J. Korean Math. Soc. **40** (2003), No. 5, pp. 901–920

THEORY OF INFINITELY NEAR SINGULAR POINTS

HEISUKE HIRONAKA

ABSTRACT. The notion of infinitely near singular points, classical in the case of plane curves, has been generalized to higher dimensions in my earlier articles ([5], [6], [7]). There, some basic techniques were developed, notably the three technical theorems which were *Differentiation Theorem*, *Numerical Exponent Theorem* and *Ambient Reduction Theorem* [7]. In this paper, using those results, we will prove the *Finite Presentation Theorem*, which the auther believes is the first of the most important milestones in the general theory of infinitely near singular points. The presentation is in terms of a *finitely generated* graded algebra which describes the total aggregate of the trees of infinitely near singular points. The totality is a priori very complex and intricate, including all possible successions of permissible blowing-ups toward the reduction of singularities. The theorem will be proven for singular data on an ambient algebraic shceme, regular and of finite type over any perfect field of any characteristics. Very interesting but not yet apparent connections are expected with many such works as ([1], [8]).

0. Introduction

To investigate the generalized notion of infinitely near singular points in the cases of all higher dimensions, it turned out in this paper at least technically that the use of partial differential operators is ubiquitously indispensable. The differentiation techniques are interesting in its own right, for instance as was shown by Jean Giraud and others in connection

Received August 12, 2002.
2000 Mathematics Subject Classification: Primary 14E15; Secondary 14J17, 32S45.
Key words and phrases: idealistic exponent, finite presentation.
The author has been supported by the Inamori Foundation during the final phase of this work. He was also helped by the stimulating conversations with many mathematical friends during the F. Pham Conference in Nice, France, especially by Dr Olivier Piltant (Palaiseau, France) for much useful information concerning with finite presentation problems on algebro-geometrical graded algebras (cf. [2] for instance).

with the theory of *maximal contact* ([3], [4]). However, it is interesting to ask if our technique could be replaced by something essentially new that is perhaps arithmetically intrinsic and algebraically abstract. At any rate, before talking about the generalization, let us review the classical cases of plane curves, which will at least illuminate the essence of our basic ideas and our own point of view that led us to higher dimensional version of infinitely near singularities.

Let k be any perfect field and $Z = \mathrm{Spec}(A)$ where $A = k[x, y]$ with two variables x and y. Let $X = \mathrm{Spec}(A/fA)$, a curve defined by an equation $f(x, y)$ in the plane Z. For a point $\xi \in X$, let $m = m_\xi(X)$ be the *multiplicity* of X at ξ, i.e., the order $\mathrm{ord}_\xi(f)$ of the defining polynomial f in the local ring A_ξ. Classically the infinitely near singular points of X at ξ means those m-fold points which appear in the strict transforms of X obtained by successive quadratic transformations, or blowing-ups, whose centers are all m-fold points corresponding to ξ. For the sake of simplicity, let us assume that ξ is an isolated m-fold point of X. Firstly, blow-up with center ξ. If we find any m-fold point ξ_1 of the strict transform $X_1 \subset Z_1$ of $X \subset Z$ which is mapped to ξ, then we know that ξ_1 is unique. Next, blow-up with center ξ_1 and if there still exists an m-fold point ξ_2 of the strict transform $X_2 \subset Z_2$ of $X_1 \subset Z_1$ then blow-up with center ξ_2. Continue the process as long as we find an m-fold point in the transform, and we find the process terminates after a finite number of succession. So, we get a unique finite succession of m-fold points, beginning with ξ, where the most important invariant is $\bar{\delta} = \bar{\delta}_\xi(X)$, which is the number of the m-fold points mapped to ξ including ξ itself. The blowing-up diagram is as follows:

$$\begin{array}{ccccccc} Z_{\bar{\delta}} \to & Z_{\bar{\delta}-1} & \to \cdots \to & Z_1 & \to & Z_0 = Z \\ & \ni \xi_{\bar{\delta}-1} & & \ni \xi_1 & & \ni \xi_0 = \xi \end{array}$$

where the ξ_i are m-fold points of the successive strict transforms $X_i \subset Z_i$ of $X_0 = X \subset Z$ and there are no m-fold points of $X_{\bar{\delta}}$ which are mapped to $\xi_{\bar{\delta}-1}$. The first naive question is to ask when two curves through ξ have the same number $\bar{\delta}$ of infinitely near singular points, or more strictly, to ask when two curves have the same blowing-up diagram as above.

Let us take a simple experimental case in which ξ is the origin $(0, 0)$ and $f = y^m - x^l$ where m, l are integers such that $1 < m < l$. We are excluding the case in which f is an m-th power of another polynomial. Incidentally, the number l/m is called the *first characteristic exponent* of X at ξ which we denote by $\delta_\xi(X)$. We have $m_\xi(X) = m$. By blowing-up with center ξ, we can have at most one point of multiplicity m of the transform of X which is mapped to ξ. If such a point should exist, then it must be the origin, say ξ_1, with respect to the coordinate system $(x, y/x) = (x_1, y_1)$, and the equation of the strict transform X_1 of X is $f_1 = y_1^m - x_1^{l-m}$ where $l - m \geq m$. Moreover, we then blow-up with center ξ_1. If we could repeat such a blowing-up r-times, then the equation of the final transform would be $f_r = y_r^m - x_r^{l-rm}$ while the only possiblity of a m-fold point ξ_r of X_r is the origin of the coordinate system $(x_r, y_r) = (x_{r-1}, y_{r-1}/x_{r-1})$. It is therefore clear that the origin is no longer an m-fold point, i.e., $r = \bar{\delta}$ in the above sense, if and only if $l - (r-1)m \geq m > l - rm$ or $l/m \geq r > l/m - 1$. We thus conclude $\bar{\delta} = [l/m] = [\delta_\xi(X)]$, where [ ] means integral part.

For a general plane curve X whose first characteristic exponent $\delta_\xi(X)$ is attached to an isolated m-fold point ξ, the number of inifinitely near singular points of X at ξ, denoted by $\bar{\delta}$, is equal to the integral part $[\delta_\xi(X)]$. The rational number $\delta_\xi(X)$ itself cannot be deduced from the number of infinitely near singular points or from the blowing-up diagram as above. But there exists a method of telling the number $\delta_\xi(X)$ exactly by the generalized blowing-up diagrams. The method is precisely what is formulated as *Numerical Exponent Theorem* in all dimensions and characteristics. Here, however, we want to show the essence of the idea in a very special case.

