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Theory of Matter with Super Light Velocity

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The matter with a super light velocity is treated as one possible model in the course of looking for the physical concept that will essentially govern the future theory of elementary particles. At first, it is investigated according to the canonical quantization to what extent this matter could be reconciled to the usual particle aspect of elementary particles. Further, taking into consideration the interaction of this matter with other particles, it is attempted to quantize this matter so as to derive the Lorentz-invariant, but quasi-causal S-matrix and to remove the free state of this matter from possible physical states.

§ 1. Introduction

Recent successive researches for the field theory of elementary particles have made clear the general features of the theory apart from the detailed structure of interactions.[1] That is, they start only from the several fundamental postulates concerning the special theory of relativity and quantum mechanics, such as the Lorentz-invariance, the micro-causality, the existence of the vacuum state, the assumption on the asymptotic behavior of field and so on.

However, these conditions are so rigid that it seems nearly impossible to seek for the new possibility of the unified and consistent future theory in these framework, although we have yet no definite evidence against it. At present time, many authors have serious doubts as to whether the above framework will remain valid in the future theory. So far, several attempts have been made to change the present field theory, for instance, the denial of the point model for elementary particles in the non-local field theory,[2] or the introduction of the indefinite metric into the Hilbert space in Heisenberg's non-linear field theory.[3]

On the other hand, experimental evidences for strange particles as shown in Nishijima-Gell-Mann's rule seem strongly to suggest that the future theory should necessarily be able to give rise to new degrees of freedom, such as the iso-spin, strangeness, etc.

Now, in order to get a clue to the modification of the present theory, it seems quite important to discover the new physical concept which will probably characterize the future theory. From such a viewpoint, it is worthwhile to pay our attention to the concept of " *B*-matter " recently introduced by Sakata and his co-workers.* They have proposed a new baryon model (named " Nagoya Model ",)

* Their works appeared in Prog. Theor. Phys. **23** (1960), 1174.

in which N, P and Λ are considered to be composed of B-matter and leptons, based on Kiev symmetry[4] in weak interactions. According to this model, B-matter could exist only by accompanying leptons, and not by itself. Further, this matter would probably violate principles of the usual quantum mechanics. While such a matter is unfamiliar to us from the customary point of view of the field quantization, it may offer some image for the future theory.

From the viewpoint on the theoretical side, it is important to seek for the new possibility, by focusing our attention on such an image as symbolized by the above model.

In this paper we shall investigate the matter with a super light velocity (hereafter we shall describe it in terms of "S-field") as one possible model in the course of looking for the above image. The relativistic invariant S-field is defined by the following wave equation,

$$(\Box+m^2)\phi(x)=0, \quad \left(\Box\equiv\Delta-\frac{\partial^2}{\partial x_0{}^2}\right) \tag{1·1}$$

with m real or by its linearized equation

$$(\gamma_\mu\partial_\mu+im)\psi(x)=0. \tag{1·2}$$ *, **

From (1·1), we easily find that the four momentum ($\boldsymbol{p}$, iE) satisfies the modified Einstein relation

$$E^2=\boldsymbol{p}^2-m^2. \tag{1·3}$$

Therefore, the group velocity of this wave with $p^2>m^2$ becomes a super light velocity:

$$v_g=c\,\partial E/\partial p=cp/E>c. \tag{1·4}$$

Although Pais and Uhlenbeck[5] once investigated a multi-mass field with complex conjugate pair of masses, μ and μ^*, they abandoned such a possibility as an unacceptable field by reason of occurrence of the solution with unbounded character, to which we shall refer in § 3. Further, Schmidt[6] recently investigated in detail the invariant solution of (1·1) and its source problem from the viewpoint of the causality. However, the essential character of the S-field will be revealed rather through the quantization procedure. In fact, it is found in § 3 and § 4, that S-field is intimately connected with the limitation of the fundamental concept of the present field theory such as the localizability of the field, the indefinite metric, the existence of the vacuum state, the micro-causality, parity indefiniteness, etc.

* More generally, the linearized equation of (1·1) is given by $(\gamma_\mu\partial_\mu+i\alpha+\beta\gamma_5)\psi(x)=0$, where α and β are arbitrary real constants. However, this equation is transformed into (1·2) by suitable linear transformation of ψ. See Appendix A.

** $a_\mu b_\mu=ab=\boldsymbol{ab}-a_0b_0$

In § 2, we shall study the propagation character of the classical S-field in connection with the results obtained by Schmidt.[6] In § 3, the quantization of the S-field is given by the orthodox canonical formalism, in order to see to what extent this field could be reconciled with the usual particle aspect for elementary particles. Further in § 4, we shall consider the possibility of the alternative quantization procedure for the S-field, taking into consideration its interaction with other particles.

§ 2. The invariant function and the propagation character of the classical wave

Let us consider the invariant solution of the source problem of (1·1),

$$(\Box+m^2)G(x)=\delta^4(x). \tag{2·1}$$

This problem has already been investigated in detail by Schmidt.[6] Therefore, in the present paper, we shall refer mainly to the problem concerning the solution with the *abnormal part*, which the above author did not take into consideration as an unphysical one.

