Progress of Theoretical Physics, Vol. 56, No. 6, December 1976

Theory of Canonical Transformations for Nonlinear Evolution Equations. I

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(Received July 5, 1976)

For nonlinear evolution equations, a canonical transformation which keeps the Hamiltonian form invariant is investigated. It is shown that the so-called Bäcklund transformation is the canonical transformation of this type. Group property of the canonical transformation and relations between infinitesimal canonical transformations and conservation laws are also investigated. Sine-Gordon equation, Korteweg-de Vries equation and modified Korteweg-de Vries equation are considered as examples.

§ 1. Introduction

A number of nonlinear evolution equations such as sine-Gordon equation, Korteweg-de Vries equation and modified Korteweg-de Vries equation are known to have many remarkable properties in common.[1] The most fundamental property is that these nonlinear evolution equations are "soliton" systems or, more precisely, complete integrable systems. The particle-like property of solitons and the possession of an infinite number of conserved quantities are no doubt related to this fundamental property. We note that the applicability of inverse scattering method leads very naturally to the complete integrability of the system.[2] We also note that the only persuasive method to introduce inverse scattering method is via the Bäcklund transformation.[3] The Bäcklund transformation is mathematically a transformation which transforms a pseudo-spherical surface into another of the same curvature.[4]

Recently, Toda and one of the authors (M.W.) have derived a canonical transformation for the exponential lattice equation and shown that the Bäcklund transformation is the canonical transformation.[5] Subsequently, the present authors have observed a similar conclusion for the sine-Gordon equation in rest frame.[6] These results clarify the dynamical significance of Bäcklund transformation and moreover suggest that the theory of canonical transformation may play a fundamental role in the theory of complete integrable system.

In this paper, we define canonical transformation for the nonlinear equation of the simple evolution type (see Eq. (2·1)) and show that the Bäcklund transformations associated with these evolution equations are the canonical transformations. In the case of the simple evolutional type equations, the independent field variable is *unique* (that is, the canonical conjugate momentum is *not* an independent variable). Hence the canonical transformation defined in this paper is essentially the transformation of 1-field variable.

In § 2, we give a general theory of the canonical transformation to the nonlinear evolution equation of the simple evolution type. Then, as a particular case, we consider the canonical transformation whereby the transformed Hamiltonian keeps the same form as the original Hamiltonian except for additional constants. In other words, this canonical transformation maps a dynamical space of the system to itself. In § 3, the method developed in § 2 is applied to the sine-Gordon equation in characteristic frame, Korteweg-de Vries equation and modified Korteweg-de Vries equation as examples. In § 4, the relationship between the canonical transformation and the Bäcklund transformation is discussed from the viewpoint of the dynamics. Further, upon rewriting the canonical transformation considered in § 3 as the form connected with identity transformation, we show that the canonical transformation connstitutes a group. In § 5, we define the infinitesimal canonical transformation and consider the physical significance of the conserved quantities and the symmetry of the evolution equation. Finally in § 6, summary and discussion are provided.

§ 2. Canonical transformation

Consider an evolution equation of the form

$$\phi_{xt} = \mathcal{K}(\phi, \phi_x, \phi_{xx}, \cdots). \tag{2·1}$$

Here $\mathcal{K}(\phi, \phi_x, \phi_{xx}, \cdots)$ is, in general, a nonlinear function of ϕ and its x-derivatives, but does not include t-derivatives of ϕ. Equation (2·1) can be derived from the Euler-Lagrange equation

$$\frac{\partial}{\partial t}\frac{\delta L}{\delta \phi_t} - \frac{\delta L}{\delta \phi} = 0\,, \tag{2·2}$$

with the Lagrangian L,

$$L = \int_{-\infty}^{\infty} dx\, \mathcal{L}\,, \tag{2·3}$$

$$\mathcal{L} = \phi_x \phi_t - \mathcal{U}(\phi, \phi_x, \phi_{xx}, \cdots). \tag{2·4}$$

In fact, from Eqs. (2·2)～(2·4), we have

$$\phi_{xt} = -\frac{1}{2}\frac{\delta U}{\delta \phi}\,, \tag{2·5}$$

where

$$U[\phi] = \int_{-\infty}^{\infty} dx\, \mathcal{U}(\phi, \phi_x, \phi_{xx}, \cdots). \tag{2·6}$$

