terms with respect to $\frac{h}{a}$ as well. The small parameter $\frac{a}{l}$ enters the problem from the observation that $X_{,1} \approx \frac{X}{l}$ and $X_{,2} \approx \frac{X}{a}$ for any quantity X.

''Zeroth'' Approximation. This is a starting point of the recursive procedure. All terms that contain the small parameter $\frac{a}{l}$ in the functional are set to zero. The resulting functional is degenerate and the general solution for its kernel (null space) is found. This defines one-dimensional variables. In our case setting all terms in Eqs. (4) containing derivatives with respect to a ''slow'' axial variable leads to an expression for nonzero strains of ''zeroth'' functional given by

$$2\gamma_{12} = v_{1,2} \quad \rho_{12} = \frac{1}{4R}v_{1,2}$$

$$\gamma_{22} = v_{2,2} + \frac{v_3}{R} \quad \rho_{22} = v_{3,22} - \left(\frac{v_2}{R}\right)_{,2}. \tag{9}$$

Since Eq. (5) is a positive-definite quadratic form of strains, for a displacement field to belong to the kernel of ''zeroth'' functional, all strains in Eq. (9) must vanish. It can be directly checked that the general solution of this problem has the form

$$v_1 = U_1 \quad v_2 = U_\alpha \dot{x}_\alpha + \theta r_n$$

$$v_3 = U_2 \dot{x}_3 - U_3 \dot{x}_2 - \theta r_\tau \tag{10}$$

where U_i and $\theta \equiv v_2/R - v_{3,2}$ are arbitrary functions of x_1. It is easy to see (using Eqs. (3)) that these one-dimensional variables correspond to motion of a cross section as a rigid body: $U_i(x_1)$ translation of a cross section in the x_i-direction, and $\theta(x_1)$ is the rotation of a cross section about x_1.

Asymptotic Recursion. Perturbation of the displacement field which was obtained at the previous step is now introduced, namely,

$$v_1 = U_1 + \hat{w}_1$$

$$v_2 = U_2 \dot{x}_2 + U_3 \dot{x}_3 + \theta r_n + \hat{w}_2 \qquad (11)$$

$$v_3 = U_2 \dot{x}_3 - U_3 \dot{x}_2 - \theta r_\tau + \hat{w}_3.$$

Substituting this displacement field, Eq. (11), into the strains, Eqs. (4), and, in turn, substituting the strains into Eq. (5), one obtains an energy functional. Only the leading terms with respect to small parameters are retained at this step, and a minimization with respect to $\hat{w}_i$ is conducted. As a result of this procedure the perturbations $\hat{w}_i$ are found as functions of one-dimensional variables and their derivatives.

In the most general case, deformations due to all four one-dimensional strain measures are of the same order (denoted by ϵ, a nondimensional constant of the order of the maximum strain in the beam). If this were not the case, any smaller deformations could be simply neglected in the main approximation. The one-dimensional strain measures are given in Eq. (1). The only problem is to determine appropriate dimensional constants that need to multiply these measures to provide a term of the order ϵ (this does not affect U_1' which is already nondimensional). As shown below, this constant must be either a or h, depending on the geometry of the contour. One can calculate the appropriate order using the expression for the one-dimensional energy for the isotropic case, since all material properties are assumed to be of the same magnitude, so the order of the one-dimensional strain measures is not affected. However, these orders will naturally fall out of our derivation. Let us emphasize that the order of perturbations is not assumed but determined during the minimization. In fact, it is easily estimated prior to the minimization by reckoning that leading quadratic and linear terms in the functional with respect to the unknown perturbation are of the same order.

