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Theory of Electron-Phonon Interaction Responsible for Current Saturation Phenomena in Semiconductors

Jiro YAMASHITA and Ki-ichi NAKAMURA*

Institute for Solid State Physics, University of Tokyo
Azabu, Tokyo
**Central Research Laboratories*
Nippon Electric Co., Ltd., Kawasaki

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A theory applicable over the whole range of ql is presented for electron-phonon interaction responsible for the current saturation phenomena in semiconductors. Transport equations are set up by the nonequilibrium Green's function technique. The equations show clearly that phonons play two different physical roles in the description of transport phenomena; one is to act on electrons as waves and the other is to interact with them as quasiparticles. As for amplification of phonons as quasiparticles, calculation shows that the radical reduction of amplification factor is not expected even when $ql<1$. In the case of high-mobility semiconductors, the quasiparticle aspect is applicable in the initial stage of amplification, but the wave aspect is expected to appear in the final stage. The amplification of waves is discussed in the nonlinear regime. To the second order approximation, it is shown that the amplification factor is decreased with increase of the wave amplitude. The domain problem is studied by using an electronic computer. It is certain that the domain formation is irrelevant to the Ridley mechanism.

§ 1. Introduction

It is well established experimentally that the current saturation phenomenon in piezoelectric semiconductors arises from the electron-phonon interaction as a result of amplification of acoustic phonons through the piezoelectric coupling. Theoretical explanations have been attempted along two distinct lines originally accepted in the theory of ultrasonic amplification. The first[1] is the classical approach based on the macroscopic equations. The rate of amplification of sound waves is evaluated in the linear approximation. The current saturation phenomenon, however, lies outside the scope of the linear theory. The second[2] is the quantum approach, in which the conventional perturbation theory is applied to the electron-phonon interaction. The Boltzmann equations are set up both for the electron system and the phonon system. They have been solved to give phonon amplification in the narrow Čerenkov cone, and subsequent decrease of the electric current, although nonlinear effects in electron-phonon interaction and in phonon-phonon interaction are not systematically taken into account.

As is well known, the classical approach is applicable when the wave num-

ber, q, of the amplified phonons satisfies the condition $ql<1$, where l is the mean free path of electrons. On the other hand, the quantum approach is valid when $ql>1$. Recently, Spector[3] presented a semiclassical treatment of the ultrasonic amplification applicable to all values of ql in the linear regime. It is a characteristic feature of his theory that phonons are not explicitly considered at all even when $ql\gg1$.

On the other hand, we have presented a unified approach of the current saturation phenomena from the quantum theoretical standpoint.[4] Hereafter, it will be referred to as I. We started with the Fröhlich Hamiltonian of the electron-phonon system in a piezoelectric semiconductor. The transport equations for electrons and phonons are derived from a hierarchy of equations for the Wigner distribution functions and their correlation functions. By dividing the phonon spectrum rather artificially into two parts at the critical phonon wave number q_c, $(q_cl=1)$, we find phonons playing two different physical roles. One is to act on electrons as waves and the other is to interact with them as quasiparticles. The first role leads to the macroscopic results $(ql<1)$; for example, the well-known Hutson-White relation.[5] The second role leads to the phonon emission in the Čerenkov cone $(ql>1)$.

The purpose of the present paper is to present a theory to unify the approaches mentioned before from the quantum theoretical standpoint. In § 2, transport equations for the electron-phonon system are set up by the nonequilibrium Green's function technique introduced by Kadanoff and Baym.[6] The results are essentially the same as those obtained in I, but there is no need to introduce the artificial cutoff in phonon spectrum, as was done in I. Thus, we see that the electron-phonon interaction is systematically tractable in all ranges of ql without any intuitive or artificial models. On the other hand, the expression for the collision terms due to the electron-phonon interaction becomes much more complicated than in I, because the damping effects are automatically introduced in the Green's function formalism.

Section 3 is concerned with the amplification of phonons as quasiparticles. If the value of $\omega_q\tau$ is much larger than one, and the damping effect is small, then the simple perturbation theory is well applicable to the electron-phonon interaction. Here, τ is the relaxation time of electrons and ω_q is the frequency of phonons. On the other hand, when $ql\ll1$, we find that the amplification factor is smaller by (ql) than the one predicted by the simple perturbation theory. Therefore, the radical reduction of amplification factor is not expected even when $ql<1$, although it is smaller by $(ql)^2$ than the amplification factor of waves. When $q_Dl\gg1$, where q_D is the Debye wave number, the phonon amplification may be described in terms of the quasiparticle aspect in the initial stage of amplification. When phonons have been heavily amplified in the narrow Čerenkov cone by the electron-phonon interaction, the damping effect becomes applicable for electrons in the special range in the momentum space, so that such a damping

effect causes supression of the amplification factor of phonons and reduction of the mean free path of electrons. This is the typical nonlinear problem in the electron-phonon interaction. Therefore, the situation is modified in the final stage of amplification, and the wave aspect is expected to appear there.

In § 4, we treat the amplification of waves in the nonlinear regime. To the second order approximation, we shall show that the initial rate of amplification, which is predicted by the Hutson-White relation, is decreased with increase of the wave amplitude. In many cases, a propagating acoustic domain is observed, accompanied by the current saturation phenomena.[7] At the present time, there is no any nonlinear theory applicable to the domain problem. We shall study this problem by using a computer. An important qualitative conclusion is that the domain formation is a consequence of the transient generation of the acoustic flux packet at the boundary and of the subsequent amplification of it, and is irrelevant to the Ridley mechanism.

§ 2. Transport equations

We start with the Fröhlich Hamiltonian:

$$H=\sum_{\boldsymbol{K}} \varepsilon_{\boldsymbol{K}} a_{\boldsymbol{K}}{}^{+} a_{\boldsymbol{K}}+\sum_{\boldsymbol{q}} \hbar\omega_{\boldsymbol{q}} b_{\boldsymbol{q}}{}^{+} b_{\boldsymbol{q}}+\sum_{\boldsymbol{K}\boldsymbol{q}} C_{\boldsymbol{q}}(a_{\boldsymbol{K}+\boldsymbol{q}}^{+} a_{\boldsymbol{K}} b_{\boldsymbol{q}}+a_{\boldsymbol{K}}{}^{+} a_{\boldsymbol{K}+\boldsymbol{q}} b_{\boldsymbol{q}}{}^{+}), \tag{2·1}$$

where the notation has the usual meaning. If we introduce the displacement operator $\varphi_{\boldsymbol{q}}$ and the conjugate operator $\pi_{\boldsymbol{q}}$ by

$$\varphi_{\boldsymbol{q}}=\left(\frac{\hbar}{2\omega_{\boldsymbol{q}}}\right)^{1/2}(b_{\boldsymbol{q}}+b_{-\boldsymbol{q}}^{+}), \tag{2·2a}$$

$$\pi_{\boldsymbol{q}}=-i\left(\frac{\hbar\omega_{\boldsymbol{q}}}{2}\right)^{1/2}(b_{\boldsymbol{q}}-b_{-\boldsymbol{q}}^{+}), \tag{2·2b}$$

we can write Eq. (2·1) as

$$H=\sum_{\boldsymbol{K}} \varepsilon_{\boldsymbol{K}} a_{\boldsymbol{K}}{}^{+} a_{\boldsymbol{K}}+\tfrac{1}{2}\sum_{\boldsymbol{q}}(\pi_{\boldsymbol{q}}{}^{+}\pi_{\boldsymbol{q}}+\omega_{\boldsymbol{q}}{}^{2}\varphi_{\boldsymbol{q}}{}^{+}\varphi_{\boldsymbol{q}})+\sum_{\boldsymbol{K},\boldsymbol{q}} \nu_{\boldsymbol{q}} a_{\boldsymbol{K}+\boldsymbol{q}}^{+} a_{\boldsymbol{K}}\varphi_{\boldsymbol{q}}, \tag{2·3}$$

where

$$\nu_{\boldsymbol{q}}=\left(\frac{2\omega_{\boldsymbol{q}}}{\hbar}\right)^{1/2} C_{\boldsymbol{q}}\,. \tag{2·4}$$

To set up the transport equations, we introduce the electron Green's function defined in the imaginary time interval $(t_0, t_0-i\beta)$, where t_0 is real and $\beta=1/k_B T$, by the equation $(\hbar=1)$

$$G(\boldsymbol{K}t;\,\boldsymbol{K}'t')=-i\frac{\langle T[Sa_{\boldsymbol{K}}(t)\,a_{\boldsymbol{K}'}^{+}(t')]\rangle}{\langle T[S]\rangle}\,. \tag{2·5}$$

The notation $\langle\cdots\rangle$ is defined by

$$\langle\cdots\rangle=\mathrm{Tr}\{e^{-\beta(H-\mu N)}\cdots\}/\mathrm{Tr}\{e^{-\beta(H-\mu N)}\}$$

and any operator is written in the interaction representation: for instance $a_{\boldsymbol{K}}(t) = e^{iHt}a_{\boldsymbol{K}}e^{-iHt}$, where μ is the chemical potential and N the operator for the total number of electrons. Also, S is the time ordered operator

$$S=\exp\left\{-i\int_{t_0}^{t_0-i\beta}dtH'(t)\right\}, \tag{2·6}$$

and

$$H'(t)=\sum_{\boldsymbol{k}}v_{\boldsymbol{k}}(t)a^{+}_{\boldsymbol{K}+\boldsymbol{k}}(t)a_{\boldsymbol{K}}(t)+\sum_{\boldsymbol{q}}J_{-\boldsymbol{q}}(t)\varphi_{\boldsymbol{q}}(t), \tag{2·7}$$

where $v_{\boldsymbol{k}}$ is the potential of an external field and $J_{\boldsymbol{q}}$ is an external source of phonons which we will use in generating the Green's function equation and then set equal to zero. The phonon Green's function is defined by

$$\begin{aligned}D(\boldsymbol{q}t;\,\boldsymbol{q}'t') &= \delta\tilde{\varphi}_{\boldsymbol{q}}(t)/\delta J_{\boldsymbol{q}'}(t')\\ &= -i\frac{\langle T[S\varphi_{\boldsymbol{q}}(t)\varphi^{+}_{\boldsymbol{q}'}(t')]\rangle}{\langle T[S]\rangle}+i\tilde{\varphi}_{\boldsymbol{q}}(t)\tilde{\varphi}^{+}_{\boldsymbol{q}'}(t'),\end{aligned} \tag{2·8}$$

where

$$\tilde{\varphi}_{\boldsymbol{q}}(t)=\frac{\langle T[S\varphi_{\boldsymbol{q}}(t)]\rangle}{\langle T[S]\rangle}. \tag{2·9}$$