Since the Lagrangian density $\mathcal{L}$ is linear with respect to ϕ_t, Hamiltonian formalism is not uniquely defined. However, we find that it is most natural to consider Hamilton's equation of motion of the form

$$\phi_{xt}=-\frac{\delta H}{\delta\phi}, \tag{2·7}$$

where the Hamiltonian H is

$$H[\phi]=\frac{1}{2}U[\phi]. \tag{2·8}$$

Note that, when $\mathcal{U}$ includes only x-derivatives of ϕ, we may replace Eq. (2·7) with Eq. (2·8) by

$$\phi_{xt}=\frac{\partial}{\partial x}\frac{\delta H}{\delta\phi_x}, \tag{2·9}$$

where

$$H[\phi_x]=\frac{1}{2}U[\phi_x]. \tag{2·10}$$

The canonical transformation may be defined as follows: Transformation, which maps ϕ into ϕ', is canonical if Pfaffian form

$$\theta=\int_{-\infty}^{\infty}dx(\phi_x d\phi-\mathcal{H}dt) \tag{2·11}$$

is invariant under the transformation. That is,

$$\int_{-\infty}^{\infty}dx(\phi_x d\phi-\mathcal{H}dt)=\int_{-\infty}^{\infty}dx(\phi_x' d\phi'-\mathcal{H}'dt)+dW[\phi,\phi';t], \tag{2·12}$$

where $W[\phi,\phi';t]$ is an arbitrary functional and will be referred to as generating functional (or generator) of the transformation. Pfaffian form (2·11) is a natural extension of the case of finite degrees of freedom.[7]

It can be shown from the canonical simplectic form $d\theta$ (d denotes an exterior derivative[7]) that if ϕ satisfies

$$\phi_{xt}=-\frac{\delta H}{\delta\phi}, \tag{2·13}$$

then ϕ' satisfies

$$\phi'_{xt}=-\frac{\delta H'}{\delta\phi'}. \tag{2·14}$$

The detail of the discussion is given in Appendix A. Since $dW[\phi,\phi';t]$ is in an exact differential form

$$dW[\phi,\phi';t]=\frac{\partial W}{\partial t}dt+\int_{-\infty}^{\infty}dx\left(\frac{\delta W}{\delta\phi}d\phi+\frac{\delta W}{\delta\phi'}d\phi'\right),$$

we obtain the transformation equations

$$\phi_x=\frac{\delta W}{\delta\phi},\quad \phi_x'=-\frac{\delta W}{\delta\phi'},\quad H'=H+\frac{\partial W}{\partial t}. \tag{2·15}$$

The canonical transformation for the case of Eq. (2·9) is discussed in Appendix B.

In general, if Hamiltonian H does not explicitly depend on time t, the generating functional $W[\phi, \phi'; t]$ takes the form[7]

$$W[\phi, \phi'; t] = S[\phi, \phi'] - Et\,, \tag{2·16}$$

where E is a constant (energy integral) determined by the boundary conditions of the flows ϕ and ϕ'. Then, the transformed Hamiltonian H' is

$$H' = H + \frac{\partial W}{\partial t} = H - E\,. \tag{2·17}$$

In particular, if H is transformed into $H' = 0$ (stationary flow), we have the Hamilton-Jacobi equation

$$H\left[\phi, \frac{\delta S}{\delta \phi}\right] = E\,. \tag{2·18}$$

Generally, in order to derive the canonical transformation we must solve Eq. (2·15). Among various types of canonical transformations, we restrict our interest to the canonical transformation such that the transformed Hamiltonian H' is of the same form as the original Hamiltonian H except for the additional constant. Equivalently, we look for the canonical transformation under which we have

$$\mathcal{H}'(\phi', \phi_x', \cdots) - \mathcal{H}(\phi, \phi_x, \cdots) = \frac{\partial}{\partial x}\mathcal{F}(\phi, \phi', \phi_x, \phi_x', \cdots)\,. \tag{2·19}$$

Furthermore we assume that the generating functional $W[\phi, \phi'; t]$ takes the form

$$W[\phi, \phi'; t] = \int_{-\infty}^{\infty} \{\phi' \phi_x + \mathcal{G}(\phi, \phi')\}\, dx - Et\,. \tag{2·20}$$

From the formulas (2·13), we have

$$\mathcal{G}(\phi, \phi') = \mathcal{G}(\phi - \phi')\,, \tag{2·21}$$

$$\phi_x + \phi_x' = \frac{\partial}{\partial \phi}\mathcal{G}(\phi - \phi') \equiv g(\phi - \phi')\,, \tag{2·22}$$

$$E = -\mathcal{F}(\phi, \phi', \phi_x, \phi_x', \cdots)\big|_{x=-\infty}^{x=\infty}\,. \tag{2·23}$$

With conditions (2·19) and (2·22) we can determine the functional form of $g(\phi - \phi')$ and obtain a canonical transformation which maps Hamiltonian to itself. Explicit applications of this method to nonlinear evolution equations are given in the next section.

§ 3. Examples

Here we shall use the method developed in the preceding section to derive canonical transformations for sine-Gordon equation, Korteweg-de Vries equation and modified Korteweg-de Vries equation.

