

# *Theory of Finite Deformations of Porous Solids*

M. A. BIOT

*Communicated by G. TEMPLE*

**1. Introduction.** The linear mechanics of fluid-saturated porous media as developed by the author was reviewed and discussed in detail in two earlier papers [1], [2]. In its final form it is based on the linear thermodynamics of irreversible processes. It is applicable to the most general case of anisotropy and includes not only the basic principles of classical thermodynamics but also the effects of coupled flows of irreversible processes. Thermoelastic dissipation is also implicit since the heat flux is nothing but one of the internal thermodynamic coordinates of the system. The solid matrix itself may be viscoelastic. Actually the theory is even more general since it takes into account the viscoelastic interaction of the fluid with the solid due to the micromechanics of fluid penetration in cracks which are much smaller than the pores. An important concept derived from the existence of a dissipation function for the fluid is that of "Viscodynamic operator" [2]. This is an operational symmetric tensor which describes the frequency-dependent behavior of the fluid. The symmetric character of this tensor leads to important conclusions in the theory of acoustic propagation.

Regarding the extension to non-linear problems a first step is constituted by a theory which introduces the non-linear superposition of a state of initial stress and incremental deformations [3]. This also leads to an analysis of finite deformations based on stress-rates. In this case the deformation is considered as a continuous sequence of incremental deformations [4]. The concepts and methods introduced in this incremental theory lead quite naturally to the next development which considers a description of finite deformation using material coordinates. In particular the concept of pressure function for a porous medium which was introduced in the theory of incremental deformations [3] provides one of the essential means by which this extension of the theory can be accomplished. The mechanics of porous media is thus brought to the same level of development of the classical theory of finite deformations in elasticity. In order to restrict the length of the paper, the theory is presented in the context of quasi-static and isothermal deformations.

It should be pointed out that in order to develop a realistic theory of porous solids, it is essential to use a *material description* of the deformation because we are dealing with solid matter whose properties such as the porous structure are transported and rotated with the material. This is in contrast with the Eulerian or *spatial description* which is more suitable when dealing with isotropic homogeneous fluids. In this connection reference should be made to recent attempts to treat the mechanics of porous media by applying theories of interacting continua which are based on a Eulerian description. In addition to presenting basic difficulties inherent in the Eulerian description of deformation of solids, the physical model of the theory of interacting continua lacks the required sophistication to account for all significant and essential properties of porous media.

**2. Equilibrium equations.** The deformation of the solid is described by the transformation

$$(2.1) \quad \xi_i = \xi_i(x_j)$$

for the initial coordinates  $x_j$  to the final coordinates  $\xi_i$ . We are dealing here with the average displacement of an element of material sufficiently large with respect to the pore size. The average local deformation is homogeneous within a differential element and defined by the differential relations

$$(2.2) \quad d\xi_i = c_{ij} dx_j$$

where

$$(2.3) \quad c_{ij} = \frac{\partial \xi_i}{\partial x_j}.$$

A measure of strain is

$$(2.4) \quad \alpha_{ij} = \alpha_{ji} = c_{ki} c_{kj}.$$

The coefficients  $c_{ij}$  represent the homogeneous transformation of a sample cube of the solid originally of unit size. We shall refer to it as the unit sample.

The *isothermal* free energy  $W$  of this unit cube after deformation is a function of the strain measure  $\alpha_{ij}$  and of the total mass of fluid  $m$  added in the pores of the sample during deformation. It may also depend on the initial location  $x_k$ .

$$(2.5) \quad W = W(\alpha_{ij}, m, x_k).$$

We also define a vector whose components  $M_i$  represent the total mass of fluid which has flowed across a material area which before deformation is a unit square perpendicular to the  $x_i$  axis. It is related to  $m$  by the relation

$$(2.6) \quad m = -\frac{\partial W}{\partial x_i}.$$

A body force field  $\mathfrak{B}_i$  per unit mass derived from a potential  $U(\xi_i)$  is acting on the medium

$$(2.7) \quad \mathfrak{B}_i = -\frac{\partial U}{\partial \xi_i}.$$

The initial mass of solid and fluid in a unit volume before deformation is denoted by  $m_0$ . The mass of this sample after deformation is  $m_0 + m$ . The equilibrium equations of the field are obtained from the principle of virtual work. We write

$$(2.8) \quad \iiint_{\Omega} \delta W \, d\Omega + \iiint_{\Omega} \delta[(m_0 + m)U] \, d\Omega = 0.$$

This equation must be verified for all variations of the six variables  $\xi_i$  and  $M_i$ . The variations are chosen different from zero only inside the volume  $\Omega$  so that boundary conditions are not involved.

Let us first vary  $\xi_i$ . Equation (2.8) yields

$$(2.9) \quad \iiint_{\Omega} \left[ f_{ii} \delta c_{ii} + (m_0 + m) \frac{\partial U}{\partial \xi_i} \delta \xi_i \right] d\Omega = 0$$

where we have put

$$(2.10) \quad f_{ii} = \frac{\partial W}{\partial c_{ii}}.$$

These quantities represent the nine force components acting on the faces of the unit sample after deformation, hence per unit initial area. Introducing the values (2.3) for  $c_{ii}$  and integrating by parts we obtain

$$(2.11) \quad \iiint_{\Omega} \left[ -\frac{\partial f_{ii}}{\partial x_i} + (m_0 + m) \frac{\partial U}{\partial \xi_i} \right] \delta \xi_i \, d\Omega = 0.$$

Hence the equilibrium equations

$$(2.12) \quad \frac{\partial f_{ii}}{\partial x_i} - (m_0 + m) \frac{\partial U}{\partial \xi_i} = 0.$$

This condition expresses the translational equilibrium of the field  $f_{ii}$ . Note that because  $W$  is independent of a rigid rotation the nine components  $f_{ii}$  satisfy identically the three equations of equilibrium of moments of an element of the medium.

Conditions of equilibrium of the fluid are obtained also from equation (2.8) this time by varying  $M_i$ . We derive

$$(2.13) \quad \iiint_{\Omega} (\psi + U) \delta m \, d\Omega = 0$$

where

$$(2.14) \quad \psi = \frac{\partial W}{\partial m}.$$

Introducing the value (2.6) of  $m$  and integrating by parts yields the fluid equilibrium condition

$$(2.15) \quad \frac{\partial}{\partial x_i} (\psi + U) = 0$$

or

$$(2.16) \quad \psi + U = \text{Const.}$$

The physical significance of  $\psi$  is clarified as follows. We consider a large reservoir of fluid at a pressure  $p_0$  which together with the unit sample of porous medium constitutes a single thermodynamic system. Its free energy per unit mass of fluid injected is

$$(2.17) \quad \psi = \frac{\partial W}{\partial m} = -\frac{p_0}{\rho_0} - \int_{p_0}^p p d\left(\frac{1}{\rho}\right) + \frac{p}{\rho}$$

where  $\rho$  is the fluid specific mass assumed to be a function of the fluid pressure  $p$ . Its initial value at pressure  $p_0$  is  $\rho = \rho_0$ . Expression (2.17) for  $\psi$  is the work done isothermally on the system to extract a unit mass of fluid from the reservoir, bring it to the pressure  $p$  and inject it in the solid at that pressure. It may be written

$$(2.18) \quad \psi = \int_{p_0}^p \frac{dp}{\rho}.$$

We shall call it the "pressure function", a term already in use for this expression in fluid mechanics. However, as a thermodynamic variable for the mixed fluid-solid system its present definition is in a broader context where it plays the role of a *chemical potential*. Actually its application is not restricted to the presence of actual pores. The fluid may be in *solution* in the solid, or may be *adsorbed*. Such phenomena are usually associated with the concept of capillary or osmotic pressures.