2.1 Phantom Step. There are some terms in the strains which are larger in magnitude than the corresponding strain component itself. Those terms are balanced by equally large terms, so that their combination is of a smaller order. We call such terms "phantom" ones. Since at each step of asymptotic procedure only the leading terms are considered, it means that those "phantom" terms are minimized to zero. This procedure is often referred to, somewhat cruelly, as "killing" excessively large terms in the energy. Substituting the displacement field of Eqs. (11) into Eqs. (4), one obtains

$$\gamma_{11} = \underset{\epsilon}{U_1'} + \underset{\epsilon}{\hat{w}_{1,1}}$$

$$2\gamma_{12} = \underset{(a/l)^{-1}\epsilon}{\dot{x}_2 U_2'} + \underset{(a/l)^{-1}\epsilon}{\dot{x}_3 U_3'} + \underset{\epsilon}{r_n \theta'} + \underset{(a/l)^{-1}\epsilon}{\hat{w}_{1,2}} + \underset{(a/l)\epsilon}{\hat{w}_{2,1}}$$

$$\gamma_{22} = \underset{\epsilon}{\hat{w}_{2,2}} + \underset{\epsilon}{\frac{\hat{w}_3}{R}}$$

$$h\rho_{11} = h[\underset{\epsilon}{\dot{x}_3 U_2''} - \underset{\epsilon}{\dot{x}_2 U_3''} - \underset{(a/l)\epsilon}{\theta'' r_\tau} + \underset{(a/l)^2\epsilon}{\hat{w}_{3,11}}]$$

$$h\rho_{12} = h\left[\frac{1}{4R}\{\dot{x}_\alpha U_\alpha' + \theta' r_n - \hat{w}_{1,2}\} - \theta' + \hat{w}_{3,12} - \frac{3}{4R}(\hat{w}_{2,1})\right]$$

$$\underset{(a/l)^{-1}\epsilon}{} \qquad \underset{\epsilon}{} \quad \underset{(a/l)^{-1}\epsilon}{} \quad \underset{\epsilon}{} \quad \underset{(a/l)\epsilon}{} \quad \underset{(a/l)\epsilon}{}$$

$$h\rho_{22} = h\left(\hat{w}_{3,2} - \frac{\hat{w}_2}{R}\right)_2. \qquad (12)$$

$$\underset{\epsilon}{} \quad \underset{\epsilon}{}$$

At this step, terms with $\rho_{\alpha\beta}$ do not enter the minimization procedure. The reason for this is that, for each term in $\rho_{\alpha\beta}$, there is a similar term in $\gamma_{\alpha\beta}$, but multiplied by $\frac{a}{h}$. So retaining only the leading terms with respect to $\frac{h}{a}$ allows us to discard terms from $\rho_{\alpha\beta}$. The most obvious "phantom" terms of order $(\frac{a}{l})^{-1}$ are present in γ_{12} which defines the orders for the warping, written underneath the individual terms in Eqs. (12), and the solution for $\hat{w}_{1,2}$

$$\hat{w}_{1,2} = -U_\alpha' \dot{x}_\alpha. \qquad (13)$$

There might be, however, some other "phantom" terms which are of order $(\frac{a}{l})^0$, but still "too large" due to the presence of another small parameter $\frac{h}{a}$. In other words they are of order $\epsilon(\frac{h}{a})^{-1}$. The presence of one of this type of terms is related to a fundamental difference between open and closed cross sections. A constraint of single-valuedness has to be satisfied around the closed contours of closed sections for certain variables; these constraints do not apply for open cross sections. In particular, this applies to the single-valuedness of $\hat{w}_1$. For open cross sections $r_n\theta'$ in γ_{12} is a "phantom" since this term is killed by adding term $-r_n\theta'$ to the right-hand side in Eq. (13). In this case the largest nonzero terms in the functional that are proportional to θ' will come from the ρ_{12} and $\theta' \approx \frac{\epsilon}{h}$. Integration with respect to circumferential coordinate of Eq. (13) yields

$$\hat{w}_1 = -U_\alpha' x_\alpha - \theta' \int_{s_0}^{s} r_n ds \qquad (14)$$

where the coefficient for θ' is called the "sectorial coordinate" and is given by $\eta(s) \equiv \int_{s_0}^{s} r_n ds$. The sectorial coordinate is, in fact, a solution of a classical St. Venant torsional problem in the shell approximation. To avoid redefining U_1, embedded in s_0 constant of integration should be chosen such that $\int \hat{w}_1 ds = 0$. It is obviously convenient to choose the origin of the Cartesian coordinates in the geometric center of the cross section, so that $\int x_2 ds = \int x_3 ds = 0$.