The general solution of (2·1) is given by

$$G(x)=-\frac{1}{(2\pi)^4}\int_C \frac{e^{ikx}}{k^2-m^2}d^4k, \tag{2·2}$$

where C denotes the integration path in complex-k_0 plane. The poles of the integrand in the above integration in the complex k_0-plane are shown in Fig. 1 by bold lines. There exist four invariant solutions with the integration paths indicated in Fig. 1, and they are written explicitly as follows,

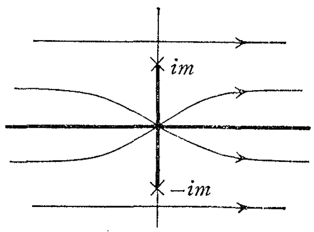

Fig. 1. Integration paths in complex-k_0 plane

$$G_\pm(x)=-\frac{1}{(2\pi)^4}\int\frac{e^{ikx}}{k^2-m^2\mp i\varepsilon}d^4k, \tag{2·3}$$

$$G_{R,A}(x)=-\frac{1}{(2\pi)^4}\int\frac{e^{i(k\pm i\eta\xi)x}}{(k\pm i\eta\xi)^2-m^2}d^4k, \tag{2·4}*$$

where $\eta>m$, ε is an infinitesimal positive number and ξ_μ a time-like unit vector in a future light cone. After the integration over k_0, or $\boldsymbol{k}$, they are rewritten as

$$G_\pm(x)=\frac{-1}{(2\pi)^2}\int_0^\infty du\frac{e^{\pm ir\sqrt{m^2+u^2}}}{r}\cos x_0 u, \tag{2·3'}$$

* See Appendix C.

$$\left.\begin{aligned} G_R(x)&=G_A(-x)\\ &=-\frac{1}{(2\pi)^3}\left\{\int\limits_{k^2>m^2} d^3k\frac{1}{\sqrt{\boldsymbol{k}^2-m^2}}\sin\sqrt{\boldsymbol{k}^2-m^2}\,x_0\,e^{ikx}\right.\\ &\left.+\int\limits_{k^2<m^2} d^3k\frac{1}{\sqrt{m^2-\boldsymbol{k}^2}}\sinh\sqrt{m^2-\boldsymbol{k}^2}\,x_0\,e^{ikx}\right\},\quad x_0>0\\ &=0,\qquad\qquad\qquad\qquad x_0<0.\end{aligned}\right\}\tag{2·4′}$$

From the above expressions, it is found that $G_{R,A}$ involves the *abnormal* part which implies the second term of (2·4′) and is an exponentially increasing function of x_0. The separation of this part from the *normal* part which means the first term of (2·4′) and is a usual oscillating function of x_0 is, of course, not Lorentz-invariant.

However, it is possible to divide both parts Lorentz-covariantly, by making use of the above time-like unit vector ξ_μ:

$$G_{R,A}(x)=G_{R,A}^{(n)}(x,\ \xi)+G_{R,A}^{(ab)}(x,\ \xi),\tag{2·5}$$

where

$$G_{R,A}^{(n)}(x,\ \xi)=-\frac{1}{(2\pi)^4}\left\{P\int\frac{e^{ikx}}{k^2-m^2}d^4k\pm i\pi\int\epsilon(k\xi)e^{ikx}\delta(k^2-m^2)d^4k\right\}.\tag{2·6}$$*

In fact, we find that $G_R^{(n)}(x,\xi)$ with $\xi_\mu=n_\mu=(0,0,0,i)$, which corresponds to the integration path in Fig. 2, is just the first term of (2·4′) and identical with the quasi-retarded function defined by Schmidt,[6] that is,

$$G_{R,A}^{(n)}(x,\ n)=-\frac{1}{(2\pi)^2}\frac{1}{r}\int_0^\infty du\cos(r\sqrt{m^2+u^2}\mp x_0u).\tag{2·7}$$

As seen from (2·7) and (2·3′), there exists the following important relations between the above functions,

$$\left.\begin{aligned} G_+^{(+)}(x)&=G_R^{(n)(+)}(x,\ n),\\ G_+^{(-)}(x)&=G_A^{(n)(-)}(x,\ n),\end{aligned}\right\}\tag{2·8}$$

where $(\pm)$'s mean the positive or negative time frequency parts. In the usual theory of Klein-Gordon particle with a real mass, the above relations hold for the exact retarded or advanced function and express a causality of theory. The physical interpretation of the above relation will be referred to in § 4.

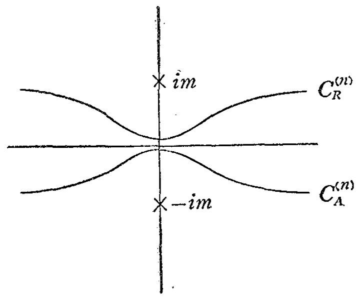

Fig. 2.

* $\epsilon(a)$ is the sign factor defined by

$$\epsilon(a)=1,\ a>0,$$
$$=-1,\ a<0.$$

Secondly, we shall study the invariant solution of (1·1). These solutions are given by the difference of solutions of the inhomogeneous equation (2·1), that is,

$$D^{(1)}(x) \equiv \{G_+(x)-G_-(x)\}/2i = \frac{-1}{2(2\pi)^3}\int d^4k\, e^{ikx}\delta(k^2-m^2)$$

$$= \frac{-1}{2(2\pi)^3}\int_{-\infty}^{\infty} d\tau \int d^4k\, e^{i(k+\tau\xi)x}\delta(k^2-m^2-\tau^2)\delta(k\xi)$$

$$= \frac{-1}{2(2\pi)^3}\int d^3p \frac{1}{2\eta\xi_0}(e^{i(p+\eta\xi)x}+e^{i(p-\eta\xi)x}), \tag{2·9}$$

with

$$p\xi=0,\quad \eta=\sqrt{p^2-m^2}=\sqrt{\mathbf{p}^2-(\mathbf{p}\cdot\boldsymbol{\xi})^2/\xi_0^2-m^2},$$

and

$$D(x) \equiv G_A(x)-G_R(x). \tag{2·10}$$

The latter D-function involves again both the normal and the abnormal part, and is also separated covariantly in a similar way as in the $G_{R,A}$-functions:

$$D(x)=D^{(n)}(x,\xi)+D^{(ab)}(x,\xi), \tag{2·11}$$

where

$$D^{(n)}(x,\xi)=G_A^{(n)}(x,\xi)-G_R^{(n)}(x,\xi)=\frac{-i}{(2\pi)^3}\int \epsilon(k\xi)e^{-ikx}\delta(k^2-m^2)d^4k,$$