It is of interest to examine the particular case where the fluid content  $m$  depends only on the pore volume and fluid density. An additional physical interpretation of  $W$  is provided in this case. We call  $v$  the increase of pore volume per unit initial volume of bulk material. Under the present assumption we may write

$$(2.19) \quad dv = d\left(\frac{m + m_0}{\rho}\right).$$

The differential of the free energy is

$$(2.20) \quad dW = f_{,i} dc_{,i} + \psi dm.$$

By putting

$$(2.21) \quad d^{\circ}W = f_{ii} dc_{ii} + p dv$$

and taking into account relations (2.17) and (2.19) we find

$$(2.22) \quad W = {}^{\circ}W + \left[ - \int_{p_0}^p p d\left(\frac{1}{\rho}\right) - \frac{p_0}{\rho_0} \right] (m_0 + m).$$

This result may be interpreted as follows. Consider the system composed of the solid matrix and a thin layer of fluid attached to the internal walls of the pores. We may call it the wetted solid. As required for the validity of equation (2.19) we must assume that the mass of fluid in this layer is constant. The pressure  $p$  is then the pressure acting on the solid across the wetting fluid layer. The term  $p dv$  represents the work done by this pressure on the wetted solid. If the free energy of the wetted solid depends only on  $c_{ii}$  and  $v$  it is represented by  ${}^{\circ}W$ . Hence

$$(2.23) \quad \begin{aligned} f_{ii} &= \frac{\partial {}^{\circ}W}{\partial c_{ii}} \\ p &= \frac{\partial {}^{\circ}W}{\partial v}. \end{aligned}$$

Note that interfacial interactions of fluid and solid are taken into account here provided it can be localized in a thin layer of constant composition and considered as belonging to the solid matrix. For example the contribution of surface tension to the rigidity of a porous solid is included quantitatively in equations (2.23). It is well known that a large percentage of the elastic properties of porous media may depend on surface tension effects, particularly if the pores are small so that the total fluid solid interfacial area is large. Hence expression (2.22) for the free energy  $W$  may be interpreted as the sum of the free energies of two *non-interacting* phases. The first term  ${}^{\circ}W$  is the free energy of the wetted solid, while the second term is the free energy of a mass  $m_0 + m$  of fluid brought to the pressure  $p$  from the reservoir at the initial pressure  $p_0$ . This mass of fluid is introduced into the pores without additional work.

*Green's tensor.* This form of the stress tensor may be defined as in the classical theory of Elasticity. It is expressed by

$$(2.24) \quad T_{ii} = \frac{\partial W}{\partial \gamma_{ii}}$$

using Green's definition of strain as

$$(2.25) \quad \gamma_{ii} = \frac{1}{2}(\alpha_{ii} - \delta_{ii}).$$

*Change of variables by contact transformation.* In many problems it is convenient to use other independent variables than  $c_{ii}$  and  $m$ . For example we may choose  $c_{ii}$  and  $\psi$  as independent variables. This may be achieved by a classical procedure using contact transformations. We define a function

$$(2.26) \quad \mathfrak{F} = W - m\psi.$$

Its differential is

$$(2.27) \quad d\mathfrak{F} = dW - m d\psi - \psi dm.$$

Substituting the value (2.20) for  $dW$  we derive

$$(2.28) \quad d\mathfrak{F} = f_{,ii} dc_{,ii} - m d\psi.$$

We may express  $\mathfrak{F}$  in terms of  $\alpha_{,ii}$  and  $\psi$

$$(2.29) \quad \mathfrak{F} = \mathfrak{F}(\alpha_{,ii}, \psi).$$

Equation (2.28) implies

$$(2.30) \quad \begin{aligned} f_{,ii} &= \frac{\partial \mathfrak{F}}{\partial c_{,ii}} \\ -m &= \frac{\partial \mathfrak{F}}{\partial \psi}. \end{aligned}$$

Relations, in terms of the fluid pressure  $p$  and the increase in pore volume  $v$  are obtained by putting

$$(2.31) \quad \mathfrak{C} = \mathfrak{W} - pv$$

applying the contact transformation method to  $\mathfrak{W}$ , *i.e.*, to the free energy of the solid-matrix expressed as a function of  $c_{,ii}$  and  $v$ . When changing the independent variables to  $c_{,ii}$  and  $p$  we obtain

$$(2.32) \quad \begin{aligned} f_{,ii} &= \frac{\partial \mathfrak{C}}{\partial c_{,ii}} \\ v &= -\frac{\partial \mathfrak{C}}{\partial p}. \end{aligned}$$

The last equation yields the increase in pore volume. Another contact transformation could be applied with the stress components (2.24) as independent variables. We introduce

$$(2.33) \quad \mathfrak{G} = W - T_{,ii}\gamma_{,ii}$$

which we express in terms of  $T_{,ii}$  and  $m$ . In this case

$$(2.34) \quad \begin{aligned} \gamma_{,ii} &= -\frac{\partial \mathfrak{G}}{\partial T_{,ii}} \\ \psi &= \frac{\partial \mathfrak{G}}{\partial m}. \end{aligned}$$

Other similar relations may easily be derived for any group of seven state variables by the appropriate contact transformation.

3. **Generalized Darcy's law.** When the equilibrium condition (2.16) of the fluid is not satisfied, the motion of the fluid relative to the solid is expressed by Darcy's law which for isotropic permeability is written

$$(3.1) \quad \dot{w}_i = -k \frac{\rho}{\eta} \frac{\partial \phi}{\partial \xi_i}$$

where  $w_i$  is the rate of volume flow per unit area of bulk material,  $k$  is the permeability coefficient,  $\rho$  the fluid specific mass,  $\eta$  the fluid viscosity. The total potential  $\phi$  is

$$(3.2) \quad \phi = \psi + U$$

where  $\psi$  is the pressure function (2.18) and  $U$  is the body force potential in accordance with equation (2.7). The form (3.1) of Darcy's law was derived by M. King Hubbert [5] with an extensive discussion of its geometrical and physical significance.

For anisotropic porosity, the author has shown [1] that the generalized form of Darcy's law is

$$(3.3) \quad \dot{w}_i = -k_{ij} \frac{\rho}{\eta} \frac{\partial \phi}{\partial \xi_j}$$

where the permeability is a symmetric tensor

$$(3.4) \quad k_{ij} = k_{ji} .$$

For a Newtonian fluid this symmetry property is a direct consequence of the existence of the well-known dissipation function of Rayleigh. For the reader unfamiliar with dissipation function properties, the relationship between the symmetry property (3.4), and the existence of a dissipation function is explained in the appendix. The author also pointed out [1], [2] that the symmetry property (3.4) is quite general and is not restricted to pure viscous behavior. It remains valid if we include physical-chemical interactions between the solid and the fluid where the viscosity concept may not be applicable. In fact, it is a consequence of general principles of irreversible thermodynamics as expressed by Onsager's reciprocity relations. These relations are valid for a large class of phenomena of mass and energy transport and imply the existence of a thermodynamic dissipation function which is a measure of the rate of entropy production. For our purpose of analyzing finite deformation we introduce a more convenient form of Darcy's law (3.3) in terms of initial coordinates before deformation. Consider an area  $S$  of the porous material before deformation. After deformation it becomes  $S'$ . The total mass flow rate across this area is

$$(3.5) \quad F = \iint_{S'} \rho (\dot{w}_1 d\xi_2 d\xi_3 + \dot{w}_2 d\xi_3 d\xi_1 + \dot{w}_3 d\xi_1 d\xi_2) .$$