3.1 *Sine-Gordon equation*

We consider the sine-Gordon equation in a characteristic frame:

$$\phi_{xt} = \sin\phi\,, \tag{3·1}$$

with the boundary condition $\phi\to 0$ (mod. 2π) as $|x|\to\infty$. The Lagrangian density for Eq. (3·1) is

$$\mathcal{L} = \phi_x\phi_t + 2(1-\cos\phi), \tag{3·2}$$

and then, from Eq. (2·8), the Hamiltonian for this system is

$$H[\phi] = -\int_{-\infty}^{\infty}(1-\cos\phi)\,dx\,. \tag{3·3}$$

The generating function for a canonical transformation may be obtained as follows. From Eq. (3·3) we have the difference between the Hamiltonian densities $\mathcal{H}'$ and $\mathcal{H}$:

$$\begin{aligned}\mathcal{H}'(\phi') - \mathcal{H}(\phi) &= \cos\phi' - \cos\phi \\ &= 2\sin\frac{1}{2}(\phi+\phi')\sin\frac{1}{2}(\phi-\phi').\end{aligned} \tag{3·4}$$

On the other hand, the right-hand side of the above equation should be

$$\begin{aligned}\mathcal{H}' - \mathcal{H} &= \frac{\partial}{\partial x}\mathcal{F}(\phi,\phi') = \mathcal{F}_\phi\phi_x + \mathcal{F}_{\phi'}\phi_x{}' \\ &= \frac{1}{2}[(\mathcal{F}_\phi - \mathcal{F}_{\phi'})(\phi_x - \phi_x{}') + (\mathcal{F}_\phi + \mathcal{F}_{\phi'})(\phi_x + \phi_x{}')],\end{aligned} \tag{3·5}$$

where $\mathcal{F}_\phi$ is $(\partial/\partial\phi)\mathcal{F}(\phi,\phi')$. Compatibility of Eqs. (2·22), (3·4) and (3·5) implies that we should take $\mathcal{F}(\phi,\phi') = \mathcal{F}(\phi+\phi')$. Then we arrive at the identity

$$\mathcal{H}' - \mathcal{H} = 2\sin\frac{1}{2}(\phi-\phi')\sin\frac{1}{2}(\phi+\phi') = g(\phi-\phi')\mathcal{F}_\phi(\phi+\phi'), \tag{3·6}$$

whence we obtain

$$g(\phi-\phi') = A\sin\frac{1}{2}(\phi-\phi'), \tag{3·7}$$

$$\mathcal{F}_\phi(\phi+\phi') = \frac{2}{A}\sin\frac{1}{2}(\phi+\phi'). \tag{3·8}$$

Here A is some constant. Consequently the generating functional is

$$W[\phi,\phi';t] = \int_{-\infty}^{\infty}\left[\phi'\phi_x - 2A\left\{\cos\frac{1}{2}(\phi-\phi') - 1\right\}\right]dx - Et \tag{3·9}$$

and the constant E is

$$E = -\frac{4}{A}\cos\frac{1}{2}(\phi+\phi')\Big|_{x=-\infty}^{x=\infty}. \tag{3·10}$$

The canonical transformation induced by the generating functional (3·10) is

$$\phi_x = -\frac{\delta W}{\delta\phi} = -\phi_x' + A\sin\frac{1}{2}(\phi-\phi'). \tag{3·11}$$

Of course Eq. (3·11) has already been found from Eq. (2·22) with Eq. (3·7).

In particular let us consider a transformation from a solution with the boundary conditions

$$\phi \to 0 \quad \text{as} \quad x \to -\infty,$$
$$\phi \to 2\pi \quad \text{as} \quad x \to +\infty \tag{3·12}$$

into a trivial solution $\phi' = 0$. Under the transformation a flow on dynamical surface with energy integral $E = -(4/A)$ is mapped into a stationary flow $E=0$. Comparison of this transformation with the Bäcklund transformation makes the solution structure clearer. We notice that the transformation (3·11) is a space part of Bäcklund transformation. Therefore we find that the original and the transformed system represent one-soliton and vacuum state respectively. In general we have the canonical transformation which transforms from the stationary flow into a dynamical flow (see § 4). This transformation procedure is equivalent to the procedure for generating N-soliton solution by the Bäcklund transformation.

3.2 *Korteweg-de Vries equation*

We consider the Korteweg-de Vries equation in the form

$$\phi_{xt} - 6\phi_x\phi_{xx} + \phi_{xxxx} = 0, \tag{3·13}$$

with the boundary condition $\phi_x \to 0$ as $|x| \to \infty$. The Lagrangian density for Eq. (3·13) is given by

$$\mathcal{L} = \phi_x\phi_t - \phi_{xx}^2 - 2\phi_x^3 \tag{3·14}$$

and then, from Eq. (2·8), the Hamiltonian for this system is

$$H[\phi] = \int_{-\infty}^{\infty}\left(\phi_x^3 + \frac{1}{2}\phi_{xx}^2\right)dx. \tag{3·15}$$

The generating functional is obtained in the following way. Require a condition that the difference between the Hamiltonian densities is a total differential with respect to x, i.e.,

$$\mathcal{H}'(\phi_x', \phi_{xx}') - \mathcal{H}(\phi_x, \phi_{xx}) = \left(\phi_x'^3 + \frac{1}{2}\phi_{xx}'^2\right) - \left(\phi_x^3 + \frac{1}{2}\phi_{xx}^2\right)$$
$$= \frac{\partial}{\partial x}\mathcal{F}(\phi, \phi', \phi_x, \phi_x'). \tag{3·16}$$