or

$$=\frac{-i}{(2\pi)^3}\int_{-\infty}^{\infty} d\tau \int d^4k\,\epsilon(\tau)e^{i(k+\tau\xi)x}\delta(k^2-m^2-\tau^2)\delta(k\xi), \tag{2·12}$$

$$=\frac{-i}{(2\pi)^3}\int d^3p\frac{1}{2\eta\xi_0}\{e^{i(p+\eta\xi)x}-e^{i(p-\eta\xi)x}\}, \tag{2·13}$$

with $p\xi=0,\ p^2\geq m^2,\ \eta=\sqrt{p^2-m^2}=\sqrt{\mathbf{p}^2-(\mathbf{p}\cdot\boldsymbol{\xi})^2/\xi_0^2-m^2}$,

$$D^{(ab)}(x,\xi)=G_A^{(ab)}(x,\xi)-G_R^{(ab)}(x,\xi)$$

$$=\frac{+1}{(2\pi)^3}\int_{-\infty}^{\infty} d\tau\int d^4k\,\epsilon(\tau)e^{i(k-i\tau\xi)x}\delta(k^2-m^2+\tau^2)\delta(k\xi) \tag{2·14}$$

$$=\frac{+1}{(2\pi)^3}\int\frac{d^3p}{2\rho\xi_0}\{e^{i(p-i\rho\xi)x}-e^{i(p+i\rho\xi)x}\}, \tag{2·15}$$

with $p\xi=0$ and $\rho=\sqrt{m^2-p^2}=\sqrt{m^2+(\mathbf{p}\cdot\boldsymbol{\xi})^2/\xi_0^2-\mathbf{p}^2}$.

Especially, for $\xi_\mu=n_\mu$

$$D^{(n)}(x,n)=\frac{-1}{(2\pi)^3}\int_{k^2>m^2}\frac{d^3k}{\sqrt{\mathbf{k}^2-m^2}}\sin\sqrt{\mathbf{k}^2-m^2}\,x_0\,e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{2·13'}$$

$$D^{(ab)}(x,n)=\frac{-1}{(2\pi)^3}\int_{k^2<m^2}\frac{d^3k}{\sqrt{m^2-\mathbf{k}^2}}\sinh\sqrt{m^2-\mathbf{k}^2}\,x_0\,e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{2·14'}$$

Of course, the normal and the abnormal parts of the invariant function are not Lorentz-invariant, while they have the following covariant property,

$$F(x,\ \xi)=F(x',\ \xi'),$$

where

$$x_\mu'=\Lambda_{\mu\nu}x_\nu,\quad \xi_\mu'=\Lambda_{\mu\nu}\xi_\nu,$$

and $\Lambda_{\mu\nu}$'s are Lorentz transformation coefficients.

The above defined function $D(x)$ satisfies the following familiar relations:

$$D(x)=0,\quad x_\mu^2>0 \tag{2·16}$$

and

$$\left.\begin{aligned}&\frac{\partial}{\partial x_0}D(\boldsymbol{x},\ x_0)|_{x_0=0}=-\delta^3(\boldsymbol{x}),\\&\frac{\partial}{\partial x_i}D(\boldsymbol{x},\ x_0)|_{x_0=0}=0.\end{aligned}\right\} \tag{2·17}$$

Of course, $D^{(n)}(x,\xi)$-function has no such a property. In fact, it is in general non-vanishing in the space-like region except the special hypersurface with $\xi x=0$.

By means of this property of D-function, we can solve the initial value problem in the usual way. The solutions of (1·2) and (1·1) are given by

$$\phi(x)=-\int_\sigma D(x-x')\overleftrightarrow{\partial}_\mu'\phi(x')d\sigma_\mu' \tag{2·18}$$

$$(A\overleftrightarrow{\partial}_\mu B\equiv(\partial_\mu A)B-A\partial_\mu B)$$

and

$$\psi(x)=\int_\sigma(\gamma_\mu\partial_\mu+m\gamma_5)D(x-x')\gamma_\mu\psi(x')d\sigma_\mu', \tag{2·19}$$

where the hypersurface σ is indicated in Fig. 3. Since the D-function vanishes in the space-like region, S-field obviously has a hyperbolic propagation character, namely, the behavior of the wave in an arbitrary world point is completely determined only by the information on the wave on a hypersurface in a past light cone. This is obvious also from the fact that the propagation character is in general determined only from the form of terms which involve the highest degrees of derivative with respect to space-time.[5)]

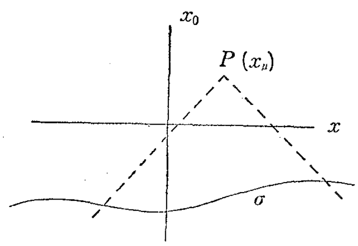

Fig. 3.

Although the above statement apparently seems to contradict the fact that S-field has a group-velocity larger than the light velocity, this situation is easily understood as follows. First of all, it is noted that the propagation character essentially indicates the propagation behavior of the spatial discontinuity of the wave. On the contrary, the wave with which the group velocity is concerned is composed of the plane waves with an infinitesimal interval of nearly adjacent wave-lengths and thus has no spatial discontinuity. Furthermore in such a wave, the field quantities at two different spatial regions at a given time are no longer independent from each other. Therefore although the group velocity of S-field seems to imply the propagation of information with a super light velocity, this is not the case.