By transforming the variables  $\xi_i$  to the initial coordinates  $x_i$  we obtain

$$(3.6) \quad F = \iint_S (\dot{M}_1 dx_2 dx_3 + \dot{M}_2 dx_3 dx_1 + \dot{M}_3 dx_1 dx_2)$$

where

$$(3.7) \quad \dot{M}_i = \rho \dot{w}_i R_{i i}$$

and  $R_{i i}$  is the cofactor of  $c_{i i}$  in the Jacobian determinant

$$(3.8) \quad \Delta = \det \left| \frac{\partial \xi_i}{\partial x_j} \right| = \det |c_{i j}|.$$

The linear relations between the differentials  $d\xi_i$  and  $dx_i$  lead to the fundamental property

$$(3.9) \quad \frac{\partial x_i}{\partial \xi_j} = \frac{R_{j i}}{\Delta}.$$

These expressions are the elements of the inverse of the matrix of  $\Delta$ .

Expression (3.6) shows that  $\dot{M}_i$  represents a mass flow vector. For example  $\dot{M}_1$  is the rate of mass flow through a displaced surface originally equal to unity and perpendicular to  $x_1$  axis. Hence the vector  $\dot{M}_i$  is the mass flow rate through the faces of a deformed parallelepiped, initially a unit cube with faces parallel to the axes  $x_i$ . We may write

$$(3.10) \quad \dot{M}_i = \frac{\partial M_i}{\partial t}.$$

Although it is a non-Cartesian vector, it nevertheless represents a directly measurable and physically meaningful quantity.

We now go back to Darcy's law (3.3), by introducing it in the value (3.7) of  $\dot{M}_i$ . We obtain

$$(3.11) \quad \dot{M}_i = -k_{i m} \frac{\rho^2}{\eta} R_{i i} \frac{\partial \phi}{\partial \xi_m}.$$

Taking into account relations (3.9) we may also write

$$(3.12) \quad \frac{\partial \phi}{\partial \xi_m} = \frac{\partial \phi}{\partial x_j} \frac{\partial x_j}{\partial \xi_m} = \frac{1}{\Delta} R_{m j} \frac{\partial \phi}{\partial x_j}.$$

Substitution of this expression in equation (3.11) yields

$$(3.13) \quad \dot{M}_i = -K_{i j} \frac{\partial \phi}{\partial x_j}$$

where

$$(3.14) \quad K_{i j} = k_{i m} \frac{\rho^2}{\eta} \frac{R_{i i} R_{m j}}{\Delta}.$$

The symmetry property  $k_{i m} = k_{m i}$  implies the same property for  $K_{i j}$ , *i.e.*,

$$(3.15) \quad K_{i j} = K_{j i}.$$



Equation (3.13) is an intrinsic formulation of Darcy's law governing the mass flow through the faces of a deformed element of the material in terms of the gradient of  $\phi$  between parallel faces. The intrinsic permeability is now represented by the directly measurable symmetric tensor  $K_{ij}$ . For a Newtonian fluid the symmetry property of this intrinsic permeability may be derived as a direct consequence of Rayleigh's dissipation function without having to use relation (3.14). When the flow involves more general chemical physical processes the symmetry property of  $K_{ij}$  is a consequence of Onsager's relations. The intrinsic permeability is of course a function of the initial location, the deformation and the fluid mass content. Hence

$$(3.16) \quad K_{ij} = K_{ij}(x_k, \alpha_{kl}, m).$$

*Power dissipated.* It is interesting to evaluate the power dissipated  $\mathcal{P}$  per unit initial volume

$$(3.17) \quad \mathcal{P} = -\dot{w}_i \rho \frac{\partial \phi}{\partial \xi_i} \Delta = k_{ij} \frac{\rho^2}{\eta} \frac{\partial \phi}{\partial \xi_i} \frac{\partial \phi}{\partial \xi_j} \Delta.$$

Expressing these quantities in terms of  $\partial \phi / \partial x_i$ , taking into account (3.7) and (3.9) we find

$$(3.18) \quad \mathcal{P} = -\dot{M}_i \frac{\partial \phi}{\partial x_i} = K_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j}$$

with

$$(3.19) \quad K_{ij} = k_{im} \frac{\rho^2}{\eta} \frac{\partial x_i}{\partial \xi_i} \frac{\partial x_j}{\partial \xi_m} \Delta.$$

This coincides with the value (3.14). Expressions (3.17) and (3.18) are invariants. They show that  $k_{ij}$  and  $K_{ij}$  are contravariant tensors which in tensor notation are written with superscripts. We may also write

$$(3.20) \quad \mathcal{P} = K_{im} \frac{\partial \phi}{\partial \xi_i} \frac{\partial \phi}{\partial \xi_j} \frac{\partial \xi_i}{\partial x_i} \frac{\partial \xi_j}{\partial x_m}.$$

Hence comparing with (3.17)

$$(3.21) \quad k_{ij} = \frac{\eta}{\rho^2 \Delta} K_{im} \frac{\partial \xi_i}{\partial x_i} \frac{\partial \xi_j}{\partial x_m}.$$

*Complete set of equations for the deformation field.* There are basically six unknowns represented by the two vectors  $\xi_i$  and  $M_i$ . An important feature of the present formulation is the holonomic character of the conservation equation (2.6). The free energy may be written as a function of the six unknowns  $\xi_i$  and  $M_i$

$$(3.22) \quad W = W\left(\alpha_{ij}, -\frac{\partial M_i}{\partial x_i}, x_k\right).$$

With  $f_{ii} = \partial W / \partial c_{ii}$ , the equilibrium equations (2.12)

$$(3.23) \quad \frac{\partial f_{ii}}{\partial x_i} - \left( m_0 - \frac{\partial M_i}{\partial x_i} \right) \frac{\partial U}{\partial \xi_i} = 0$$

provide three equations for the six unknowns. Three more equations are provided by Darcy's law (3.13), *i.e.*,

$$(3.24) \quad \frac{\partial M_i}{\partial t} = -K_{ii} \frac{\partial \phi}{\partial x_i}.$$

The six equations (3.23) and (3.24) constitute a complete set with  $x_i$  and  $t$  as independent variables. Another complete set of equations may be obtained which involves only four unknowns  $\xi_i$  and  $\psi$ . The function  $\mathcal{F}$  defined by equation (2.26) is a function of  $c_{ii}$  and  $\psi$ . We write the equilibrium equations (2.12)

$$(3.25) \quad \frac{\partial f_{ii}}{\partial x_i} - (m_0 + m) \frac{\partial U}{\partial \xi_i} = 0$$

with  $f_{ii}$  and  $m$  expressed by equations (2.30) in terms of  $c_{ii}$  and  $\psi$ . By applying the divergence operation on equation (3.24), and taking into account relation (2.6) we derive

$$(3.26) \quad \frac{\partial m}{\partial t} = \frac{\partial}{\partial x_i} \left( K_{ii} \frac{\partial \phi}{\partial x_i} \right).$$

We now have a set of four equations (3.25) and (3.26) for the four unknowns  $\xi_i$  and  $\psi$ .