Using the assumed form of the canonical transformation (2·22), we have

$$\mathcal{H}'-\mathcal{H}=\frac{\partial}{\partial x}\left\{-\frac{1}{4}g'(\phi-\phi')(\phi_x-\phi_x')^2\right\}$$
$$+(\phi_x-\phi_x')\left\{\frac{1}{4}g^2(\phi-\phi')g''(\phi-\phi')+g^2(\phi-\phi')\right\}$$
$$-(\phi_x-\phi_x')\phi_x\phi_x'\{g''(\phi-\phi')-1\}, \tag{3·17}$$

where

$$g'(z)=\frac{\partial}{\partial z}g(z). \tag{3·18}$$

It is clear that the second term can be expressed in a total x-differential. Therefore if

$$g''(\phi-\phi')=1, \tag{3·19}$$

then the condition (3·16) is satisfied. Equation (3·19) yields

$$g(\phi-\phi')=\frac{1}{2}(\phi-\phi')^2+B(\phi-\phi')+A, \tag{3·20}$$

where A and B are integral constants. Without loss of generality, we may take $B=0$. Substituting Eq. (3·20) into (3·17), we obtain

$$\mathcal{H}'-\mathcal{H}=\frac{\partial}{\partial x}\left[-\frac{1}{4}(\phi-\phi')(\phi_x-\phi_x')^2-\frac{3}{80}(\phi-\phi')^5\right.$$
$$\left.-\frac{1}{4}A(\phi-\phi')^3-\frac{3}{4}A^2(\phi-\phi')\right]. \tag{3·21}$$

Thus we obtain the generating functional

$$W[\phi,\phi';t]=\int_{-\infty}^{\infty}\left\{\phi_x\phi'+\frac{1}{6}(\phi-\phi')^3+A(\phi-\phi')+C\right\}dx-Et, \tag{3·22}$$

where C is an arbitrary constant, and a constant E is

$$E=\left\{\frac{1}{4}(\phi-\phi')(\phi_x-\phi_x')^2+\frac{3}{80}(\phi-\phi')^5+\frac{1}{4}A(\phi-\phi')^3\right.$$
$$\left.+\frac{3}{4}A^2(\phi-\phi')\right\}\Bigg|_{x=-\infty}^{x=\infty}. \tag{3·23}$$

The transformation equation is given by

$$\phi_x=\frac{\delta W}{\delta\phi}=-\phi_x'+\frac{1}{2}(\phi-\phi')^2+A. \tag{3·24}$$

3.3 *Modified Korteweg-de Vries equation*

We consider the modified Korteweg-de Vries equation in the form

$$\phi_{xt}+6\phi_x^2\phi_{xx}+\phi_{xxxx}=0, \tag{3·25}$$

with the boundary condition $\phi_x \to 0$ as $|x| \to \infty$. The Lagrangian density for Eq. (3·25) is given by

$$\mathcal{L} = \phi_x \phi_t - \phi_{xx}^2 + \phi_x{}^4 , \tag{3·26}$$

and then, from Eq. (2·8), the Hamiltonian for this system is

$$H[\phi] = \frac{1}{2} \int_{-\infty}^{\infty} (\phi_{xx}^2 - \phi_x{}^4)\, dx . \tag{3·27}$$

A similar calculation in the case of the Korteweg-de Vries equation leads to

$$\begin{aligned} \mathcal{H}' - \mathcal{H} &= \frac{1}{2}(\phi_{xx}'^2 - \phi_x'^4) - \frac{1}{2}(\phi_{xx}^2 - \phi_x{}^4) \\ &= \frac{\partial}{\partial x}\left\{ -\frac{1}{4} g'(\phi - \phi')(\phi_x - \phi_x')^2 \right\} \\ &\quad + \frac{1}{4}(\phi_x - \phi_x')\{ g^2(\phi-\phi') g''(\phi - \phi') + 2g^3(\phi - \phi')\} \\ &\quad - (\phi_x - \phi_x')\phi_x \phi_x' \{ g''(\phi - \phi') + g(\phi - \phi')\} , \end{aligned} \tag{3·28}$$

where remember that $\phi_x + \phi_x' = g(\phi - \phi')$. The second term can be expressed in a total x-derivative form. Therefore if

$$g''(\phi - \phi') + g(\phi - \phi') = 0 , \tag{3·29}$$

the difference between $\mathcal{H}'$ and $\mathcal{H}$ is written in the form of a total x-derivative, i.e.,

$$\mathcal{H}'(\phi_x', \phi_{xx}') - \mathcal{H}(\phi_x, \phi_{xx}) = \frac{\partial}{\partial x}\mathcal{F}(\phi, \phi', \phi_x, \phi_x') . \tag{3·30}$$

From the condition (3·29), we obtain

$$g(\phi - \phi') = A \sin(\phi - \phi') , \tag{3·31}$$

where A is an arbitrary constant. Substituting Eq. (3·31) into Eq. (3·28), we have

$$\begin{aligned} \mathcal{H}' - \mathcal{H} = \frac{\partial}{\partial x}\Big\{ & -\frac{1}{4} A (\phi_x - \phi_x')^2 \cos(\phi - \phi') \\ & - \frac{1}{4} A^3 \cos(\phi - \phi') + \frac{1}{12} A^3 \cos^3(\phi - \phi') \Big\} . \end{aligned} \tag{3·32}$$