So, once again for simplicity and clarity of the idea, we take the case of a plane curve X defined by an equation of the form $f = y^m - x^l$. Let $\xi = (0,0)$. Pick an indeterminate t with respect to A and let $Z'_0 = \mathrm{Spec}(A'_0)$ with $A'_0 = A[t] = k[x, y, t]$. The equation f for the curve X may also be viewed as an element of A'_0 and defines a cylinder X'_0 over X, i.e., the product of X and the affine line $\mathrm{Spec}(k[t])$. Let L_0 be the line $\mathrm{Spec}\big(A'_0/(x,y)A'_0\big)$, along which X_0 has multiplicity $m = m_\xi(X)$. Let ξ'_0 be the k-point of L_0 with $t = 0$, or $\xi'_0 = (0,0,0)$. Now take the blowing-up $Z'_1 \to Z'_0$ with center ξ'_0 and we get strict transforms X'_1, L_1 of X'_0, L_0 respectively. Let ξ'_1 be the point of L_1 which corresponds to ξ'_0. It is the origin of (x_1, y_1, t) with $x_1 = x/t$ and $y_1 = y/t$, which is one of the coordinate charts for Z'_1. Now the transform X'_1 is defined by the equation $f_1 = f/t^m = y_1^m - x_1^l t^{l-m}$ and clearly has the same multiplicity m. Take the blowing-up $Z'_2 \to Z'_1$ with center ξ'_1 and we get strict transforms X'_2, L_2 of X'_1, L_1 respectively. Let ξ'_2 be the point of L_2 corresponding to ξ'_1, where X'_2 is defined by the equation $f_2 = f_1/t^m = y_2^m - x_2^l t^{2(l-m)}$ and blow-up with center ξ'_2. We thus repeat the process r-times and get a point $\xi'_r \in L_r \subset X'_r \subset Z'_r$ where X'_r is defined by an equation $f_r = y_r^m - x_r^l t^{r(l-m)}$. Now, fix any one $r \gg 1$ and let Y_r be the last exceptional curve in X'_r, that is $\mathrm{Spec}(A'_r/(y_r, t)A'_r)$ where $A'_r = k[x_r, y_r, t]$. Take the blowing-up $Z'_{r,1} \to Z'_r$ with center Y_r. Let η_r be the generic point of Y_r. Then all the points of the strict transform $X'_{r,1}$ of X'_r which correspoind to η_r are contained in the affine chart with coordinate system $(x_r, y_{r,1}, t)$ of $Z'_{r,1}$, where $y_{r,1} = y_r/t$, and the equation of $X'_{r,1}$ is $f_{r,1} = y_{r,1}^m - x_r^l t^{r(l-m)-m}$. Hence, so long as $r(l-m) - m > 0$, there exists one and only one point $\eta_{r,1} \in X'_{r,1}$ corresponding to η_r which is the generic point of the unique smooth curve $Y_{r,1}$ corresponding to Y_r. If $X'_{r,1}$ has multiplicity m along $Y_{r,1}$ or equivalently $r(l-m) - m \geq m$, then we take the blowing-up $Z'_{r,2} \to Z'_{r,1}$ with center $Y_{r,1}$ and obtain $\eta_{r,2} \in Y_{r,2} \subset X'_{r,2}$ with the equation $f_{r,2} = y_{r,2}^m - x_r^l t^{r(l-m)-2m}$ with $y_{r,2} = y_{r,1}/t$. We repeat this process untill the exponent of t goes down to less than m. After s-times repetition, we would have $\eta_{r,s} \in Y_{r,s} \subset X'_{r,s}$ with the equation $f_{r,s} = y_{r,s}^m - x_r^l t^{r(l-m)-sm}$ with $y_{r,s} = y_{r,s-1}/t$. The maximal number for s called $s(r)$ which is a function of r is determined by the condition $0 \leq r(l-m) - s(r)m < m$. In other words, $0 \leq l/m - (s(r)/r + 1) < 1/r$ which implies $\lim_{r\to\infty}(s(r)/r + 1) = l/m = \delta_\xi(X)$. In other words, the *first characteristic exponent* $\delta_\xi(X)$ itself is determined by the blowing-up diagrams:

$$\begin{array}{ccccc} Z'_{r,s(r)} \to & Z'_{r,s(r)-1} & \to \cdots \to & Z'_{r,0} = Z'_r \\ & \cup & & \cup \\ & Y_{r,s(r)-1} & & Y_{r,0} = Y_r \end{array}$$

combined with

$$\begin{array}{ccccc} Z'_r \to & Z'_{r-1} & \to \cdots \to & Z'_1 & \to Z'_0 \\ & \ni & & \ni & \ni \\ & \xi'_{r-1} & & \xi'_1 & \xi'_0 \end{array}$$

for all $r \gg 1$.

It can be proven with a little harder work that the same result as above can be obtained for an arbitrary plane curve.

1. Idealistic exponents and their equivalences

In order to state our general *Numerical Exponent Theorem*, we need a generalization of the notion of infinitely near singular points, or rather a general notion of *permissible blowing-up diagrams* in all dimensions. To begin with, we are assumed to be given a smooth algebraic scheme Z over a perfect field k of any characteristics. An *idealistic exponent* $E = (J, b)$ on Z is nothing but a pair of a coherent ideal sheaf J on Z and a positive integer b. When Z is an affine scheme, say $Z = \mathrm{Spec}(A)$, we will identify J with the ideal in A which generates J. We will consider a finite system of indeterminates $t = (t_1, t_2, \ldots, t_a)$ and let $Z[t]$ denote the product of Z and $\mathrm{Spec}(k[t])$ over k. We also let $E[t]$ denote the pair $(J[t], b)$ where $J[t]$ denotes the ideal sheaf on $Z[t]$ generated by J with respect to the canonical projection.

DEFINITION 1.1. A local sequence of smooth blowing-ups over Z, called *LSB* over Z for short, means a diagram of the following type:

$$(1.1)\quad \begin{array}{ccccccc} Z_r \to & U_{r-1} \subset Z_{r-1} & \to \cdots \to & U_1 \subset Z_1 & \to & U_0 \subset Z_0 = Z \\ & \cup & & \cup & & \cup \\ & D_{r-1} & & D_1 & & D_0 \end{array}$$

where U_i is an open subscheme of Z_i, D_i is a smooth closed subscheme of U_i and the arrows mean that $Z_{i+1} \to U_i$ is the blowing-up with center $D_i, \forall i$.

DEFINITION 1.2. We now want to define the notion of permissibility of *LSB* for a given idealistic exponent $E = (J, b)$ on Z. For this to be done inductively, it is enough to have two notions for a single blowing-up, one being that of permissibility for a blowing-up and the other being that of the transform by a permissible blowing-up. For an open subset $U_0 \subset Z$, we simply replace E by its restriction $E \,|\, U_0 = (J\,|U_0, b)$. So it is enough to consider the case of $Z = U_0$. First of all, we define $\mathrm{Sing}(E)$, called the *singular locus of* E, to be the following closed subset of Z:

$$\mathrm{Sing}(E) = \{\eta \in Z \,|\, \mathrm{ord}_\eta(J) \geq b\}.$$

A blowing-up $Z_1 \to Z$ with center D_0 is said to be *permissible* for E if D_0 is smooth and contained in $\mathrm{Sing}(E)$ and the transform $E_1 = (J_1, b)$ of E is defined by saying that $J_1 P^b$ is equal to the ideal sheaf on Z_1 generated by J with respect to the blowing-up morphism $Z_1 \to Z$, where P denotes the ideal sheaf of the exceptional divisor, i.e., the locally principal ideal sheaf on Z_1 generated by the ideal defining $D_0 \subset Z$. Note that J_1 is uniquely determined by the above equality and that it exists as an ideal sheaf in the structural sheaf $\mathcal{O}_{Z_1}$ of Z_1.

DEFINITION 1.3. For a pair of idealistic exponents $E_i = (J_i, b_i), i = 1, 2$, we define the *inclusion*:

$$E_1 \subset E_2$$

meaning to satisfy the following condition: *Pick any finite system of indeterminates $t = (t_1, \dots, t_a)$ and let $E_i[t] = (J_i[t], b_i), i = 1, 2$. If any LSB over $Z[t]$ in the sense of Definition 1.1 is permissible for $E_1[t]$ then it is also permissible for $E_2[t]$.*

The *equivalence*:

$$E_1 \sim E_2$$

will mean that $E_1 \subset E_2$ and $E_1 \supset E_2$ at the same time. The *equivalence* of the form $E_1 \cap E_2 \sim E_3$ will mean that an LSB over $Z[t]$ for any t is permissible for $E_3[t]$ if and only if it is so for both $E_1[t]$ and $E_2[t]$. An idealistic exponent E may be understood as the totality of all permissible LSB for $E[t]$ for all t.

DEFINITION 1.4. For an idealistic exponent $E = (J, b)$ on Z, we define its *order* at a point $\xi \in Z$ as follows: $\mathrm{ord}_\xi(E) = \mathrm{ord}_\xi(J)/b$ *if* $\mathrm{ord}_\xi(J) \geq b$, *and* $\mathrm{ord}_\xi(E) = 0$ *if* $\mathrm{ord}_\xi(J) < b$. So we have

$$\mathrm{Sing}(E) = \{\xi \in Z \mid \mathrm{ord}_\xi(E) \geq 1\}.$$

What follows are most of the important basic facts about idealistic exponents. They will be later refered to as Basic 1, Basic 2, and so on. Their proofs are either easy to reconstruct or found in my paper [7].

