

## Theory of Gas Bubble Dynamics in Oscillating Pressure Fields

MILTON S. PLESSET AND DIN-YU HSIEH\*

*California Institute of Technology, Pasadena, California*

(Received August 18, 1960)

The behavior of a permanent gas bubble in a liquid with an oscillating pressure field is analyzed with a linearized theory. If the assumption is made that conditions within the bubble are uniform, the thermodynamic relations found are as expected; i.e., at low frequencies the bubble behaves isothermally and at high frequencies the behavior becomes adiabatic. However, a more detailed analysis, which allows the bubble interior to vary not only in time but also in space, leads to an average isothermal behavior for the bubble even in the high-frequency limit.

### INTRODUCTION

THE problem which will be considered in this paper is the dynamic behavior of a gas bubble in a liquid with an oscillatory pressure field. In particular, the steady-state motion will be sought. In one of the earliest papers concerned with gas bubble dynamics, Rayleigh<sup>1</sup> discussed the free oscillations of a gas bubble. These free oscillations would not persist in a model which included heat conduction and viscous effects, and they will be disregarded in the present treatment. The existence of a steady-state solution for the case of an oscillatory disturbance is only a conjecture even for a simplified model. An obvious simplification which is required is the exclusion of any deviation from equilibrium such as is represented by the growth of a bubble in a superheated<sup>2</sup> or supersaturated<sup>3</sup> liquid. Even with this simplification other effects produce a non-steady behavior. An example of such a one-sided effect is the diffusion of mass into a gas bubble in an oscillating pressure field.<sup>4</sup> It will be shown here, however, that a steady state is actually attained provided the disturbance is small enough so that the problem can be linearized. One-sided effects such as "rectified" mass diffusion are then of second order.

The problem of the oscillating pressure disturbance is related to the problem of wave propagation in a liquid containing gas bubbles. The problem of the scattering of sound waves by a gas bubble and the propagation of shock waves through a gassy

liquid have already been treated.<sup>5,6</sup> The thermodynamic behavior of the bubble under the influence of a wave, which was not considered in these investigations, will be examined here. Furthermore, for long wavelengths the spherically symmetric problem treated here is a good approximation to the behavior of a gas bubble when an acoustic wave passes over it.

The thermodynamic behavior of the bubble which is of principal interest here may be considered by qualitative physical arguments. If  $D_1$  is the thermal diffusivity of the liquid, then one may define the diffusion length  $R_{D_1} = (D_1/\omega)^{1/2}$  which is the characteristic length for the conduction of heat with an oscillatory disturbance of angular frequency  $\omega$ . Now for any increment of temperature  $\Delta T$  in the bubble, the corresponding increase in internal energy is

$$\Delta h_2 = \frac{4}{3} \pi R_0^3 \rho_2 C_2 \Delta T,$$

where  $R_0$  is the radius of the bubble,  $\rho_2$  is the gas density, and  $C_2$  is the specific heat of the gas at constant volume. On the other hand, the flow of heat from the bubble during, for example, the compression half-cycle is approximately

$$\Delta h_1 \cong k_1 \frac{\Delta T}{R_{D_1}} 4\pi R_0^2 \frac{\pi}{\omega},$$

where  $k_1$  is the coefficient of thermal conductivity of the liquid. If  $\rho_1$  is the density of liquid and  $C_1$  its specific heat, then  $D_1 = k_1/\rho_1 C_1$ , and one may write

$$\Delta h_1 \cong 4\pi R_0^2 \rho_1 C_1 (D_1/\omega)^{1/2} \Delta T.$$

One then has

<sup>5</sup> Lord Rayleigh, *The Theory of Sound* (Dover Publications, New York, 1945), Vol. II, pp. 282-284.

<sup>6</sup> I. J. Campbell and A. S. Pitcher, Proc. Roy. Soc. (London) **A243**, 534 (1958).

\* International Nickel Company, Inc. Fellow.

<sup>1</sup> Lord Rayleigh, Phil. Mag. **34**, 94 (1917).

<sup>2</sup> M. S. Plesset and S. A. Zwick, J. Appl. Phys. **25**, 493 (1954).

<sup>3</sup> P. S. Epstein and M. S. Plesset, J. Chem. Phys. **18**, 1505 (1954).

<sup>4</sup> This one-sided diffusion of mass under these conditions has been treated by the authors elsewhere.

$$\frac{\Delta h_1}{\Delta h_2} \cong 3\pi \frac{\rho_1 C_1 (D_1/\omega)^{\frac{3}{2}}}{\rho_2 C_2 R_0}$$

If

$$\frac{3\pi \rho_1 C_1 (D_1/\omega)^{\frac{3}{2}}}{\rho_2 C_2 R_0} \gg 1$$

so that  $\Delta h_2 \ll \Delta h_1$ , then only an insignificant part of the transferred energy is available for the increase of the internal energy, or equivalently for raising the temperature of the bubble. This condition corresponds simply to the situation in which heat diffusion is so rapid that temperature changes in the bubble cannot take place. The thermodynamic process is thus essentially isothermal. On the other hand, if  $\Delta h_1 \ll \Delta h_2$ , corresponding arguments lead to the expectation that the bubble behavior would then be adiabatic.

It will be shown that quite similar results are obtained from a detailed analysis if the assumption is made that the temperature within the bubble is uniform. This assumption of uniform bubble interior is equivalent to the assumption that the coefficient of heat conductivity of the gas is very large. The coefficients of heat conductivity of gases are actually smaller than those for the common liquids, so that the assumption of uniform bubble interior is not justified. When the analysis is extended to allow nonuniform bubble temperatures, it is then found that the high-frequency limit does not lead to adiabatic behavior.

The model used in the calculations consists of a spherical bubble of perfect gas in an incompressible inviscid liquid of infinite extent. With this model it can be seen that potential flow may be assumed for the liquid. It may be remarked also that for this spherically symmetric problem the irrotationality of the flow will not be destroyed by the introduction of viscous effects. Further, in the linearized theory, the addition of the viscous effect makes only a slight modification for a moderately viscous liquid so that the essential features of the problem are not affected. This behavior is due in part to the fact that heat conduction plays the role of a damping factor in the system.

**BUBBLE WITH UNIFORM INTERIOR**

**A. Linearized Formulation of the Problem**

The Bernoulli equation for the potential flow of an incompressible, inviscid fluid is

$$\frac{P_1}{\rho_1} + \frac{1}{2}(\nabla\varphi)^2 - \frac{\partial\varphi}{\partial t} = K(t),$$

where  $\rho_1$  here denotes the density of the liquid,  $P_1$  is the pressure in the liquid,  $\varphi$  is the velocity potential such that the liquid velocity at any point is  $\mathbf{q}_1 = -\nabla\varphi$ , and  $K(t)$  is as indicated a function of time only. For the spherically symmetric case,  $\varphi$  can be written as

$$\varphi = R^2 \dot{R}/r,$$

where  $r = R$  gives the location of the bubble boundary and  $\dot{R}$  is  $dR/dt$ . If  $P_\infty(t)$  is the pressure in the liquid at infinity, the Bernoulli equation can be re-expressed as

$$\frac{P_1}{\rho_1} + \frac{1}{2} \frac{R^4 \dot{R}^2}{r^4} - \frac{1}{r} \left( 2R\dot{R}^2 + R^2 \frac{d^2R}{dt^2} \right) = \frac{P_\infty(t)}{\rho_1}. \tag{1}$$