Thus we obtain the generating functional

$$W[\phi, \phi'; t] = \int_{-\infty}^{\infty} [\phi_x \phi' - A\{\cos(\phi - \phi') - 1\}]\, dx - Et , \tag{3·33}$$

where a constant E is

$$E=\left[\frac{1}{4}A(\phi_x-\phi_x')^2\cos(\phi-\phi')+\frac{1}{4}A^3\{\cos(\phi-\phi')-1\}\right.$$
$$\left.+\frac{1}{12}A^3\{\cos^3(\phi-\phi')-1\}\right]\Bigg|_{x=-\infty}^{x=\infty}. \tag{3·34}$$

The transformation equation is given by

$$\phi_x=\frac{\delta W}{\delta\phi}=-\phi_x'+A\sin(\phi-\phi'). \tag{3·35}$$

To close this section, we remark that the canonical transformation for the form $\phi_{xt}+\phi_x^n\phi_{xx}+\phi_{xxxx}=0$ with $n>2$ does *not* exist in our scheme.

§ 4. Canonical transformation, Bäcklund transformation and canonical transformation groups

In the preceding section we have obtained explicitly canonical transformations which keep the Hamiltonian form invariant for sine-Gordon equation, Korteweg-de Vries equation and modified Korteweg-de Vries equation. We observe that the canonical transformations found there are the space parts of the Bäcklund transformations. The time part of the Bäcklund transformation can be derived using Hamilton's equation and the canonical transformation. Let us observe it taking sine-Gordon equation as an example. The canonical transformation (3·11) is

$$\phi_x+\phi_x'=A\sin\frac{1}{2}(\phi-\phi'). \tag{4·1}$$

Differentiating both sides of this equation with respect to time t and taking Hamilton's equation, $\phi_{xt}=\sin\phi$, into account, we have

$$\frac{A}{2}(\phi_t-\phi_t')\cos\frac{1}{2}(\phi-\phi')=\sin\phi+\sin\phi'$$
$$=2\sin\frac{1}{2}(\phi+\phi')\cos\frac{1}{2}(\phi-\phi'). \tag{4·2}$$

Therefore we obtain

$$\phi_t-\phi_t'=\frac{4}{A}\sin\frac{1}{2}(\phi+\phi'). \tag{4·3}$$

A pair of Eqs. (4·1) and (4·3) are the well-known Bäcklund transformation for the sine-Gordon equation. Similarly we obtain Bäcklund transformations for Korteweg-de Vries equation and modified Korteweg-de Vries equation from the canonical transformations and Hamilton's equations. Then, we conclude that the Bäcklund transformation is a canonical transformation which keeps Hamiltonian form invariant.

This result implies that the procedure of constructing N-soliton solution by the Bäcklund transformation is interpreted as a canonical transformation between the dynamical states, stationary state (vacuum state) and N-soliton state.

In the following we consider the canonical transformation from the viewpoint of group theory. We shall show that the canonical transformation constitutes a group (canonical transformation group) and the group is an Abelian group. For further discussion, it is convenient to rewrite the canonical transformations considered in the previous section as the forms connected with the identity transformation. Again we consider the sine-Gordon equation only. A similar argument is possible for Korteweg-de Vries equation and modified Korteweg-de Vries equation. The canonical transformations for these equations connected with the identity transformations are presented in Appendix C.

Since sine-Gordon equation is invariant under the transformation $\phi\to-\phi$, we may rewrite the generating functional (3·9) as the following form:

$$W_a[\phi,\phi';t]=\int_{-\infty}^{\infty}\left[\phi\phi_x'-2a\left\{\cos\frac{1}{2}(\phi+\phi')-1\right\}\right]dx-E_a t\,, \tag{4·4}$$

where a is a continuous transformation parameter. Then the canonical transformation, $\mathrm{T}_a;\phi_x\to\phi_x'$, is given by

$$\phi_x=\frac{\delta W_a}{\delta\phi}=\phi_x'+a\sin\frac{1}{2}(\phi+\phi'). \tag{4·5}$$

⟨Theorem⟩ The transformation (4·5) constitutes a group, so that the following properties hold:

i) *Closure*: If the transformations, $\mathrm{T}_a;\phi_x\to\phi_x'$ and $\mathrm{T}_b;\phi_x'\to\phi_x''$, are canonical. Their successive (*product*) transformation, $\mathrm{T}=\mathrm{T}_b\mathrm{T}_a;\phi_x\to\phi_x''$, is also canonical.

ii) *Associative law*: Three canonical transformations, T_a, T_b and T_c, satisfy the associative law:

$$\mathrm{T}_c(\mathrm{T}_b\mathrm{T}_a)=(\mathrm{T}_c\mathrm{T}_b)\mathrm{T}_a\,. \tag{4·6}$$

iii) *Identity*: There is a unique identity element corresponding to the identity transformation, T_0; $\phi_x=\phi_x'$.

iv) *Unique inverse*: To every element $\mathrm{T}_a;\phi_x\to\phi_x'$, there is a unique element $\mathrm{T}_a^{-1};\phi_x'\to\phi_x$ corresponding to the inverse transformation of T_a.