1. $(J^e, eb) \sim (J, b)$ for every positive integer e.
2. For every common multiple m of b_1 and b_2, we have
$$(J_1, b_1) \cap (J_2, b_2) \sim (J_1^{\frac{m}{b_1}} + J_2^{\frac{m}{b_2}}, m).$$
In particular if $b_1 = b_2 = b$ $(= m)$ and $J_1 \subset J_2$ then we have $(J_1, b) \supset (J_2, b)$.
3. We always have
$$(J_1 J_2, b_1 + b_2) \supset (J_1, b_1) \cap (J_2, b_2).$$
The reversed inclusion does not hold in general. However, if $\mathrm{Sing}(J_i, b_i + 1)$ are both *empty* for $i = 1, 2$, then the left hand side becomes *equivalent* to the right hand side. Moreover, we always have
$$\begin{aligned} & (J, b) \subset (J_k, b_k), 1 \leq k \leq r, \\ \Rightarrow \quad & (J, b) \sim (J^r, rb) \subset \Big(\prod_{1 \leq k \leq r} J_k \,,\, \sum_{1 \leq k \leq r} b_k \Big). \end{aligned}$$
4. Let us compare two idealistic exponents having the same ideal but different b's, say $F_1 = (J, b_1)$ and $F_2 = (J, b_2)$ with $b_1 > b_2$. Then we have
   (a) $F_1 \subset F_2$.

(b) For any LSB permissble for F_1, and hencé so for F_2, their final transforms differ only by a locally principal non-zero factor supported by exceptional divisors. To be precise, their final transforms being denoted by $F_1^* = (J_1^*, b_1)$ and $F_2^* = (J_2^*, b_2)$, we have $J_2^* = MJ_1^*$ where M is a positive power product of the ideals of the strict transforms of the exceptional divisors created by the blowing-ups belonging to the LBS.

5. We have $(J_1, b) \supset (J_2, b)$ if J_1 is contained in the *integral closure* of J_2 in the sense of *integral dependence* (after Oscar Zariski) defined in the theory of ideals. Recall the definition: For ideals $H_i, i = 1, 2$, in a commutative ring B, H_1 is integral over H_2 in the sense of the *ideal theory* if and only if $\sum_{a\geq 0} H_1^a T^a$ is integral over $\sum_{a\geq 0} H_2^a T^a$ in the sense of the *ring theory*, where T is an indeterminate over B. In our case, since Z is normal, if $\rho : \tilde{Z} \longrightarrow Z$ is any proper birational morphism such that $\tilde{Z}$ is normal and $J_2\mathcal{O}_{\tilde{Z}}$ is locally non-zero principal, then the direct image $\rho_*(J_2\mathcal{O}_{\tilde{Z}})$ is equal to the *integral closure* of J_2, where $\mathcal{O}_{\tilde{Z}}$ denotes the structural sheaf of $\tilde{Z}$. As an example of such ρ, we could take the *normalized blowing-up* of J_2, i.e., the blowing-up followed by normalization.

6. Here is one of the most important technical facts about idealistic exponents with respect to blowing-ups. We cite it as a theorem as follows.

   Let $\mathcal{O}_Z$ denote the structural sheaf (i.e., the sheaf of functions) of the scheme Z and let $Diff_Z^{(i)}$ denote the sheaf of the differential operators from $\mathcal{O}_Z$ into itself whose orders are $\leq i$. The sheaf has a natural structure of a left O_Z-module.

THEOREM 1.1 (Diff Theorem). (*cf. Theorem 1, section 8,* [7]) *If $\mathcal{D}$ is any left $\mathcal{O}_Z$-submodule of $Diff_Z^{(i)}$ then we have*

$$(J, b) \subset (\mathcal{D}J, b - i)$$

or equivalently

$$(J, b) \cap (\mathcal{D}J, b - i) \sim (J, b).$$

7. There is another important fact about idealistic exponents that was hinted earlier by experimental examples, and here is a general statement.

THEOREM 1.2 (Numerical Exponent Theorem). (*cf. Proposition 8, section 2.* [7]) *If $E_1 \subset E_2$ in the sense of Definition 1.3, then we have*

$$\mathrm{ord}_\zeta(E_1) \leq \mathrm{ord}_\zeta(E_2)$$

for every point $\zeta \in Z$. *It follows that*

$$(E_1) \sim (E_2) \quad \Rightarrow \quad \mathrm{ord}_\zeta(E_1) = \mathrm{ord}_\zeta(E_2)$$

for every point $\zeta \in Z$.

8. Finally we have a theorem that is technically useful for cutting down the dimension of the ambient scheme. Let W be any closed subscheme of Z and define

$$Red_W(E) = \bigcap_{0 \le j < b} \left((Diff_Z^{(j)} J)\mathcal{O}_W, b-j\right)$$

which is obviously an idealistic exponent on W.

THEOREM 1.3 (Ambient Reduction Theorem). (*cf. Th.* 5, *section* 8, [7]) *An* LSB *over* W *is permissible for* $Red_W(E)$ *if and only if it is so for* E *when it is naturally extended to an* LSB *with the same centers over* Z.

Here the natural extension of LSB is done according to the general fact that the blowing-up of a subscheme is embedded into the blowing-up of the ambient scheme with the same center, as the former is identified with the one induced by the latter into the strict transform of the subscheme.

2. Main theorem "finite presentation"

We are now ready to state the main theorem of this paper. Let Z be a smooth algebraic scheme over a perfect field k. We will assume that Z is connected and hence irreducible, because we loose no generality by doing so. Given an idealistic exponent $E = (J, b)$ on Z, we define a graded $\mathcal{O}_Z$-algebra

$$\wp(E) = \sum_{0 \le a < \infty} J_{\max}(a)\, T^a \tag{2.1}$$

to be the sheaf of graded $\mathcal{O}_Z$-algebras on Z associated with:

$$U \longmapsto \wp_U(E) = \sum_{0 \le a < \infty} J_{\max}(a)_U T^a \quad \text{for each affine open } U \subset Z$$

where T is an indeterminate and $J_{\max}(a)_U$ is an ideal in the affine ring A_U of $Z|U$, defined by the following property: for every integer $a \ge 0$ and for an ideal I in A_U, we have

$$(I, a) \supset (J_U, b) = E_U \quad \Longleftrightarrow \quad I \subset J_{\max}(a)_U$$

where E_U denotes the restriction of E to U and the first inclusion is in the sense of Definition 1.3 while the second in the set-theoretical sense.

It should be noted that if $E \sim F$ with another idealistic exponent F on Z then $\wp(E) = \wp(F)$, which is clear from the above definition. It should also be noted that if there are two ideals $I_i, i = 1, 2$, in A_U such that $(I_i, a) \supset E_U$ for both $i = 1, 2$, then we have $(I_1 + I_2, a) \supset E_U$ by Basic 2. Therefore, for each $a \geq 0$, we have the unique maximal one among all those ideals I which have the above property. The maximal one is $J_{\max}(a)_U$ and $J_{\max}(a)_U$ itself has the property. It should be understood that $J_{\max}(0)_U$ is the unit ideal, irrespective of U. Moreover, the property implies that $(J_{\max}(a)_U, a) \supset E_U$ and $J_{\max}(a_1)_U J_{\max}(a_2)_U \subset J_{\max}(a_1 + a_2)_U$ for every $a > 0$ and $a_j > 0, j = 1, 2$, by Basic 3. This is why $\wp(E)$ is an $\mathcal{O}_Z$-algebra. Let us note that $J_{\max}(a)_U \subset A_U$ is *integrally closed* in the sense of *ideal theory* thanks to Basic 5 and the above definition. We will later see a stronger result than this.