(A) *Equation of motion for* $G(\boldsymbol{K}t;\,\boldsymbol{K}'t')$

The equation of motion of the electron Green's function is written as

$$\begin{aligned}&\left(i\frac{\partial}{\partial t}-\varepsilon_{\boldsymbol{K}}\right)G(\boldsymbol{K}t;\,\boldsymbol{K}'t')-\sum_{\boldsymbol{k}}v_{\boldsymbol{k}}(t)G(\boldsymbol{K}-\boldsymbol{k},\,t;\,\boldsymbol{K}'t')\\ &\quad-\sum_{\boldsymbol{q}}\nu_{\boldsymbol{q}}R(\boldsymbol{K}-\boldsymbol{q},\,t;\,\boldsymbol{K}'\boldsymbol{t}';\,\boldsymbol{q}t)=\delta_{\boldsymbol{K}\boldsymbol{K}'}\delta(t-t'),\end{aligned} \tag{2·10}$$

where $R(\boldsymbol{K}t;\,\boldsymbol{K}'t';\,\boldsymbol{K}''t'')$ is a mixed electron-phonon Green's function of the form

$$R(\boldsymbol{K}t;\,\boldsymbol{K}'t';\,\boldsymbol{K}''t'')=-i\frac{\langle T[S\varphi_{\boldsymbol{K}''}(t'')a_{\boldsymbol{K}}(t)a^{+}_{\boldsymbol{K}'}(t')]\rangle}{\langle T[S]\rangle}. \tag{2·11}$$

We use the following identity[8] to rewrite Eq. (2·10):

$$R(\boldsymbol{K}t;\,\boldsymbol{K}'t';\,\boldsymbol{K}''t'')=\tilde{\varphi}_{\boldsymbol{K}''}(t'')G(\boldsymbol{K}t;\,\boldsymbol{K}'t')+i\frac{\delta G(\boldsymbol{K}t;\,\boldsymbol{K}'t')}{\delta J_{-\boldsymbol{K}''}(t'')}. \tag{2·12}$$

Substitution of Eq. (2·12) into Eq. (2·10) yields

$$\begin{aligned}&\left(i\frac{\partial}{\partial t}-\varepsilon_{\boldsymbol{K}}\right)G(\boldsymbol{K}t;\,\boldsymbol{K}'t')-\sum_{\boldsymbol{K}''}U_{\boldsymbol{K}-\boldsymbol{K}''}(t)G(\boldsymbol{K}''t;\,\boldsymbol{K}'t')\\ &\quad-i\sum_{\boldsymbol{K}''}\nu_{\boldsymbol{K}-\boldsymbol{K}''}\frac{\delta G(\boldsymbol{K}''t;\,\boldsymbol{K}'t')}{\delta J_{\boldsymbol{K}''-\boldsymbol{K}}(t)}=\delta_{\boldsymbol{K}\boldsymbol{K}'}\delta(t-t'),\end{aligned} \tag{2·13}$$

where

$$U_{\boldsymbol{k}}(t)=v_{\boldsymbol{k}}(t)+\nu_{\boldsymbol{k}}\tilde{\varphi}_{\boldsymbol{k}}(t). \tag{2·14}$$

We apply the method of functional derivative to the last term on the left side of Eq. (2·13):

$$\frac{\delta G(\boldsymbol{K}''t;\ \boldsymbol{K}'t')}{\delta J_{\boldsymbol{K}''-\boldsymbol{K}}(t)}=-\sum_{\boldsymbol{K}_1\boldsymbol{K}_2}\int_{t_0}^{t_0-i\beta}dt_1dt_2G(\boldsymbol{K}''t;\ \boldsymbol{K}_1t_1)\frac{\delta G^{-1}(\boldsymbol{K}_1t_1;\ \boldsymbol{K}_2t_2)}{\delta J_{\boldsymbol{K}''-\boldsymbol{K}}(t)}. \tag{2·15}$$

Substituting Eq. (2·15) into Eq. (2·13), we obtain

$$\left(i\frac{\partial}{\partial t}-\varepsilon_{\boldsymbol{K}}\right)G(\boldsymbol{K}t;\ \boldsymbol{K}'t')-\sum_{\boldsymbol{K}''}U_{\boldsymbol{K}-\boldsymbol{K}''}(t)G(\boldsymbol{K}''t;\ \boldsymbol{K}'t')$$

$$-\sum_{\boldsymbol{K}_2}\int_{t_0}^{t_0-i\beta}dt_2\Sigma(\boldsymbol{K}t;\ \boldsymbol{K}_2t_2)G(\boldsymbol{K}_2t_2;\ \boldsymbol{K}'t')=\delta_{\boldsymbol{K}\boldsymbol{K}'}\delta(t-t'), \tag{2·16}$$

where the electron self-energy $\Sigma(\boldsymbol{K}t;\ \boldsymbol{K}'t')$ is given by

$$\Sigma(\boldsymbol{K}t;\ \boldsymbol{K}'t')=i\sum_{\boldsymbol{K}''\boldsymbol{K}_1\boldsymbol{K}_2}\int_{t_0}^{t_0-i\beta}dt_1dt_2\nu_{\boldsymbol{K}-\boldsymbol{K}''}\nu_{\boldsymbol{K}_2}G(\boldsymbol{K}''t;\ \boldsymbol{K}_1t_1)$$

$$\times\Gamma(\boldsymbol{K}_1t_1;\ \boldsymbol{K}'t';\ \boldsymbol{K}_2t_2)D(\boldsymbol{K}_2t_2;\ \boldsymbol{K}''-\boldsymbol{K},\ t), \tag{2·17}$$

and the vertex function Γ is defined by

$$\Gamma(\boldsymbol{K}_1t_1;\ \boldsymbol{K}_2t_2;\ \boldsymbol{K}_3t_3)=-\frac{1}{\nu_{\boldsymbol{K}_3}}\frac{\delta G^{-1}(\boldsymbol{K}_1t_1;\ \boldsymbol{K}_2t_2)}{\delta\tilde{\varphi}_{\boldsymbol{K}_3}(t_3)}. \tag{2·18}$$

By defining the Fourier transform

$$G(\boldsymbol{r}t;\ \boldsymbol{r}'t')=\frac{1}{V}\sum_{\boldsymbol{K}\boldsymbol{K}'}e^{i\boldsymbol{K}\boldsymbol{r}}G(\boldsymbol{K}t;\ \boldsymbol{K}'t')e^{-i\boldsymbol{K}'\boldsymbol{r}'}, \tag{2·19}$$

Eq. (2·16) can be expressed in the configuration space as

$$\left[i\frac{\partial}{\partial t_1}+\frac{1}{2m^*}\nabla_1^2-U(\boldsymbol{r}_1t_1)\right]G(\boldsymbol{r}_1t_1;\ \boldsymbol{r}_2t_2)$$

$$-\int_V d\boldsymbol{r}_3\int_{t_0}^{t_0-i\beta}dt_3\Sigma(\boldsymbol{r}_1t_1;\ \boldsymbol{r}_3t_3)G(\boldsymbol{r}_3t_3;\ \boldsymbol{r}_2t_2)=\delta(x_1-x_2), \tag{2·20a}$$

where $\delta(x-x')=\delta(\boldsymbol{r}-\boldsymbol{r}')\delta(t-t')$ and $\Sigma(\boldsymbol{r}t;\ \boldsymbol{r}'t')$ is defined as

$$\Sigma(\boldsymbol{r}t;\ \boldsymbol{r}'t')=\frac{1}{V}\sum_{\boldsymbol{K}\boldsymbol{K}'}e^{i\boldsymbol{K}\boldsymbol{r}}\Sigma(\boldsymbol{K}t;\ \boldsymbol{K}'t')e^{-i\boldsymbol{K}'\boldsymbol{r}'}. \tag{2·21}$$

Also, $U(\boldsymbol{r}t)$ is given by

$$U(\boldsymbol{r}t)=\sum_{\boldsymbol{k}}v_{\boldsymbol{k}}(t)e^{i\boldsymbol{K}\boldsymbol{r}}+\sum_{\boldsymbol{q}}\nu_{\boldsymbol{q}}\tilde{\varphi}_{\boldsymbol{q}}(t)e^{i\boldsymbol{q}\boldsymbol{r}}. \tag{2·22}$$

Similarly we have

$$\left[-i\frac{\partial}{\partial t_2}+\frac{1}{2m^*}\nabla_2^2-U(\boldsymbol{r}_2t_2)\right]G(\boldsymbol{r}_1t_1;\ \boldsymbol{r}_2t_2)$$
$$-\int_V d\boldsymbol{r}_3\int_{t_0}^{t_0-i\beta}dt_3G(\boldsymbol{r}_1t_1;\ \boldsymbol{r}_3t_3)\Sigma(\boldsymbol{r}_3t_3;\ \boldsymbol{r}_2t_2)=\delta(x_1-x_2). \qquad (2\cdot20\text{b})$$

From Eqs. (2·20a) and (2·22), we find that $\tilde{\varphi}_{\boldsymbol{q}}(t)$ has a very clear physical meaning: it determines the potential produced by the phonon field. The equation of motion of $\tilde{\varphi}_{\boldsymbol{q}}(t)$ is written easily from Eq. (2·9) as

$$\left(\frac{\partial^2}{\partial t^2}+\omega_q^2\right)\tilde{\varphi}_{\boldsymbol{q}}(t)=-\nu_q\int_V d\boldsymbol{r}n(\boldsymbol{r}t)e^{-i\boldsymbol{q}\boldsymbol{r}}, \qquad (2\cdot23)$$

where $n(\boldsymbol{r}t)$ is the density of conduction electrons.