On the other hand, for a closed cross section, $r_n\theta'$ in γ_{12} is not a "phantom"! The requirement of single-valuedness for $\hat{w}_1$ prevents the possibility of displacement field as in Eq. (14); only the last term creates a problem, since $\oint r_n\theta' ds$ is not zero. As a result, terms proportional to $r_n\theta'$ do enter the functional, which implies that $a\theta' \approx \epsilon$. Then the terms with θ' in ρ_{12} will be of order $\epsilon(\frac{h}{a})$ and can be neglected. Therefore, for the closed sections the equivalent of the last term in Eq. (14) belongs to the next step of approximation.

There is another "phantom" term that is also of the form $\epsilon(\frac{h}{a})^{-1}$. If $a \approx R$ then

$$\gamma_{22}(\hat{w}_2, \hat{w}_3) \approx \left(\frac{h}{a}\right)^{-1} h\rho_{22}(\hat{w}_2, \hat{w}_3). \qquad (15)$$

Thus, minimization of the main terms in the functional simply renders

$$\gamma_{22} = \hat{w}_{2,2} + \frac{\hat{w}_3}{R} = 0. \qquad (16)$$

However, each individual term in Eq. (16) is not zero, but rather of order $\epsilon(\frac{h}{a})^{-1}$ and is undetermined at this step. The second equation for these unknowns stems from ρ_{22} and due to Eq. (15) will be provided in the next approximation. If a and R are not of the same order, then orders of γ_{22} and ρ_{22} for a given displacement field are uncoupled, and no "phantom" terms are present. In particular this is the case when no curvature is present (i.e., $R = \infty$). However, formulas for classical stiffnesses will have the same form in both cases, as shown below.

2.2 Classical Approximation. At this step terms of order ϵ^2 in the functional are recovered. Displacement field obtained at the previous step is perturbed again. Denoting these perturbations as w_i, one can write

$$v_1 = U_1 - x_\alpha U'_\alpha - \underline{\theta' \eta} + w_1$$

$$v_2 = U_\alpha \dot{x}_\alpha + \theta r_n + \hat{w}_2 + w_2 \qquad (17)$$

$$v_3 = U_2 \dot{x}_3 - U_3 \dot{x}_2 - \theta r_\tau + \hat{w}_3 + w_3.$$

This is the most general form of the perturbed displacement field. As described above, the underlined term is present only for open cross sections, while $\hat{w}_\alpha$ are present only if $a \approx R$. The latter terms are still unknown, but connected by Eq. (16). Substitution of Eqs. (17) into Eqs. (4) leads to the following expressions for strains:

$$\gamma_{11} = \underset{\epsilon}{U'_1} - \underset{\epsilon}{x_\alpha U'_\alpha} - \underset{(a/l)\epsilon a/h}{\theta'' \eta} + \underset{(a/l)\epsilon}{w_{1,1}}$$

$$2\gamma_{12} = \underset{\epsilon}{\underline{\underline{r_n \theta'}}} + \underset{\epsilon}{w_{1,2}} + \underset{(a/l)\epsilon a/h}{\hat{w}_{2,1}}$$

$$\gamma_{22} = \underset{\epsilon}{w_{2,2} + \frac{w_3}{R}}$$

$$h\rho_{11} = h[\underset{\epsilon}{x_3 U''_2} - \underset{\epsilon}{x_2 U''_3} - \underset{(a/l)\epsilon}{\theta'' r_\tau} + \underset{(a/l)^2 \epsilon}{\hat{w}_{3,11} + w_{3,11}}]$$

$$h\rho_{12} = h\left[\underset{\epsilon}{-\theta'} + \underset{\epsilon}{\frac{\theta' r_n + w_{1,2}}{4R}} + \underset{(a/l)\epsilon}{\hat{w}_{3,12} + w_{3,12}} - \underset{(a/l)^2 \epsilon}{\frac{3(\hat{w}_{2,1} + w_{2,1})}{4R}} \right]$$

$$h\rho_{22} = h\left[\underset{\epsilon}{\left(\hat{w}_{3,2} - \frac{\hat{w}_2}{R} \right)_{,2}} + \underset{\epsilon}{\left(w_{3,2} - \frac{w_2}{R} \right)_{,2}} \right]. \qquad (18)$$