In fact, the above statement is concretely confirmed in the following way. For simplicity, let us take a flat σ-surface perpendicular to the time axis as in Fig. 4. In general, the right-hand side of (2·18) is divided into four components, which come from the devision of the D-function into the normal and the abnormal parts as in (2·11) and from the separation of the integral region over the σ-plane into the inside and the outside regions of the light cone as shown in Fig. 4. This devision is written as

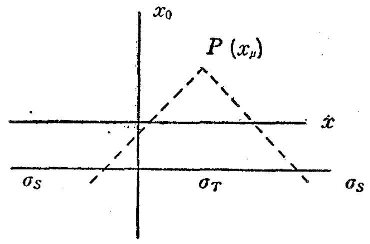

Fig. 4.

$$\phi(x)=\phi_S^{(n)}(x)+\phi_T^{(n)}(x)+\phi_S^{(ab)}(x)+\phi_T^{(ab)}(x), \tag{2·20}$$

where

$$\left.\begin{aligned}\phi_{S(T)}^{(n)}(x)&=\int_{\sigma_S(\sigma_T)} D^{(n)}(x-x',\ n)\overleftrightarrow{\partial}_0\phi(x')d^3x',\\ \phi_{S(T)}^{(ab)}(x)&=\int_{\sigma_S(\sigma_T)} D^{(ab)}(x-x',\ n)\overleftrightarrow{\partial}_0\phi(x')d^3x',\end{aligned}\right\} \tag{2·21}$$

and $\sigma_S(\sigma_T)$ denotes the space-like (the time-like) region of the σ-surface. From the relation of (2·16),

$$\phi_S^{(n)}(x)+\phi_S^{(ab)}(x)=0. \tag{2·22}$$

Now, the wave of which the group velocity can be defined is composed only of the plane waves whose wave vectors satisfy $\boldsymbol{p}^2>m^2$. On the other hand, $D^{(ab)}(x,\ n)$ involves only the plane wave with $\boldsymbol{p}^2<m^2$ as shown in (2·14′). Thus, for the wave under consideration, one finds that the following relation is satisfied,

$$\phi_S^{(ab)}(x)+\phi_T^{(ab)}(x)=0. \tag{2·23}$$

Then, from (2·22) and (2·23),

$$\phi_S^{(n)}(x)=\phi_T^{(ab)}(x),$$

or

$$\int_{\sigma_S} D^{(n)}(x-x',\ n)\overleftrightarrow{\partial}_0{}'\phi(x')d^3x'=\int_{\sigma_T} D^{(ab)}(x-x',\ n)\overleftrightarrow{\partial}_0{}'\phi(x')d^3x'. \tag{2·24}$$

Thus, making use of (2·22) and (2·23), we obtain two different expressions for $\phi(x)$,

$$\phi(x)=\phi_T^{(n)}(x)+\phi_T^{(ab)}(x)=\int_{\sigma_T} D(x-x')\overleftrightarrow{\partial_0}'\phi(x')d^3x', \qquad (2\cdot25)$$

and

$$\phi(x)=\phi_T^{(n)}(x)+\phi_S^{(n)}(x)=\int_{\sigma(\sigma_S+\sigma_T)} D^{(n)}(x-x',\ n)\overleftrightarrow{\partial_0}'\phi(x')d^3x'. \qquad (2\cdot26)$$

In the first expression, the information only in the past time-like region contributes to $\phi(x)$, contrary to the second expression in which the information in the space-like region is necessary in addition to the past time-like region. Namely, if we describe the propagation of S-field only in terms of the normal D-function, it becomes necessary to take into account the information in the space-like region, which is however completely determined by $\phi_T^{(ab)}(x)$, namely the information in the time-like region, as shown in (2·24). Thus, the apparently acausal behavior of the S-field is explained by the fact that the behavior of the wave under consideration is not independent in a space-like region, but is imposed by the condition such as (2·24)*.

We have hitherto confined ourselves only to the propagation character of classical wave. In the quantized theory, the causal property for the energy propagation is not completely determined only by the field equation, but connected essentially with the definition of the vacuum state. We shall deal with this problem in the next section.

§ 3. The quantization of S-field ——*The canonical formalism*——

The Lagrange functions, which lead to the fundamental equation of S-field

$$(\Box+m^2)\phi(x)=0, \qquad (1\cdot1)$$

or

$$(\gamma_\mu\partial_\mu+im)\psi(x)=0 \qquad (1\cdot2)$$

are given by

$$\mathcal{L}(x)=\phi^*(x)(\Box+m^2)\phi(x), \qquad (3\cdot1)^\dagger$$

or

$$\mathcal{L}(x)=\psi^\dagger(x)(\gamma_\mu\partial_\mu+im)\psi(x), \qquad (3\cdot2)$$

where

$$\psi^\dagger(x)=\psi^*(x)\gamma_4\gamma_5=\bar{\psi}(x)\gamma_5. \qquad (3\cdot3)$$

* It fact, the well-known fact that the phase velocity in the usual Klein-Gordon field becomes larger than the light velocity is essentially based on the same reason. In this case, it is enough to separate the D-function into two parts so that one of them covers completely the interval of wave-lengths of the wave under consideration.

† * denotes the complex conjugate.

The hermiticity of the above Lagrangian is obvious. According to the usual canonical formalism, we obtain the following commutation relations for $\phi(x)$ and $\psi(x)$,

$$\left[\phi(\boldsymbol{x},\ x_0),\ \frac{\partial}{\partial x_0}\phi^*(\boldsymbol{x}',\ x_0)\right]=\delta^3(\boldsymbol{x}-\boldsymbol{x}'),\tag{3·4}$$

$$\{\psi(\boldsymbol{x},\ x_0),\ \psi^*(\boldsymbol{x}',\ x_0)\}=\gamma_5\delta(\boldsymbol{x}-\boldsymbol{x}'),\tag{3·5}$$

and their covariant expressions

$$[\phi(x),\ \phi^*(x')]=iD(x-x'),\tag{3·4'}$$

$$\{\psi(x),\ \bar{\psi}(x')\}=-i\gamma_5(\gamma_\mu\partial_\mu-m\gamma_5)D(x-x'),\tag{3·5'}*$$

by making use of the equation of motion and the property of D-function. D-function in the above commutation relation involves the normal and the abnormal parts, as shown in the preceding section, so the canonical formalism based on the local independence of fields makes necessary to introduce the abnormal component in the field operators themselves.