(B) *Equation of motion for* $D(\boldsymbol{q}t;\ \boldsymbol{q}'t')$

Applying the same method as in (A), we get

$$\left(-\frac{\partial^2}{\partial t^2}-\omega_q^2\right)D(\boldsymbol{q}t;\ \boldsymbol{q}'t')$$
$$-\sum_{\boldsymbol{q}''}\int_{t_0}^{t_0-i\beta}dt''\Pi(\boldsymbol{q}t;\ \boldsymbol{q}''t'')D(\boldsymbol{q}''t'';\ \boldsymbol{q}'t')=\delta_{\boldsymbol{q}\boldsymbol{q}'}\delta(t-t'), \qquad (2\cdot24)$$

where the phonon self-energy $\Pi(\boldsymbol{q}t;\ \boldsymbol{q}'t')$ is given by

$$\Pi(\boldsymbol{q}t;\ \boldsymbol{q}'t')=-i\sum_{\boldsymbol{K}\boldsymbol{K}''\boldsymbol{K}'''}\int_{t_0}^{t_0-i\beta}dt''dt'''\nu_{-\boldsymbol{q}}\nu_{\boldsymbol{q}'}G(\boldsymbol{K}t;\ \boldsymbol{K}''t'')$$
$$\times\Gamma(\boldsymbol{K}''t'';\ \boldsymbol{K}'''t''';\ \boldsymbol{q}'t')G(\boldsymbol{K}'''t''';\ \boldsymbol{K}-\boldsymbol{q},\ t). \qquad (2\cdot25)$$

Introducing the Fourier transform

$$D(\boldsymbol{r}t;\ \boldsymbol{r}'t')=\frac{1}{V}\sum_{\boldsymbol{q},\boldsymbol{q}'}e^{i\boldsymbol{q}\boldsymbol{r}}D(\boldsymbol{q}t;\ \boldsymbol{q}'t')e^{-i\boldsymbol{q}'\boldsymbol{r}'},$$

we can write Eq. (2·24) in the configuration space as

$$\left(-\frac{\partial^2}{\partial t_1^2}+s^2\nabla_1^2\right)D(\boldsymbol{r}_1t_1;\ \boldsymbol{r}_2t_2)$$
$$-\int_V d\boldsymbol{r}_3\int_{t_0}^{t_0-i\beta}dt_3\Pi(\boldsymbol{r}_1t_1;\ \boldsymbol{r}_3t_3)D(\boldsymbol{r}_3t_3;\ \boldsymbol{r}_1t_1)=\delta(x_1-x_2). \qquad (2\cdot26\text{a})$$

Similarly we have

$$\left(-\frac{\partial^2}{\partial t_2^2}+s^2\nabla_2^2\right)D(\boldsymbol{r}_1t_1;\ \boldsymbol{r}_2t_2)$$

$$-\int_V d\boldsymbol{r}_3 \int_{t_0}^{t_0-i\beta} dt_3 D(\boldsymbol{r}_1 t_1;\ \boldsymbol{r}_3 t_3)\Pi(\boldsymbol{r}_3 t_3;\ \boldsymbol{r}_2 t_2)=\delta(x_1-x_2). \qquad (2\cdot 26\text{b})$$

Equations (2·20) and (2·26) are the basic equations, with which we start to derive the transport equations.

(C) *Calculation of the self-energies Σ and Π*

First of all, we calculate Eq. (2·17). We define

$$G_0^{-1}(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2)=\left\{\left(i\frac{\partial}{\partial t_1}-\varepsilon_{\boldsymbol{K}_1}\right)\delta_{\boldsymbol{K}_1\boldsymbol{K}_2}-U_{\boldsymbol{K}_1-\boldsymbol{K}_2}(t_1)\right\}\delta(t_1-t_2). \qquad (2\cdot 27)$$

Then, Eq. (2·20a) can be written as

$$G^{-1}(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2)=G_0^{-1}(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2)-\Sigma(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2). \qquad (2\cdot 28)$$

Substitution of Eq. (2·28) into Eq. (2·18) yields

$$\Gamma(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2;\ \boldsymbol{K}_3 t_3)=-\frac{1}{\nu_{\boldsymbol{K}_3}}\frac{\delta}{\delta\tilde{\varphi}_{\boldsymbol{K}_3}(t_3)}G_0^{-1}(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2)$$
$$+\frac{1}{\nu_{\boldsymbol{K}_3}}\frac{\delta}{\delta\tilde{\varphi}_{\boldsymbol{K}_3}(t_3)}\Sigma(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2). \qquad (2\cdot 29)$$

Adopting the Migdal approximation, we neglect the second term on the right side of Eq. (2·29). Using Eq. (2·27) in Eq. (2·29), we have

$$\Gamma(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2;\ \boldsymbol{K}_3 t_3)=\delta_{\boldsymbol{K}_1-\boldsymbol{K}_2,\boldsymbol{K}_3}\delta(t_1-t_2)\delta(t_1-t_3). \qquad (2\cdot 30)$$

Substituting Eq. (2·30) into Eq. (2·17), we obtain

$$\Sigma(\boldsymbol{K}_1 t_1;\ \boldsymbol{K}_2 t_2)=i\sum_{\boldsymbol{q},\boldsymbol{q}'}\nu_{\boldsymbol{q}}\nu_{-\boldsymbol{q}'}G(\boldsymbol{K}_1+\boldsymbol{q},\ t_1;\ \boldsymbol{K}_2+\boldsymbol{q}',t_2)D(\boldsymbol{q}'t_2;\ \boldsymbol{q}t_1). \qquad (2\cdot 31)$$

Secondly, calculation of Eq. (2·25) leads easily to

$$\Pi(\boldsymbol{q}t;\ \boldsymbol{q}'t')=-i\sum_{\boldsymbol{K},\boldsymbol{K}'}\nu_{-\boldsymbol{q}}\nu_{\boldsymbol{q}'}G(\boldsymbol{K}t;\ \boldsymbol{K}'t')G(\boldsymbol{K}'-\boldsymbol{q}',\ t';\ \boldsymbol{K}-\boldsymbol{q},\ t). \qquad (2\cdot 32)$$

(D) *Basic transport equations*

As shown by Kadanoff and Baym, the transport equations are derived from Eqs. (2·20) and (2·26) in terms of the real time Green's functions, $g^{\gtrless}$ and $d^{\gtrless}$, obtained by a trivial analytical continuation:

$$\lim_{t_0\to-\infty}G^{\gtrless}(\boldsymbol{r}_1 t_1;\ \boldsymbol{r}_2 t_2)=g^{\gtrless}(\boldsymbol{r}_1 t_1;\ \boldsymbol{r}_2 t_2), \qquad (2\cdot 33)$$

$$\lim_{t_0\to-\infty}D^{\gtrless}(\boldsymbol{r}_1 t_1;\ \boldsymbol{r}_2 t_2)=d^{\gtrless}(\boldsymbol{r}_1 t_1;\ \boldsymbol{r}_2 t_2), \qquad (2\cdot 34)$$

where the notations $>$ and $<$ are the same as in KB. We now introduce the relative coordinates and the center-of-mass coordinates as follows:

$$\begin{aligned} \boldsymbol{\xi} &= \boldsymbol{r}_1 - \boldsymbol{r}_2\,, & \tau &= t_1 - t_2\,, \\ \boldsymbol{r} &= \tfrac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_2)\,, & t &= \tfrac{1}{2}(t_1 + t_2)\,. \end{aligned} \tag{2·35}$$

We also define the following Fourier transforms:

$$g^{\gtrless}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t) = \pm i \int_V d\boldsymbol{\xi} \int_{-\infty}^{\infty} d\tau e^{-i\boldsymbol{K}\cdot\boldsymbol{\xi}+i\varepsilon\tau} g^{\gtrless}(\boldsymbol{\xi}\tau;\, \boldsymbol{r}t), \tag{2·36}$$

$$d^{\gtrless}(\boldsymbol{q}\omega;\, \boldsymbol{r}t) = i \int_V d\boldsymbol{\xi} \int_{-\infty}^{\infty} d\tau e^{-i\boldsymbol{q}\cdot\boldsymbol{\xi}+i\omega\tau} d^{\gtrless}(\boldsymbol{\xi}\tau;\, \boldsymbol{r}t), \tag{2·37}$$

where we have written $g^{>}(\boldsymbol{r}_1 t_1;\, \boldsymbol{r}_2 t_2) = g^{>}(\boldsymbol{\xi}\tau;\, \boldsymbol{r}t)$ and so on. Further, $\Sigma^{\gtrless}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t)$ and $\Pi^{\gtrless}(\boldsymbol{q}\omega;\, \boldsymbol{r}t)$ satisfy the same Fourier transforms as Eqs. (2·36) and (2·37), respectively.

If the potential U defined by Eq. (2·22) has wavelengths much longer than the thermal wavelength of electrons and frequencies much smaller than the collision rate, we can derive readily from Eqs. (2·20) the following Kadanoff-Baym equation for electrons:

$$\begin{aligned} &[\varepsilon - \varepsilon_{\boldsymbol{K}} - U(\boldsymbol{r}t) - \mathrm{Re}\,\Sigma(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t),\, g^{\gtrless}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t)] + [\mathrm{Re}\, g(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t),\, \Sigma^{\gtrless}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t)] \\ &\quad = \Sigma^{<}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t) g^{>}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t) - \Sigma^{>}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t) g^{<}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t). \end{aligned} \tag{2·38}$$

Similarly, we have for phonons

$$\begin{aligned} &[\omega^2 - \omega_q^{\,2} - \mathrm{Re}\,\Pi(\boldsymbol{q}\omega;\, \boldsymbol{r}t),\, d^{\gtrless}(\boldsymbol{q}\omega;\, \boldsymbol{r}t)] + [\mathrm{Re}\, d(\boldsymbol{q}\omega;\, \boldsymbol{r}t),\, \Pi^{\gtrless}(\boldsymbol{q}\omega;\, \boldsymbol{r}t)] \\ &\quad = \Pi^{<}(\boldsymbol{q}\omega;\, \boldsymbol{r}t) d^{>}(\boldsymbol{q}\omega;\, \boldsymbol{r}t) - \Pi^{>}(\boldsymbol{q}\omega;\, \boldsymbol{r}t) d^{<}(\boldsymbol{q}\omega;\, \boldsymbol{r}t). \end{aligned} \tag{2·39}$$

The self-energies $\Sigma^{\gtrless}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t)$ and $\Pi^{\gtrless}(\boldsymbol{q}\omega;\, \boldsymbol{r}t)$ can be expressed as follows. From Eq. (2·31), we have in the real time domain

$$\Sigma^{>}(\boldsymbol{K}_1 t_1;\, \boldsymbol{K}_2 t_2) = i \sum_{\boldsymbol{q}_1 \boldsymbol{q}_2} \nu_{\boldsymbol{q}_1} \nu_{-\boldsymbol{q}_2} g^{>}(\boldsymbol{K}_1 t_1;\, \boldsymbol{K}_2 t_2) d^{<}(\boldsymbol{q}_2 t_2;\, \boldsymbol{q}_1 t_1), \tag{2·40}$$

or in the configuration space from Eq. (2·21)

$$\Sigma^{>}(\boldsymbol{r}_1 t_1;\, \boldsymbol{r}_2 t_2) = i \sum_{\boldsymbol{q}_1 \boldsymbol{q}_2} \nu_{\boldsymbol{q}_1} \nu_{-\boldsymbol{q}_2} g^{>}(\boldsymbol{r}_1 t_1;\, \boldsymbol{r}_2 t_2) e^{i\boldsymbol{q}_2 \boldsymbol{r}_2} d^{<}(\boldsymbol{q}_2 t_2;\, \boldsymbol{q}_1 t_1) e^{-i\boldsymbol{q}_1 \boldsymbol{r}_1}. \tag{2·41}$$