Note that the still unknown $\hat{w}_\alpha$ are present along with w_α—they are distinct, so that $\frac{h}{a}\hat{w}_\alpha \approx w_\alpha$. This allows one to neglect the latter with respect to former in $\rho_{\alpha\beta}$. Of course, when terms due to $\hat{w}_\alpha$ vanish, terms due to w_α have to be retained—this is the case for γ_{22} (or for $\rho_{\alpha\beta}$ when $\hat{w}_\alpha$ themselves vanish—see the previous step). Underlined terms exist only for open sections while double-underlined term only for closed cross sections. Let us keep in Eqs. (18) only terms of order ϵ, denote them with bars and sort the result into two arrays: those containing the one-dimensional strain measures ($\bar{\psi}^T \equiv \{\bar{\gamma}_{11}, h\bar{\rho}_{11}, \bar{\rho}_{12}\}$) and those with only unknown quantities which will be found in the process of minimization ($\bar{\phi}^T \equiv \{\bar{\gamma}_{12}, \bar{\gamma}_{22}, h\bar{\rho}_{22}\}$). This provides the motivation for writing strain the energy density in the form Eq. (8) and resembles the semi-inversion procedure that was used in Reissner and Tsai [1]. Depending on the geometry of the cross section, the following distinct cases can be identified.

2.3 Strips and Open Cross Sections. Ironically, strips represent the only case where all three components of $\bar{\psi}$ are needed. If we align the larger dimension of the strip along with x_2 then $x_3 = 0$ and U_3 drops from the $\bar{\gamma}_{11}$, therefore the largest term with U_3 comes from $\bar{\rho}_{11}$. The double-underlined term in Eqs. (18) is absent (no constraint of single-valuedness), so the largest terms with θ comes from $\bar{\rho}_{12}$. The resulting orders follow as $aU''_2 \approx hU''_3 \approx h\theta' \approx \epsilon$, so $\bar{\psi}^T = \{U'_1 - x_2 U''_2, hU''_3, -h\theta'\}$, or in matrix form $\bar{\psi} = T_{\text{strip}}(s)\alpha$, where T_{strip} is a 3×4 matrix.

For open cross sections U_3 does not drop out from the $\bar{\gamma}_{11}$ so $\bar{\rho}_{11}$ can be neglected and $aU''_3 \approx \epsilon$. Thus, the known strains depend on the one-dimensional strain measures as $\bar{\psi}^T = \{U'_1 - x_2 U''_2 - x_3 U''_3, 0, -h\theta'\}$, or in matrix form, $\bar{\psi} = T_{\text{open}}(s)\alpha$.

There is no constraint on $\bar{\phi}$, so minimization is straightforward, yielding

$$\bar{\phi}_i = -P_{ij}^{-1} S_{jk} \bar{\psi}_k. \qquad (19)$$

Substituting the result into Eqs. (8), we obtain the final expression for the classical strain energy, given by

$$C = \int T^T (Q - SP^{-1}S)T ds \qquad (20)$$

where T is either T_{strip} or T_{open}, depending on the cross section in question.

2.4 Closed Cross Sections. As described above, $\bar{\gamma}_{12}$ contains nonzero terms of order θ' neglected so that $\bar{\psi}^T = \{U'_1 - x_2 U''_2 - x_3 U''_3, 0, 0\}$. Here the 3×4 matrix T, which connects $\bar{\psi}$ and α, effectively becomes a column matrix $T = \{1, -x_2, -x_3, 0\}$; the $\bar{\phi}_i$ are not arbitrary and proper constraints have to be imposed if the minimization is conducted in terms of these unknowns. For n-celled sections there are $4 \times n$ such constraints—four constraints per each cell. Single-cell formulas are derived below, but the procedure is equally applicable for multiple cells as well.