In particular, at the bubble wall where  $r = R$ , one has

$$R \frac{d^2R}{dt^2} + \frac{3}{2} \dot{R}^2 = \frac{P_e - P_\infty}{\rho_1}, \tag{2}$$

where  $P_e$  is the pressure in the liquid, or the external pressure, at  $r = R$ .<sup>1,7</sup> Let  $P_2$  be the pressure of the gas inside the bubble which is assumed here to be uniform, and let  $\sigma$  be the surface tension constant. Then

$$P_2 = P_e + (2\sigma/R),$$

and the dynamic equation, Eq. (2), becomes

$$R \frac{d^2R}{dt^2} + \frac{3}{2} \dot{R}^2 = \frac{1}{\rho_1} \left[ P_2 - \frac{2\sigma}{R} - P_\infty(t) \right]. \tag{2a}$$

Let  $T_1$  be the temperature at any point in the liquid. Then the heat equation, which expresses the conservation of energy, assumes the following form:

$$\nabla^2 T_1 = \frac{1}{D_1} \left( \frac{\partial T_1}{\partial t} + \mathbf{q}_1 \cdot \nabla T_1 \right), \tag{3}$$

where  $D_1 = k_1/\rho_1 C_1$  is the thermal diffusivity of the liquid,  $k_1$  is the coefficient of thermal conductivity of the liquid, and  $C_1$  is its specific heat. The gas in the bubble is assumed to be perfect and the temperature  $T_2$  is uniform throughout the bubble. In this case

$$P_2(R)R^3 = NT_2, \tag{4}$$

where  $N$  is a constant and is actually equal to  $(3/4\pi)nB$ . Here  $n$  is the number of mols of gas in the bubble and  $B$  is the universal gas constant. It should be noted that the requirement of continuity of temperature gives  $T_2 = T_1$  ( $r = R$ ).

Because of the assumed uniformity inside the gas bubble, the problem of the gas-liquid system

<sup>7</sup> M. S. Plesset, J. Appl. Mech. 16, 277 (1949).

is equivalent to a problem in the liquid only with the state of the gas serving as a boundary condition at the bubble wall. This boundary has an equation of motion given by Eq. (2a). The boundary condition for the heat equation, Eq. (3), at the bubble wall can be formulated from the energy consideration for the gas bubble as

$$MC_2 \frac{dT_2}{dt} = -4\pi R^2 P_2(R) \frac{dR}{dt} + 4\pi R^2 k_1 \left( \frac{\partial T_1}{\partial r} \right)_{r=R}, \quad (5)$$

where  $M$  is the total mass of gas which is assumed to be constant, and  $C_2$  is the specific heat of the gas at constant volume. Since the temperature at infinity is taken to be constant,

$$T_1(\infty, t) = T_\infty, \quad (6)$$

where  $T_\infty$  is the fixed temperature at a distance from the bubble.

It will be supposed that the disturbance begins at a certain instant of time, say  $t = 0$ , so that there is a state of equilibrium for  $t \leq 0$ . Therefore, for  $t \leq 0$ :

$$R(t) = R_0, \quad (7)$$

$$\dot{R}(t) = \frac{d^2 R}{dt^2}(t) = 0, \quad (8)$$

$$P_\infty(t) = P_0, \quad (9)$$

and

$$T_2(t) = T_1(r, t) = T_\infty, \quad (10)$$

where  $R_0$  is the equilibrium radius of the bubble and  $P_0$  is the equilibrium pressure in the liquid.

The disturbance is introduced by a perturbing oscillating pressure at infinity which may be expressed as

$$P_\infty(t) = P_0[1 + \epsilon(t)] = P_0[1 + \epsilon_0 e^{i\omega t}], \quad t > 0. \quad (11)$$

The complex quantity in Eq. (11) is introduced for convenience in the computations and only the real part of the expression has physical significance. Since all the subsequent operations are linear, the physically significant solution is just the real part of the solution with complex quantities.

The linearization procedure is carried out with respect to the equilibrium configuration and is based on the smallness of  $\epsilon_0$  in comparison with unity. From the condition of equilibrium for  $t \leq 0$  one has the following relations:

$$P_2(R_0) - P_0 = \frac{2\sigma}{R_0}; \quad (12)$$

$$T_2(R_0) = T_\infty; \quad (13)$$

and

$$P_2(R_0)R_0^3 = NT_\infty. \quad (14)$$

One now writes

$$R = R_0(1 + x); \quad (15)$$

$$T_1 = T_\infty(1 + \theta_1); \quad T_2 = T_\infty(1 + \theta_2); \quad (16)$$

and,

$$P_2 = P_2(R_0) \left( 1 + \frac{p}{a} \right) = P_0(a + p); \quad (17)$$

where  $x, \theta_1, \theta_2$ , and  $p$ , as well as  $\epsilon$ , are small quantities compared with unity. In Eq. (17), the constant  $a$  is given by

$$a = 1 + W,$$

where

$$W = 2\sigma/P_0R_0.$$

$W$  is sometimes called the Weber number. The linearization of the dynamic equation, Eq. (2a), gives

$$\frac{d^2 x}{dt^2} - \alpha W x = \alpha(p - \epsilon), \quad (18)$$

where

$$\alpha = P_0/\rho_l R_0^2.$$

The linearization of the gas equation of state, Eq. (4), gives

$$p = a\theta_2 - 3ax. \quad (19)$$

The linearization of the heat equation, Eq. (3), leads to the ordinary heat equation without convection:

$$\nabla^2 \theta_1 = \frac{1}{D_1} \frac{\partial \theta_1}{\partial t}, \quad (20)$$

and the boundary condition, Eq. (5), becomes

$$\beta \frac{d\theta_2}{dt} = -aP_0R_0 \frac{dx}{dt} + k_1 T_\infty \left( \frac{\partial \theta_1}{\partial r} \right)_{r=R_0}, \quad (21)$$

where

$$\beta = MC_2 T_\infty / 4\pi R_0^2.$$

One has in a similar way

$$x = \dot{x} = \frac{d^2 x}{dt^2} = \theta_1 = \theta_2 = 0, \quad \text{for } t \leq 0, \quad (22)$$

and

$$\theta_1(\infty, t) = 0, \quad \text{for all } t. \quad (23)$$

### B. The Formal Solution

The problem is now reduced to solution of the heat equation, Eq. (20), with boundary conditions given by Eqs. (21) and (23). In the boundary

condition of Eq. (21), it may be kept in mind that  $\theta_2(t) = \theta_1(R_0, t)$ . The equation of motion for the bubble wall, Eq. (18), may be written in the form

$$\frac{d^2x}{dt^2} + \alpha(3 + 2W)x = \alpha[(1 + W)\theta_2 - \epsilon], \quad (24)$$

with the initial conditions given by Eq. (22). Since the problem has spherical symmetry, Eq. (20) may be written

$$\frac{\partial^2}{\partial r^2}(r\theta_1) = \frac{1}{D_1} \frac{\partial}{\partial t}(r\theta_1). \quad (25)$$

One may introduce the Laplace transform of  $\theta_1$  as follows.