Let us introduce the auxiliary definitions as follows:

$$\begin{aligned} \wp_U(E)(\mu) &= \sum_{0 \leq a < \infty} J_{\max}(a\mu)_U T^{a\mu} \\ \wp(E)(\mu) &= \sum_{0 \leq a < \infty} J_{\max}(a\mu) T^{a\mu} \quad \text{for each integer } \mu \geq 1. \end{aligned}$$

For every $a > 0$ and for every $\mu > 0$, we have $(J_{\max}(a)_U)^\mu \subset J_{\max}(a\mu)_U$ which implies that every element of $J_{\max}(a)_U T^a$ is integral over $\wp_U(E)(\mu)$ in the sense of the ring theory. In short, $\wp_U(E)$ is integral over $\wp_U(E)(\mu)$. It is easy to see that all those $\wp_U(E)(\mu), \mu \geq 1$, have the same field of fractions which is $\mathbf{K}(T)$ where $\mathbf{K}$ denotes the function field of Z. We claim that, in the sense of the ring theory,

(2.2) $\quad \wp_U(E)$ is the integral closure of $\wp_U(E)(\mu)$ in $\mathbf{K}(T)$, $\forall \mu \geq 1$.

In fact, $A_U[T]$ is integrally closed and any polynomial equation in $\mathbf{K}[T]$ splits into its homogeneous parts in terms of the variable T. Hence the integral closure of $\wp_U(E)(\mu)$ is contained in $A_U[T]$ and graded in terms of the non-negative powers of T. So, what we need to prove is that

$$J_{\max}(a)_U = \{h \in A_U | hT^a \text{ is integral over } \wp_U(E)(\mu)\}.$$

To prove this, let H denote the right hand side which is an ideal in A_U. First of all it is clear that we have $H \supset J_{\max}(a)_U$ because $(J_{\max}(a)_U)^\mu \subset J_{\max}(a\mu)_U$ and hence $(hT^a)^\mu \in J_{\max}(a\mu)_U T^{a\mu}, \forall h \in J_{\max}(a)_U$. As for the converse inclusion, pick any $h \in H$. We then have a monic equation of the form

$$(hT^a)^l + M_1(hT^a)^{l-1} + \cdots + M_l = 0 \text{ where}$$
$$M_j \in J_{\max}(aj)_U T^{aj}, \forall j, \text{ and } M_j = 0 \text{ if } aj \not\equiv 0 \mod \mu.$$

The equation also means that hT^a is also integral over $A_U[M_1, \ldots, M_l]$. Let Δ be the product of those aj subject to the conditions: $1 \le j \le l$ and $aj \equiv 0 \mod \mu$. Write $\Delta = a\delta\mu$ with a positive integer δ. All those non-zero M_j are integral over $A_U[J_{\max}(a\delta\mu)_U T^{\delta\mu}]$ and hence so is hT^a. It follows that $(hT^a)^{\delta\mu}$ is integral over $A_U[J_{\max}(a\delta\mu)_U T^{\delta\mu}]$. This implies that $h^{\delta\mu}$ is integral over $J_{\max}(a\delta\mu)_U$ in the sense of *ideal theory*. We know that the latter is integrally closed and hence we get $h^{\delta\mu} \in J_{\max}(a\delta\mu)_U$. Here the number δ can be replaced by any one of its positive integral multiples. Since H is finitely generated, we may assume that $H^{\delta\mu} \subset J_{\max}(a\delta\mu)_U$. Thus $(H, a) \sim (H^{\delta\mu}, a\delta\mu) \supset (J_{\max}(a\mu)_U, a\mu) \supset E_U$ which implies $H \subset J_{\max}(a)_U$. Hence $H = J_{\max}(a)_U$, which is the assertion (2.2).

Now we claim the main theorem of this paper as follows:

FINITE PRESENTATION THEOREM. *The ideal sheaves $J_{\max}(a)$ are all coherent on Z for all integers $a \ge 0$ and $\wp(E)$ is locally finitely generated as O_Z-algebra. Therefore, on each affine open subset of the ambient scheme Z, $\wp(E)$ is finitely generated as k-algebra.*

Note that if we take an affine open subset $U = \mathrm{Spec}(A_U)$ of Z, then we have $\wp(E)(U) = \sum_a J_{\max}(a)_U T^a$. This is a finitely generated as k-algebra if and only if it is so as A_U-algebra because A_U itself is a finitely generated k-algebra.

The rest of the paper is devoted to a proof of the theorem. For this purpose, we need some technical definitions and lemmas.

DEFINITION 2.1. For an idealisitic exponent $F = (H, b)$ on Z, we say that F is *Diff-generated* by a system of idealisitic exponents $G_j = (I_j, b_j), 0 \le j \le r$, if for every integer $\mu > 0$ and for every integer $i, 0 \le i < b\mu$, we have

$$(2.3) \qquad Diff_Z^{(i)} H^\mu \quad \subset \sum_{\sum_j e_j b_j \ge b\mu - i} \Big(\prod_j I_j^{e_j} \Big).$$

Here the big summation is taken for all those systems $(e_0, \ldots, e_r)$ of integers $e_j \ge 0, \forall j$, which are subject to the condition $\sum_{0 \le j \le r} e_j b_j \ge b\mu - i$.

REMARK 2.1. The conditions (2.3) for $\mu > 1$ are consequences of the one for $\mu = 1$. In fact, for $\mu > 1$, we have

$$\begin{aligned} Diff_Z^{(i)} H^\mu &\subset \sum_{i_k \geq 0, \forall k, (\sum_{k=1}^{\mu} i_k) = i} \prod_{k=1}^{\mu} Diff_Z^{(i_k)} H \\ &\subset \sum_{i_k \geq 0, \forall k, (\sum_{k=1}^{\mu} i_k) = i} \prod_{k=1}^{\mu} \Big(\sum_{\sum_j e_{kj} b_j \geq b - i_k} \Big(\prod_j I_j^{e_{kj}} \Big) \Big) \\ &\subset \sum_{i_k \geq 0, \forall k, (\sum_{k=1}^{\mu} i_k) = i} \Big(\sum_{\sum_j e_{kj} b_j \geq b - i_k} \Big(\prod_j I_j^{\sum_k e_{kj}} \Big) \Big) \\ &\subset \sum_{\sum_j e_j b_j \geq b\mu - i} \Big(\prod_j I_j^{e_j} \Big) \end{aligned}$$

where $e_j = \sum_k e_{kj}$. The claim is proven.

REMARK 2.2. If $(H_k, c_k), k = 1, 2$, are all *Diff-generated* by one system $G_j = (I_j, b_j), 0 \leq j \leq r$, as above, then so by the same system are the following:

$$(H_1 H_2, c_1 + c_2), \ (H_1^m, mc_1), \forall m > 0, \text{ and}$$
$$(Diff_Z^{(l)} H_1, c_1 - l), \forall l, 0 \leq l \leq b_1.$$

In fact, for the first one, the proof is by:

$$\begin{aligned} & Diff_Z^{(i)}(H_1 H_2) \\ \subset & \sum_{i_1 + i_2 = i} Diff_Z^{(i_1)}(H_1) Diff_Z^{(i_2)}(H_2) \\ \subset & \sum_{i_1 + i_2 = i} \Big(\sum_{\sum_j e_{1j} b_j \geq c_1 - i_1} \Big(\prod_j I_j^{e_{1j}} \Big) \Big) \Big(\sum_{\sum_j e_{2j} b_j \geq c_2 - i_2} \Big(\prod_j I_j^{e_{2j}} \Big) \Big) \\ \subset & \sum_{\sum_j e_j b_j \geq (c_1 + c_2) - i} \Big(\prod_j I_j^{e_j} \Big). \end{aligned}$$

For the last one, the proof is immediate thanks to Remark 2.1. The others follow.

DEFINITION 2.2. We say that $F = (H, b)$ is *Diff-full* if for every integer $i, 0 \leq i < b$,

$$\big(Diff_Z^{(i)} H\big)^b \text{ is contained in the integral closure of } H^{b-i}$$

in the sense of the ideal theory, which is equivalent to saying that if $\phi : \tilde{Z} \longrightarrow Z$ is the normalized blowing-up of the ideal sheaf H then $(Diff_Z^{(i)} H)^b \mathcal{O}_{\tilde{Z}} \subset H^{b-i}\mathcal{O}_{\tilde{Z}}$.