Using the Fourier transforms (2·36) and (2·37), we can write Eq. (2·41) as

$$\Sigma^{\gtrless}(\boldsymbol{K}\varepsilon;\, \boldsymbol{r}t) = \sum_{\boldsymbol{q}} \int_V d\boldsymbol{r}' \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} S(\boldsymbol{q},\, \boldsymbol{R}) g^{\gtrless}(\boldsymbol{K}+\boldsymbol{q},\, \varepsilon+\omega;\, \boldsymbol{r}t) d^{\gtrless}(\boldsymbol{q}\omega;\, \boldsymbol{r}'t), \tag{2·42}$$

where

$$S(\boldsymbol{q},\, \boldsymbol{R}) = \frac{1}{V} \sum_{\boldsymbol{\kappa}} \nu_{\boldsymbol{q}+\boldsymbol{\kappa}/2} \nu_{\boldsymbol{q}-\boldsymbol{\kappa}/2} e^{i\boldsymbol{\kappa}\cdot\boldsymbol{R}}, \tag{2·43}$$

with $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$. Similarly, we have

$$\Pi^{\gtrless}(\boldsymbol{q}\omega;\boldsymbol{r}t)=\sum_{\boldsymbol{K}}\int_{V}d\boldsymbol{r}'\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}S(\boldsymbol{q},\boldsymbol{R})g^{\gtrless}(\boldsymbol{K}+\boldsymbol{q},\varepsilon+\omega;\boldsymbol{r}t)g^{\gtrless}(\boldsymbol{K}\varepsilon;\boldsymbol{r}'t). \quad (2\cdot44)$$

According to KB, we note that the Σ's and Π's on the left sides of Eqs. (2·38) and (2·39) describe the kinetic effects of the electron-phonon interaction, i.e. how the interaction changes the energy-momentum relation of free electrons to the more complex spectrum. Because those effects are not important in the present investigation, we employ the approximations $\Sigma^{\gtrless}=\mathrm{Re}\,\Sigma=0$ and $\Pi^{\gtrless}=\mathrm{Re}\,\Pi=0$ on the left side of Eqs. (2·38) and (2·39).

Before deriving the transport equations, we consider the effect of the electron-electron interaction. This effect may be taken into account by adding to the potential U defined by Eq. (2·22)

$$U_c(\boldsymbol{r}t)=\sum_{\boldsymbol{k}}\frac{4\pi e^2}{V\epsilon_0 k^2}\int_{V}d\boldsymbol{r}'\rho(\boldsymbol{r}'t)e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}, \quad (2\cdot45)$$

and by replacing a factor $\nu_{\boldsymbol{q}}$ of the S-function (2·43) by

$$\nu_{\boldsymbol{q}}^{*}=\nu_{\boldsymbol{q}}/\epsilon_{\boldsymbol{q}}, \quad (2\cdot46)$$

where $\rho=n-n_0$, ϵ_0 is the static dielectric constant and $\epsilon_q=1+(q_D/q)^2$ is the wave-number-dependent dielectric constant, in which $q_D=(4\pi n_0 e^2/\epsilon_0 k_B T)^{1/2}$ and n_0 is the equilibrium density of conduction electrons. The replacement (2·46) corresponds to the random-phase approximation.

Now we introduce the electron distribution function $f_{\boldsymbol{K}}(\boldsymbol{r}t)$ defined as

$$f_{\boldsymbol{K}}(\boldsymbol{r}t)=\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}g^{<}(\boldsymbol{K}\varepsilon;\boldsymbol{r}t), \quad (2\cdot47)$$

and assume the following forms for $g^{\gtrless}$ and $d^{\gtrless}$:

$$\begin{aligned} g^{>}(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)&=a(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)[1-f(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)],\\ g^{<}(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)&=a(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)f(\boldsymbol{K}\varepsilon;\boldsymbol{r}t). \end{aligned} \quad (2\cdot48a)$$

$$\begin{aligned} d^{>}(\boldsymbol{q}\omega;\boldsymbol{r}t)&=\chi(\boldsymbol{q}\omega;\boldsymbol{r}t)[1+N(\boldsymbol{q}\omega;\boldsymbol{r}t)],\\ d^{<}(\boldsymbol{q}\omega;\boldsymbol{r}t)&=\chi(\boldsymbol{q}\omega;\boldsymbol{r}t)N(\boldsymbol{q}\omega;\boldsymbol{r}t). \end{aligned} \quad (2\cdot48b)$$

The functions $a(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)$ and $f(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)$ are interpreted as the local spectral function and the local distribution function for electrons, and $\chi(\boldsymbol{q}\omega;\boldsymbol{r}t)$ and $N(\boldsymbol{q}\omega;\boldsymbol{r}t)$ are those for phonons.

Thus, using Eqs. (2·42) and (2·48) in Eq. (2·38) and writing explicitly the factors of $\hbar$, we get

$$\frac{\partial f_{\boldsymbol{K}}}{\partial t}+\frac{1}{\hbar}(e\boldsymbol{E}_0-\nabla U)\frac{\partial f_{\boldsymbol{K}}}{\partial \boldsymbol{K}}+\boldsymbol{v}_{\boldsymbol{K}}\frac{\partial f_{\boldsymbol{K}}}{\partial \boldsymbol{r}}$$

$$=\sum_{\boldsymbol{q}}\int_{V}d\boldsymbol{r}'\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{d\omega}{2\pi}S(\boldsymbol{q},\boldsymbol{R})a(\boldsymbol{K}+\boldsymbol{q},\varepsilon+\hbar\omega;\boldsymbol{r}t)a(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)\chi(\boldsymbol{q}\omega;\boldsymbol{r}'t)$$

$$\times\{f(\boldsymbol{K}+\boldsymbol{q},\,\varepsilon+\hbar\omega;\,\boldsymbol{r}t)\,[1-f(\boldsymbol{K}\varepsilon;\,\boldsymbol{r}t)\,][N(\boldsymbol{q}\omega;\,\boldsymbol{r}'t)+1]$$
$$-f(\boldsymbol{K}\varepsilon;\,\boldsymbol{r}t)\,[1-f(\boldsymbol{K}+\boldsymbol{q},\,\varepsilon+\hbar\omega;\,\boldsymbol{r}t)\,]N(\boldsymbol{q}\omega;\,\boldsymbol{r}'t)\},\qquad\text{(I)}$$

where $S(\boldsymbol{q},\,\boldsymbol{R})$ is expressed as

$$S(\boldsymbol{q},\,\boldsymbol{R})=\frac{1}{V}\sum_{\kappa}\nu^*_{q+\kappa/2}\nu^*_{q-\kappa/2}e^{i\kappa\boldsymbol{R}}\qquad(2\cdot49)$$

using Eq. (2·46). The potential U is written as the sum of two terms: $U=U_c+U_p$, where the first term represents the Coulomb potential and the second term represents the potential produced by the phonon field. As mentioned in Eq. (2·22), U_p is given by

$$U_p=\sum_{q}\nu_q\tilde{\varphi}_q(t)\,e^{iqr}.\qquad(2\cdot50)$$

Although the same notation as in the imaginary time domain has been used here, it should be interpreted as $\lim_{t_0\to-\infty}\tilde{\varphi}_q(t)$. The equation of motion of $\tilde{\varphi}_q$ is given by Eq. (2·23):

$$\left(\frac{\partial^2}{\partial t^2}+\omega_q{}^2\right)\tilde{\varphi}_q(t)=-\nu_q\int_V d\boldsymbol{r}n(\boldsymbol{r}t)\,e^{-iq\boldsymbol{r}}.\qquad\text{(IIa)}$$

We should notice that the electron-phonon interaction constant in Eq. (IIa) is not modified by the electron-electron interaction. Using Eqs. (2·44) and (2·48b) in Eq. (2·39), the transport equation for phonons is written as

$$\left(\frac{\partial}{\partial t}+s^2\boldsymbol{q}\cdot\nabla\right)\chi(\boldsymbol{q}\omega;\,\boldsymbol{r}t)N(\boldsymbol{q}\omega;\,\boldsymbol{r}t)$$
$$=\frac{1}{2}\sum_{\boldsymbol{K}}\int_V d\boldsymbol{r}'\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}S(\boldsymbol{q},\,\boldsymbol{R})\,a(\boldsymbol{K}+\boldsymbol{q},\,\varepsilon+\hbar\omega;\,\boldsymbol{r}'t)\,a(\boldsymbol{K}\varepsilon;\,\boldsymbol{r}'t)\chi(\boldsymbol{q}\omega;\,\boldsymbol{r}t)$$
$$\times\{f(\boldsymbol{K}+\boldsymbol{q},\,\varepsilon+\hbar\omega;\,\boldsymbol{r}'t)\,[1-f(\boldsymbol{K}\varepsilon;\,\boldsymbol{r}'t)\,][N(\boldsymbol{q}\omega;\,\boldsymbol{r}t)+1]$$
$$-f(\boldsymbol{K}\varepsilon;\,\boldsymbol{r}'t)\,[1-f(\boldsymbol{K}+\boldsymbol{q},\,\varepsilon+\hbar\omega;\,\boldsymbol{r}'t)\,]N(\boldsymbol{q}\omega;\,\boldsymbol{r}t)\}.\qquad\text{(III)}$$

We have already determined the potential U. However, it is sometimes convenient to express U in terms of $\alpha_q(t)=\tilde{b}_q(t)$ instead of $\tilde{\varphi}_q(t)$. The equation of motion of α_q is written as

$$\left(\frac{\partial}{\partial t}+i\omega_q\right)\alpha_q(t)=-i(C_q/\hbar)\int_V d\boldsymbol{r}n(\boldsymbol{r}t)\,e^{-iq\boldsymbol{r}}.\qquad\text{(IIb)}$$