$$w_1 \equiv \mathcal{L}\{\theta_1(r, t)\} \equiv \int_0^\infty \theta_1 e^{-st} dt,$$

and similarly,

$$w_2 \equiv \mathcal{L}\{\theta_2\}; \quad z \equiv \mathcal{L}\{x\}; \quad \mu \equiv \mathcal{L}\{\epsilon\}.$$

Using the initial and boundary conditions, one obtains the following equations:

$$\frac{d^2}{dr^2}(rw_1) = \frac{s}{D_1}(rw_1), \quad (26)$$

with

$$w_1(r = \infty) = 0; \quad (27)$$

$$\beta s w_2 = -P_0 R_0 (1 + W) s z + k_1 T_\infty \left( \frac{dw_1}{dr} \right)_{r=R_0}, \quad (28)$$

with

$$w_2 = (w_1)_{r=R_0}; \quad (29)$$

and

$$[s^2 + \alpha(3 + 2W)]z = \alpha[(1 + W)w_2 - \mu]. \quad (30)$$

The solution is readily found to be

$$w_1 = \frac{\alpha E s \mu R_0}{P(s^{\frac{1}{2}}) r} \exp[-(r - R_0)(s/D_1)^{\frac{1}{2}}], \quad (31)$$

where  $P$  is a polynomial defined by

$$P(u) \equiv (\beta u^2 + Au + B)(u^{\frac{1}{2}} + \omega^{\frac{1}{2}}) + \alpha E(1 + W)u^2. \quad (32)$$

The constants which have been introduced have the following definitions:

$$\omega_1^2 = \alpha(3 + 2W); \quad A = k_1 T_\infty / D_1^{\frac{1}{2}}; \\ B = k_1 T_\infty / R_0; \quad E = P_0 R_0 (1 + W).$$

When  $\epsilon = \epsilon_0 e^{i\omega t}$ , its Laplace transform is

$$\mu \equiv \mathcal{L}\{\epsilon\} = \frac{\epsilon_0}{s - i\omega}$$

so that Eq. (31) then becomes

$$w_1 = \frac{\alpha E \epsilon_0 s}{(s - i\omega) P(s^{\frac{1}{2}})} \frac{R_0}{r} \exp[-(r - R_0)(s/D_1)^{\frac{1}{2}}]. \quad (33)$$

Using Eq. (33), one now finds from Eqs. (28) through (30) that

$$z = -\frac{\alpha \epsilon_0 (\beta s + A s^{\frac{1}{2}} + B)}{(s - i\omega) P(s^{\frac{1}{2}})}. \quad (34)$$

The formal solutions to the problem are obtained by inversion of Eqs. (33) and (34). Let the roots of  $P(u) = 0$  be  $-a_1, -a_2, -a_3, -a_4, -a_5,$  and  $-a_6$ . Also define  $a_7 = \omega^{\frac{1}{2}} e^{i\pi/4}$  and  $a_8 = -\omega^{\frac{1}{2}} e^{i\pi/4} = \omega^{\frac{1}{2}} e^{-i3\pi/4}$ . Then Eqs. (33) and (34) can be expressed, in principle, by partial fractions:

$$w_1(r; s) = \frac{R_0}{r} \exp[-(r - R_0)(s/D_1)^{\frac{1}{2}}] \cdot \sum_{i=1}^8 \frac{b_i}{s^{\frac{1}{2}} + a_i} \quad (35)$$

$$z(s) = \sum_{i=1}^8 \frac{c_i}{s^{\frac{1}{2}} + a_i}. \quad (36)$$

It may be observed that  $\sum_{i=1}^8 b_i = \sum_{i=1}^8 c_i = 0$ ; the inversion proceeds in the usual way<sup>8</sup> to give

$$\theta_1(r, t) = -\frac{R_0}{r} \sum_{i=1}^8 a_i b_i \cdot \text{Erf} c[(r - R_0)/2(D_1 t)^{\frac{1}{2}} + a_i t^{\frac{1}{2}}] \cdot \exp[a_i(r - R_0)/D_1^{\frac{1}{2}} + a_i^2 t], \quad (37)$$

so that

$$\theta_2(t) = -\sum_{i=1}^8 a_i b_i \text{Erf} c(a_i t^{\frac{1}{2}}) \exp(a_i^2 t), \quad (38)$$

and

$$x(t) = -\sum_{i=1}^8 a_i c_i \text{Erf} c(a_i t^{\frac{1}{2}}) \exp(a_i^2 t). \quad (39)$$

One also has

$$p(t) = (1 + W)(\theta_2 - 3x) \\ = (1 + W) \sum_{i=1}^8 a_i (3c_i - b_i) \text{Erf} c(a_i t^{\frac{1}{2}}) \exp(a_i^2 t). \quad (40)$$

The notation used here is that

$$\text{Erf} c(x) = 2\pi^{-\frac{1}{2}} \int_x^\infty \exp(-t^2) dt.$$

<sup>8</sup> See for example *Tables of Integral Transforms*, edited by A. Erdélyi (McGraw-Hill Book Company, Inc., New York, 1954).

**C. Asymptotic Time Behavior and Thermodynamic Relations**

The formal solutions just presented have little practical significance since it is difficult to find the roots  $a_i$ , except by numerical methods when specific values for the physical constants are used. In addition, even if the roots are located, it is not a simple matter to visualize the behavior of the error functions with complex arguments. The determination of the asymptotic behavior of the solutions, however, is not only dictated by these considerations, but also because it yields the steady-state solution which is of particular physical significance for thermodynamic relations.

As a preliminary step, the roots of the polynomial  $P(u)$  must be located. Since all the coefficients in this polynomial are real and positive, it may be shown that all its roots,  $-a_1 \cdots -a_6$ , lie in the sector

$$|\arg(-a_i)| > \pi/4, \text{ or } |\arg a_i| < 3\pi/4, \quad (41)$$

by the principle of the argument in the theory of functions (see Appendix I). This result is essential for the boundedness of the asymptotic expressions.

The asymptotic behavior may be found by two different methods. One may start with the formal solutions and use the relations of equivalence between error functions and confluent hypergeometric functions. Then the asymptotic formulas are obtained from the known asymptotic expansions of the confluent hypergeometric functions. Alternatively, one may proceed directly from the inversion integral of the transformed solution and change the contour of integration by use of Cauchy's residue theorem. The asymptotic expressions are then obtained by application of the method of steepest descent. Both methods give, of course, the same result. The second procedure is presented in Appendix II. The asymptotic expressions follow:

$$\theta_1(r, t) = \frac{\alpha\epsilon_0 E\omega}{P(\omega^{1/2}e^{i\pi/4})} \frac{R_0}{r} \exp[-(r - R_0)/(i\omega/D_1)^{1/2}] \cdot \exp[i(\omega t + \pi/2)] + O(t^{-5/2}); \quad (42)$$

$$\theta_2(r, t) = \frac{\alpha\epsilon_0 E\omega}{P(\omega^{1/2}e^{i\pi/4})} \cdot \exp[i(\omega t + \pi/2)] + O(t^{-5/2}); \quad (43)$$

$$x(t) = -\frac{\alpha\epsilon_0[\beta\omega + A\omega^{1/2}e^{-i\pi/4} - iB]}{P(\omega^{1/2}e^{i\pi/4})} \cdot \exp[i(\omega t + \pi/2)] + O(t^{-1}); \quad (44)$$

and

$$p(t) = \frac{\alpha\epsilon_0(1 + W)[(E + 3\beta)\omega + 3A\omega^{1/2}e^{-i\pi/4} - 3iB]}{P(\omega^{1/2}e^{i\pi/4})} \cdot \exp[i(\omega t + \pi/2)] + O(t^{-1}). \quad (45)$$

The forms of these steady-state solutions are typical for a problem of a linear forced oscillation with damping. The damping is introduced indirectly here by the effect of heat conduction.