**Extension to non-Newtonian fluids.** For an isotropic fluid with non-linear properties, the rate dependent stress is expressed in the form (see appendix)

$$(3.27) \quad \sigma'_{ij} = \frac{\partial}{\partial e'_{ij}} \varphi(I'_1, I'_2, I'_3)$$

where  $I'_1, I'_2, I'_3$  are the three invariants of the strain-rate tensor  $e'_{ij}$ . The function  $\varphi$  plays the role of a dissipation function. As explained in the appendix this leads to a generalized non-linear form of Darcy's law

$$(3.28) \quad \frac{\partial D'}{\partial \dot{w}_i} + \rho \frac{\partial \phi}{\partial \xi_i} = 0.$$

It may be formulated in terms of the mass flow rate  $\dot{M}_i$ , and the initial coordinates  $x_i$ . We write equations (3.28) in the form

$$(3.29) \quad \frac{\partial D'}{\partial \dot{w}_i} + \rho \frac{\partial \phi}{\partial x_i} \frac{\partial x_i}{\partial \xi_i} = 0$$

or

$$(3.30) \quad \frac{\partial D'}{\partial \dot{w}_i} \frac{\partial \xi_i}{\partial x_i} + \rho \frac{\partial \phi}{\partial x_i} = 0.$$

We also derive

$$(3.31) \quad \frac{\partial D'}{\partial \dot{w}_i} \frac{\partial \xi_i}{\partial x_i} = \frac{\partial D'}{\partial \dot{M}_i} \frac{\partial \dot{M}_i}{\partial \dot{w}_i} \frac{\partial \xi_i}{\partial x_i}.$$

Taking into account relations (3.7) and (3.9) we obtain

$$(3.32) \quad \frac{\partial \dot{M}_i}{\partial \dot{w}_i} = \rho R_{,i1} = \rho \Delta \frac{\partial x_i}{\partial \xi_i}.$$

Hence (3.31) becomes

$$(3.33) \quad \frac{\partial D'}{\partial \dot{w}_i} \frac{\partial \xi_i}{\partial x_i} = \rho \Delta \frac{\partial D'}{\partial \dot{M}_i}.$$

Substitution of this value in equation (3.30) yields

$$(3.34) \quad \frac{\partial \mathfrak{D}}{\partial \dot{M}_i} + \frac{\partial \phi}{\partial x_i} = 0$$

where  $\mathfrak{D} = \Delta D'$  is the dissipation function per unit initial volume. Equations (3.34) represent Darcy's law referred to initial coordinates  $x_i$ . The power dissipated per unit initial volume is

$$(3.35) \quad \mathcal{P} = -\dot{M}_i \frac{\partial \phi}{\partial x_i} = \dot{M}_i \frac{\partial \mathfrak{D}}{\partial \dot{M}_i}.$$

Direct evaluation of this dissipated power also provides a method of obtaining Darcy's law (3.34). The dissipated power may be calculated on the basis of expression (A.10) of the appendix.

**4. Variational principles and Lagrangian equations.** Variational principles and the associated Lagrangian formulation were introduced and applied extensively in the linear theory [1], [2], [3]. The same formulation may be extended to finite deformations. We first write Darcy's law (3.24) in the form

$$(4.1) \quad \frac{\partial \phi}{\partial x_i} = -\Lambda_{,i} \dot{M}_i$$

where  $\Lambda_{,i} = \Lambda_{,i}$  is a "resistivity" tensor which is the inverse of  $K_{,ij}$ . Following the procedure developed earlier [6] we write the six equations (3.23) and (4.1) in the form of a variational scalar product as

$$(4.2) \quad \iiint_{\Omega} \left[ -\frac{\partial f_{,ij}}{\partial x_i} + \left( m_0 - \frac{\partial M_i}{\partial x_i} \right) \frac{\partial U}{\partial \xi_i} \right] \delta \xi_i \, d\Omega$$

$$+ \iiint_{\Omega} \left( \frac{\partial \phi}{\partial x_i} + \Lambda_{,i} \dot{M}_i \right) \delta M_i \, d\Omega = 0.$$

This variational equation must be valid for arbitrary variations  $\delta\xi_i$ ,  $\delta M_i$ . The integrals are over the volume  $\Omega$  in the space  $x_i$  of the initial state. Integration by parts of equation (4.2) yields the variational principle

$$(4.3) \quad \delta V + \iiint_{\Omega} \Lambda_{i,i} \dot{M}_i \delta M_i d\Omega = \int_S f_i \delta\xi_i dS - \int_S \phi n_i \delta M_i dS$$

where

$$(4.4) \quad V = \iiint_{\Omega} \left[ W + \left( m_0 - \frac{\partial M_i}{\partial x_i} \right) U \right] d\Omega$$

is the total free energy of the system including the body force energy. The surface integrals are extended over the boundary  $S$  of the initial volume  $\Omega$ . The vector  $n_i$  denotes the unit outward normal to the initial boundary, while

$$(4.5) \quad f_i = n_i f_{ij}$$

is the force acting at the deformed boundary per unit initial area. Hence  $f_i \delta\xi_i$  is the virtual work of these boundary forces. The quantity  $n_i \delta M_i$  is the variation of the outward mass flow across the deformed boundary per unit initial area.

We now represent the unknown fields  $\xi_i$  and  $M_i$  by  $n$  generalized coordinates  $q_i$ ,

$$(4.6) \quad \begin{aligned} \xi_i &= \xi_i(q_1, q_2 \cdots q_n, x_i, t) \\ M_i &= M_i(q_1, q_2 \cdots q_n, x_i, t). \end{aligned}$$

With variations  $\delta q_i$ , the variational principle (4.3) becomes

$$(4.7) \quad \frac{\partial V}{\partial q_i} \delta q_i + \iiint_{\Omega} \Lambda_{i,i} \dot{M}_i \frac{\partial M_i}{\partial q_i} \delta q_i d\Omega = Q_i \delta q_i$$

where

$$(4.8) \quad Q_i = \iint_S f_i \frac{\partial \xi_i}{\partial q_i} dS - \iint_S \phi n_i \frac{\partial M_i}{\partial q_i} dS$$

is a generalized force at the boundary. Finally from

$$(4.9) \quad \dot{M}_i = \frac{\partial M_i}{\partial q_i} \dot{q}_i + \frac{\partial M_i}{\partial t}$$

we derive

$$(4.10) \quad \frac{\partial \dot{M}_i}{\partial \dot{q}_i} = \frac{\partial M_i}{\partial q_i}.$$

Hence because of the reciprocity property  $\Lambda_{i,i} = \Lambda_i$ , we may write

$$(4.11) \quad \iiint_{\Omega} \Lambda_{i,i} \dot{M}_i \frac{\partial M_i}{\partial q_i} d\Omega = \iiint_{\Omega} \Lambda_i \dot{M}_i \frac{\partial \dot{M}_i}{\partial q_i} d\Omega = \frac{\partial D}{\partial \dot{q}_i}$$

where  $D$  is a dissipation function

$$(4.12) \quad D = \frac{1}{2} \iiint_{\Omega} \Lambda_i \dot{M}_i \dot{M}_i \, d\Omega.$$

Since the variation principle (4.7) must be satisfied for arbitrary variations  $\delta q_i$ , we derive the conditions

$$(4.13) \quad \frac{\partial V}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = Q_i.$$

They constitute the  $n$  Lagrangian equations for the  $n$  unknown generalized coordinates  $q_i$ .