LEMMA 2.1. *Assume that $F = (H, b)$ is Diff-generated by a system of idealisitic exponents $G_j = (I_j, b_j), 0 \leq j \leq r$, in the sense of Definition 2.1 and that $G_j \supset F$ in the sense of Definition 1.3 for all $j, 0 \leq j \leq r$. Then, for every positive integer μ, we have*

$$(1)\quad F \sim (H^\mu, b\mu) \sim \bigcap_{\sum_j e_j b_j \geq b\mu} \Big(\prod_j I_j^{e_j}\ ,\ b\mu\Big)$$

and moreover for every smooth subscheme W of Z we have

$$(2)\quad Red_W(H^\mu, b\mu) \sim \bigcap_{0 \leq j < r} (I_j \mathcal{O}_W,\ b_j).$$

Proof. For every integer $\mu \geq 0$,

$$\begin{aligned}
(H^\mu, b\mu) &\sim \bigcap_{\left(\sum_j e_j b_j\right) \geq b\mu} \Big(\prod_j H^{b_j e_j}, b\big(\sum_j e_j b_j\big)\Big) \\
&\subset \bigcap_{\left(\sum_j e_j b_j\right) \geq b\mu} \Big(\prod_j I_j^{b e_j}, b\big(\sum_j e_j b_j\big)\Big) \\
&\sim \bigcap_{\left(\sum_j e_j b_j\right) \geq b\mu} \Big(\prod_j I_j^{e_j},\ \ \big(\sum_j e_j b_j\big)\Big) \\
&\subset \bigcap_{\left(\sum_j e_j b_j\right) \geq b\mu} \Big(\prod_j I_j^{e_j}, b\mu\Big)
\end{aligned}$$

where the first inclusion is by the second assertion of Basic 3, the last by Basic 4 and the equivalences by Basic 1. On the other hand, by the *Diff-generation* assumption, we have

$$H^\mu \subset \sum_{\left(\sum_j e_j b_j\right) \geq b\mu} \Big(\prod_j I_j^{e_j}\Big)$$

which implies the reversed inclusion:

$$(H^\mu, b\mu) \supset \Big(\sum_{\left(\sum_j e_j b_j\right) \geq b\mu} \Big(\prod_j I_j^{e_j}\Big),\ b\mu\Big).$$

We have thus obtained (1). Now for (2), we have

$$\begin{aligned}
&Red_W(H^\mu, b\mu)\\
&= \bigcap_{0\le i<b\mu} \Big(Diff_Z^{(i)}(H^\mu)\mathcal{O}_W,\ b\mu - i\Big)\\
&\supset \bigcap_{0\le i<b\mu} \Big(\big(\sum_{(\sum_j e_jb_j)\ge b\mu-i} \prod_j I_j^{e_j}\big)\mathcal{O}_W,\quad b\mu - i\Big)\\
&= \bigcap_{0\le i<b\mu} \Big(\sum_{(\sum_j e_jb_j)\ge b\mu-i} \big(\prod_j I_j^{e_j}\mathcal{O}_W\big)\ ,\quad b\mu - i\Big)\\
&\sim \bigcap_{0\le i<b\mu,\ \left(\sum_j e_jb_j\right)\ge b\mu-i} \Big(\prod_j I_j^{e_j}\mathcal{O}_W,\ b\mu - i\Big)\\
&\supset \bigcap_{0\le i<b\mu,\ (\sum_j e_jb_j)\ge b\mu-i} \Big(\prod_j I_j^{e_j}\mathcal{O}_W,\ \sum_j e_jb_j\Big)\\
&\supset \bigcap_{0\le i<b\mu,\ (\sum_j e_jb_j,)\ge b\mu-i} \Big(\bigcap_j (I_j^{e_j}\mathcal{O}_W,\ e_jb_j)\ \Big)\\
&\sim \bigcap_{0\le i<b\mu,\ (\sum_j e_jb_j)\ge b\mu-i} \Big(\bigcap_j (I_j\mathcal{O}_W,\ b_j)\Big) \quad\sim\quad \bigcap_j (I_j\mathcal{O}_W,\ b_j)
\end{aligned}$$

where the first inclusion between idealisitic exponents is due to the reversed inclusion of ideals by the *Diff-generation*, while the second inclusion by Basic 4 and the last by the Basic 3. On the other hand, since $F \subset G_j$ we get $Red_W(F) \subset Red_W(G_j)$ for every j by the *Ambient Reduction Theorem* and Definition 1.3. Hence we have

$$\begin{aligned}
Red_W(F) \quad &\subset \quad \textstyle\bigcap_j Red_W(G_j)\\
&= \textstyle\bigcap_j \bigcap_{0\le k<b_j} \big((Diff_Z^{(k)} I_j)\mathcal{O}_W, b_j - k\ \big)\\
&\subset \quad \textstyle\bigcap_j (I_j\mathcal{O}_W, b_j)
\end{aligned}$$

which shows the converse to the preceeding inclusion and (2) is proven. □

LEMMA 2.2. *Any given idealistic exponent $E = \big(J, b\big)$ on Z is Diff-generated by the following system of idealisitic exponents:*

$$\Big\{\ \Big(Diff_Z^{(i)} J\ ,\ b_j\Big), 0 \le i < b\Big\}$$

where $b_i = b - i$. Moreover, define

$$E^\sharp = (J^\sharp, b^\sharp)$$

where

$$J^\sharp = \sum_{0\le i<b} \left(Diff_Z^{(i)} J\right)^{b!/(b-i)} \quad \textit{and} \quad b^\sharp = b!$$

and we assert that $E^\sharp$ *is Diff-full.*

Proof. As for the first claim, the problem is local. Namely, it is enough to prove the inclusion of the type of Definition 2.1 *locally* at every point $\xi \in \mathrm{Sing}(E)$. It should be noted that at a point outside $\mathrm{Sing}(E)$ one of the ideals $Diff_Z^{(i)}J$, $0 \le i < b$, is the unit ideal and the claim is trivial. Let us pick a regular system of parameters $x = (x_1, \dots, x_n)$ in the local ring of Z at ξ and define the elementary differential oporators $\partial_\alpha, \alpha \in \mathbb{Z}_0^n$, where $n = \dim_\xi Z$, by the conditions:

$$\partial_\alpha x^\beta = \begin{cases} \binom{\beta}{\alpha} x^{\beta-\alpha} & \text{if } \beta \in \alpha + \mathbb{Z}_0^n \\ 0 & \text{if } \beta \notin \alpha + \mathbb{Z}_0^n. \end{cases}$$

Let μ be any positive integer. For every integer $i, 0 \le i < b\mu$, pick any one ∂_α with $|\alpha| = i$. Then

$$\partial_\alpha J_\xi^\mu \quad \subset \quad \sum_{\alpha=\sum_{1\le k\le\mu}\alpha_k, \alpha_k\in\mathbb{Z}_0^n} \left(\prod_{1\le k\le\mu} \partial_{\alpha_k} J_\xi\right)$$

and for each $(\alpha_1, \dots, \alpha_\mu)$ we have

$$\prod_{1\le k\le\mu} \partial_{\alpha_k} J_\xi \quad \subset \prod_{i\ge j\ge 0} (Diff_Z^{(j)} J_\xi)^{e_j} \quad \subset \prod_{\min(i,b-1)\ge j\ge 0} (Diff_Z^{(j)} J_\xi)^{e_j}$$

where e_j is the number of those α_k such that $|\alpha_k| = j$. Here an important point is that $\sum_{\min(i,b-1)\ge j\ge 0} e_j b_j = \sum_{\min(i,b-1)\ge j\ge 0} e_j(b-j) \ge \sum_{i\ge j\ge 0} e_j(b-j) = (\sum_{i\ge j\ge 0} e_j)b - \sum_{i\ge j\ge 0} e_j j = b\mu - |\alpha| = b\mu - i$. This numerical inequality and the last product ideal are unaffected if we extend the range of j to $b-1 \ge j \ge 0$ in the case of $i < b-1$ and let $e_j = 0$ for $j > i$. Therefore, thanks to the numerical inequality, we get

$$\prod_{1\le k\le\mu} \partial_{\alpha_k} J_\xi \quad \subset \quad \sum_{(\sum_{0\le j<b} e_j b_j)\ge b\mu-i} \left(\prod_{0\le j<b} (Diff_Z^{(j)} J_\xi)^{e_j}\right).$$