Then, the potential U is given by

$$U=U_c+U_p,\qquad\text{(IVa)}$$
$$U_p=\sum_{q}C_q(\alpha_q e^{iq\boldsymbol{r}}+\alpha_q{}^*e^{-iq\boldsymbol{r}}),\qquad\text{(IVb)}$$
$$U_c=\sum_{k}(4\pi e^2/V\epsilon_0k^2)\int_V d\boldsymbol{r}'\rho(\boldsymbol{r}'t)\,e^{i\boldsymbol{k}(\boldsymbol{r}-\boldsymbol{r}')}.\qquad\text{(IVc)}$$

Equations (I), (II) and (III) represent one form of our basic transport equations. From Eq. (I), one can see easily that phonons play two different physical roles in the description of transport phenomena. The potential U_p on the left side implies that phonons act on electrons as a wave field, whose amplitude and phase are determined by Eq. (II). If the collision term of Eq. (I) is expressed by the relaxation time approximation, we see easily that the distribution function varies at distances of electron mean free path $l=v_{\rm th}\tau$, where $v_{\rm th}$ is the thermal velocity of electrons. Therefore, from Eq. (II), we find that the wave field is caused by long-wave phonons ($q<q_c=l^{-1}$), and that the criterion for the validity of Eq. (I) is $K_{\rm th}l>1$, where $K_{\rm th}$ is the thermal wave number of electrons. This is nothing but the condition for the validity of the Boltzmann approach. On the other hand, the right side of Eq. (I) implies that phonons interact with electrons as quasiparticles and transfer them from one energy-momentum configuration to another. It should be noted that the transport equations are nonlocal in the sense that their collision terms at a given point in space relate the electron or phonon spectral functions and distribution functions at other points. The function $S(\boldsymbol{q}, \boldsymbol{R})$ is characteristic of the nonlocal property of the transport equations.

§ 3. Amplification of phonons

The purpose of this section is to study the role of phonons as quasiparticles in the non-ohmic conduction. Putting $N(\boldsymbol{q}\omega; \boldsymbol{r}t)=N_0(\boldsymbol{q}\omega)+\xi(\boldsymbol{q}\omega; \boldsymbol{r}t)$ in Eq. (I) and adopting the relaxation time approximation for collision with the phonon distribution $N_0(\boldsymbol{q}\omega)$ in thermal equilibrium, we obtain the following equation of motion for the current defined as $\boldsymbol{v}=V^{-1}\sum \boldsymbol{v}_{\boldsymbol{K}}f_{\boldsymbol{K}}$:

$$\left(\tau\frac{\partial}{\partial t}+1\right)\boldsymbol{v}=\boldsymbol{v}_0+\boldsymbol{v}_w+\boldsymbol{v}_p\,, \tag{3·1}$$

where $\boldsymbol{v}_0=ne\tau\boldsymbol{E}_0/m^*$ is the ohmic current, and

$$\boldsymbol{v}_w=-\frac{\tau}{m^*}n\nabla U-D\nabla n\,, \tag{3·2}$$

$$\boldsymbol{v}_p=-\frac{\tau}{V}\sum_{\boldsymbol{K},\boldsymbol{q}}\frac{\hbar\boldsymbol{q}}{2m^*}\int_V d\boldsymbol{r}'\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\frac{d\omega}{2\pi}S(\boldsymbol{q},\boldsymbol{R})a(\boldsymbol{K}+\boldsymbol{q},\varepsilon+\hbar\omega;\boldsymbol{r}t)a(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)$$

$$\times[f(\boldsymbol{K}+\boldsymbol{q},\varepsilon+\hbar\omega;\boldsymbol{r}t)-f(\boldsymbol{K}\varepsilon;\boldsymbol{r}t)]\chi(\boldsymbol{q}\omega;\boldsymbol{r}'t)\xi(\boldsymbol{q}\omega;\boldsymbol{r}'t)\,. \tag{3·3}$$

Here, $D=k_BT\tau/m^*$ is the diffusion constant. It is needless to say that the non-ohmic current arises from the phonon amplification of two aspects.

From Eq. (III), the amplification factor of phonons is given by

$$A(\boldsymbol{q}\omega)=(1/2\omega)\sum_{\boldsymbol{K}}\int_V d\boldsymbol{r}'(d\varepsilon/2\pi)S(\boldsymbol{q},\boldsymbol{R})a(\boldsymbol{K}+\boldsymbol{q},\varepsilon+\hbar\omega;\boldsymbol{r}'t)a(\boldsymbol{K}\varepsilon;\boldsymbol{r}'t)$$

$$\times[f(\boldsymbol{K}+\boldsymbol{q},\ \varepsilon+\hbar\omega;\ \boldsymbol{r}'t)-f(\boldsymbol{K}\varepsilon;\ \boldsymbol{r}'t)]. \tag{3·4}$$

We find that the phonon amplification of quasiparticle nature may be interpreted as the population inversion process of electrons in momentum space. For the piezoelectric coupling

$$C_q=\left(\frac{4\pi e\beta}{\epsilon_0}\right)\left(\frac{\hbar}{2V\rho_i sq}\right)^{1/2}, \tag{3·5}$$

$S(\boldsymbol{q},\ \boldsymbol{R})$ is expressed as

$$S(\boldsymbol{q},\ \boldsymbol{R})=\left(\frac{4\pi e\beta}{\epsilon_0\sqrt{\rho_i}}\right)\left[\delta(\boldsymbol{R})-\frac{2q_D{}^2}{\pi}\frac{\cos(2\boldsymbol{q}\cdot\boldsymbol{R})}{R}e^{-2q_DR}\right.$$
$$\left.+\frac{q_D{}^4}{\pi^3}\int_{-\infty}^{\infty}d\boldsymbol{\kappa}\frac{e^{2i\boldsymbol{\kappa}\boldsymbol{R}}}{\{|\boldsymbol{q}+\boldsymbol{\kappa}|^2+q_D{}^2\}\{|\boldsymbol{q}-\boldsymbol{\kappa}|^2+q_D{}^2\}}\right]. \tag{3·6}$$

It is difficult to evaluate Eq. (3·6). However, we can see that when $q\gg q_D$, $S(\boldsymbol{q},\ \boldsymbol{R})$ shows the damped oscillation as a function of R, whose period is of the order of $1/q$, and in the limit $q\to 0$, $S(\boldsymbol{q},\ \boldsymbol{R})$ decreases exponentially over a spatial range of the screening length $q_D{}^{-1}$. We have examined again the behavior of $S(\boldsymbol{q},\ \boldsymbol{R})$ with respect to R by approximating $\nu_q{}^*=0$ for $q<q_D$ and $\nu_q{}^*=\text{const}$ for $q>q_D$. The result is that $S(\boldsymbol{q},\ \boldsymbol{R})$ is an oscillatory function with a period of the order of $(q+q_D)^{-1}$ when $\boldsymbol{R}$ is in the direction of $\boldsymbol{q}$. Therefore, it is reasonable to approximate the period of $S(\boldsymbol{q},\ \boldsymbol{R})$ with respect to R as $(q+q_D)^{-1}$. Since the electron distribution varies at distances of the electron mean free path l, as mentioned in § 2, the following two cases should be distinguished in the calculation of Eq. (3·4). In the case $q_Dl\gg 1$, $S(\boldsymbol{q},\ \boldsymbol{R})$ is equivalent to $\nu_q{}^{*2}\delta(\boldsymbol{R})$, so that there remains no nonlocal property in Eq. (3·4). This case corresponds to high-mobility semiconductors such as InSb at 77°K. On the other hand, in the case $q_Dl<1$, there appears nonlocal property in Eq. (3·4), when $ql<1$.

Let us consider now the case $q_Dl\gg 1$. As for the spectral function, it may be reasonable to assign the functional form to it:

$$a(\boldsymbol{K}\varepsilon;\ \boldsymbol{r}t)=\frac{\Gamma(\boldsymbol{K}\varepsilon;\ \boldsymbol{r}t)}{(\varepsilon-\varepsilon_{\boldsymbol{K}})^2+\frac{1}{4}\Gamma(\boldsymbol{K}\varepsilon;\ \boldsymbol{r}t)^2}. \tag{3·7}$$

On the other hand, the distribution function is assumed to be given by a displaced Fermi function or a displaced Maxwell function. Then, the amplification factor is expressed as

$$A(\boldsymbol{q}\omega)=A_0(\boldsymbol{q}\omega)\,(v_d\cos\theta-s_q)/s \tag{3·8}$$

with

$$A_0(\boldsymbol{q}\omega)=\left(\frac{V}{2\pi^2}\right)\left(\frac{\hbar\nu_q{}^{*2}}{2\omega k_BT}\right)\left(\frac{m^*s}{\hbar^2}\right)\int_{-\infty}^{\infty}\frac{d\varepsilon}{2\pi}\int_{-\infty}^{\infty}dK\,K\tan^{-1}\left[\frac{2}{\Gamma}(\varepsilon_{K+q}-\varepsilon-\hbar\omega)\right]$$
$$\times\frac{\Gamma}{(\varepsilon-\varepsilon_K)^2+\frac{1}{4}\Gamma^2}f_0(\varepsilon)\,[1-f_0(\varepsilon+\hbar\omega)], \tag{3·9}$$

where $s_q=\omega/q$, $\boldsymbol{v}_d$ is the drift velocity of electrons and f_0 is the distribution function in thermal equilibrium ($\boldsymbol{v}_d=0$). When k_BT/Γ is much larger than one, a Lorentzian function in Eq. (3·9) may be replaced by $2\pi\delta(\varepsilon-\varepsilon_K)$ and A_0 is easily evaluated as

$$A_0(\boldsymbol{q}\omega)=\left(\frac{\pi}{2}\right)^{1/2}K^2\omega_p{}^2\left(\frac{m^*s^2}{k_BT}\right)^{3/2}\left(\frac{1}{\omega}\right)\left(\frac{q^2}{q^2+q_D{}^2}\right)^2 p(a) \tag{3·10}$$

for the piezoelectric coupling, where K is the electromechanical coupling constant, ω_p is the plasma frequency, and $p(a)$ is defined by

$$p(a)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\kappa\frac{a}{a^2(\kappa+\kappa_0)^2+1}\exp(-\kappa^2), \tag{3·11}$$

with $a=(8/3)^{1/2}(ql)$ and $\kappa_0=(3/2)^{1/2}|q/2-m^*s_q/\hbar|/K_{\text{th}}$. The function $p(a)$ is equal to $\exp(-\kappa_0{}^2)$, when $a\gg1$. This condition is equivalent to $\omega_q\tau\gg1$, because v_{th}/s is of the order 10^2. Thus, the amplification factor of phonon of $\omega=\omega_q$ is just the same as the one obtained by the simple perturbation theory. On the other hand, $p(a)=\pi^{-1/2}a$ when $ql\ll1$, so that the amplification factor is smaller by (ql) than the one obtained from the simple perturbation theory. This means that even when $ql<1$, the quasiparticle aspect of phonon is not expected to disappear radically, although its amplification factor is smaller by $(ql)^2$ than that of waves. Our present question is which aspect contributes dominantly to the phonon amplification, when $q_Dl\gg1$. Noting from Eq. (3·8) and the Hutson-White relation (4·8) that the amplification factors have maximum values at $q\sim q_D$, we find that the quasiparticle aspect is more dominant by $(q_Dl)^2$ than the wave one. Therefore, we expect that the phonon amplification in high-mobility semiconductors may be described in terms of the particle aspect in the initial stage of amplification. As will be pointed out later, however, the situation will be modified in the final stage (the current saturation state), because the electron mean free path is decreased by the Čerenkov emission of phonons and then the condition $q_Dl\gg1$ will break down.