Let us consider most general case when $R \approx a$ (the other cases are analogous with obvious simplifications and lead to the same constraints). We denote $\Xi \equiv h\hat{w}_{3,2} - h\hat{w}_2/R$, so that $\Xi_{,2} = \phi_3$. Clearly $\oint \bar{\phi}_3 ds \equiv \oint \Xi_{,2} ds = 0$. Three other constraints stem from the requirement of single-valuedness of displacements in Cartesian coordinates, such that $\oint u_{i,2} ds = 0$. Note the analogy between the imposed constraints and the introduction of one-dimensional variables Eqs. (10). First, $0 = \oint u_{1,2} ds = \oint \hat{w}_{1,2} ds$, so that $\oint \bar{\phi}_1 ds = \theta' \oint r_n ds$. The other two constraints are a bit less straightforward. Using Eqs. (3) the following relations can be written for $\hat{w}_\alpha$:

$$\oint [\hat{w}_{2,2}\dot{x}_2 + \hat{w}_2 \ddot{x}_2 + \hat{w}_{3,2}\dot{x}_3 + \hat{w}_3 \ddot{x}_3] ds = 0$$

$$\oint [\hat{w}_{2,2}\dot{x}_3 + \hat{w}_2 \ddot{x}_3 - \hat{w}_{3,2}\dot{x}_2 - \hat{w}_3 \ddot{x}_2] ds = 0. \qquad (21)$$

Taking advantage of Eqs. (2) this can be rewritten as

$$\oint \left[\dot{x}_2 \left(\hat{w}_{2,2} + \frac{\hat{w}_3}{R} \right) + \dot{x}_3 \left(\hat{w}_{3,2} - \frac{\hat{w}_2}{R} \right) \right] ds = 0$$

$$\oint \left[\dot{x}_3 \left(\hat{w}_{2,2} + \frac{\hat{w}_3}{R} \right) + \dot{x}_2 \left(\hat{w}_{3,2} - \frac{\hat{w}_2}{R} \right) \right] ds = 0. \qquad (22)$$

Recalling Eq. (16), one finds that

$$\oint \dot{x}_\alpha \Xi ds = 0 \quad \text{or} \quad \oint x_\alpha \phi_3 ds = 0.$$

Therefore, for a single-cell cross section functional to be minimized has the form

$$2\Lambda = \oint [\bar{\psi}_1^2 Q_{11} + 2\bar{\phi}_i S_{i1} \bar{\psi}_1 + \bar{\phi}_i P_{ij} \bar{\phi}_j + 2\lambda_1 (\bar{\phi}_1 + \theta' r_n)$$

$$+ 2\bar{\phi}_3 (\lambda_\alpha x_\alpha + \lambda_4)] ds \qquad (23)$$

where λ_a are Lagrange multipliers; here and below $a = 1, \ldots, 4$. For multiple-cell cross sections, such a set of four Lagrange multipliers has to be introduced for each cell, while minimization should be conducted over the whole cross section.

Then the solution is given by

$$\bar{\phi}_i = -c_i \bar{\psi}_1 - P_{ij}^{-1} t_j \quad \text{where} \quad c_i \equiv P_{ij}^{-1} S_{j1}. \qquad (24)$$

Here $t^T = \{\lambda_1, 0, (\lambda_\alpha x_\alpha + \lambda_4)\} \equiv \hat{T}(s)\lambda$. We can rewrite Eqs. (24) explicitly in terms of α and λ, yielding

$$\bar{\phi}_i = -c_i T_a \alpha_a - P_{ij}^{-1} \hat{T}_{ja} \lambda_a. \qquad (25)$$

Substituting Eqs. (25) into expressions for constraints, we obtain

$$-\alpha_a \oint [c_1 T_a - E_a] ds = \lambda_a \oint [P_{1j}^{-1} \hat{T}_{ja}] ds$$

$$-\alpha_a \oint [\{1, x_\alpha\} c_3 T_a] ds$$

$$= \lambda_a \oint [\{1, x_\alpha\} P_{3j}^{-1} \hat{T}_{ja}] ds \qquad (26)$$

here $E = \{0, 0, 0, r_n\}$. These are four linear equations for λ in terms of the one-dimensional strain measures: $F\lambda = J\alpha$, $\lambda = F^{-1}J\alpha$. Substituting the result into Eqs. (24) we obtain the solution for $\bar{\phi}$ as

$$\bar{\phi} = -(cT + P^{-1}\hat{T}F^{-1}J)\alpha \equiv Y(s)\alpha. \qquad (27)$$

Finally, substituting Eq. (27) into Eq. (23) yields the stiffness matrix:

$$C = \oint T^T Q T - Y^T P Y + LE \qquad (28)$$

where $L_a = F_{1b}^{-1}J_{ba}$ (LE corresponds to the term $\lambda_1 \theta' r_n$).