In considering the thermodynamic behavior of the bubble one may notice that the energy flow condition, Eq. (21),

$$k_1 T_\infty \left( \frac{\partial \theta_1}{\partial r} \right)_{r=R_0} = \beta \frac{d\theta_2}{dt} + P_0 R_0 (1 + W) \frac{dx}{dt} \quad (21')$$

is an expression of the first law of thermodynamics

$$\delta Q = C_2 dT + P dV, \quad (46)$$

and the relative magnitudes of  $C_2 dT$  and  $\delta Q$  indicate the tendency of the system to be isothermal or adiabatic. From the asymptotic formulas, one gets

$$\beta \frac{d\theta_2}{dt} = \frac{g}{3} \frac{\rho_2 C_2}{\rho_1 C_1} R_0^2 \frac{i\omega}{D_1} \exp[i(\omega t + \pi/2)], \quad (47)$$

and

$$k_1 T_\infty \left( \frac{\partial \theta_1}{\partial r} \right)_{r=R_0} = -g[1 + R_0(i\omega/D_1)^{1/2}] \exp[i(\omega t + \pi/2)], \quad (48)$$

where

$$g = \frac{\alpha E \epsilon_0 k_1 T_\infty \omega}{R_0 P(\omega^{1/2}e^{i\pi/4})}$$

and

$$\beta = \frac{1}{3} \rho_2 C_2 R_0 T_\infty.$$

The comparison of Eqs. (47) and (48) leads to the following conclusion regarding the average behavior of the thermodynamic process of the oscillating bubble: the process is isothermal if

$$\frac{\rho_2 C_2}{\rho_1 C_1} R_0^2 \frac{\omega}{D_1} \ll 1$$

and adiabatic if the foregoing inequality holds in the reverse sense.

The conclusion just given has been limited to the *average* behavior since a corresponding result cannot be obtained for the instantaneous behavior for nonzero frequencies because of the phase difference between  $\theta_2$  and  $x$ . With the basic equation of state for a perfect gas one must expect a phase difference between these state variables for any thermodynamic process.

BUBBLE WITH NONUNIFORM INTERIOR

A. Linearized Formulation

When the requirement is no longer imposed that conditions are uniform throughout the bubble, the problem is more complex than the previous case although the formulation of the problem in the surrounding liquid is not changed. Equation (2a) still applies for the motion of the bubble wall, and Eq. (3) again determines the heat flow in the liquid region,  $r \geq R$ . The gas inside the bubble is again assumed to be a perfect gas, but the equation of state is now satisfied locally only because of the nonuniformity of the interior. In place of Eq. (4), one now has

$$P_2(r, t) = B\rho_2(r, t)T_2(r, t), \quad r \leq R, \quad (49)$$

where  $B$  is the ratio of the perfect gas constant to the gram molecular weight of the gas. The conservation of mass in the gas region is expressed by the continuity equation

$$\frac{\partial \rho_2}{\partial t} + \rho_2 \nabla \cdot \mathbf{q}_2 + \mathbf{q}_2 \cdot \nabla \rho_2 = 0, \quad r \leq R. \quad (50)$$

With the neglect of viscous effects, the equation of motion in the interior region is

$$\rho_2 \left[ \frac{\partial \mathbf{q}_2}{\partial t} + (\mathbf{q}_2 \cdot \nabla) \mathbf{q}_2 \right] = -\nabla P_2, \quad r \leq R, \quad (51)$$

and the energy equation takes the following form:

$$\begin{aligned} \nabla^2 T_2 = \frac{1}{D_2} \left[ \frac{\partial T_2}{\partial t} + \mathbf{q}_2 \cdot \nabla T_2 \right] \\ + \frac{1}{k_2} (\nabla \cdot \mathbf{q}_2) P_2, \quad r \leq R. \end{aligned} \quad (52)$$

The last term in Eq. (52) represents the increment in internal energy due to compressibility. The coefficient of heat conduction of the gas is denoted by  $k_2$  and the coefficient of thermal diffusivity by  $D_2 = k_2/\rho_2 C_2$ .

The pressure at a distance from the bubble is prescribed as  $P_\infty(t)$ , and the boundary condition is imposed that all physical quantities must be finite at  $r = 0$  and remain finite as  $r \rightarrow \infty$ . Further, the pressure, the particle velocity, the temperature, and the heat flux must be continuous at the bubble wall,  $r = R$ . These conditions of continuity are

$$P_1(R, t) = P_2(R, t) - (2\sigma/R),$$

$$q_1(R, t) = \dot{R}(t) = q_2(R, t),$$

$$T_1(R, t) = T_2(R, t),$$

and

$$k_1 \left( \frac{\partial T_1}{\partial r} \right)_{r=R} = k_2 \left( \frac{\partial T_2}{\partial r} \right)_{r=R}.$$

As for the previous problem of the uniform interior, the initial conditions are described by saying that the system is in equilibrium for  $t < 0$  and at  $t = 0$  a perturbation in  $P_\infty$  sets the system into motion.

The problem is linearized in the same way as before with the following definitions:

$$R = R_0(1 + x),$$

$$T_1 = T_\infty(1 + \theta_1),$$

$$T_2 = T_\infty(1 + \theta_2),$$

$$P_\infty(t) = P_0[1 + \epsilon(t)],$$

$$P_2 = P_0(a + p),$$

and

$$\rho_2 = \rho_0(1 + \eta),$$

where  $x, \theta_1, \theta_2, p,$  and  $\eta$  are small quantities compared with one and are of the same order as the perturbing term  $\epsilon$  in the pressure. When these quantities are substituted in the governing equations of the system, the linearized equations are obtained upon neglect of second-order terms. In this way, the equation of motion, Eq. (2a), becomes

$$\frac{d^2 x}{dt^2} - \alpha W x = \alpha [p(R_0, t) - \epsilon]. \quad (53)$$

As before  $\alpha = P_0/\rho_1 R_0^2$  and  $W = 2\sigma/P_0 R_0$ . The equilibrium condition gives

$$aP_0 = P_2(R_0, t) = P_0 + (2\sigma/R_0)$$

so that again

$$a = 1 + W.$$

Equation (25) applies here as the linearized energy equation in the liquid. The equilibrium condition in the bubble is

$$aP_0 = B\rho_0 T_\infty,$$

and the linearized local equation of state is

$$p/a = \eta + \theta_2. \quad (54)$$

The linearized form of the continuity equation [Eq. (50)] is

$$\frac{\partial \eta}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_2) = 0, \quad (55)$$

while Eq. (51) becomes

$$\rho_0 (\partial q_2 / \partial t) = -P_0 (\partial p / \partial r). \quad (56)$$