In the case $q_Dl<1$, the nonlocal property should be taken into account in Eq. (3·4) when $ql<1$. Although a quantitative result has not been obtained as yet, the radical reduction of the amplification factor does not seem to be expected even when $ql<1$. However, we can find that the maximum value of the amplification factor of waves is larger by $(q_Dl)^{-2}$ than that of quasiparticle. It is certain that the phonon amplification may be described in terms of wave aspect in low-mobility semiconductors such as CdS.

Since $\Gamma(\boldsymbol{K}\varepsilon)$ in the spectral function represents the line width, it may be written as $\Gamma(\boldsymbol{K}\varepsilon)=\hbar/\tau(\boldsymbol{K}\varepsilon)$. There are two mechanisms wich mainly contribute to the relaxation time $\tau(\boldsymbol{K}\varepsilon)$. One is the interaction between conduction electrons and thermal phonons, which determines the mean free path of electrons in the Ohmic state. If τ is assumed to have the order of the magnitude of 10^{-14} sec,

the magnitude of $C=k_BT\tau/\hbar$ is nearly equal to one at 300°K. These figures roughly correspond to the case of CdS. Therefore, the damping effect due to the interaction between electrons and thermal phonons is not serious, though not negligibly small. In the case of GaAs and InSb at 77°K, the value of C seems to be much larger than one, so that the damping effect is expected to be negligible in the Ohmic case. The situation may be different when phonons are heavily amplified in the Čerenkov cone, because the damping by the piezoelectric interaction cannot be disregarded. According to the usual experimental condition, the initial drift velocity is assumed to be as twice as the final velocity, to which the drift velocity is reduced by the scattering of amplified phonons. Conduction electrons as a whole are scattered by amplified phonons as efficiently as thermal phonons, because the final velocity becomes half of the initial value. We must note, however, that the amplified phonons interact selectively with electrons in the limited range in $\boldsymbol{K}$-space. Since excess phonons exist only in the very narrow Čerenkov cone, it may be allowed to assume that the wave number vector $\boldsymbol{q}$ of the excess phonons is almost parallel to the field direction, which is chosen as the z-axis. Then, the conservation law leads to the result that the transition in the emission of $\boldsymbol{q}$ is allowed, if the following condition is nearly satisfied: $K_z=(q_z/2)\rightarrow K_z'=-(q_z/2)$. There is no special limitation for K_x and K_y. As shown in the previous paper,[2] the number of excess phonons is very large only in the narrow range of q, so that the electrons which interact with these excess phonons are scattered much more frequently than the average. Thus, the damping constant $\Gamma(\boldsymbol{K}\varepsilon)$ becomes quite large for these electrons and their transition probability must be reduced. Finally, the further increase of excess phonons must be checked when excess phonons are increased to a certain limiting number. This is a typical nonlinear process in the electron-phonon interaction.

In these circumstances, it may be possible that the mean free path of these electrons is much reduced as compared with the average. In the case of GaAs the value of ql seems to be a little larger than one for the most effectively amplified phonons ($q=5\times10^5$/cm), so that the quasiparticle picture is expected to be applicable in the initial stage of the phonon amplification. It is possible, however, that the situation is modified in the final stage, where the mean free path of the electrons with $K_z\sim10^5$/cm is much reduced to break the condition $ql>1$. Therefore, the situation seems to be very complicated in the case of GaAs or InSb at 77°K. We conjecture that the current saturation phenomena in such a case is explained by the two aspects of phonons.

The phonon-phonon interaction was discussed qualitatively in the previous paper[2] and it is not discussed in the present paper. An important effect of the phonon-phonon interaction is to replace the sound velocity in the factor $(v_d(\boldsymbol{r}t)-s)$ by the effective sound velocity which is sometimes much larger than the sound velocity. The fact that the critical drift velocity in GaAs and InSb is considerably faster than the sound velocity will find the best explanation in the

idea of the effective sound velocity. This fact is a strong support of the idea that phonon amplification process is probably the cause of an initiation of the current instability.

However, it is not quite certain whether the Čerenkov emission picture is appropriate, or the wave picture is suitable for explanation of processes concerning the high-field domain, because these processes take place in the well-developed stage of the phonon flux.

§ 4. Amplification of lattice waves

In § 3, we have mentioned that in low-mobility semiconductors the phonon amplification may be described in terms of the wave aspect. Further, this aspect is expected to appear in the well-developed stage of amplification, even in high-mobility semiconductors. The linear theory of wave amplification is well established and is known as the Hutson-White theory. In this section, we want to study the wave amplification in the nonlinear regime.

If the phonon amplification of quasiparticle nature is disregarded, then the following equations are easily derived from Eqs. (I) and (3·1)

$$\frac{\partial n}{\partial t}+\operatorname{div}(n\boldsymbol{v}_d)=0\,, \tag{4·1}$$

$$\boldsymbol{v}_d=\boldsymbol{v}_{d0}-\frac{\tau}{m^*}\nabla U-\frac{D}{n}\nabla n\,, \tag{4·2}$$

where $\partial\boldsymbol{v}_d/\partial t$ has been neglected, $\boldsymbol{v}_d=\boldsymbol{v}/n$ is the drift velocity of electrons, $\boldsymbol{v}_{d0}=(e\tau/m^*)\boldsymbol{E}_0$ is the Ohmic drift velocity, and U is defined by Eq. (IV). Thus, Eqs. (4·1), (4·2) and (IIb) are regarded as a set of coupled equations which we want to solve.

(a) *Linear approximation*

The linear theory will first be reviewed briefly, because it gives a starting point for study in the nonlinear regime. In the limit of the linear approximation, Eqs. (4·1) and (4·2) are reduced to

$$\frac{\partial n}{\partial t}+n_0\operatorname{div}\boldsymbol{v}_d+\boldsymbol{v}_{d0}\nabla n=0\,, \tag{4·3}$$

$$\boldsymbol{v}_d=\boldsymbol{v}_{d0}-\frac{\tau}{m^*}\nabla U-\frac{D}{n_0}\nabla n\,. \tag{4·4}$$

Let us solve the problem by using the Fourier series:

$$\begin{aligned}
n(\boldsymbol{r}t)&=n_0+\sum_{\boldsymbol{q}}n(\boldsymbol{q})\exp\{i(\boldsymbol{q}\boldsymbol{r}-\nu_{\boldsymbol{q}}t-i\gamma_{\boldsymbol{q}}t)\}+\text{c.c.}\,,\\
\boldsymbol{v}_d(\boldsymbol{r}t)&=\boldsymbol{v}_{d0}+\sum_{\boldsymbol{q}}\boldsymbol{v}(\boldsymbol{q})\exp\{i(\boldsymbol{q}\boldsymbol{r}-\nu_{\boldsymbol{q}}t-i\gamma_{\boldsymbol{q}}t)\}+\text{c.c.}\,,\\
U_p(\boldsymbol{v}t)&=\sum_{\boldsymbol{q}}C_q\alpha(\boldsymbol{q})\exp\{i(\boldsymbol{q}\boldsymbol{r}-\nu_{\boldsymbol{q}}t-i\gamma_{\boldsymbol{q}}t)\}+\text{c.c.}
\end{aligned} \tag{4·5}$$

Then, the amplification factor of waves of wave number $\boldsymbol{q}$ is written easily as

$$\gamma_{\boldsymbol{q}}=(VC_q^2 n_0\tau q^2/m^*\hbar)\,(\boldsymbol{q}\boldsymbol{v}_{d0}-\omega_q)/[(\boldsymbol{q}\boldsymbol{v}_{d0}-\omega_q)^2+(Dq^2+\tau\omega_p^2)^2]. \tag{4·6}$$

At the same time, the renormalized frequency is given by

$$\nu_{\boldsymbol{q}}-\omega_{\boldsymbol{q}}=-(VC_q^2 n_0\tau q^2/m^*\hbar)\,(Dq^2+\tau\omega_p^2)/[(\boldsymbol{q}\boldsymbol{v}_{d0}-\omega_q)^2+(Dq^2+\tau\omega_p^2)^2]. \tag{4·7}$$

For the piezoelectric coupling (3·5), we can write Eq. (4·6) as

$$\gamma_{\boldsymbol{q}}=(K^2\omega_c/2)\,(v_{d0}\cos\theta/s-1)/[(v_{d0}\cos\theta/s-1)^2+(\omega_c/\omega_q)^2(1+\omega_q^2/\omega_c\omega_D)^2], \tag{4·8}$$

where $\omega_c=\tau\omega_p^2$, and $\omega_D=s^2/D$. This is the famous Hutson-White relation. In the linear approximation, the average value of $n\boldsymbol{v}_d$ is $n_0\boldsymbol{v}_{d0}$, and in the next approximation it is evaluated as

$$\langle n\boldsymbol{v}_d\rangle=n_0\boldsymbol{v}_{d0}-(\hbar\tau/Vm^*)\sum_{\boldsymbol{q}}2\gamma_{\boldsymbol{q}}\boldsymbol{q}|\alpha(\boldsymbol{q})|^2\exp(2\gamma_{\boldsymbol{q}}t). \tag{4·9}$$

The last term is identical with the acoustoelectric current derived by Hutson.