In order to give the explicit representation for field operators, it is necessary to remark several characters concerning this abnormal component of field operators. Let us assume the following form for the solution of (1·1):

$$\exp(ip_\mu x_\mu).\tag{3·6}$$

Then, the four vector p_μ satisfies

$$p_\mu p_\mu=\boldsymbol{p}^2-p_0^2=m^2.\tag{3·7}$$

From (2·2), one finds that the solutions are divided into two classes with distinct characters. The first class of the solutions is characterized by the real four vector p_μ which necessarily leads to $\boldsymbol{p}^2\geq m^2$. In this case the condition $\boldsymbol{p}^2\geq m^2$ is Lorentz invariant,** but the sign of p_0 is not, because the (real) four vector p_μ is a space-like vector. The second class of the solution is characterized by the four vector p_μ, where $\boldsymbol{p}^2<m^2$ and thus p_0 is pure imaginary. However, this statement is not Lorentz-invariant, because this four vector p_μ is in general transformed into complex four vector in other reference system. Thus, it is necessary to specify the reference system, in which the second class of solution is defined by the conditions $\boldsymbol{p}^2<m^2$ and p_0 to be pure imaginary. It is obvious that the first and the second classes in the above division correspond to the normal and the abnormal parts of the invariant functions in the previous section, because the latter part involves the

* It should be noted that the above quantization of a spinor field in connection with the equation of motion violates the invariance under the space inversion. The detailed discussion is left for Appendix A.

** In fact, if p_μ forms a real four vector, then $\boldsymbol{p}^2\geq m^2$. Taking account of the property of the Lorentz transformation, the real property of p_μ remains invariant in any other reference system, so the relation $\boldsymbol{p}^2\geq m^2$ also holds invariantly.

factor which is in general exponentially increasing both in space-like and time-like directions, and so has no physical reality.

Accordingly, in order to deal with the normal and the abnormal parts in a unified and Lorentz-covariant way, it is convenient to introduce a time-like unit vector ξ_μ in a future light cone: $\xi_\mu\xi_\mu=-1, \xi^0>0$. Then the four vector p_μ in (3·7) for both parts is described in the following form, respectively:

N) *Normal part*:

$$p_\mu=k_\mu+\tau\xi_\mu, \tag{3·8}$$

where we assume

$$k_\mu\xi_\mu=0 \quad \text{or} \quad k_0=\boldsymbol{k}\cdot\boldsymbol{\xi}/\xi_0. \tag{3·9}$$

Then, τ, which is assumed to be real, is determined from (3·7) as follows,

$$\tau=\pm\sqrt{k_\mu^2-m^2}=\pm\sqrt{\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{\xi})^2/\xi_0^2-m^2}, \tag{3·10}$$

with the condition

$$k_\mu k_\mu\geq m^2. \tag{3·11}$$

Therefore, the normal part is characterized by $\boldsymbol{k}$ and the sign of τ which is Lorentz-invariant.

Ab) *Abnormal part*:

$$p_\mu=k_\mu+i\tau\xi_\mu, \tag{3·8'}$$

under the conditions (3·9). Then, τ, which is assumed to be real, is determined as

$$\tau=\pm\sqrt{m^2-k_\mu^2}=\pm\sqrt{m^2-\boldsymbol{k}^2-(\boldsymbol{k}\cdot\boldsymbol{\xi})^2/\xi_0^2}, \tag{3·12}$$

with the condition

$$0\leq k_\mu^2<m^2. \tag{3·13}$$

Therefore, the abnormal part is also characterized by the $\boldsymbol{k}$ and the invariant sign of τ.

Thus, the field operator subject to the commutation relation

$$[\phi(x),\ \phi^*(x')]=iD(x-x'), \tag{3·4'}$$

is explicitly represented by

$$\phi(x,\ \xi)=\phi^{(n)}(x,\ \xi)+\phi^{(ab)}(x,\ \xi), \tag{3·14}$$

$$\left.\begin{aligned}\phi^{(n)}(x,\ \xi)&=\sum_{\substack{p\xi=0\\ p^2\geq m^2}}\frac{1}{\sqrt{2\eta\xi_0V}}\{a(p,\ \xi)e^{i(p+\eta\xi)x}+b^*(p,\ \xi)e^{i(p-\eta\xi)x}\},\\ \phi^{(ab)}(x,\ \xi)&=\sum_{\substack{p\xi=0\\ m^2>p^2\geq 0}}\frac{1}{\sqrt{2\rho\xi_0V}}\{c(p,\ \xi)e^{i(p-i\rho\xi)x}+d(p,\ \xi)e^{i(p+i\rho\xi)x}\},\end{aligned}\right\} \tag{3·15}$$

where

$$\left.\begin{aligned}\eta&=\sqrt{p_\mu^2-m^2}=\sqrt{\boldsymbol{p}^2-(\boldsymbol{p}\cdot\boldsymbol{\xi})^2/\xi_0^2-m^2},\\ \rho&=\sqrt{m^2-p_\mu^2}=\sqrt{m^2-\boldsymbol{p}^2+(\boldsymbol{p}\cdot\boldsymbol{\xi})^2/\xi_0^2}.\end{aligned}\right\}\tag{3·16}$$

The above commutation relation (3·4′) is reduced to the following two relations,

$$[\phi^{(n)}(x,\xi),\ \phi^{(n)*}(x',\xi)]=iD^{(n)}(x-x',\xi),\tag{3·17}$$

and

$$[\phi^{(ab)}(x,\xi),\ \phi^{(ab)*}(x',\xi)]=iD^{(ab)}(x-x',\xi),\tag{3·18}$$

assuming the commutativity between the normal and the abnormal components. Substituting the expression (3·14) and (3·15) into the relation (3·17) and (3·18), respectively, and taking into consideration the explicit form of $D^{(n)}$ and $D^{(ab)}$-functions, we arrive at the final relations;

$$[a(p,\xi),\ a^*(p,\xi)]=[b(p,\xi),\ b^*(p',\xi)]=\delta_{pp'},\tag{3·19}$$

under the conditions $p_\mu^2\geq m^2$, $p\xi=0$,

and
$$[c(p,\xi),\ d^*(p',\xi)]=i\delta_{pp'},\tag{3·20}$$

under the conditions $0\leq p_\mu^2<m^2$, $p\xi=0$, and all other commutators vanish.