*Lagrangian equations and irreversible thermodynamics.* The equilibrium conditions may be written

$$(4.14) \quad \frac{\partial V}{\partial q_i} = Q_i.$$

The departure from equilibrium is measured by the “disequilibrium force”

$$(4.15) \quad X_i = Q_i - \frac{\partial V}{\partial q_i}.$$

For a thermodynamic system whose behavior obeys Onsager’s relations when disturbed from equilibrium the author has shown [7] that the equations of the system are

$$(4.16) \quad X_i = \frac{\partial D}{\partial \dot{q}_i}$$

where  $D$  is a quadratic form in  $\dot{q}_i$ , which represents the dissipated energy proportional to the entropy production. While developed in the context of linear systems the result is obviously valid for the non-linear case where  $V$  is arbitrary and the coefficients in the quadratic form  $D$  may be functions of  $q_i$ . Equations (4.15) and (4.16) are identical to the Lagrangian equations (4.13).

*Lagrangian equations for the case of a non-Newtonian fluid.* We have shown that Darcy’s law in this case is expressed by equation (3.34). The variational principal (4.3) must be replaced by

$$(4.17) \quad \delta V + \iiint_{\Omega} \frac{\partial \mathfrak{D}}{\partial \dot{M}_i} \delta M_i \, d\Omega = \iint_S f_i \delta \xi_i \, dS - \iint_S \phi n_i \delta M_i \, dS$$

where  $\mathfrak{D}$  is the dissipation function per unit initial volume of porous medium. By proceeding as above we obtain the Lagrangian equations (4.13) where the dissipation function is now

$$(4.18) \quad D = \iiint_{\Omega} \mathfrak{D} \, d\Omega.$$

Note that  $\mathfrak{D}$  is a function of  $\alpha_{ij}$ ,  $m$ , and  $\dot{M}_i$ .

**5. Isotropy and second order theory.** We consider the case of isotropic bulk properties of the porous medium. This case may be formulated by considering the three invariants

$$(5.1) \quad \begin{aligned} I_1 &= \alpha_{11} + \alpha_{22} + \alpha_{33}, \\ I_2 &= \begin{vmatrix} \alpha_{22} & \alpha_{23} \\ \alpha_{32} & \alpha_{33} \end{vmatrix} + \begin{vmatrix} \alpha_{33} & \alpha_{31} \\ \alpha_{13} & \alpha_{11} \end{vmatrix} + \begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix}, \\ I_3 &= \det |\alpha_{ij}|. \end{aligned}$$

The free energy in this case is a function of  $I_1$ ,  $I_2$ ,  $I_3$  and  $m$ ,

$$(5.2) \quad W = W(I_1, I_2, I_3, m).$$

The functional dependence of the permeability on the deformation is also restricted by the condition of isotropy. The intrinsic permeability is expressed as

$$(5.3) \quad K_{ij} = F_1 \frac{\partial I_1}{\partial \alpha_{ij}} + F_2 \frac{\partial I_2}{\partial \alpha_{ij}} + F_3 \frac{\partial I_3}{\partial \alpha_{ij}}$$

where  $F_1$ ,  $F_2$ ,  $F_3$  are functions of the three invariants (5.1) and  $m$ . Note that when  $K_{ij}$  has the value (5.3) the expression  $K_{i;\alpha_{ij}}$  is an invariant. Since  $\alpha_{ij}$  is covariant, expression (5.3) for  $K_{ij}$  has the correct contravariant property and obeys the same transformation laws as (3.15). That it represents an isotropic property can be seen by referring  $\alpha_{ij}$  to its principal directions in which case (5.3) takes the form

$$(5.4) \quad K_{11} = f(\alpha_{11}, \alpha_{22} + \alpha_{33}, \alpha_{22}\alpha_{33}, m)$$

and two others obtained by cyclic permutation.

The medium has been assumed statistically homogeneous, but of course this is not required since  $W$  and  $K_{ij}$  may also be functions of  $x_i$ . We shall consider in particular the second theory where

$$(5.5) \quad \xi_i = x_i + u_i$$

and  $u_i$  is a first order displacement. According to equation (2.25) Green's strain tensor is

$$(5.6) \quad \gamma_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i} \frac{\partial u_k}{\partial x_j} \right).$$

The free energy for an isotropic medium may be written

$$(5.7) \quad W = W(\mathfrak{J}_1, \mathfrak{J}_2, \mathfrak{J}_3, m)$$

where  $\mathfrak{J}_1$ ,  $\mathfrak{J}_2$ ,  $\mathfrak{J}_3$  are the invariants obtained from expressions (5.1) replacing  $\alpha_{ij}$  by  $\gamma_{ij}$ . In a second order theory the relations

$$(5.8) \quad \begin{aligned} f_{ij} &= \frac{\partial W}{\partial c_{ij}}, \\ \psi &= \frac{\partial W}{\partial m} \end{aligned}$$

must contain all first and second order terms in  $(\partial u_i / \partial x_j)$  and  $m$ . This requires  $W$  to be of the general form

$$(5.9) \quad \begin{aligned} W = C_1 \mathfrak{I}_1^2 + C_2 \mathfrak{I}_2 + C_3 \mathfrak{I}_1^3 + C_4 \mathfrak{I}_1 \mathfrak{I}_2 + C_5 \mathfrak{I}_3 + C_6 m^2 + C_7 m^3 \\ + C_8 \mathfrak{I}_1 m + C_9 \mathfrak{I}_1^2 m + C_{10} \mathfrak{I}_1 m^2 + C_{11} \mathfrak{I}_2 m \end{aligned}$$

where  $C_1, C_2, \dots$  are constants. Hence the free energy is defined by eleven constants in this case. That eleven constants are required to define second order elastic properties of a porous medium was already derived earlier by the author [1]. Putting  $m = 0$  yields the five coefficients of the classical second order elasticity theory.

A second order theory also requires the functional dependence of the permeability to be expressed to the first order. From equation (5.3) we derive

$$(5.10) \quad K_{ij} = K \delta_{ij} + 2\beta_1 e_{ij} + (\beta_2 e_{ik} \delta_{jk} + \beta_3 m) \delta_{ij}$$

where  $\beta_1, \beta_2$  and  $\beta_3$  are constants and

$$(5.11) \quad e_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

is the first order strain.

**6. Viscoelastic behavior.** In addition to the dissipation represented by Darcy's law, there is another contribution due to the dissipation in the fluid by the rate of deformation of the solid. It is generally much smaller than the former. However while remaining small it may become dominant when Darcy's flow is negligible or disappears. This will be the case for nearly homogeneous deformations with small gradients of the potential  $\phi$ . There are also special frequencies in wave propagation where this happens. It is important to note that this additional dissipation is uncoupled from Darcy's law so that the two effects may be evaluated independently. This can be seen by evaluating the combined dissipation in the fluid. The cross product terms disappear because of symmetry considerations since they are products of the vector  $\dot{w}_i$  and the strain rate tensor  $e'_{ij}$ . We must assume of course a fluid of linear viscosity. The property may also be considered as a consequence of Curie's theorem. However the conclusion is not valid if the fluid is non-Newtonian with strong non-linearity.