Although there was no need to use the distribution function in the derivation of Eqs. (4·6) and (4·7), it may be interesting to see what function is to be used for $n(\boldsymbol{r}t)$ in order to get the same relations. Such an investigation will be useful for the second order approximation theory, which will be discussed in (c). It is not a hard task to prove that the distribution function

$$\begin{aligned} n(\boldsymbol{r}t)&=(n_0/N)\exp[-(U_p+U_c)/k_BT]\exp(U_{ac}/k_BT)\\ &\cong n_0[1-(U_p+U_c-U_{ac})/k_BT] \end{aligned} \tag{4·10}$$

satisffies the relations (4·3) and (4·4), if we define $\gamma_{\boldsymbol{q}}$ and $\nu_{\boldsymbol{q}}$ by Eqs. (4·6) and (4·7), where N is the normalization constant. Here, U_{ac} is defined by

$$-e\boldsymbol{E}_{ac}=\nabla U_{ac}=(m^*/\tau)\,(\boldsymbol{v}_{d0}-\boldsymbol{v}_d). \tag{4·11}$$

It is quite easy to show that the field $\boldsymbol{E}_{ac}$ is the same as the acoustoelectric field given by Eq. (4·9).

(b) *Stationary state*

It is believed that a stationary state is realized at the final stage. Then, all electrons are trapped in the potential trough, which is moving to the field direction with the renormalized sound velocity s^*. The amplification of sound waves is expected to be stopped. At present, it is not clear experimentally what state is realized in the final stage of the amplification process. The problem of stationary state was discussed first by Gurevich.[9] On the basis of a simple approach, we want to point out here that such a stationary state is conceivable.

For simplicity, the one-dimensional case will be considered. Let us start with the assumption that the density distribution is given by

$$n(xt)=(n_0/N)\exp\{-[U_p(x-s^*t)+U_c(x-s^*t)-U_{ac}(x-s^*t)]/k_BT\}. \tag{4·12}$$

On the other hand, the acoustoelectric potential U_{ac} is given by $U_{ac}=(m^*/\tau)$

$\times(v_{d0}-s^*)(x-s^*t)$. Then, it is easily shown that Eqs. (4·1) and (4·2) are satisfied by the distribution function (4·12) and the additionary assumption: $v_d=s^*$. The physical picture is obvious. All electrons are trapped in the potential valley and the valley itself is moving to the x-direction with the sound velocity. Thus, this solution corresponds to the so-called "bunching". In fact, from Eq. (IIb), we see easily that when $n(xt)=n(x-s^*t)$, the potential U_p is of the functional form $U_p(x-s^*t)$ and its amplitude does not increase with time.

(c) *Second order approximation*

The initial rate of amplification must be decreased with time. In order to see this situation, let us proceed to the second order approximation. Here, we shall introduce two simplifying assumptions: (i) the coefficient $\alpha(q)$ is taken as a real quantity and is assumed to be positive, and (ii) $C_q=C$ (a constant). Then, we see that the real part of $n(q)$ becomes negative and the imaginary part of $n(q)$ becomes positive. Further, in usual conditions, the real part of $n(q)$ is much larger than the imaginary part. Let us start with the distribution function

$$n(xt)=n_0[1-(U_p+U_c-U_{ac})/k_BT+(U_p+U_c-U_{ac})^2/2(k_BT)^2]. \quad (4\cdot13)$$

Using Eq. (IVb), we have the following equation after some calculation:

$$\begin{aligned} n_q(t)=&-(n_0C/k_BT)\alpha_q(t)[1-(C/2k_BT)\alpha_q^{-1}(t)\sum_{q'}\alpha_{q'}(t)\alpha_{q-q'}(t)+\cdots] \\ &+i(n_0/Dq)v_q(t)[1-(C/2k_BT)v_q^{-1}(t)\sum_{q'}\alpha_{q'}(t)v_{q-q'}(t)+\cdots] \\ &-(q_D/q)^2n_q(t)[1-(m^*\omega_p^2/2k_BT)n_0^{-1}n_q^{-1}(t)\sum_{q'}q'^{-2}n_{q'}(t)n_{q-q'}(t)+\cdots], \end{aligned} \quad (4\cdot14)$$

where some leading terms alone are written down. Here, the correction terms in each bracket have the order of the magnitude $(C/k_BT)\sum\alpha_q(t)$, so that the linear approximation will break down at the stage when the bunching of electrons becomes appreciable. It is quite conceivable that the bunching phenomenon lies outside the linear theory. Since $\alpha_q(t)$ is positive and the real part of $n_q(t)$ is negative, we may write Eq. (4·14) in the following form:

$$[1+(q_D/q)^2(1+\theta_3)]n_q(t)=-(n_0C/k_BT)(1-\theta_1)\alpha_q(t)+i(n_0/D_q)(1-\theta_2)v_q(t), \quad (4\cdot15)$$

where θ_1, θ_2 and θ_3 are some time-dependent positive quantities which are increased with time. If these quantities are neglected, Eq. (4·15) is identical with an equation derived in the linear approximation. Then, we have an approximate formula for $\gamma_q(t)$,

$$\begin{aligned} \gamma_q(t)=&(VC_q^2n_0\tau q^2/m^*\hbar)(1-\theta_1)(1-\theta_2)(qv_{d0}-\omega_q) \\ &\div[(qv_{d0}-\omega_q)^2(1-\theta_2)^2+(Dq^2+\tau\omega_p^2(1+\theta_3))^2]. \end{aligned} \quad (4\cdot16)$$

Equation (4·16) means that $\gamma_q(t)$ decreases rapidly when the bunching of electrons begins to play.

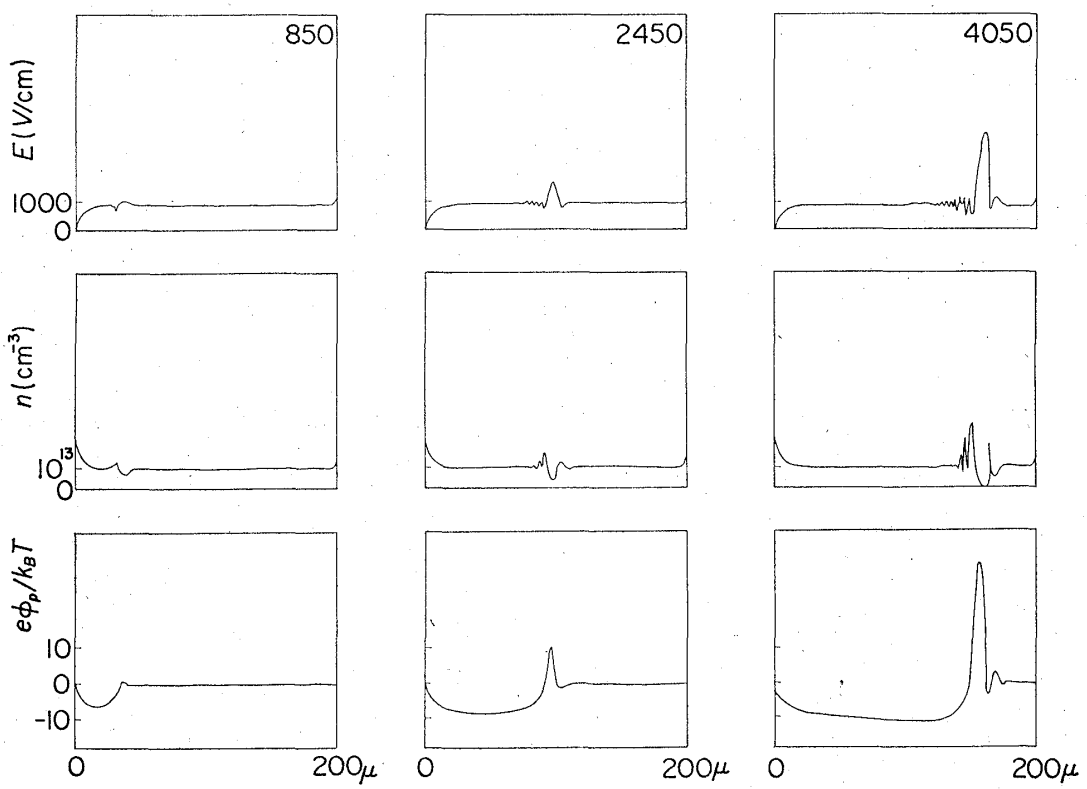
In the second order approximation, the different modes of the Fourier components are coupled each other and, as a result, the initial rate of amplification is decreased with time. Since the change of the rate is dependent on q, and the high-frequency part of the spectrum is expected to be cut off, the dispersion curve of the amplified wave is expected to shift to the side of lower frequency.

(d) *Computer experiments on the domain motion*

In many cases, a propagating acoustic domain is observed accompanied by the current saturation phenomena. At the present time, however, there is not any nonlinear theory applicable to the domain phenomena. In this subsection, we want to study the domain problem*) using a computer. For this problem, it is convenient to introduce the total electric field $\boldsymbol{E}=\boldsymbol{E}_0-e^{-1}\nabla U$ and to use Eqs. (2·50) and (IIa). Then, the basic equations

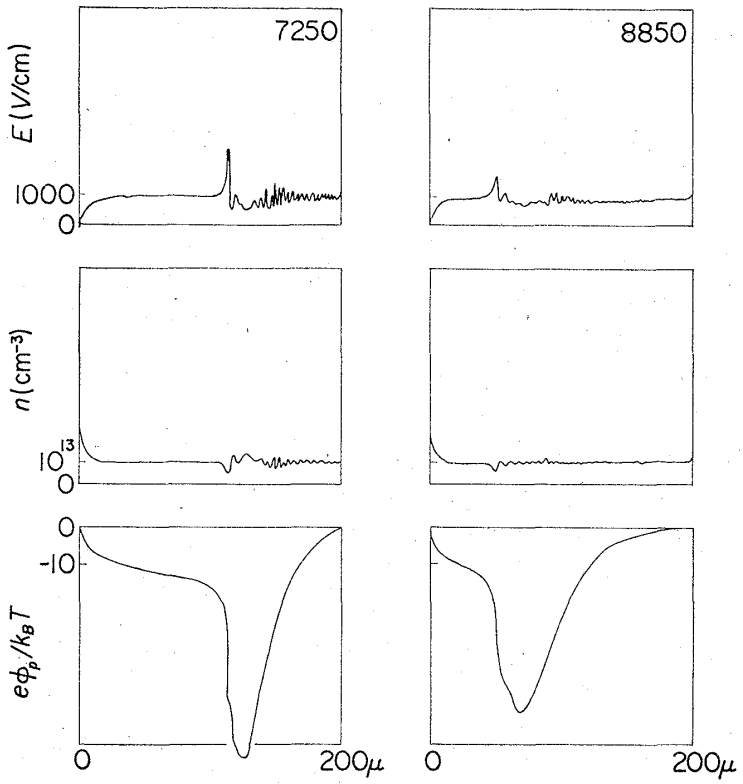
$$\partial n/\partial t+\operatorname{div} n=0\,, \tag{4·17a}$$