It should be worth-while to note that the total field $\phi(x,\xi)$ has also an explicit dependence on the time-like unit vector ξ_μ, as well as its component fields $\phi^{(n)}(x,\xi)$ and $\phi^{(ab)}(x,\xi)$, and exhibits the oscillatory behavior only in the hypersurface indicated by $\xi x=$const. Thus, we find that there exist continuously many numbers of the unitary non-equivalent representations, corresponding to ξ_μ, each of which is subject to the same commutation relation (3·4′). This fact is approved later in this section directly from the spectrum of the displacement operator for $\phi(x,\xi)$.

However, each representation specified by ξ_μ is not Lorentz-invariant. In fact, the Lorentz transformation is represented by the unitary operator $U(\Lambda)$, which satisfies

$$U(\Lambda)\phi(x,\xi)U^{-1}(\Lambda)=\phi'(x,\xi)=\phi(\Lambda^{-1}x,\Lambda^{-1}\xi).\tag{3·21}$$

This relation implies the following equations,

$$\left.\begin{aligned}U(\Lambda)a(p,\xi)U^{-1}(\Lambda)&=a(\Lambda^{-1}p,\Lambda^{-1}\xi)\quad(=a'(p,\xi)),\\ U(\Lambda)b(p,\xi)U^{-1}(\Lambda)&=b(\Lambda^{-1}p,\Lambda^{-1}\xi)\quad(=b'(p,\xi)),\end{aligned}\right\}\tag{3·22}$$

under the conditions $p\xi=0$ and $p_\mu^2\geq m^2$, and similar relations for the $c(p,\xi)$ and $d(p,\xi)$'s. Thus, we find that the Lorentz-transformation in general gives rise to the mixing of each irreducible representation (corresponding to ξ_μ) of the canonical commutation relation (3·4′). The group theoretical structure under the Lorentz transformation is given by Appendix B.

The displacement operators $P_\mu(\xi)$'s for the respective field $\phi(x,\xi)$ are easily obtained from the following relation,

$$-i\frac{\partial\phi(x,\xi)}{\partial x_\mu}=[\phi(x,\xi),\ P_\mu(\xi)]. \tag{3·23}$$

By making use of the commutation relations (3·19) and (3·20), the explicit form of $P_\mu(\xi)$'s is determined as follows:

$$P_\mu(\xi)=P_\mu^{(n)}(\xi)+P_\mu^{(ab)}(\xi), \tag{3·24}$$

$$P_\mu^{(n)}(\xi)=\sum_{\substack{p\xi=0\\ p^2\geqq m^2}}\{(\eta+\xi p)_\mu a^*(p,\xi)a(p,\xi)+(\eta\xi-p)_\mu b^*(p,\xi)b(p,\xi)\}, \tag{3·25}$$

$$P_\mu^{(ab)}(\xi)=\sum_{\substack{p\xi=0\\ m^2>p^2\geqq 0}}\{(\rho\xi+ip)_\mu d^*(p,\xi)c(p,\xi)+(\rho\xi-ip)_\mu c^*(p,\xi)d(p,\xi)\}, \tag{3·26}$$

where η and ρ are given by (3·16). Further, the projection of $P_\mu(\xi)$ in the ξ_μ-direction, $W(\xi)$, has a simple form, namely

$$W(\xi)=-\xi_\mu P_\mu(\xi)=W^{(n)}(\xi)+W^{(ab)}(\xi), \tag{3·24'}$$

$$W^{(n)}(\xi)=\sum\eta\{a^*(p,\xi)a(p,\xi)+b^*(p,\xi)b(p,\xi)\}, \tag{3·25'}$$

$$W^{(ab)}(\xi)=\sum\rho\{d^*(p,\xi)c(p,\xi)+c^*(p,\xi)d(p,\xi)\}. \tag{3·26'}$$

The behavior of $P_\mu(\xi)$ under the Lorentz-transformation is given by

$$P_\mu'(\xi')=\Lambda_{\mu\nu}P_\nu(\xi)=U(\Lambda)P_\mu(\xi')U^{-1}(\Lambda), \tag{3·27}$$

making use of the relation (3·22).

In order to obtain the spectrum of the above displacement operators, we must determine the explicit representation of operators, a's, b's, c's and d's. For its purpose, we must assume the existence of the "vacuum" state besides the commutation relations (3·20).

First, let us focus our attention on the normal part, and require for the vacuum state $|0\rangle$,

$$a(p,\xi)|0\rangle=0,\quad b(p,\xi)|0\rangle=0,\quad \langle 0|0\rangle=1, \tag{3·28}$$

for all p_μ and ξ_μ under the conditions $p\xi=0$, $p_\mu^2\geqq m^2$.*

This choice of vacuum state, of course, is Lorentz invariant as seen from (3·22), and guarantees the positive integer eigenvalue for the number operators $a^*(p,\xi)a(p,\xi)=n^+(p,\xi)$ and $b^*(p,\xi)b(p,\xi)=n^-(p,\xi)$. Although, as easily seen from expressions (3·25) and (3·25'), this vacuum state is the lowest eigenstate of $W^{(n)}(\xi)$, it is, in general, not the case for $P^{(n)}(\xi)$. Further, the spectrum of the $P^{(n)}(\xi)$ has an explicit dependence on ξ_μ's. The latter fact indicates directly the unitary non-equivalence of representations $\phi(x,\xi)$ with different ξ_μ's, mentioned early in this section.

* If we take $a(p,\xi)|0\rangle=0$, $b^*(p,\xi)|0\rangle=0$ instead of (3·28), we must encounter with the indefinite metric in Hilbert space.