Finally, the energy equation [Eq. (52)] assumes the form

$$\frac{1}{r} \frac{\partial}{\partial r} \left[ r^2 \left( \frac{\partial \theta_2}{\partial r} - \frac{P_0 \alpha q_2}{k_2 T_\infty} \right) \right] = \frac{1}{D_2} \frac{\partial \theta_2}{\partial t} \tag{57}$$

For boundary conditions at the bubble wall

$$q_1(R_0, t) = R_0 \dot{x} = q_2(R_0, t),$$

$$\theta_1(R_0, t) = \theta_2(R_0, t),$$

and

$$k_1 \left( \frac{\partial \theta_1}{\partial r} \right)_{R_0} = k_2 \left( \frac{\partial \theta_2}{\partial r} \right)_{R_0}.$$

It should be remarked that the quantities  $q_1$  and  $q_2$  have been treated in the linearization process as being of the same order of magnitude as  $R_0 \dot{x}$ . Some question might be raised regarding the validity of this linearization for high-frequency oscillations. There should be no difficulty, however, with this point so long as the amplitude of the perturbation can be made arbitrarily small. As the frequency increases, deviations from the theory could appear for other reasons, such as departures from instantaneous validity of the equation of state, or a lack of constancy of the physical coefficients.

**B. The Asymptotic Solutions**

As in the previous case, the problem is solved by application of Laplace transforms. The transformed functions are defined as follows:

$$z = \mathcal{L}\{x\}; \quad w_1 = \mathcal{L}\{\theta_1\}; \quad w_2 = \mathcal{L}\{\theta_2\}; \quad f = \mathcal{L}\{p\};$$

$$g = \mathcal{L}\{\eta\}; \quad v = \mathcal{L}\{q_2\}; \quad \mu = \mathcal{L}\{\epsilon\}.$$

It may be noted that

$$\mathcal{L}\{R_0 \dot{x}\} = R_0 s z = v(R_0) = \mathcal{L}\{q_2(R_0, t)\}.$$

The initial conditions

$$x(r, 0) = \dot{x}(r, 0) = \theta_1(r, 0) = \theta_2(r, 0)$$

$$= q_2(r, 0) = \eta(r, 0) = p(r, 0) = 0$$

lead to the following transformed equations:

$$(s^2 - \alpha W)z = \alpha[f(R_0) - \mu]; \tag{58}$$

$$\frac{d^2}{dr^2}(r w_1) = \frac{s}{D_1}(r w_1), \quad \text{for } r \geq R_0; \tag{59}$$

$$f = a(g + w_2), \quad \text{for } r \leq R_0; \tag{60}$$

$$s g + \frac{1}{r^2} \frac{d}{dr}(r^2 v) = 0, \quad \text{for } r \leq R_0; \tag{61}$$

$$\rho s v = -P_0(df/dr); \tag{62}$$

$$\frac{1}{r} \frac{d^2}{dr^2}(r w_2) - \frac{P_0 a}{k_2 T_\infty} \frac{1}{r^2} \frac{d}{dr}(r^2 v) = \frac{s}{D_2} w_2. \tag{63}$$

Equations (61) and (62) give

$$g = \frac{P_0}{\rho_0 s^2} \frac{1}{r} \frac{d^2}{dr^2}(r f). \tag{64}$$

One then obtains from Eq. (60)

$$w_2 = \frac{f}{a} - g = \frac{f}{a} - \frac{P_0}{\rho_0 s^2} \frac{1}{r} \frac{d^2}{dr^2}(r f). \tag{65}$$

The relations of Eqs. (62) and (65) give with Eq. (63)

$$\frac{d^4}{dr^4}(r f) - \left[ \frac{\rho_0 s^2}{P_0 a} + \left( \frac{P_0 a}{k_2 T_\infty} + \frac{1}{D_2} \right) s \right] \frac{d^2}{dr^2}(r f)$$

$$+ \frac{\rho_0 s^3}{P_0 a D_2}(r f) = 0.$$

This last equation may be written in the form

$$\left( \frac{d^2}{dr^2} - \alpha_1^2 \right) \left( \frac{d^2}{dr^2} - \alpha_2^2 \right) (r f) = 0, \tag{66}$$

where

$$\alpha_{1,2}(s) = \frac{1}{2} \left( \left( \frac{P_0 a}{k_2 T_\infty} + \frac{1}{D_2} \right) s + \frac{\rho_0 s^2}{P_0 a} \right. \\ \left. \pm \left[ \left( \frac{P_0 a}{k_2 T_\infty} + \frac{1}{D_2} \right) s + \frac{\rho_0}{P_0 a} s^2 \right]^2 - \frac{4 \rho_0}{P_0 a D_2} s^3 \right)^{\frac{1}{2}}. \tag{67}$$

The general solution for  $f$  may be now written as

$$f = \frac{1}{r} [A_1 \cosh \alpha_1 r + A_2 \sinh \alpha_1 r \\ + A_3 \cosh \alpha_2 r + A_4 \sinh \alpha_2 r].$$

Equation (64) then determines  $g$  to be

$$g = \frac{P_0}{\rho_0 s^2} \frac{1}{r} [\alpha_1^2 A_1 \cosh \alpha_1 r + \alpha_1^2 A_2 \sinh \alpha_1 r \\ + \alpha_2^2 A_3 \cosh \alpha_2 r + \alpha_2^2 A_4 \sinh \alpha_2 r],$$

since  $f$  is to be finite at  $r = 0$ ,  $A_1 = -A_3$ . Since  $g$  is also finite at  $r = 0$ , and since in general  $\alpha_1 \neq \alpha_2$ , one also has  $A_1 = A_3 = 0$ . Thus

$$f = \frac{1}{r} [A_2 \sinh \alpha_1 r + A_4 \sinh \alpha_2 r]; \tag{68}$$

$$g = \frac{P_0}{\rho_0 s^2} \frac{1}{r} [\alpha_1^2 A_2 \sinh \alpha_1 r + \alpha_2^2 A_4 \sinh \alpha_2 r]. \tag{69}$$

From Eq. (62) one obtains

$$v = -\frac{P_0}{\rho_0 s} \left[ A_2 \left( \frac{\alpha_1 \cosh \alpha_1 r}{r} - \frac{\sinh \alpha_1 r}{r^2} \right) \right. \\ \left. + A_4 \left( \frac{\alpha_2 \cosh \alpha_2 r}{r} - \frac{\sinh \alpha_2 r}{r^2} \right) \right]. \tag{70}$$

It is easily verified that  $v \rightarrow 0$  as  $r \rightarrow 0$ . Equation (65) determines  $w_2$  to be

$$w_2 = \frac{1}{r} \left[ A_2 \left( \frac{1}{a} - \frac{P_0 \alpha_1^2}{\rho_0 s^2} \right) \sinh \alpha_1 r + A_4 \left( \frac{1}{a} - \frac{P_0 \alpha_2^2}{\rho_0 s^2} \right) \sinh \alpha_2 r \right]. \quad (71)$$