Essentially what is involved here is a behavior analogous to a non-linear viscoelastic solid. This will be rigorously the case if  $m = 0$ , *i.e.*, if the fluid content of the pores does not change, as would be obtained for sealed pores. However

The last equations are linear differential equations for  $q_i$ . If  $\alpha_{i,j}$  and  $m$  are given functions of time, the expressions

$$(6.7) \quad \alpha_i = -A_i - B_i$$

are known functions of time. The last of equations (6.6) may then be solved for  $q_i$ . The solution is

$$(6.8) \quad q_i = \hat{\mathfrak{L}}_{i,j} \alpha_i$$

where  $\hat{\mathfrak{L}}_{i,j}$  is the integral operator defined as

$$(6.9) \quad \hat{\mathfrak{L}}_{i,j} \alpha_i = \sum C_{ij}^{(s)} \int_0^t \exp [\lambda_s(t' - t)] \alpha_i dt'$$

The relaxation constants  $\lambda_s$  are non-negative. The solution (6.8) was derived by the author for linear irreversible thermodynamics in the context of viscoelasticity [8] and of heat conduction [6]. The non-negative character of  $\lambda_s$  is a consequence of the fact that  $a_{i,j}$  and  $b_{i,j}$  are symmetric and define respectively non-negative and positive-definite forms.

By substituting the value (6.8) for  $q_k$  into equations (6.6) we obtain expressions for  $f_{i,j}$  and  $\psi$  as relaxation functionals of  $c_{i,j}$  and  $m$ . The particular interest of this result lies in the fact that the experimental non-linear behavior of porous and viscoelastic media tends to follow expressions (6.6).

In many cases an adequate description of the material is obtained by putting

$$(6.10) \quad \frac{\partial B_k}{\partial \dot{c}_{i,j}} = \frac{\partial D_1}{\partial \dot{m}} = \frac{\partial B_k}{\partial \dot{m}} = 0.$$

Hence

$$(6.11) \quad f_{i,j} = \frac{\partial W_1}{\partial c_{i,j}} + \frac{\partial D_1}{\partial \dot{c}_{i,j}} + \frac{\partial A_k}{\partial c_{i,j}} \hat{\mathfrak{L}}_{ki} \alpha_i,$$

$$\psi = \frac{\partial W_1}{\partial m} + \frac{\partial A_k}{\partial m} \hat{\mathfrak{L}}_{ki} \alpha_i.$$

The term  $\partial D_1 / \partial \dot{c}_{i,j}$  represents a viscous resistance while the terms containing  $\hat{\mathfrak{L}}_{ki} \alpha_i$  represent a relaxation. They are simply superposed, as additional terms, upon the static equilibrium stress-strain relations

$$(6.12) \quad f_{i,j} = \frac{\partial W_1}{\partial c_{i,j}},$$

$$\psi = \frac{\partial W_1}{\partial m}.$$

Note that the second of equations (6.11) implies a relaxation of  $\psi$ , hence of the fluid pressure due to a deformation or a change in fluid content. Physically this corresponds to a delayed penetration of the fluid into very small micropores or cracks. Also this may be associated with a squeezing effect of the viscous fluid in thin interstitial gaps between grains. These effects were pointed out



and discussed earlier in the linear context [1], [2]. For small strain and small values of the volumetric fluid content  $\zeta$  ( $\zeta = m/\rho_0$ ,  $\rho_0 =$  initial fluid density) the equations may be linearized to the form

$$(6.13) \quad \begin{aligned} \tau_{ij} &= \hat{A}_{ij}^{\mu\nu} e_{\mu\nu} + \hat{M}_{ij} \zeta \\ p &= \hat{M}_{ij} e_{ij} + \hat{M} \zeta \end{aligned}$$

where  $\tau_{ij}$  is the stress and  $p$  is the increment of fluid pressure. The operators  $\hat{A}_{ij}^{\mu\nu} = \hat{A}_{\mu\nu}^{ij}$  are defined by

$$(6.14) \quad \hat{A}_{ij}^{\mu\nu} = \int_0^t \sum_s A_{ij}^{(s)\mu\nu} \exp[\lambda_s(t' - t)] d. + A_{ij}^{\mu\nu} + A_{ij}^{\mu\nu} \frac{d}{dt}.$$

The operators  $\hat{M}_{ij}$  and  $\hat{M}$  are given by similar expressions. Equations (6.13) coincide with those derived from the linear thermodynamic theory of porous media [1], [2]. This linear theory includes not only the effect of the fluid viscosity but also the viscoelastic properties of the solid matrix itself. Note that expressions (6.6) for finite deformations are also applicable to a viscoelastic solid if we assume that the internal degrees of freedom satisfy linear thermodynamics.

As already stated for a Newtonian fluid, the dissipation involved in this viscous relaxation effects expressed by equations (6.6) are not coupled to Darcy's law. It is therefore possible to include these effects in the general analysis of the deformation field by using the values (6.6) for  $f_{ij}$  and  $\psi$  into the equilibrium equations (3.23) and in Darcy's law (3.24).

**7. Simplified equations for cylindrical and spherical symmetry.** The preceding general analysis uses the metric tensor (2.4) which constitutes one way to express the property that the free energy is rotation-independent. In practice this causes many difficulties. However, the use of the metric tensor may be avoided when local rotation is absent from the deformation field. This leads to drastic simplification without loss of rigor. We shall consider two such cases with axial and spherical symmetry. The first case assumes an axis of symmetry  $z$  with a radial coordinate  $r$  in the plane perpendicular to  $z$ . The deformation is defined by a radial horizontal displacement  $u(r, t)$  function only of  $r$  and the time  $t$ . Hence the deformation is the same in all horizontal planes. We also assume that the body force is negligible. The finite principal strains in the radial and circumferential directions are respectively

$$(7.1) \quad \begin{aligned} \epsilon_1 &= \frac{\partial u}{\partial r}, \\ \epsilon_2 &= \frac{u}{r}. \end{aligned}$$

The corresponding principal stresses  $\tau_1$  and  $\tau_2$  are defined as the forces per unit initial area. The principle of virtual work is written

$$(7.2) \quad \int (\tau_1 \delta\epsilon_1 + \tau_2 \delta\epsilon_2)r \, dr = 0$$

which must be true for arbitrary variations  $\delta u$ . We choose variations  $\delta u$  which are different from zero only within the limits of integration. Substituting the values (7.1) for  $\epsilon_1$  and  $\epsilon_2$  and integrating by parts yields the equilibrium condition for the stress field

$$(7.3) \quad \frac{\partial \tau_1}{\partial r} + \frac{\tau_1 - \tau_2}{r} = 0.$$

Let the medium be isotropic. The free energy per unit initial volume is of the form

$$(7.4) \quad W = W(J_1, J_2, m)$$

with

$$(7.5) \quad \begin{aligned} J_1 &= \epsilon_1 + \epsilon_2, \\ J_2 &= \epsilon_1\epsilon_2 \end{aligned}$$

as required by the assumption of isotropy. The stresses and the pressure function are

$$(7.6) \quad \begin{aligned} \tau_1 &= \frac{\partial W}{\partial \epsilon_1} = \frac{\partial W}{\partial J_1} + \epsilon_2 \frac{\partial W}{\partial J_2}, \\ \tau_2 &= \frac{\partial W}{\partial \epsilon_2} = \frac{\partial W}{\partial J_1} + \epsilon_1 \frac{\partial W}{\partial J_2}, \\ \psi &= \frac{\partial W}{\partial m}. \end{aligned}$$