This being true for all α and $(\alpha_1, \dots, \alpha_\mu)$ as above, we conclude that

$$Diff_Z^{(i)} J_\xi^\mu \quad \subset \quad \sum_{(\sum_{0\le j<b} e_j b_j)\ge b\mu-i} \left(\prod_{0\le j<b} (Diff_Z^{(j)} J_\xi)^{e_j}\right).$$

This being true for every $\xi \in \mathrm{Sing}(E)$, the same inclusion holds when the suffix ξ is dropped. The first assertion of the lemma is thus proven. Next, to prove the second assertion, let $\rho : \tilde{Z} \longrightarrow Z$ be the normalized blowing-up of the ideal sheaf $J^{\#}$. Let us pick any point $\tilde{\zeta} \in \tilde{Z}$ and let $\zeta = \rho(\tilde{\zeta}) \in Z$. Since the pull back $J^{\#}\mathcal{O}_{\tilde{Z}}$ is locally non-zero principal everywhere, there exists an index ι such that

1. $(Diff_Z^{(\iota)} J)^{b!/(b-\iota)}\mathcal{O}_{\tilde{Z},\tilde{\zeta}}$ is non-zero principal, say $= (h_{\tilde{\zeta}})\mathcal{O}_{\tilde{Z},\tilde{\zeta}}$
2. $(J^{\sharp}\mathcal{O}_{\tilde{Z}})_{\tilde{\zeta}} = (h_{\tilde{\zeta}})\mathcal{O}_{\tilde{Z},\tilde{\zeta}}$ and
3. $h_{\tilde{\zeta}}$ divides $(Diff_Z^{(k)} J)^{b!/(b-k)}\mathcal{O}_{\tilde{Z},\tilde{\zeta}}, \forall k, 0 \le k < b$.

We will later refer to these as *properties* of the chosen $h_{\tilde{\zeta}}$. Now, by the definition of $J^{\sharp}$, we have the following inclusions for $0 \le m < b!$

$$\begin{aligned}
&\left(Diff_Z^{(m)} J^{\sharp}\right)\mathcal{O}_{\tilde{Z},\tilde{\zeta}} \\
\subset\ & \sum_{0\le j<b} \left(Diff_Z^{(m)} \left(Diff_Z^{(j)} J\right)^{b!/(b-j)}\right)\mathcal{O}_{\tilde{Z},\tilde{\zeta}} \\
=\ & \sum_{0\le j<b} \left(\sum_{(\sum_{1\le k\le b!/(b-j)} m_k)=m} \ \prod_{1\le k\le b!/(b-j)} Diff_Z^{(m_k)} Diff_Z^{(j)} J\right)\mathcal{O}_{\tilde{Z},\tilde{\zeta}} \\
=\ & \sum_{0\le j<b} \left(\sum_{(\sum_{1\le k\le b!/(b-j)} m_k)=m} \ \prod_{1\le k\le b!/(b-j)} Diff_Z^{(m_k+j)} J\right)\mathcal{O}_{\tilde{Z},\tilde{\zeta}} \\
\subset\ & \sum_{0\le j<b} \left(\left\{\sum\prod\right\} Diff_Z^{(m_k+j)} J\right)\mathcal{O}_{\tilde{Z},\tilde{\zeta}}
\end{aligned}$$

where the sum inside $(\ )\mathcal{O}_{\tilde{Z},\tilde{\zeta}}$ ranges over all the systems $(m_1, \ldots, m_{b!/(b-j)})$ with integers $m_k \ge 0$ such that $\sum_k m_k = m$, while $\left\{\sum\prod\right\}$ means that the sum-product has the same ranges as above but the term $Diff_Z^{(m_k+j)} J$ is replaced by the unit ideal if and only if $m_k + j \ge b$. Note that for any j and for any system $(m_1, \ldots, m_{b!/(b-j)})$, it is impossible that $m_k + j \ge b$ for all k, for if otherwise we would have

$$m = \sum_k m_k \ge \sum_k (b-j) = \big(b!/(b-j)\big)(b-j) = b!$$

which contradicts the assumption $m < b!$. Moreover note that for every $(m_1, \ldots, m_{b!/(b-j)})$, say $= (m)$, appearing in the above sum-product we have

$$b! - m = \big(b!/(b-j)\big)(b-j) - \sum_k m_k = \sum_k \big((b-j) - m_k\big)$$

$$= \sum_{k} \bigl(b - (m_k + j)\bigr) \leq \sum_{k: m_k + j < b} \bigl(b - (m_k + j)\bigr).$$

Call this last number $B_{(m)}$. Then, thanks to the above *properties* of the chosen $h_{\tilde{\zeta}}$, the summand, localized at $\tilde{\zeta}$,

$$\Bigl(\prod_{k: m_k + j < b} Diff_Z^{(m_k + j)} J \Bigr) \mathcal{O}_{\tilde{Z}, \tilde{\zeta}}$$

in the above sum-product $(\{\ \})$ should be divisible by $h_{\tilde{\zeta}}^{B_{(m)}/b!}$, which means, more rigorously, that

$$\Bigl(\prod_{k: m_k + j < b} Diff_Z^{(m_k + j)} J \Bigr)^{b!} \mathcal{O}_{\tilde{Z}, \tilde{\zeta}}$$

is divisible by $h_{\tilde{\zeta}}^{B_{(m)}}$ and hence by $h_{\tilde{\zeta}}^{b! - m}$. We thus conclude that $\Bigl(Diff_Z^{(m)} J^\sharp\Bigr)^{b!} \mathcal{O}_{\tilde{Z}, \tilde{\zeta}}$ is dibisible by $h_{\tilde{\zeta}}^{b! - m}$. But this $h_{\tilde{\zeta}}^{b! - m}$ is the generator of $(J^\sharp \mathcal{O}_{\tilde{Z}})_{\tilde{\zeta}}^{b! - m}$ by the *properties* of $h_{\tilde{\zeta}}$. Namely, with $b^\sharp = b!$,

$$\Bigl(Diff_Z^{(m)} J^\sharp\Bigr)^{b^\sharp} \mathcal{O}_{\tilde{Z}, \tilde{\zeta}} \quad \subset \quad (J^\sharp)^{b^\sharp - m} \mathcal{O}_{\tilde{Z}, \tilde{\zeta}}.$$

This being true for all m and for all $\tilde{\zeta}$, we conclude that $\Bigl(Diff_Z^{(m)} J^\sharp\Bigr)^{b^\sharp}$ is contained in the integral closure of $(J^\sharp)^{b^\sharp - m}$ for all m, i.e., $E^\sharp = (J^\sharp, b^\sharp)$ is *Diff-full.* The proof of the lemma is now all done. □

LEMMA 2.3. *Given $E = (J, b)$ on Z, let $E^\sharp = (J^\sharp, b^\sharp)$ be the same as in Lemma 2.2. Then for every discrete valuation ring R of rank one with $\hat{\zeta} = \max(R)$ and for every morphism $\phi : \mathrm{Spec}(R) \to Z$ such that the pull back $J^\sharp R$ of $J^\sharp$ by ϕ is not the unit ideal of R, we have $\phi(\hat{\zeta}) \in \mathrm{Sing}(E)$ and $\mathrm{ord}_{\hat{\zeta}}(J^\sharp R) \geq \mathrm{ord}_{\phi(\hat{\zeta})}(J^\sharp) \geq b^\sharp$.*