From the present point of view, the derivation in Berdichevsky et al. [4] is equivalent to setting $\bar{\phi}_3$ to zero. It can be shown using Eqs. (24) and (26) that this assumption is appropriate for so-called circumferentially uniform sections (CUS) (i.e., when material constants can be taken outside the integral and satisfying three constraints on $\bar{\phi}_3$ renders it zero). However, there are some cases that the influence of this term does make a difference. To demonstrate this let us consider a box-beam with geometry and material properties taken from Smith and Chopra [9]. Two following configurations are considered:

outer dimensions: height $b = 0.53$ in.

width $a = 0.953$ in.

wall thickness: $h = 0.03$ in.

material properties: $E_l = 20.6 \times 10^6$ psi

$E_t = 1.42 \times 10^6$ psi

$G_{lt} = 8.7 \times 10^5$ psi

$G_{tn} = 6.96 \times 10^5$ psi

$\nu_{lt} = \nu_{tn} = 0.42$

antisymmetric: right and upper wall layup: $(\Theta)_3/(-\Theta)_3$

antisymmetric: left and lower wall layup: $(-\Theta)_3/(\Theta)_3$

$\qquad (29)$

symmetric: right and left wall layup: $(\Theta)_3/(-\Theta)_3$

symmetric: upper and lower wall layup: $(-\Theta)_3/(\Theta)_3$.

Both antisymmetric and symmetric layups exhibit essentially no elastic coupling, and the one-dimensional stiffness matrices are diagonal.

The torsional rigidity can be significantly overestimated if $\bar{\phi}_3$ is disregarded. This can be observed by comparing the results of Berdichevsky et al. [4] with the present ones and with the numerical results obtained from VABS Cesnik and Hodges [3]. The results of Berdichevsky et al. [4] are far too stiff in torsion relative to VABS results while the present theory exhibits excellent agreement with VABS. Indeed, for $h = 0.03$ the difference is less than three percent (see Figs. 2 and 3). With decreasing thickness (leaving the other dimensions the same) analytical results converge to the numerical results. In fact by $h = 0.006$ in the analytical, results exceed the precision of 1000 six-noded finite elements in VABS. One should recall here that finite elements with large aspect ratios are notoriously fickle. We also note that the difference between
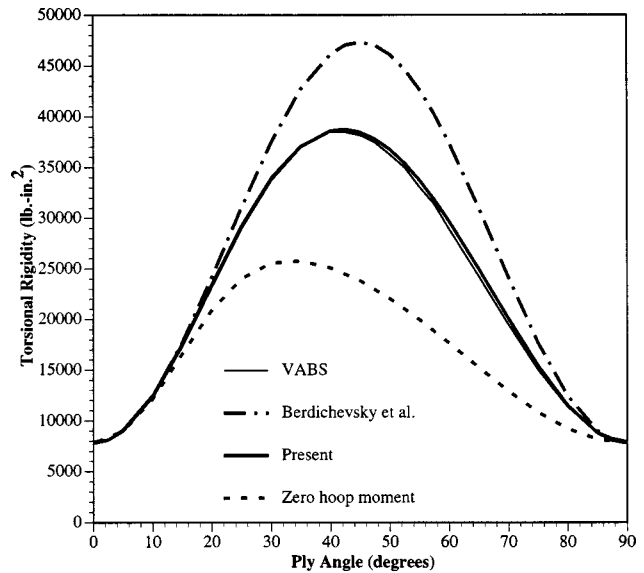

Fig. 2 Torsional rigidity, antisymmetric layup $h = 0.03$ in

the results of Berdichevsky et al. [4] and those of the present asymptotically correct theory is practically independent of thickness, as can be seen Fig. 4.