The generalized Darcy's law takes the form

$$(7.7) \quad \dot{M} = -K \frac{\partial \psi}{\partial m}$$

where  $M$  is the total mass of fluid which has flowed through an area initially equal to unity and perpendicular to the radial direction. Conservation of mass is expressed by

$$(7.8) \quad m = -\frac{1}{r} \frac{\partial}{\partial r} (rM).$$

The intrinsic permeability  $K$  is a function of  $\epsilon_1$ ,  $\epsilon_2$  and  $m$ .

$$(7.9) \quad K = K(\epsilon_1, \epsilon_2, m).$$

Substitution of the values (7.6) and (7.8) into the equation of equilibrium (7.3) and Darcy's law (7.7) yields two equations for the unknowns  $u$  and  $M$ . The Lagrangian equations are obtained by writing  $u$  and  $M$  in terms of the generalized coordinates  $q_i$ .

$$(7.10) \quad \begin{aligned} u &= u(q_1, q_2 \cdots q_n, r, t), \\ M &= M(q_1, q_2 \cdots q_n, r, t). \end{aligned}$$

We consider a system within initial cylindrical boundaries at  $r = a$  and  $r = b$ . The values of  $V$ ,  $D$  and  $Q_i$  to be used in the Lagrangian equations (4.13) are

$$(7.11) \quad \begin{aligned} V &= \int_a^b W r \, dr, \\ D &= \frac{1}{2} \int_a^b \frac{M^2}{K} r \, dr, \\ Q_i &= b\tau_b \frac{\partial u_b}{\partial q_i} - a\tau_a \frac{\partial u_a}{\partial q_i} - b\psi_b \frac{\partial M_b}{\partial q_i} + a\psi_a \frac{\partial M_a}{\partial q_i}. \end{aligned}$$

The subscripts indicate the values of the variables at the boundaries. For example  $\tau_a$  is the value of  $\tau_1$  at  $r = a$ .

A similar analysis is valid for the case of spherical symmetry. In this case  $u$  denotes the radial displacement at the distance  $r$  from the origin. It is a function only of  $r$  and the time  $t$ . The principal stresses per unit initial area are  $\tau_1$  along  $r$  and  $\tau_2 = \tau_3$  in a plane tangent to the spherical surface centered at the origin. The corresponding principal strains are  $\epsilon_1$  and  $\epsilon_2 = \epsilon_3$  and are given in terms of  $r$  by the same expressions (7.1). The principle of virtual work yields

$$(7.12) \quad \int (\tau_1 \delta\epsilon_1 + 2\tau_2 \delta\epsilon_2) r^2 \, dr = 0.$$

We derive the equilibrium condition for the stress

$$(7.13) \quad \frac{\partial \tau_1}{\partial r} + \frac{2(\tau_1 - \tau_2)}{r} = 0.$$

For an isotropic material the free energy is

$$(7.14) \quad W = W(J_1, J_2, J_3, m)$$

where

$$(7.15) \quad \begin{aligned} J_1 &= \epsilon_1 + \epsilon_2 + \epsilon_3 = \epsilon_1 + 2\epsilon_2, \\ J_2 &= \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1 + \epsilon_1\epsilon_2 = 2\epsilon_1\epsilon_2 + \epsilon_2^2, \\ J_3 &= \epsilon_1\epsilon_2\epsilon_3 = \epsilon_1\epsilon_2^2. \end{aligned}$$

We express  $W$  in terms of the three variables  $\epsilon_1$ ,  $\epsilon_2$ ,  $m$  and write the stress-strain relations

$$(7.16) \quad \begin{aligned} \tau_1 &= \frac{\partial W}{\partial \epsilon_1}, \\ \tau_2 &= \frac{1}{2} \frac{\partial W}{\partial \epsilon_2} \\ \psi &= \frac{\partial W}{\partial m}. \end{aligned}$$

Darcy's law is given by equation (7.7) as for the cylindrical case and the conservation equation is

$$(7.17) \quad m = -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 M)$$

where  $M$  is now the mass of fluid which has flowed through a spherical surface, centered at the origin, per unit initial area. Substitution of the values (7.16) and (7.17) for  $\tau_1, \tau_2, \psi$  and  $m$  into the equilibrium equation (7.13) and Darcy's law (7.7) yields two equations for the unknowns  $u$  and  $M$ .

Generalized coordinates may be introduced by expressions formally identical to (7.10) where  $r$  is now a spherical coordinate. For a material initially bounded by two spheres  $r = a$  and  $r = b$  the Lagrangian equations are obtained using the values

$$(7.18) \quad \begin{aligned} V &= \int_a^b W r^2 dr, \\ D &= \frac{1}{2} \int_a^b \frac{M^2}{K} r^2 dr, \\ Q_i &= b^2 \tau_b \frac{\partial u_b}{\partial q_i} - a^2 \tau_a \frac{\partial u_a}{\partial q_i} - b^2 \psi_b \frac{\partial M_b}{\partial q_i} + a^2 \psi_a \frac{\partial M_a}{\partial q_i}. \end{aligned}$$

The values of  $\tau_i$  at the boundaries are  $\tau_a$  and  $\tau_b$  with subscripts denoting similarly the boundary values of other variables.

**Appendix.**

*Properties of the dissipation function.* The existence of a dissipation function and its properties in connection with the Lagrangian equations of a fluid with Newtonian viscosity are well known. Use of these properties has been made earlier in the theory of porous media [1], [2]. They are briefly recalled here.

The velocity field  $v_i$  of a Newtonian fluid of specific mass  $\rho$  is referred to Cartesian coordinates. When inertia forces are neglected, the stress field  $\sigma_{ij}$  in the fluid satisfies the equilibrium equations

$$(A.1) \quad \frac{\partial \sigma_{ij}}{\partial \xi_j} - \rho \frac{\partial U}{\partial \xi_i} = 0.$$

The body force potential per unit mass is denoted by  $U$ . The stress  $\sigma_{ij}$  is the sum of two terms

$$(A.2) \quad \sigma_{ij} = \sigma'_{ij} - p \delta_{ij}$$

where  $\sigma'_{ij}$  is a viscous stress while  $p$  is a hydrostatic stress depending only on volume changes. The viscous stress is expressed by

$$(A.3) \quad \sigma'_{ij} = 2\eta e'_{ij} - \frac{2}{3}\eta e' \delta_{ij}$$

with a viscosity coefficient  $\eta$  and strain-rate components

$$(A.4) \quad e'_{ii} = \frac{1}{2} \left( \frac{\partial v_i}{\partial \xi_i} + \frac{\partial v_i}{\partial \xi_i} \right),$$

$$e' = e'_{11} + e'_{22} + e'_{33}.$$

Note that  $\sigma'_{ij}$  is the stress deviator. Introducing the value (A.2) of  $\sigma_{ij}$  the equilibrium equations (A.1) become

$$(A.5) \quad \frac{\partial \sigma'_{ij}}{\partial \xi_i} - \rho \frac{\partial \phi}{\partial \xi_i} = 0$$

where  $\phi = \psi + U$ . If this equation is satisfied we may obviously write

$$(A.6) \quad \iiint_{\Omega'} \left( \frac{\partial \sigma'_{ij}}{\partial \xi_i} - \rho \frac{\partial \phi}{\partial \xi_i} \right) \delta v_i \, d\Omega' = 0$$

where  $\delta v_i$  are arbitrary variations of the velocity field. If  $\delta v_i$  are chosen to vanish at the boundary of the volume of integration  $\Omega'$ , then integration by parts in equation (A.6) yields

$$(A.7) \quad \iiint_{\Omega'} \left( \sigma'_{ij} \delta e'_{ij} + \rho \frac{\partial \phi}{\partial \xi_i} \delta v_i \right) d\Omega' = 0.$$