$$\boldsymbol{v}=n\mu\boldsymbol{E}-D\nabla n\,, \tag{4·17b}$$

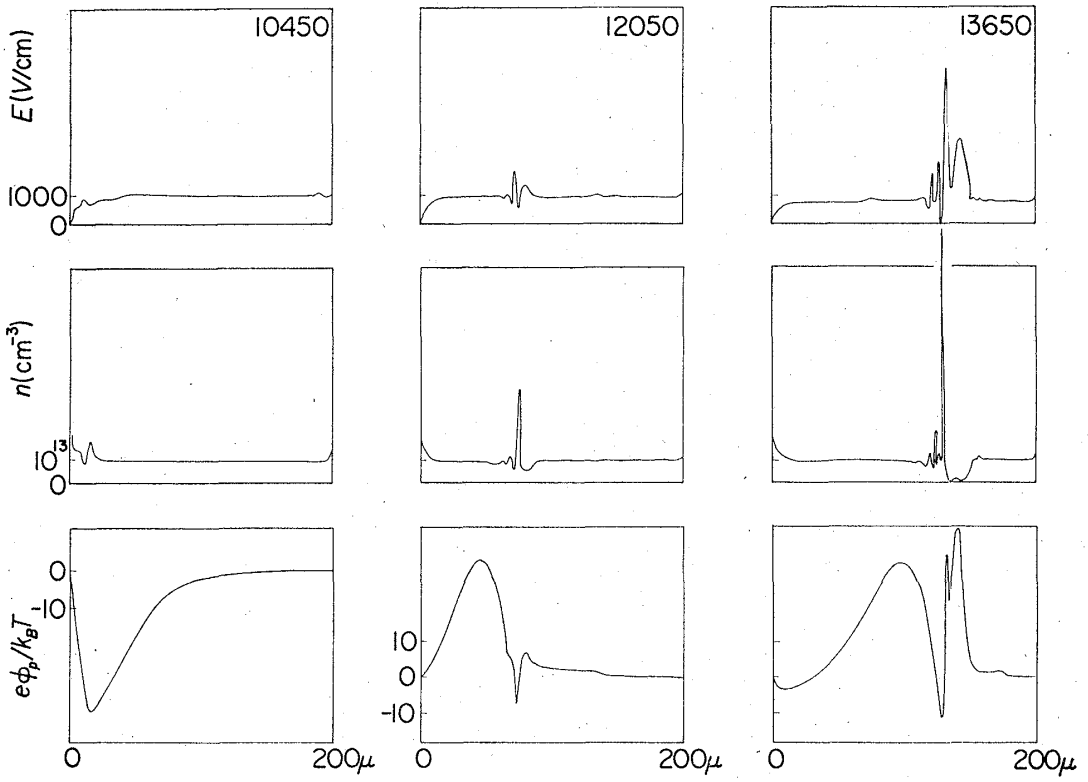


(a) First one-way trip

*) Friedman has recently discussed the domain problem on the basis of the particle aspect: Phys. Rev. **163** (1967), 712.



(b) Return trip



(c) Second one-way trip

Fig. 1. Computed electric field, electron density and piezoelectric potential distributions in the sequence of time. The number written in the right edge of each figure denotes the time in the unit of 2.2286×10^{-11} sec. When $t=11250$, 12050, 12850 and 13650, the drift velocity of electrons in the accumulation layer on the second one-way trip (c) is estimated as $v_d=2.24$, 1.64, 1.48 and 1.45×10^5 cm/sec, respectively.

$$\mathrm{div}\, \boldsymbol{E} + \nabla^2 \varphi_p = 4\pi e^* (n - n_0), \tag{4·17c}$$

$$\Box \varphi_p = 4\pi e_p^* (n - n_0), \tag{4·17d}$$

where φ_p is the piezoelectric potential, $e^* = e/\epsilon_0$ and $e_p^* = K^2 e^*$. Here, n_0 is the density of the background of the positive charge which usually depends on $\boldsymbol{r}$. Equations (4·17) are nothing but the classical equations adopted in the macroscopic theory.

A one-dimensional crystal of length L is considered. Equations (4·17) are solved numerically with the following initial and boundary conditions:

$$\rho(x, 0) = 0; \qquad \rho(0, t) = \rho(L, t) = 0,$$

$$\int_0^L E(xt)\, dx = LE_0,$$

$$\varphi_p(x, 0) = \varphi_p'(x, 0) = 0; \qquad \varphi_p(0, t) = \varphi_p(L, t) = 0,$$

where the prime denotes the differentiation with respect to time. The values of the parameters used in the calculation are: $L = 2 \times 10^{-2}$ cm, $E_0 = 1000$ V/cm, $K^2 = 0.3$, $s = 1.75 \times 10^5$ cm/sec, $\mu = 300$ cm^2/V sec, $\epsilon_0 = 9$ and $D = 8$ cm^2/sec. The donor density is taken as $n_0 = 10^{13}$/cm^3 inside the crystal and as $n_0 = 10^{14}$/cm^3 at each end. For convenience of calculation, we have chosen the sample shorter than that used in the usual experiments, and instead have taken the coupling constant several times as large as $K \sim 0.1$ in the actual CdS crystal, in order to get adequate amplification of waves through the sample. Also, the value of the applied field has been determined by the equation $E_0 = (s/\mu)[1 + (q_D l)\,(4k_B T/3m^* s^2)^{1/2}]$, which gives the maximum gain of amplification of waves with the frequency $\omega_m = (\omega_c \omega_D)^{1/2}$ in the Hutson-White relation (4·8).

The computed electron density, electric field and piezoelectric potential distributions for successive instants in time are shown in Fig. 1. The sequence shows that a packet of the electron density fluctuation is nucleated near the cathode and its amplitude grows up, as it propagates with the average velocity $v_D = 1.5 \times 10^5$ cm/sec which is nearly equal to the sound velocity estimated by the formula $s^* = s(1 - K^2/2)$ in the linear theory. In the initial stage, the packet is composed of a depletion layer and an accumulation layer, and after a while some additional fluctuations appear on both sides of the packet. As for the field distribution, it is found that a high-field domain appears in the depletion layer and a low-field domain in the accumulation layer. As soon as the packet reaches the anode, it is reflected and is attenuated on its return trip (Fig. 1(b)). Then, a new packet is again nucleated near the cathode and moves towards the anode (Fig. 1(c)). One finds that its amplitude grows up to a level higher than that on the first one-way trip. In the final stage, where the deep depletion layer and the sharp accumulation layer are realized, the values of the electric field outside the domain becomes smaller than that of the applied field and the value of the total

electric current through the crystal reduces to about half of its ohmic value. Figure 2 shows that the current on the second trip decreases more rapidly with time than that on the first one-way trip and tends to saturate.

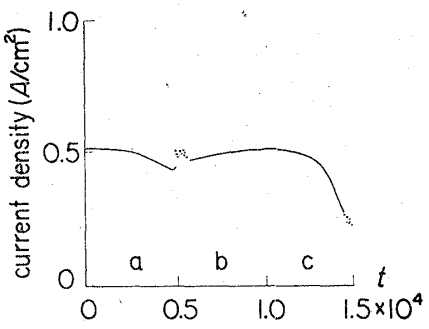


Fig. 2. Computed decay of current with time. The unit of time is the same as that in Fig. 1. a, b and c denote the first one-way, return and second one-way trip, respectively.

We are interested in the mechanism of the domain formation. It is instructive to consider the behaviors of the piezoelectric potential distribution. In the early stage, a potential valley is formed near the cathode (Fig. 1(a)). As will be shown later, this valley is produced by the density distribution arising from the diffusion of electrons at the boundary and gives rise to an accumulation layer through the piezoelectric field acting on the wall of the valley in the direction opposite to the applied field. As the valley becomes broad, a potential barrier appears at the edge of the valley and moves towards the anode with increase of its height. As soon as the barrier reaches the anode and disappears, a large potential valley arises from there and goes back to the cathode with decrease of its depth (Fig. 1(b)). This means that the phase of the reflected wave shifts by π from that of the incoming wave because of the boundary condition $\varphi_p(L, t)=0$. When the valley approaches the cathode, the density distribution returns to the initial distribution. The potential distribution keeps the shape of valley for some time after the nucleation of the new packet of the density fluctuation near the cathode. Shortly, a potential trough is formed by interference between the reflected wave at the cathode and the incoming wave and moves towards the anode with increase of its depth, accompanied by the potential barriers before and behind it (Fig. 1(c)). The back barrier height increases in the initial stage but decreases with the gradual increase of the front barrier height. Electrons are accumulated in the potential trough through the piezoelectric fields on its wall and are depleted in the front barrier. As written in Fig. 1, the drift velocity of electrons in the potential trough is decreased with time and approaches the sound velocity.

Using Eqs. (4·17), we can understand the formation of the packet of density fluctuation in the initial stage. If we assume $\rho=\rho_0 e^{-\alpha x}$ for the density distribution arising from the diffusion of electrons at the boundary, we get

$$\varphi_p(xt) = -(4\pi e_p^* \rho_0/\alpha^2)\left[(1-e^{-\alpha x}) - \tfrac{1}{2}(1-e^{-\alpha(x+st)}) - \tfrac{1}{2}\,\mathrm{sgn}(x-st)\,(1-e^{-\alpha|x-st|})\right]. \qquad (4\cdot 18)$$

The potential distribution given by Eq. (4·18) takes the shape of a valley whose wall moves with the sound velocity. The piezoelectric field $E_p=-\nabla\varphi_p$ on the wall is produced in the direction opposite to the applied field, and then the drift velocity of electrons becomes smaller there than that in the other region. So,

electrons are piled up on the wall and are depleted in front of the wall. This leads to the packet of the density waves shown in Fig. 1(a), and subsequently such a density distribution gives rise to the potential barrier. From the point of view mentioned here, we find that the domain formation is a consequence of the transient generation of the acoustic flux packet at the boundary and of the subsequent amplification of it. It is almost certain that the domain formation is irrelevant to the Ridley mechanism.[10] The process described by our computer experiment is the same as the round trip process observed by McFee[11] in photoconducting CdS.

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Note added in proof:

During the course of publication of this paper an interesting paper was written by Tien [Phys. Rev. **171** (1968), 970]. The results of his computor calculation are similar in many respects to those of the present paper in sections 4 and 5.