It has to be emphasized that while the cases where the theory from Berdichevsky et al. [4] breaks down are quite rare, it might actually create a false sense of security: For the considered symmetric case torsional rigidity is overpredicted by a factor of two! On the other hand, another quite obvious approximation would be to set the hoop bending moment to zero. This can be interpreted as a thin-walled equivalent of the so-called "uniaxial stress" assumption (when all stresses in the cross-sectional plane are set to zero) that is quite common in beam theories, e.g., Rand [10] and Kim and White [11]. As can be observed from Figs. 2 and 3, this assumption leads to an underprediction of their torsional rigidity. It has to be added that for the specific cases considered in Berdichevsky et al. [4], the differences between our results and theirs are negligible. Thus, for the sake of brevity, the excellent correlations published therein with experimental and numerical data need not be repeated here.
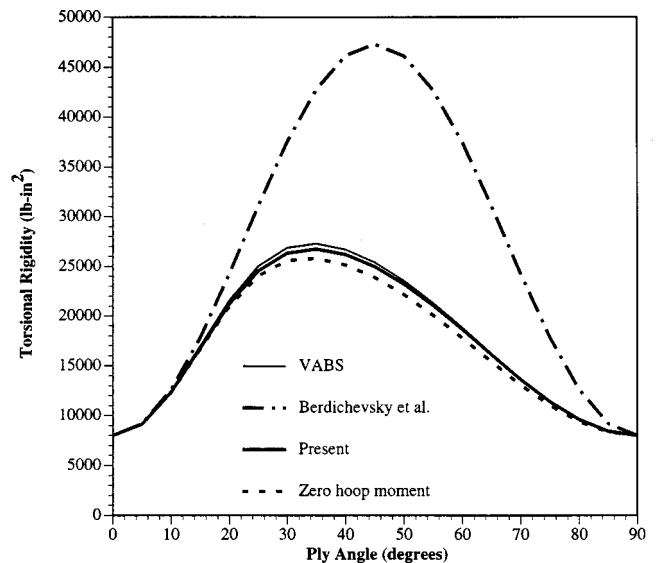

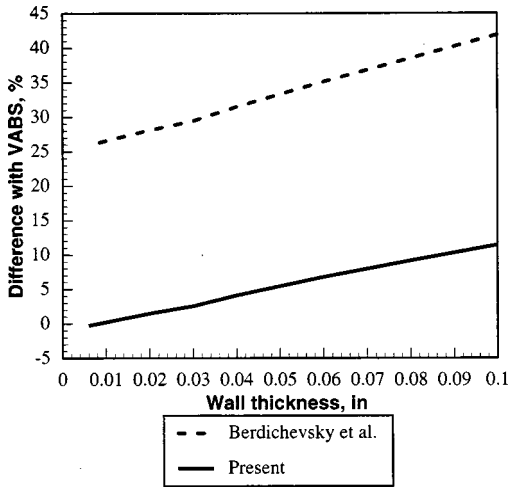**Fig. 3 Torsional rigidity, symmetric layup $h = 0.03$ in**

Fig. 4 Difference in torsional rigidity, $\Theta = 60$ deg

The general conclusion can be drawn that, while for most layups either ''no bending shell strain measures'' or ''no hoop moment'' might work quite satisfactorily, only the present theory can insure correct results for all the cases.

Strain Field. Let us emphasize that for all types of cross sections—even in ''classical'' approximation—the cross section is *not* rigid in its own plane! The in-plane strains are not zero but are given by Eqs. (19) and (27). By the same token, unless one deals with isotropy or similarly restricted case, the shear strain γ_{12} is nonzero and essential to the analysis, even without resorting to Timoshenko-like theories. On the other hand, within the precision of this approximation, γ_{11}, ρ_{11}, and ρ_{12} have very simple expressions, since they are given by appropriate components of $\bar{\psi}$.

Recovering Displacements. When there is no curvature $\bar{\phi}^T = \{w_{1,2}, w_{2,2}, hw_{3,22}\}$, so once $\bar{\phi}$ is obtained this relationship can be integrated. In order to preserve the definition of one-dimensional variables one has to eliminate rigid-body motions from this warping (i.e., $\int w_i ds = \int (w_2 x_3 - w_3 x_2) ds = 0$), this allows one to define w_i uniquely which then should be substituted into Eqs. (17) to obtain the full displacement field. However, when $R \approx a$ only $\hat{w}_i$, and w_1 can be obtained, whereas knowing $\bar{\phi}_2$ is not sufficient to recover w_2 and w_3 individually. Thus, the full displacement field cannot be recovered in this case. The latter situation is similar to the one described in Berdichevsky and Misyura [12].