For the abnormal component, the more complicated situation occurs. The commutation relation (3·20) under the consideration takes the following form,

$$[c,\ d^*]=i. \tag{3·29}$$

For the definition of the vacuum state several possibilities are conceivable. At first, let us consider the following definition of the vacuum state:

$$\left.\begin{aligned} c\,|0\rangle&=0,\\ d\,|0\rangle&=0,\end{aligned}\right\} \tag{3·30)*}$$

and

$$\langle 0|0\rangle=1.$$

This definition of the vacuum state, however, necessarily gives rise to the indefinite metric in Hilbert space. In order to deal with this case, it is convenient to introduce the following notation:

$$N^+=-id^*c,\quad N^-=ic^*d, \tag{3·31}$$

and

$$|n+,\ m-\rangle=\frac{1}{\sqrt{n!\,m!}}d^{*n}c^{*m}|0\rangle. \tag{3·32}$$

Then, making use of (3·29) and (3·30), we obtain

$$\left.\begin{aligned} N^+|n+,\ m-\rangle&=n\,|n+,\ m-\rangle,\\ N^-|n+,\ m-\rangle&=m\,|n+,\ m-\rangle,\end{aligned}\right\} \tag{3·33}$$

and

$$\langle n'+,\ m'-|n+,\ m-\rangle=(-i)^m(+i)^n\delta_{m'n}\delta_{n'm}. \tag{3·34}$$

Namely operators N^+ and N^-, which are adjoint operators to each other, take the positive integer eigenvalue associated with the eigenvector $|n+,\ m-\rangle$. Further, as seen from (3·34), the eigenstate $|n+,\ m-\rangle$, which has a vanishing norm by itself, has a finite scalar product with other states with different indices. This is, however, a familiar feature in theories with an indefinite metric.

On the other hand, the displacement operator $P_\mu^{(ab)}(\xi)$ or its projection $W^{(ab)}(\xi)$ in the direction ξ_μ can be expressed by the operators N^+ and N^-:

$$P_\mu^{(ab)}(\xi)=-\sum_{\substack{p\xi=0\\ m^2<p^2\geqq 0}}\{(p-i\rho\xi)_\mu N^+(p,\ \xi)+(p+i\rho\xi)_\mu N^-(p,\ \xi)\}, \tag{3·35}$$

---

* Let us consider the following cases instead of (3·30),

$$\left.\begin{aligned} c|0\rangle&=0,\\ d^*|0\rangle&=0,\end{aligned}\right\}\quad\text{or}\quad\left.\begin{aligned} c^*|0\rangle&=0,\\ d|0\rangle&=0.\end{aligned}\right\}$$

Since, as seen from the above commutation relation, either case leads to the vanishing norm of the vacuum state, we shall take such a possibility out of consideration.

$$W^{(ab)}(\xi)=\sum_{\substack{p\xi=0\\ m^2>p^2\geq 0}} ip\{N^+(p,\xi)-N^-(p,\xi)\}. \tag{3·36}$$

Therefore $P_\mu^{(ab)}(\xi)$ has, in general, complex eigenvalues and $W^{(ab)}(\xi)$ has pure imaginary eigenvalues associated with the eigenstate $|n+, m-\rangle$ with a vanishing norm.

Another quantization of the abnormal part is also conceivable, in which the indefinite metric no longer occurs. For this purpose, it is convenient to transform the commutation relation (3·29) into the diagonal form. This can be done through the linear transformation of operators c and d:

$$\left.\begin{aligned}\bar{c}&=\alpha c+\beta d,\\ \bar{d}^*&=\gamma c+\delta d,\end{aligned}\right\} \tag{3·37}$$

under the condition

$$\alpha\delta^*=\beta\gamma^*. \tag{3·38}$$

For instance, if we take $\alpha=\gamma=1/\sqrt{2}$, $\beta=-\delta=i/\sqrt{2}$, namely

$$\left.\begin{aligned}\bar{c}&=(c+id)/\sqrt{2},\\ \bar{d}^*&=(c-id)/\sqrt{2},\end{aligned}\right\} \tag{3·37′}$$

then

$$\left.\begin{aligned}[\bar{c},\ \bar{c}^*]&=1,\\ [\bar{d},\ \bar{d}^*]&=1,\end{aligned}\right\} \tag{3·39}$$

and all other commutators vanish.

Thus, as in the normal case, under the following definition of the vacuum state

$$\left.\begin{aligned}\bar{c}|0\rangle(=(c+id)|0\rangle)&=0,\\ \bar{d}|0\rangle(=(c^*+id^*)|0\rangle)&=0,\end{aligned}\right\} \tag{3·30′}$$*

the basic states $|n+, m-\rangle$, defined by

$$|n+, m-\rangle=\frac{1}{\sqrt{n!\,m!}}\bar{c}^{*n}\bar{d}^{*m}|0\rangle, \tag{3·40}$$

span the Hilbert space with a definite metric and the operators $N^+=\bar{c}^*\bar{c}$ and $N^-=\bar{d}^*\bar{d}$ become number operators with positive integer eigenvalues. However, in this case, the displacement operators $P_\mu^{(ab)}(\xi)$'s or its projection $W^{(ab)}(\xi)$ in the

* If we take the vacuum state as

$$\bar{c}|0\rangle(=(c+id)|0\rangle)=0,$$
$$\bar{d}^*|0\rangle(=(c-id)|0\rangle)=0.$$

instead of (3·30′), this again leads to the indefinite metric. In fact, this vacuum state is just identical with the preceding one in (3·30).

direction ξ_μ can no longer be commutable with the number operator. In fact, as easily proved, both $P_\mu^{(ab)}(\xi)$ (or $W^{(ab)}(\xi)$) and the commutation relation can never be diagonalized at the same time. In terms of $\bar{c}$ and $\bar{d}$, $P_\mu^{(ab)}(\xi)$ and $W^{(ab)}(\xi)$ are expressed as

$$P_\mu^{(ab)}(\xi)=\sum\{p_\mu(\bar{d}\bar{d}^*-\bar{c}^*\bar{c})-i\rho\xi_\mu(\bar{d}\bar{c}-\bar{c}^*\bar{d}^*)\}, \qquad (3\cdot41)$$

$$W^{(ab)}(\xi)=\sum i\rho(\bar{c}^*\bar{d}^*-\bar{d}\bar{c}). \qquad (3\cdot42)$$