The corresponding solution in the liquid is from Eq. (59)

$$w_1 = \frac{A_0}{r} \exp [-(s/D_1)^{1/2} r], \quad (72)$$

where the requirement that  $w_1$  remain finite as  $r \rightarrow \infty$  has been imposed. The constants  $A_0$ ,  $A_2$ , and  $A_4$  which are functions of the parameter  $s$  are determined by the equation of motion of the bubble wall [Eq. (58)] together with the following boundary conditions at  $r = R$ :

$$\begin{aligned} w_1(R_0) &= w_2(R_0); \\ k_1 \left( \frac{dw_1}{dr} \right)_{R_0} &= k_2 \left( \frac{dw_2}{dr} \right)_{R_0}; \\ v(R) &= szR_0. \end{aligned}$$

One finds in this way

$$\begin{aligned} A_2 &= \frac{1}{\Delta(s)} \frac{\alpha \mu s R_0}{s^2 - \alpha W} \left( \frac{1}{a} - \frac{P_0 \alpha_2^2}{\rho_0 s^2} \right) \{ k_2 (\alpha_2 \cosh \alpha_2 R_0 - \sinh \alpha_2 R_0 / R_0) + k_1 [(s/D_1)^{1/2} + 1/R_0] \sinh \alpha_2 R_0 \}; \\ A_4 &= -\frac{1}{\Delta(s)} \frac{\alpha \mu s R_0}{s^2 - \alpha W} \left( \frac{1}{a} - \frac{P_0 \alpha_1^2}{\rho_0 s^2} \right) \{ k_2 (\alpha_1 \cosh \alpha_1 R_0 - \sinh \alpha_1 R_0 / R_0) + k_1 [(s/D_1)^{1/2} + 1/R_0] \sinh \alpha_1 R_0 \}; \end{aligned}$$

where

$$\Delta(s) = \Delta_1(\alpha_1, \alpha_2) - \Delta_1(\alpha_2, \alpha_1),$$

and

$$\begin{aligned} \Delta_1(\alpha_1, \alpha_2) &= \left[ \frac{P_0 \alpha_1}{\rho_0 R_0 s} \cosh \alpha_1 R_0 - \left( \frac{P_0}{\rho_0 R_0 s^2} - \frac{\alpha s}{s^2 - \alpha W} \right) \right. \\ &\quad \cdot \sinh \alpha_1 R_0 \left. \right] \left[ \frac{1}{a} - \frac{P_0 \alpha_2^2}{\rho_0 s^2} \right] \\ &\quad \cdot \{ k_2 (\alpha_2 \cosh \alpha_2 R_0 - \sinh \alpha_2 R_0 / R_0) \\ &\quad + k_1 [(s/D_1)^{1/2} + 1/R_0] \sinh \alpha_2 R_0 \}. \end{aligned}$$

The transforms have been determined so that in principle the quantities of interest such as  $p$ ,  $\theta_1$ ,  $\theta_2$ ,  $q_2$ , or  $\eta$  could be obtained by inversion. Only the asymptotic behavior for large  $t$  will be considered here. In particular, for the inversion of  $f$  to give  $p$

$$\begin{aligned} p(r, t) &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(r; s) e^{st} ds, \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{1}{r} [A_2(s) \sinh \alpha_1 r + A_4(s) \sinh \alpha_2 r] e^{st} ds, \end{aligned} \quad (73)$$

where the path of integration is to the right of all the singularities of the integrand. It may be remarked that the integrand in Eq. (73) is symmetric with respect to the interchange of  $\alpha_1$  and  $\alpha_2$  as it should be.

An examination of the integrand of Eq. (73) apparently indicates that it has three branch points, namely at  $s = 0$  and at the two roots  $s_1$  and  $s_2$  of the expression

$$\left[ \left( \frac{P_0}{k_2 T_\infty} + \frac{1}{D_2} \right)^2 + \frac{\rho_0 s}{P_0 a} \right]^2 - \frac{4\rho_0 s}{P_0 a D_2} = 0.$$

The values  $\alpha_1$  and  $\alpha_2$  are interchanged when a circle around either  $s_1$  or  $s_2$  is completed and, because of the symmetry of the integrand in Eq. (73) with respect to the interchange of  $\alpha_1$  and  $\alpha_2$ , it may be seen that the points  $s_{1,2}$  are only apparent branch points.

The perturbation in the external pressure may be now specified to be

$$\epsilon = \epsilon_0 e^{i\omega t} \quad (74)$$

so that

$$\mu = \mathcal{L}\{\epsilon\} = \epsilon_0 / (s - i\omega). \quad (75)$$

It is clear that, for the integration indicated in Eq. (73), the point  $i\omega$  is a pole inside the contour  $\Gamma$  shown in Fig. 1. It is not easy to see whether there are any other poles of this integrand inside  $\Gamma$ . The physical argument may be made, however, that in the steady state the only mode of oscillation which can persist is one with the forcing frequency. For this to be the case, the only other poles of the

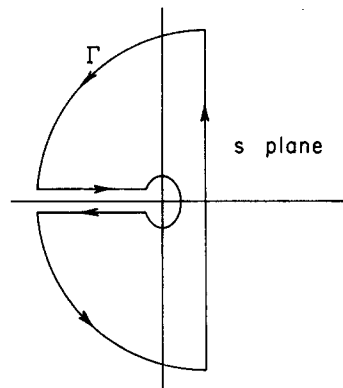


FIG. 1.



integrand if any other poles exist, must lie in the left half of the  $s$  plane so as to yield exponentially decaying terms. The branch integral from the branch point  $s = 0$  is  $O(1/t)$  as  $t \rightarrow \infty$  and by Cauchy's theorem one obtains then

$$p(r, t) = \frac{1}{r} [B_2(i\omega) \sinh \alpha_1(i\omega)r + B_4(i\omega) \sinh \alpha_2(i\omega)r] e^{i\omega t} + O(1/t), \text{ as } t \rightarrow \infty, \quad (76)$$

where

$$B_2(i\omega) = -\frac{R_0}{\Delta(i\omega)} \frac{\alpha \epsilon_0 i\omega}{\omega^2 + \alpha W} \left[ \frac{1}{a} + \frac{P_0 \alpha_2^2(i\omega)}{\rho_0 \omega^2} \right] \cdot \left\{ k_2 \left[ \alpha_2(i\omega) \cosh \alpha_2(i\omega)R_0 - \frac{\sinh \alpha_2(i\omega)R_0}{R_0} \right] + k_1 [(i\omega/D_1)^{\frac{1}{2}} + 1/R_0] \sinh \alpha_2(i\omega)R_0 \right\},$$

and

$$B_4(i\omega) = \frac{R_0}{\Delta(i\omega)} \frac{\alpha \epsilon_0 i\omega}{\omega^2 + \alpha W} \left[ \frac{1}{a} + \frac{P_0 \alpha_1^2(i\omega)}{\rho_0 \omega^2} \right] \cdot \left\{ k_2 \left[ \alpha_1(i\omega) \cosh \alpha_1(i\omega)R_0 - \frac{\sinh \alpha_1(i\omega)R_0}{R_0} \right] + k_1 [(i\omega/D_1)^{\frac{1}{2}} + 1/R_0] \sinh \alpha_1(i\omega)R_0 \right\}.$$

The quantities  $\alpha_1(i\omega)$ ,  $\alpha_2(i\omega)$ , and  $\Delta(i\omega)$  are obtained from the expressions given in the foregoing by replacing the argument  $s$  by  $i\omega$ .