Proof We shall prove that the transformation (4·5) satisfies the properties i)~iv). First we confirm closure property. Since T_a is a canonical transformation, there is a generating functional $W_a[\phi,\phi';t]$:

$$\begin{aligned}dW_a&=\int_{-\infty}^{\infty}dx\left(\frac{\delta W_a}{\delta\phi}d\phi+\frac{\delta W_a}{\delta\phi'}d\phi'\right)+\frac{\partial W_a}{\partial t}dt\\&=\int_{-\infty}^{\infty}dx(\phi_x d\phi-\phi_x'd\phi')-(H-H')dt\,.\end{aligned}$$

Similarly, for T_b,

$$dW_b = \int_{-\infty}^{\infty} dx(\phi_x' d\phi' - \phi_x'' d\phi'') - (H' - H'')dt.$$

Therefore we have

$$dW = \int_{-\infty}^{\infty} dx(\phi_x d\phi - \phi_x'' d\phi'') - (H - H'')dt$$
$$= d(W_a + W_b), \tag{4·7}$$

which indicates that the product transformation, $T = T_b T_a$, is also canonical. The generating functional for the product transformation, $T = T_b T_a$, is

$$W[\phi, \phi''; t] = W_a[\phi, \phi'; t] + W_b[\phi', \phi''; t]. \tag{4·8}$$

By using the transformations

$$\phi_x = \phi_x' + a \sin\frac{1}{2}(\phi + \phi'), \tag{4·9a}$$

$$\phi_x' = \phi_x'' + b \sin\frac{1}{2}(\phi' + \phi''), \tag{4·9b}$$

ϕ' (and ϕ_x') on the right-hand side of Eq. (4·8) can be eliminated. Note that $W[\phi, \phi''; t]$ depends on *two* parameters a and b symmetrically.

Property ii) is obvious since the generating functionals for both sides of Eq. (4·6) are the same; $W_a + W_b + W_c$. The identity transformation is induced from the generating functional

$$W_0[\phi, \phi'; t] = \int_{-\infty}^{\infty} \phi\phi_x' dx, \tag{4·10}$$

which is obtained from the functional (4·4) by setting $a = 0$. The inverse transformation of T_a, T_a^{-1}; $\phi_x' \to \phi_x$, is given by

$$dW_a^{-1} = \int_{-\infty}^{\infty} dx(\phi_x' d\phi' - \phi_x d\phi) - (H' - H)dt = -dW_a. \tag{4·11}$$

Therefore, the generating functional for the inverse transformation is $W_a^{-1} = -W_a$ (that is, $T_a^{-1} = T_{-a}$).

Then, the properties i)~iv) are confirmed. Moreover, it is clear that $T_a T_b = T_b T_a$. Therefore the group is Abelian. Above discussion leads to a conclusion: The Bäcklund transformation considered as a canonical transformation makes an Abelian group.

§ 5. Infinitesimal canonical transformation

We define the infinitesimal canonical transformation using the canonical transformation connected with the identity transformation. Let ε be an infinitesimal parameter:

$$\phi_x - \phi_x' \sim O(\varepsilon). \tag{5·1}$$

The generating functional for infinitesimal canonical transformation is given by

$$W_\varepsilon[\phi, \phi'; t] = \int_{-\infty}^{\infty} \phi\phi_x' dx + \varepsilon J[\phi, \phi'; t]. \tag{5·2}$$

In fact, generator (5·2) yields

$$\phi_x = \frac{\delta W_\varepsilon}{\delta\phi} = \phi_x' + \varepsilon\frac{\delta J}{\delta\phi}, \tag{5·3a}$$

$$\phi_x' = -\frac{\delta W_\varepsilon}{\delta\phi'} = \phi_x - \varepsilon\frac{\delta J}{\delta\phi'}. \tag{5·3b}$$

The compatibility condition requires

$$J[\phi, \phi'; t] = J[\phi + \phi'; t]. \tag{5·4}$$

The functional $J[\phi, \phi'; t]$ will be referred to as a generator of the infinitesimal canonical transformation.

In particular, if ε is taken to be the displacement da of the continuous parameter a, Eq. (5·3) becomes

$$\frac{d}{da}\phi_x = -\frac{\delta J}{\delta\phi}. \tag{5·5}$$

If we take $J=H$ and $a=t$, Eq. (5·5) is Hamilton's equation. This indicates the system is invariant under time translation. For the nonlinear evolution equations presented in § 3, it is easily shown that there is a conserved quantity

$$P = \frac{1}{2}\int_{-\infty}^{\infty} \phi_x^2 dx. \tag{5·6}$$

If we take $J=P$ and $a=x$, then Eq. (5·5) indicates the invariance under space translation. From this, P is interpreted as the momentum of the system.