Proof. Let C be the closed irreducible reduced subscheme of Z whose generic point is $\zeta = \phi(\hat{\zeta})$. Let $\pi : Z_1 \to Z$ be the blowing-up with center C. We then have a morphism $\psi : \mathrm{Spec}(R) \to Z_1$ such that $\phi = \pi \circ \psi$. Let η be the generic point of $\pi^{-1}(\zeta)$, i.e., that of the exceptional divisor for π, and let $\bar{\eta} = \psi(\hat{\zeta}) \in Z_1$. Note that $\bar{\eta}$ is a smooth point of Z_1 and also such of the exceptional divisor. The local ring $\mathcal{O}_{Z_1, \eta}$ is a discrete valuation ring of rank one and $\mathrm{ord}_\eta(h) = \mathrm{ord}_\zeta(h), \forall h \in \mathcal{O}_{Z, \zeta}$. We thus have

$$\mathrm{ord}_{\hat{\zeta}}(J^\sharp R) \geq \mathrm{ord}_{\bar{\eta}}(J^\sharp \mathcal{O}_{Z_1}) \geq \mathrm{ord}_\eta(J^\sharp \mathcal{O}_{Z_1}) = \mathrm{ord}_\zeta(J^\sharp).$$

So it is enough to prove that if $\mathrm{ord}_\zeta(J^\sharp) > 0$ then $\mathrm{ord}_\zeta(J^\sharp) \geq b!$, i.e., $\zeta \in \mathrm{Sing}(E^\sharp)$. Assume $\mathrm{ord}_\zeta(J^\sharp) < b!$. We then have $\mathrm{ord}_\zeta J < b$ for if otherwise we would have $\mathrm{ord}_\zeta(Diff_Z^{(i)} J) \geq b - i, \forall i$ and $\mathrm{ord}_\zeta(J^\sharp) \geq b!$. Let $e = \mathrm{ord}_\zeta J$. Then $\mathrm{ord}_\zeta(Diff_Z^{(e)} J) = 0$, i.e., $Diff_Z^{(e)} J_\zeta$ must be the unit ideal in $\mathcal{O}_{Z,\zeta}$. It follows that $\mathrm{ord}_\zeta(J^\sharp) = 0$. This proves the lemma. □

LEMMA 2.4. *If $E = (J, b)$ is Diff-full, then for every smooth subscheme $W \subset Z$ we have $Red_W(E) \sim (J\mathcal{O}_W, b)$.*

Proof. By the *Diff-fullness*, $(Diff_Z^{(j)} J)^b$ is integrally dependent upon J^{b-j}. It follows that $(Diff_Z^{(j)} J)^b \mathcal{O}_W$ is integrally dependent upon $J^{b-j}\mathcal{O}_W$. Hence

$$\begin{aligned} Red_W(E) &= \bigcap_{0 \leq j \leq b-1} \left((Diff_Z^{(j)} J)\mathcal{O}_W, b - j\right) \\ &\sim \bigcap_{0 \leq j \leq b-1} \left((Diff_Z^{(j)} J)^b \mathcal{O}_W, b(b - j)\right) \\ &\supset \bigcap_{0 \leq j \leq b-1} \left(J^{b-j}\mathcal{O}_W, b(b - j)\right) \sim (J\mathcal{O}_W, b) \end{aligned}$$

where the first equality is by definition and the inclusion is by the integral dependence. But the first intersection has the term with $j = 0$, which is nothing but $(J\mathcal{O}_W, b)$. Hence the reversed inclusion is trivially true. The proof is done. □

3. Proof of the main theorem

First of all we remark that $E = (J, b)$ may be replaced by any other idealistic exponent $F = (K, c) \sim E$, because we have the implication $E \sim F \Rightarrow \wp(E) = \wp(F)$. Hece we may replace E by $E^\sharp$ of Lemma 2.2 because $E^\sharp \sim E$ by the *Diff Theorem* and by Definition 1.3. Thus, we may and will assume:

i) E itself is Diff-full in the sense of Definition 2.2. (The *Diff-fullness* of $E^\sharp$ is by Lemma 2.2)

ii) there exist $G_j = (I_j, b_j) \supset E, 1 \leq j \leq r$, such that E is Diff-generated by the system $G_j, 1 \leq j \leq r$, in the sense of Definition 2.1. (The *Diff-generation* is by Lemma 2.1, where $G_j = \left(Diff_Z^{(j)} J, b-j\right)$, $0 \leq j < b$, and the inclusions are by *Diff Theorem*) *iii) for every discrete rank one valuation ring R and for every morphism $\phi : \mathrm{Spec}(R) \to Z$ with $JR \neq R$ we have that* $\mathrm{Im}(\phi) \cap \mathrm{Sing}(E)$ *is not empty* (this is by Lemma 2.3 for $E^\sharp$).

Let $\rho : \tilde{Z} \longrightarrow Z$ be the normalized blowing-up of the ideal sheaf J, so that $\tilde{Z}$ is normal and $J\mathcal{O}_{\tilde{Z}}$ is locally non-zero principal. For brevity, we will write $D_j = Diff_Z^{(j)} J,\ 0 \leq j < b$. Since E is *Diff-full*, we have $D_j^b \mathcal{O}_{\tilde{Z}} \subset J^{b-j}\mathcal{O}_{\tilde{Z}}$ and hence $D_j^b\mathcal{O}_{\tilde{Z}}$ is divisible by $J^{b-j}\mathcal{O}_{\tilde{Z}}$ because the last ideal is locally non-zero principal. Let us make clear what we want to prove under the assumotions *i)*, *ii)* and *iii)*. Following the notation in the definition of $\wp(E)$, we want to prove:

$$(\flat) \quad J_{\max}(b\mu) = \rho_*(J^\mu \mathcal{O}_{\tilde{Z}}) \text{ for every integer } \mu \geq 0.$$

Before going to prove $(\flat)$, let us first see that if it is proven then the main theorem follows. This implication is seen as follows. The question is local in Z and we will assume that Z is affine, say $Z = \mathrm{Spec}(A)$. We have $J_{\max}(a)^b \subset J_{\max}(ba)$ by their definition and we know that $\rho_*(J^a\mathcal{O}_{\tilde{Z}})$ is *integral* over J^a in the sense of the *ideal theory.* If $(\flat)$ is proven, then for $\forall g \in J_{\max}(a)$, g^b is *integral* over J^a in the sense of the *ideal theory.* This is equivalent to saying that $(gT^a)^b$ is *integral* over the graded algebra $\sum_{\mu\geq 0} J^\mu T^{b\mu}$ in the sense of the *ring theory.* Let $P(E) = \sum_{\mu \geq 0} J^\mu T^{b\mu}$. In view of Basic 5 and the fact (2.2), we can conclude:

> $\wp(E)$ is equal to the integral closure of the k-algebra $P(E)$ in the field of fractions $\mathbf{K}$ of $A[T]$.

Here, since $\mathbf{K}$ is finitely generated as a *field* over k and $P(E)$ is finitely generated as k-*algebra*, it follows from the general theory of commutative algebra that the integral closure $\wp(E)$ of $P(E)$ in $\mathbf{K}$ is a finite $P(E)$-module and hence $\wp(E)$ is finitely generated as k-algebra. Thus $(\flat)$ is all that remains to be proven.