In a similar way, one obtains asymptotic formulas for the relative temperature oscillations in the bubble,  $\theta_2$ , and for the relative oscillations in radius,  $x$ , as follows:

$$\theta_2(r, t) \cong \frac{1}{r} \left\{ B_2(i\omega) \left[ \frac{1}{a} + \frac{P_0 \alpha_1^2(i\omega)}{\rho_0 \omega^2} \right] \sinh \alpha_1(i\omega)r + B_4(i\omega) \left[ \frac{1}{a} + \frac{P_0 \alpha_2^2(i\omega)}{\rho_0 \omega^2} \right] \sinh \alpha_2(i\omega)r \right\} e^{i\omega t}, \quad (77)$$

$$x(t) \cong \frac{P_0}{\rho_0 R_0 \omega^2} \cdot \left\{ B_2(i\omega) \left[ \frac{\alpha_1(i\omega) \cosh \alpha_1(i\omega)R_0}{R_0} - \frac{\sinh \alpha_1(i\omega)R_0}{R_0^2} \right] + B_4(i\omega) \left[ \frac{\alpha_2(i\omega) \cosh \alpha_2(i\omega)R_0}{R_0} - \frac{\sinh \alpha_2(i\omega)R_0}{R_0^2} \right] \right\} e^{i\omega t} \quad (78)$$

It is evident that even the asymptotic formulas are of such complexity that it is difficult to obtain physical conclusions regarding details of the process.

Of interest here, however, is the thermodynamic behavior in the steady state, and for this purpose it is sufficient to examine average values over the bubble interior rather than instantaneous, local values. The relation equivalent to the first law of thermodynamics

$$\delta Q - P dV = C_v dT$$

is here

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left[ r^2 \left( \frac{\partial \theta_2}{\partial r} - \frac{P_{0a}}{k_2 T_\infty} q_2 \right) \right] = \frac{1}{D_2} \frac{\partial \theta_2}{\partial t}. \quad (79)$$

Equation (79) may be integrated over the entire gas volume. When this integration is performed, one finds after a lengthy but straightforward algebraic manipulation the following results.

$$\frac{C_v dT}{\delta Q} \cong 0(\omega), \text{ as } \omega \rightarrow 0, \quad (80)$$

and

$$\frac{C_v dT}{\delta Q} \cong \left( \frac{k_2 \rho_2 C_2}{k_1 \rho_1 C_1} \right)^{\frac{1}{2}}, \text{ as } \omega \rightarrow \infty. \quad (81)$$

For low frequencies it follows that the bubble behavior is isothermal, just as was found with the previous approximation of uniform bubble interior. However, the adiabatic behavior obtained with the previous approximation is not found in the present solution. The adiabatic limit in Eq. (81) is implied only if  $k_2/k_1 \rightarrow \infty$ . In the actual physical situation, however,  $k_2$  is of the same order as  $k_1$ , or even smaller. For example, for air  $k_2 = 2.5 \times 10^3$  erg/(cm sec °C) and for water  $k_1 = 6 \times 10^4$  erg/(cm sec °C). Hence  $(k_2 \rho_2 C_2 / k_1 \rho_1 C_1)^{\frac{1}{2}}$  is usually a small number. This result shows that even at high frequencies the average thermodynamic behavior of a gas bubble in an oscillating pressure field is very nearly isothermal rather than adiabatic. It may be remarked in conclusion that a study of the propagation of sound in a liquid with a homogeneous distribution of bubbles indicates a speed in agreement with isothermal rather than adiabatic bubble behavior. The details of this analysis will be presented elsewhere.

**ACKNOWLEDGMENT**

This work was supported by the Office of Naval Research.

**APPENDIX I**

Consider the polynomial

$$P(u) = (\beta u^2 + Au + B)(u^4 + \omega_1^2) + \alpha E(1 + W)u^2,$$

where  $\beta, A, B, \omega, \alpha, E, W$  are all positive real constants. Let us denote  $G = \alpha E(1 + W) > 0$ , then

$$P(u) = (\beta u^2 + Au + B)(u^4 + \omega_1^2) + Gu^2.$$

Since  $\beta, A, B, \omega$ , and  $G$  are all positive real constants, it is clear that  $P(u) = 0$  has no positive real root. Let

$$u = ve^{i\pi/4} = (1/2^{1/2})(1 + i)v.$$

Thus

$$\begin{aligned} P(ve^{i\pi/4}) &= [\beta v^2 + (A/2^{1/2})(1 + i)v + B](-v^4 + \omega_1^2) + iGv^2, \\ &= M(v) + iN(v), \end{aligned}$$

where

$$M(v) = -\left(\frac{A}{2^{1/2}}v + B\right)(v^4 - \omega_1^2),$$

and

$$N(v) = \left(\beta v^2 + \frac{A}{2^{1/2}}v\right)(-v^4 + \omega_1^2) + Gv^2.$$

Now if  $P(ve^{i\pi/4}) = 0$  has any real root, for this real  $v$ ,  $M(v)$  and  $N(v)$  must vanish separately. Now the only real roots of  $M(v) = 0$  are  $v = +\omega_1^{1/2}$ ,  $-\omega_1^{1/2}$  and  $-2^{1/2}B/A$ , since  $N(+\omega_1^{1/2}) \neq 0$ . Therefore we may conclude in particular that  $P(ve^{i\pi/4}) = 0$  has no positive real root. In other words,  $P(u)$  has no zero along the line  $00_2$  shown in Fig. 2.

Now the principle of the argument says that, if  $f(z)$  is regular inside a closed contour and is not zero at any point on the contour, then

$$N = \frac{1}{2\pi} \Delta_c \arg f(z),$$

where  $N$  is the number of zeros of  $f(z)$  inside  $C$  and  $\Delta_c \arg f(z)$  is the variation of the argument of  $f(z)$  round the contour  $C$ .

Now consider the contour  $00_10_20$  which consists of the positive real axis  $00_1$ , the infinite arc  $0_10_2$  and the line  $00_2$  which is obtained from rotating  $00_1$  by an angle  $\pi/4$ . Since  $P(u)$  is real and positive along  $00_1$ , then

$$\Delta_{00_1} \arg P(u) = 0.$$

We express  $u$  by its absolute value and argument, i.e.  $u = Re^{i\theta}$ . Then

$$P(u) = \beta R^9 e^{i9\theta} \left[ 1 + O\left(\frac{1}{R}\right) \right], \text{ as } R \rightarrow \infty.$$

Hence

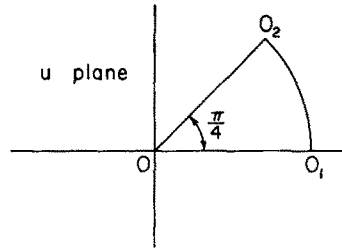


FIG. 2.

$$\Delta_{0_10_2} \arg P(u) = 6\pi/4 = 3\pi/2.$$

Along  $00_2$

$$\arg P(u) = \tan^{-1} [N(v)/M(v)].$$

For  $v = 0$ ,

$$M(0) = B\omega_1^2,$$

$$N(0) = 0;$$

and let us take  $\arg P(0) = 0$ .