Nonlinear evolution equations considered in § 3 are known to possess an infinite number of conservation laws. We shall present a systematic way to derive these conservation laws from the canonical transformation. First, assuming that ϕ_x' is analytic near $a=0$, we expand it in a Taylor series in a:

$$\phi_x' - \phi_x = \sum_{n=1}^{\infty} a^n f_n, \quad f_n \equiv \frac{1}{n!}\frac{\partial^n \phi_x'}{\partial a^n}\bigg|_{a=0}. \tag{5·7}$$

The function f_n is given by the expansion of the generating functional

$$W_a[\phi, \phi'; t] = \int_{-\infty}^{\infty} \phi\phi_x' dx + G[\phi, \phi'; t]. \tag{5·8}$$

Since

$$\phi_x' - \phi_x = -\frac{\delta G}{\delta\phi} = \sum_{n=1}^{\infty} a^n \frac{\delta G_n}{\delta\phi}, \quad \frac{\delta G_n}{\delta\phi} \equiv -\frac{1}{n!}\frac{\partial^n}{\partial a^n}\frac{\delta G}{\delta\phi}\bigg|_{\phi_x'=\phi_x}, \tag{5·9}$$

we obtain

$$f_n = -\frac{1}{n!}\frac{\partial^n}{\partial a^n}\frac{\delta G}{\delta \phi}\bigg|_{\phi_x'=\phi_x}. \qquad (5\cdot10)$$

Next, we put the equation of ϕ' into the form of a conservation law:

$$\frac{\partial}{\partial t}\mathscr{D}(\phi') + \frac{\partial}{\partial x}\mathscr{F}(\phi') = 0. \qquad (5\cdot11)$$

With Eq. (5·7), we have

$$\mathscr{D}(\phi') = \sum_{n=0}^{\infty} a^n \mathscr{D}_n(\phi), \quad \mathscr{F}(\phi') = \sum_{n=0}^{\infty} a^n \mathscr{F}_n(\phi). \qquad (5\cdot12)$$

Substituting Eq. (5·11) into Eq. (5·12) and equating the coefficients of the same powers of a, we obtain an infinite number of conservation laws:

$$\frac{\partial}{\partial t}\mathscr{D}_n(\phi) + \frac{\partial}{\partial x}\mathscr{F}_n(\phi) = 0. \qquad (5\cdot13)$$

Here $\mathscr{D}_n(\phi)$ and $\mathscr{F}_n(\phi)$ are conserved density and conserved flux, respectively. This derivation is based on the fact that our canonical transformation is the invariant transformation. Recently, invariant transformation for nonlinear evolution equations has been discussed by Kumei[8] in some detail. We can choose any conservation law for Eq. (5·11). Corresponding to the choices, we obtain conserved quantities in different forms.

§ 6. Summary and discussion

We briefly summarize the results obtained in the present study. In § 2, we have introduced a canonical transformation whereby the transformed Hamiltonian keeps the same form as the original Hamiltonian and proposed a method to derive generating functional for the transformation. The method has been applied to nonlinear evolution equations in § 3. The canonical transformation thus obtained is found to be the Bäcklund transformation in § 4. This result not only brings a new (and simple) method to derive Bäcklund transformations for nonlinear evolution equations but also clarifies the dynamical signicance of the Bäcklund transformation. Furthermore, we have proved that the canonical transformation constitutes an invariant Abelian group in § 4. In § 5, we have defined the infinitesimal canonical transformation, which was used to provide a systematic method for derivation of an infinite number of conservation laws.

To conclude this paper, we discuss a future perspective of our formalism. We have considered only sine-Gordon equation, Korteweg-de Vries equation and modified Korteweg-de Vries equation as examples. However we expect our results to remain valid for any completely integrable Hamiltonian system. Much work on the theory of soliton and the present work have clarified the mutual relationships among complete integrability, applicability of inverse scattering method, the Bäcklund transformation, the existence of an infinite number of conservation laws and the canonical transformation. But there is no unified theory. We hope that we shall be able to present a unified theory of complete integrable system starting with the canonical transformation (or infinitesimal canonical transformation) in future. Another future problem is whether we can quantize our canonical transformation. If it is possible, there may be a relation between the annihilation-creation operator for soliton[9] and the canonical transformation of the nonlinear field.

Appendix A

Here we show the consistency of the dynamics considered in § 2.

First we see that Hamilton's equation is obtained by the canonical simplectic form or the exterior derivative of the Pfaffian form defined in § 2. Let us evaluate the exterior derivative of the Pfaffian form

$$\theta=\int_{-\infty}^{\infty}dx(\phi_x d\phi-\mathcal{H}dt). \tag{A·1}$$

The exterior differential form $d\theta$ is given by

$$\omega\equiv d\theta=\int_{-\infty}^{\infty}dx\left\{d\phi_x\wedge d\phi-\sum_{i=0}(-1)^i\frac{\partial^i}{\partial x^i}\frac{\partial\mathcal{H}}{\partial\phi_i}d\phi\wedge dt\right\}$$

or

$$\omega=\int_{-\infty}^{\infty}dx\left\{d\phi_x\wedge d\phi-\sum_{i=1}(-1)^{i-1}\frac{\partial^{i-1}}{\partial x^{i-1}}\frac{\partial\mathcal{H}}{\partial\phi_i}d\phi_x\wedge dt\right\},$$

where $\phi_i\equiv(\partial/\partial x)^i\phi$, and the symbol $\wedge$ denotes an exterior product.