Let us now proceed to prove $(\flat)$. Let $\tilde{\eta}_i, 1 \leq i \leq s$ be the generic points of the subscheme of $\tilde{Z}$ defined by the ideal $J\mathcal{O}_{\tilde{Z}}$. Since $\tilde{Z}$ is normal and $J\mathcal{O}_{\tilde{Z}}$ is locally principal, they are all smooth points of $\tilde{Z}$. We can find an open affine subscheme $\tilde{U} = \mathrm{Spec}(\tilde{A})$ of $\tilde{Z}$ such that we have $\tilde{\eta}_i \in \tilde{U}, \forall i$, and $\tilde{U}$ is smooth. Since $\tilde{A}$ is finitely generated as k-algebra, we can choose a finite set of indeterminates $t = (t_1, \ldots, t_r)$ such that there exists a surjective k-algebra homomorphism $\lambda : k[t] \to \tilde{A}$. Combined with the canonical inclusion $A \hookrightarrow \tilde{A}$, λ naturally extends to a surjective homomorphism $\Lambda : A[t] \longrightarrow \tilde{A}$. Let B be the kernel of Λ and let $W = \mathrm{Spec}(A[t]/B)$, which is a smooth subscheme of $Z[t]$. It is naturally isomorphic to $\tilde{U}$. By definition, we have

$$\begin{aligned} Red_W(E[t]) &= \bigcap_{0\le j<b} \left((Diff^{(j)}_{Z[t]} J[t])\mathcal{O}_W, b-j \right) \\ &= \bigcap_{0\le j<b} \left(((Diff^{(j)}_{Z} J)[t])\mathcal{O}_W, b-j \right) \\ &= \bigcap_{0\le j<b} \left((Diff^{(j)}_{Z} J)\mathcal{O}_{\tilde{U}}, b-j \right) \end{aligned}$$

where the last equality is by the isomorphism $A[t]/B \simeq \tilde{A}$. By *iii*), the images $\eta_i = \rho(\tilde{\eta}_i)$ are all in $\mathrm{Sing}(E)$ and hence $\mathrm{ord}_{\tilde{\eta}_i}(J\mathcal{O}_{\tilde{U}}) \ge \mathrm{ord}_{\eta_i}(J) \ge b, \forall i$. Viewing $\tilde{\eta}_i$ as points of $W \subset Z[t]$ as well as of $\tilde{U}$, we see that

$$\begin{aligned} \mathrm{ord}_{\tilde{\eta}_i}(Red_W(E[t])) &= \min_{0\le j<b}\{\mathrm{ord}_{\tilde{\eta}_i}((Diff^{(j)}_Z J)\mathcal{O}_{\tilde{U}})/(b-j)\} \\ &\ge \min_{0\le j<b}\{\mathrm{ord}_{\eta_i}(Diff^{(j)}_Z J)/(b-j)\} \quad \ge \quad 1 \end{aligned}$$

and hence we have $\tilde{\eta}_i \in \mathrm{Sing}(Red_W(E[t])), \forall i$. Now pick any idealistic exponent $F = (H, c) \sim E$ on Z. We then have

$$\begin{aligned} \mathrm{ord}_{\tilde{\eta}_i}(H\mathcal{O}_{\tilde{U}})/c &= \mathrm{ord}_{\tilde{\eta}_i}(H[t]\mathcal{O}_W)/c \ \ge\ \mathrm{ord}_{\tilde{\eta}_i}\big(Red_W(F[t])\big) \\ &= \mathrm{ord}_{\tilde{\eta}_i}\big(Red_W(E[t])\big) \ =\ \mathrm{ord}_{\tilde{\eta}_i}(J\mathcal{O}_{\tilde{U}})/b,\ \forall i. \end{aligned}$$

Here the first equality is by $A[t]/B \simeq \tilde{A}$ and the second inequality is by the definition of Red_W expressed as an intersection of idealistic exponents including $(H[t], c)$ itself. The equality before the last, follows $F \sim E$ by the *Numerical Exponent Theorem* and the *Ambient Reduction Theorem.* Finally the last equality is by Lemma 2.4 thanks to the assumption i). Now, apply the above inequality to the case of $F = (J_{\max}(b\mu), b\mu), \mu > 0$, and we get

$$\mathrm{ord}_{\tilde{\eta}_i}(J_{\max}(b\mu)\mathcal{O}_{\tilde{U}})/b\mu \ \ge\ \mathrm{ord}_{\tilde{\eta}_i}(J\mathcal{O}_{\tilde{U}})/b,\ \forall i$$

which implies $J_{\max}(b\mu)\mathcal{O}_{\tilde{U},\tilde{\eta}_i}$ is divisible by $J^{\mu}\mathcal{O}_{\tilde{U},\tilde{\eta}_i}, \forall i$, because the local ring is a discrete rank one valuation ring. Since $J^{\mu}\mathcal{O}_{\tilde{U}}$ is locally non-zero principal everywhere on a normal scheme and the $\{\tilde{\eta}_i\}$ are all the generic points of $\mathrm{Spec}(\mathcal{O}_{\tilde{Z}}/J\mathcal{O}_{\tilde{Z}})$, it follows that $J_{\max}(b\mu)\mathcal{O}_{\tilde{Z}}$ is divisible by $J^{\mu}\mathcal{O}_{\tilde{Z}}$. In particular, we have

$$J_{\max}(b\mu)\mathcal{O}_{\tilde{Z}} \subset J^{\mu}\mathcal{O}_{\tilde{Z}}, \forall \mu \ge 1.$$

However, by the maximality of $J_{\max}$, we have

$$J_{\max}(b\mu) \supset J^{\mu} \text{ and } J_{\max}(b\mu) = \rho_*(J_{\max}(b\mu)\mathcal{O}_{\tilde{Z}}), \forall \mu \ge 1$$

and hence the above converse inclusion implies

$$J_{\max}(b\mu)\mathcal{O}_{\tilde{Z}} = J^{\mu}\mathcal{O}_{\tilde{Z}} \text{ and } J_{\max}(b\mu) = \rho_*(J^{\mu}\mathcal{O}_{\tilde{Z}}), \forall\mu \geq 1.$$

This proves ($\mathfrak{b}$). We complete the proof of the theorem with an additional remark which shows the coherency of $J_{\max}(a), \forall a$. The replacement of (J, b) by $(J^{\sharp}, b^{\sharp})$, called $\sharp$-*operation* below, is compatible with any localization of the affine ring A, that is with the restriction from an open affine set of Z to any smaller one. Moreover, we saw that $\wp(E)$ is the integral closure of $P(E)$ in the function field $\mathbf{K}$ of the scheme Z. The *integral closure* is also compatible with any localization. The coherency is clear. The Main Theorem is now all established.

REMARK 3.1. From the above proof of the *Main Theorem* using the $\sharp$-*operation* (cf. Lemma 2.2), it is seen that if $E = (J, b)$ and $\wp(E) = \sum_{0 \leq a \leq \infty} J_{\max}(a)T^a$ then the ideals $J_{\max}(kb^{\sharp})$ is *integral* over $J_{\max}(b^{\sharp})^k$ in the sense of *ideal theory* for all integers $k > 0$ where $b^{\sharp}$ is the a priori number $b!$.

References

[1] S. Abhyankar, *Desingularization of plane curves*, Summer Institute on Algebraic Geometry, Arcata 1981, Proc. Symp. Pure Appl. Math. 40, AMS.

[2] V. Cossart, C. Galindo, et O. Piltant, *Un example effectif de gradue non noetherien associe a une valuation divisorielle*, Ann. Inst. Fourier, Grenoble, 50 **1** (2000), 105–112.

[3] J. Giraud, *Sur la theorie du contact maximal*, Math. Z. **137** (1974), 286–310.

[4] ———, *Contact maximal en caracteristique positive*, Ann. Sci. École Norm. Sup. (4) **8** (1975), 201–234.

[5] H. Hironaka, *Gardening of infinitely near singularities*, Proc. Nordic Summer School in Math., Oslo (1970), pp. 315–332.

[6] ———, *Introduction to the theory of infinitely near singular points*, Memorias de Matematica del Instituto "Jorge Juan", no. 28, Consejo Superior de Investigaciones Cientificas, Madrid, 1974.

[7] ———, *Idealistic exponents of singularity*, Algebraic Geometry, Johns Hopkins Univ. Press, Baltimore, Md. (1977) pp. 52–125 (J. J. Sylvester Symposium, Johns Hopkins Univ., 1976).

[8] B. Youssin, *Newton Polyhedra without coordinates*, Mem. Amer. Math. Soc. **433** (1990), 1–74, 75–99.

The Japan association for Math. Science
Coop Olympia 506, Jingumae 6-chome 35-3
Shibuya-ku Tokyo 150-0001, Japan