For  $v \rightarrow \infty$ ,

$$M(v) \sim -(A/2^{1/2})v^5,$$

$$N(v) \sim -\beta v^6.$$

Therefore

$$\arg P(ve^{i\pi/4}) = \tan^{-1} v, \text{ as } v \rightarrow +\infty,$$

and

$$\arg P(ve^{i\pi/4}) = n/2\pi, \text{ as } v \rightarrow +\infty,$$

where  $n$  may take any positive or negative integral value, and  $n$  is determined by the number of infinities of  $N(v)/M(v)$ , and the way it jumps.

In our case, along  $00_2$ , there is only one infinity of  $N(v)/M(v)$ , namely,  $v = +\omega_1^{1/2}$ , and  $N(v)/M(v)$  changes from  $+\infty$  to  $-\infty$ , as  $v$  passes  $v = +\omega_1^{1/2}$ , in the direction of increasing  $v$ . Therefore we conclude that

$$\arg P(ve^{i\pi/4}) = \frac{3}{2}\pi, \text{ as } v \rightarrow +\infty.$$

Hence

$$\Delta_{0_20} \arg P(u) = 0 - \frac{3}{2}\pi = -\frac{3}{2}\pi.$$

Thus

$$\Delta_c \arg P(u) = \Delta_{00_10_20} \arg P(u) = 0.$$

Therefore there is no zero of  $P(u)$  for  $0 \leq \arg u \leq \pi/4$ . Since the complex roots of a polynomial equation with real coefficients appear in pairs of complex conjugates, it thus follows that there is no zero of  $P(u)$  for  $-\pi/4 \leq \arg u \leq 0$ .

Thus  $P(u) = 0$  has no root for which  $|\arg u| \leq \pi/4$ .

APPENDIX II

From Eqs. (33) and (34), we have

$$\theta_2(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\alpha\epsilon_0 E s e^{st}}{(s-i\omega)P(s^{\frac{1}{2}})} ds, \quad (1)$$

and

$$x(t) = \frac{1}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} - \frac{\alpha\epsilon_0(\beta s + A s^{\frac{1}{2}} + B)e^{st}}{(s-i\omega)P(s^{\frac{1}{2}})} ds, \quad (2)$$

where the path of integration is to the right of all the singularities of the integrands concerned, Fig. 3.

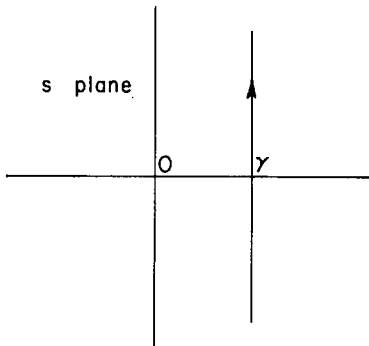


FIG. 3.

Let

$$s = u^2$$

and

$$u = \xi + i\eta.$$

Then in the  $u$  plane, the integral (1) becomes

$$\theta_2(t) = \frac{1}{2\pi i} \int_C \frac{2\alpha\epsilon_0 E u^3 \exp(u^2 t) du}{(u^2 - i\omega)P(u)},$$

where  $C$  is the branch of the hyperbola  $\xi^2 - \eta^2 = \gamma$  that lies in the right half-plane (Fig. 4). This

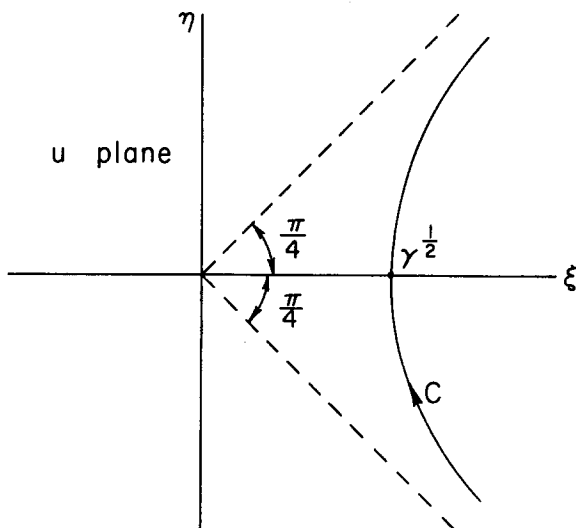


FIG. 4.

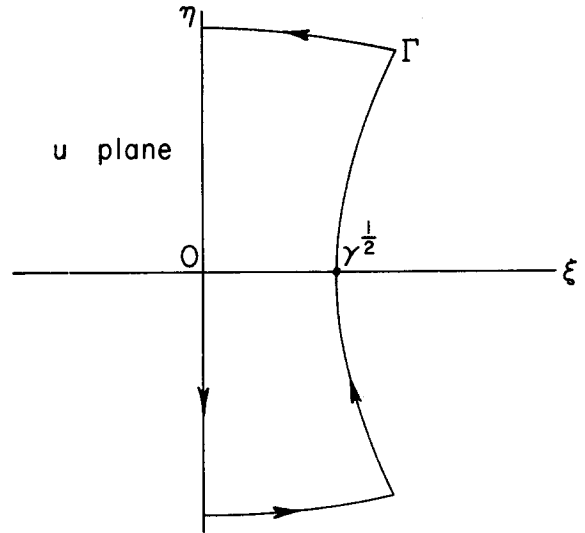


FIG. 5.

hyperbola has asymptotes

$$\xi = \pm\eta.$$

Consider the closed contour  $\Gamma$  (Fig. 5); due to the factor  $\exp(u^2 t)$ , the previous integral becomes, after applying Cauchy's theorem:

$$\theta_2(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{2\alpha\epsilon_0 E u^3 \exp(u^2 t) du}{(u^2 - i\omega)P(u)} + \sum R,$$

where  $\sum R$  denotes the sum of residues of the integrand at its poles in region I. As mentioned before, all the roots of  $P(u)$ , i.e.,  $-a_i$ , have arguments of absolute magnitude greater than  $\pi/4$ ; thus it is easy to see that, due to the factor  $\exp(u^2 t)$ , no significant contribution can result from them for large  $t$ . The residue due to the pole  $u = \omega^{\frac{1}{2}} e^{i\pi/4}$ , however, is equal to

$$\frac{\alpha\epsilon_0 E i \omega e^{i\omega t}}{P(\omega^{\frac{1}{2}} e^{i\pi/4})}.$$

Finally

$$\frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{2\alpha\epsilon_0 E u^3 \exp(u^2 t) du}{(u^2 - i\omega)P(u)} \sim 0(t^{-5/2}),$$

for large  $t$ , either by the method of steepest descent or Watson's lemma. Thus

$$\theta_2(t) = \frac{\alpha\epsilon_0 E \omega}{P(\omega^{\frac{1}{2}} e^{i\pi/4})} e^{i(\omega t + \pi/2)} + 0(t^{-5/2}), \text{ as } t \rightarrow \infty.$$

Similarly

$$x(t) = - \frac{\alpha\epsilon_0(\beta\omega + A\omega^{\frac{1}{2}} e^{-i\pi/4} - iB)}{P(\omega^{\frac{1}{2}} e^{i\pi/4})} e^{i(\omega t + \pi/2)} + 0(t^{-\frac{3}{2}}), \text{ as } t \rightarrow \infty.$$