If the differential two-form ω is an integral invariant (that is, θ is a relative integral invariant), then we obtain the Hamiltonian flow

$$\phi_{xt}=-\sum_{i=0}(-1)^i\frac{\partial^i}{\partial x^i}\frac{\partial\mathcal{H}}{\partial\phi_i}=-\frac{\delta H[\phi]}{\delta\phi} \tag{A·2}$$

or

$$\phi_t=\sum_{i=1}(-1)^{i-1}\frac{\partial^{i-1}}{\partial x^{i-1}}\frac{\partial\mathcal{H}}{\partial\phi_i}=\frac{\delta H[\phi_x]}{\delta\phi_x}. \tag{A·3}$$

Thus the Hamiltonian flow is uniquely determined by the Pfaffian form (A·1). Note that the exterior derivative of (A·1) is equivalent to the invariant variational problem (Hamilton's principle).

The conservation of the Hamiltonian in the autonomous system is easily proved by using Eqs. (B·2) and (B·3), i.e.,

$$\frac{dH}{dt}=\int_{-\infty}^{\infty}dx\sum_i\frac{\partial\mathcal{H}}{\partial\phi_i}\phi_{i,t}=\frac{1}{2}\int_{-\infty}^{\infty}dx\left\{\frac{\delta H}{\delta\phi}\phi_t+\frac{\delta H}{\delta\phi_x}\phi_{xt}\right\}$$

$$=\frac{1}{2}\int_{-\infty}^{\infty}dx\left\{\frac{\delta H}{\delta\phi}\frac{\delta H}{\delta\phi_x}-\frac{\delta H}{\delta\phi_x}\frac{\delta H}{\delta\phi}\right\}=0\,. \tag{A·4}$$

The Poisson bracket is defined by

$$[F,G]=-\frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{x}\left\{\frac{\delta F}{\delta\phi(x)}\cdot\frac{\delta G}{\delta\phi(y)}-\frac{\delta G}{\delta\phi(x)}\frac{\delta F}{\delta\phi(y)}\right\}dy\,dx \tag{A·5}$$

or

$$[F,G]=\frac{1}{2}\int_{-\infty}^{\infty}\left\{\frac{\delta F}{\delta\pi(x)}\cdot\frac{\partial}{\partial x}\frac{\delta G}{\delta\pi(x)}-\frac{\delta G}{\delta\pi(x)}\cdot\frac{\partial}{\partial x}\frac{\delta F}{\delta\pi(x)}\right\}dx\,, \tag{A·6}$$

where F and G are arbitrary functionals with ϕ or $\pi=\phi_x$. In particular, we have

$$[\pi(x),\phi(x')]=-\delta(x-x'),$$

where $\delta(z)$ is the Dirac delta function. Thus, the pair of $\pi(x)$ and $\phi(x)$ are formally interpreted as a canonical set.

Appendix B

The canonical transformation for the system

$$\phi_{xt}=\frac{\partial}{\partial x}\frac{\delta H}{\delta\phi_x} \tag{B·1}$$

is defined as follows. The Legendre transformation (integration by parts) of the Pfaffian form (2·12) yields

$$\int_{-\infty}^{\infty}dx(\phi d\phi_x+\mathcal{H}dt)=\int_{-\infty}^{\infty}dx(\phi' d\phi_x'+\mathcal{H}'dt)-d\widetilde{W}[\phi_x,\phi_x';t]. \tag{B·2}$$

Therefore the canonical transformation is given by the formulas

$$\phi=-\frac{\delta\widetilde{W}}{\delta\phi_x},\quad \phi'=\frac{\delta\widetilde{W}}{\delta\phi_x'}\,. \tag{B·3}$$

Appendix C

We list the examples of the generating functionals of the forms connected with the identity transformation.

i) Sine-Gordon equation:

$$W_a[\phi,\phi';t]=\int_{-\infty}^{\infty}\left[\phi\phi_x'-2a\left\{\cos\frac{1}{2}(\phi+\phi')-1\right\}\right]dx-E_at\,. \tag{C·1}$$

ii) Korteweg-de Vries equation:

$$\widetilde{W}_a[\phi_x,\phi_x';t]=\int_{-\infty}^{\infty}\left[\phi\phi_x'-\frac{1}{3}\left\{2(\phi_x+\phi_x')+\frac{4}{a^2}\right\}^{3/2}\right]dx-E_at\,. \tag{C·2}$$

In this case we must consider the canonical transformation given in Appendix B.

iii) Modified Korteweg-de Vries equation:

$$W_a[\phi, \phi'; t] = \int_{-\infty}^{\infty} [\phi\phi_x' - 2a\{\cos(\phi+\phi') - 1\}] dx - E_a t\,. \qquad \text{(C·3)}$$

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