We now introduce an important property of  $\sigma'_{ij}$ . Equation (A.3) shows that it may be written in the form

$$(A.8) \quad \sigma'_{ij} = \frac{\partial \varphi}{\partial e'_{ij}}$$

where

$$(A.9) \quad \varphi = \eta e'_{ij} e'_{ij} - \frac{1}{3} \eta e'^2$$

is the dissipation function of the fluid and

$$(A.10) \quad 2\varphi = \sigma'_{ij} e'_{ij}$$

represents the power dissipated per unit volume. With the value (A.8) for  $\sigma'_{ij}$  equation (A.7) becomes a variational principle

$$(A.11) \quad \iiint_{\Omega'} \left( \delta \varphi + \rho \frac{\partial \phi}{\partial \xi_i} \delta v_i \right) d\Omega' = 0.$$

We shall apply this result to derive the generalized form of Darcy's law by considering the motion of a viscous fluid through a rigid porous solid. We assume a Poiseuille type flow with negligible inertia forces and consider a small region  $R$  of the material sufficiently large relative to the pores so that properties may be averaged. In this region the microvelocity field relative to the solid may be expressed as a linear function of the average rate of volume flow  $\dot{w}_i$  through the region. We may write

$$(A.12) \quad v_i = v_{ij}(\xi_k) \dot{w}_j$$

where the coefficients  $v_{,i}$  are functions of the local coordinates  $\xi_k$ . From (A.12) we derive the local strain rate distribution

$$(A.13) \quad e'_{i,j} = \frac{1}{2} \left( \frac{\partial v_{,ik}}{\partial \xi_j} + \frac{\partial v_{,jk}}{\partial \xi_i} \right) \dot{w}_k .$$

This expression neglects the terms containing  $\partial \dot{w}_i / \partial \xi_j$ . This is consistent with the basic assumption that  $\dot{w}_i$  is sufficiently uniform throughout a region  $R$  which is large relative to the pore size. The major contribution to the dissipation is therefore due to  $\dot{w}_i$  and the effect of  $\partial \dot{w}_i / \partial \xi_j$  may be neglected. The local dissipation function  $\varphi$  is then evaluated using (A.13). We also introduce the following averaged values, over the domain  $R$

$$(A.14) \quad D' = \frac{1}{R} \iiint_R \varphi \, dR = \frac{1}{2} \eta b_{,ij} \dot{w}_i \dot{w}_j ,$$

$$\frac{1}{R} \iiint_R \rho \frac{\partial \phi}{\partial \xi_i} \delta v_i \, dR = \rho \frac{\partial \phi}{\partial \xi_i} \delta \dot{w}_i .$$

In evaluating the second integral we have replaced  $\rho \partial \phi / \partial \xi_i$  by a constant average value in  $R$ . The variational principle (A.11) is applicable to the fluid contained in a certain volume  $\Omega$  of the porous solid. The volume occupied by the fluid is  $\Omega'$ . However we may extend the integration to the volume  $\Omega$  by putting  $v_i = 0$  in the regions occupied by the solid. The validity of the variational principle (A.11) requires  $\delta v_i = 0$  at the boundaries of  $\Omega'$ . This condition is automatically satisfied at the fluid-solid interfaces because of the fluid viscosity. In addition we impose the condition  $\delta \dot{w}_i$  at the boundaries of  $\Omega$ . This insures that  $\delta v_i$  also vanishes at boundaries of the fluid which are not in contact with the solid. Introducing the average values (A.14) the variational principle (A.11) becomes

$$(A.15) \quad \iiint_{\Omega} \left( \eta b_{,ij} \dot{w}_i + \rho \frac{\partial \phi}{\partial \xi_i} \right) \delta \dot{w}_i \, d\Omega = 0 .$$

This being valid for arbitrary variations  $\delta \dot{w}_i$  inside the volume  $\Omega$ , we derive

$$(A.16) \quad \eta b_{,ij} \dot{w}_i + \rho \frac{\partial \phi}{\partial \xi_i} = 0 .$$

Solving for  $\dot{w}_i$  this result may be written

$$(A.17) \quad \dot{w}_i = -k_{,ij} \frac{\rho}{\eta} \frac{\partial \phi}{\partial \xi_j}$$

with the symmetry property

$$(A.18) \quad k_{,ij} = k_{,ji} .$$

This result established Darcy's law (3.3).

**Non-Newtonian fluids.** The concept of dissipation function may be extended to non-Newtonian isotropic fluids. This is immediately evident since the varia-

tional principle is based solely on the fact that the relation between fluid stress and the strain rate may be written in the form

$$(A.19) \quad \sigma_{ii} = \frac{\partial \varphi}{\partial e'_{ii}} - p \delta_{ii}$$

where

$$(A.20) \quad \varphi = \varphi(I'_1, I'_2, I'_3)$$

is a function of the three invariants of the strain-rate. They are obtained by replacing  $\alpha_{ii}$  by  $e'_{ii}$  in expressions (5.1). Again we express the microvelocity field as a function of  $\dot{w}_i$ . Since the fluid is now non-linear the distribution of the microvelocity field depends on the magnitude of  $\dot{w}_i$ .

From relation (A.19) we derive the same variational principle as (A.11). We may also write average values (A.14). However in this case the average dissipation function

$$(A.21) \quad D' = D'(\dot{w}_i)$$

is not a quadratic form of  $\dot{w}_i$ . Application of the variational principle using the average values (A.14) yields the condition

$$(A.22) \quad \iiint_{\Omega} \left( \frac{\partial D'}{\partial \dot{w}_i} + \rho \frac{\partial \phi}{\partial \xi_i} \right) \delta \dot{w}_i d\Omega = 0.$$

Hence with arbitrary variations  $\delta \dot{w}_i$  we obtain

$$(A.23) \quad \frac{\partial D'}{\partial \dot{w}_i} + \rho \frac{\partial \phi}{\partial \xi_i} = 0.$$

This is the generalized form of Darcy's law for non-Newtonian fluids.

#### REFERENCES

1. M. A. BIOT, *Mechanics of deformation and acoustic propagation in porous media*, Journal of Applied Physics **33** (1962), 1482-1498.
2. M. A. BIOT, *Generalized theory of acoustic propagation in porous dissipative media*, Journal of the Acoustical Society of America **34** (1962), 1254-1264.
3. M. A. BIOT, *Theory of stability and consolidation of a porous medium under initial stress*, J. Math. Mech. **4** (1963), 521-542.
4. M. A. BIOT, *Mechanics of incremental deformations*, John Wiley & Sons, New York, 1965.
5. M. KING HUBBERT, *The theory of ground water motion*, Journal of Geology **XLVIII** (1940) 785-944.
6. M. A. BIOT, *Variational principles in heat transfer*, Clarendon Press, Oxford, 1970.
7. M. A. BIOT, *Linear thermodynamics and the mechanics of solids*, Proceedings of the Third U. S. National Congress of Applied Mechanics, American Society of Mechanical Engineers, New York, 1958, 1-18.
8. M. A. BIOT, *Theory of stress-strain relations in anisotropic viscoelasticity and relaxation phenomena*, Journal of Applied Physics. **25** (1954), 1385-1391.

This research was sponsored by the A. F. Office of Scientific Research (SREM), through the European Office of Aerospace Research, OAR, United States Air Force, under contract F 44620-71-C-0092.

*Date communicated:* JULY 8, 1971