2.5 Second-Order Terms. The next step of the asymptotic procedure allows us to obtain terms in the strain energy up to $\epsilon^2 (\frac{a}{l})^2$. While generally this is a cumbersome procedure, it turns out that sometimes these terms are very significant—and easily calculated. This can be clearly seen from Eqs. (18). There are two terms present in γ_{11} and γ_{12} which are of order $\epsilon(\frac{a}{l})(\frac{a}{h})$. While we neglected those terms in the ''classical'' approximation, they clearly can be quite large. We perturb the ''classical'' displacement field in a manner similar to the previous step in which w_i was introduced into the displacement field. This led to the presence of the unknown $\bar{\phi}$ in the strain field. Here we introduce $\tilde{w}_i$, which in turn leads to $\tilde{\phi}$ in the strain. Let us note that due to the Euler-Lagrange equations for w_i, the leading cross terms between w_i and $\tilde{w}_i$ vanish in the functional. The term from γ_{11} exists for open cross sections only; it is zero for strips, since $\eta = 0$. No constraints are imposed on $\tilde{\phi}$, so the problem is similar to the unconstrained problem for closed sections in which $\tilde{\phi}_i = c_i \theta'' \eta$. This leads to the one-dimensional strain energy per unit length

$$2F_{\text{Vlasov}} = \alpha_a C_{ab} \alpha_b + 2 M_a \alpha_a \theta'' + \Gamma \theta''^2 \tag{30}$$

where C_{ab} is given by Eq. (20) for open cross sections and

$$M_a = \int \eta s_{1i} T_{ia} ds$$

$$\Gamma = \int \eta^2 (Q_{11}^2 - c_i P_{i1}) ds. \tag{31}$$

Note that M_a does not have a contribution from $\tilde{w}_i$, since terms of order ϵ are correctly obtained using only classical warping. This generalizes the formulas provided in Volovoi et al. [5] where I-beams were treated as an assembly of strips rather than as a contour, and the results were extensively correlated with three-dimensional simulations. For isotropy the formulas obtained reduce to Vlasov theory.

Finally, let us consider the term $\hat{w}_{2,1}$ from γ_{12}, which is of order $\epsilon(\frac{a}{l})(\frac{a}{h})$ if $R \approx a$. If the cross section is open, there is no constraint on $\tilde{w}$, so by choosing $\tilde{w}_{1,2} = \tilde{\phi}_1 - \hat{w}_{2,1}$ (here $\tilde{\phi}_1$ refers to the solution for Vlasov correction), this term can be killed. Theoretically, for closed sections this is not true if $\oint \hat{w}_{2,1} ds$ is not zero. Value of this integral depends on the constraints imposed on classical warping. Using constraints that are chosen so that warping does not affect the definition of one-dimensional variables, it can be shown that for a closed contour of a constant curvature $\oint \hat{w}_{2,1} ds = 0$. For a general geometry this is not so, but constraints can be adjusted appropriately. Therefore, this term is not expected to play a significant role.

3 Conclusions

Using small parameters $\frac{a}{h}$ and $\frac{h}{a}$, which are inherent to thin-walled beams, and without appeal to any *ad hoc* geometric assumptions whatsoever, asymptotically correct theories are derived for thin-walled anisotropic beams. These theories include closed-form expressions for cross-sectional stiffness constants as well as recovering relations for strain (and displacement when possible).

It is noted that the term ''asymptotical correctness'' concerning an approximate solution denotes its agreement to a specified order in a small parameter with an asymptotic expansion of the exact solution in that parameter. Asymptotical correctness is the most important characteristic of any approximate solution.

The resulting Vlasov-like theory for beams with open cross sections is a generalization of the previously published theory for I-beams in Volovoi et al. [5]. However, unlike any existing theory for closed sections, the effects of shell-bending strain measures are included herein and their importance is demonstrated. It is shown that the Vlasov effect for strips and beams with closed cross section is negligible.

Unlike most treatments of thin-walled beams in the literature, the present results are simultaneously obtained for open and closed-section anisotropic beams, including strip-beams. The significant differences entailed by these different geometries are shown to be naturally resolved within the same asymptotic framework. Now that an asymptotically correct theory is in place for thin-walled beams, one can undertake critical assessment of previously published theories of thin-walled beams.

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