Thus, in this case the vacuum state is no longer an eigenstate of $P_\mu^{(ab)}(\xi)$ and $W^{(ab)}(\xi)$. Further, we find that the expectation values of $P_\mu^{(ab)}(\xi)$ and $W^{(ab)}(\xi)$ associated with the state $|n+, m-\rangle$, (3·40), take the following values, respectively,

$$\langle n+, m-|P_\mu^{(ab)}(\xi)|n+, m-\rangle=(m-n+1)p_\mu, \qquad (3\cdot43)$$

$$\langle n+, m-|W^{(ab)}(\xi)|n-, m-\rangle=0. \qquad (3\cdot44)$$

Finally, we shall state about the vacuum expectation value of the anti-commutation relations of $\phi^{(n)}(x, \xi)$ or $\phi^{(ab)}(x, \xi)$. For the normal part, we can easily show

$$\langle 0|\{\phi^{(n)}(x, \xi),\ \phi^{(n)*}(x', \xi)\}|0\rangle=D^{(1)}(x-x'), \qquad (3\cdot45)$$

on account of the second expression of $D^{(1)}(x)$-function in (2·9). Namely the vacuum expectation value of the anticommutator of $\phi^{(n)}(x, \xi)$ has no ξ_μ-dependence and is Lorentz-invariant. On the other hand, concerning the abnormal component, the following relation holds:

$$\langle 0|\{\phi^{(ab)}(x, \xi),\ \phi^{(ab)*}(x', \xi)\}|0\rangle=iD^{(ab)}(x-x', \xi), \qquad (3\cdot46)$$

with respect to the vacuum state associated with the first definition (3·30). On the other hand, the vacuum state concerning the second definition (3·30′), as mentioned above, is not displacement invariant, therefore the corresponding expression of the anticommutator is no longer displacement invariant, that is,

$$\langle 0|\{\phi^{(ab)}(x, \xi),\ \phi^{(ab)*}(x', \xi)\}|0\rangle=\sum_{\substack{p\xi=0\\ m^2>p^2\geqq 0}}\frac{1}{\rho\xi_0}e^{ip(x-x')}\cosh\rho\xi(x+x'). \qquad (3\cdot47)$$

At the end of this section, we shall mention some remark on the problem of causality. In the previous section, we have defined the "quasi" causal function (2·3) or (2·3′), which satisfies an important relation (2·8) associated with a causality. In the case of the usual Klein-Gordon-type field, the corresponding causal function is connected with the vacuum expectation value of the chronological product of the field operators and guarantees the causality of the theory. Namely

$$\begin{aligned}\Delta_F(x-x')&=2\langle 0|P(\phi(x), \phi^*(x'))|0\rangle=\langle 0|\{\phi(x), \phi^*(x')\}|0\rangle\\&+\epsilon(x_0-x_0')\langle 0|[\phi(x), \phi^*(x')]|0\rangle=\Delta^{(1)}(x-x')+i\epsilon(x_0-x_0')\Delta(x-x').\end{aligned} \qquad (3\cdot48)$$

Here, with respect to the Lorentz-invariance of the causal function, it is important that the second term in (3·48) is a function vanishing in a space-like region. On the other hand, the "quasi" causal function G_+, (2·3) under our consideration can be written, in a form similar to (3·48), by

$$G_+(x)=\frac{-i}{2(2\pi)^3}\int e^{ipx}\delta(p^2-m^2)\,d^4p-\frac{1}{(2\pi)^4}P\int\frac{e^{ipx}}{p^2-m^2}d^4p=iD^{(1)}(x)+\bar{G}(x), \tag{3·49}$$

where $\bar{G}(x)$ is given by

$$\begin{aligned}\bar{G}(x)&=\frac{-1}{(2\pi)^4}P\int d^4p\,\frac{e^{ipx}}{p^2-m^2}\\&=-\frac{1}{2(2\pi)^2}\frac{1}{r}\int_0^\infty\{\cos(r\sqrt{m^2+u^2}-x_0u)+\cos(r\sqrt{m^2+u^2}+x_0u)\}\,du\\&=-\frac{1}{4\pi}\delta(r^2-x_0^2)+\begin{cases}0, & (r^2<x_0^2)\\ \dfrac{m}{8\pi}\dfrac{I_1(m\sqrt{r^2-x_0^2})}{\sqrt{r^2-x_0^2}}, & (r^2>x_0^2)\end{cases}\end{aligned} \tag{3·50}$$

as shown by Schmidt.[6] The first term in (3·49) could be connected to the vacuum expectation value as in (3·45). However, since the above $\bar{G}$ function is Lorentz-invariant, but non-vanishing function in a space-like region, it is impossible to factorize the chronological sign factor $\epsilon(x_0)$ from this function. Therefore, we can say that it is impossible to connect the quasi-causal function directly to the vacuum expectation value of the chronological product of the field operators. The latter situation, in connection with the failure in the definition of the relativistic vacuum state, makes it difficult to decide whether the quantized S-field has a rigorous causal property for the energy propagation, so long as we pay attention only to S-field without any interaction. This problem will be discussed from another viewpoint in the next section.

As the conclusion of this section, we can state the following facts. Under the formalism of the canonical quantization and the requirement of the Lorentz-invariance, S-field cannot be described in terms of the simple local field and has no familiar particle aspect. In fact, it is impossible to define the Lorentz-invariant and lowest energy state; that is, the vacuum state. In other words this implies that the concept of "energy" or "momentum" of the S-field by itself has no objectivity.

§ 4. Interaction of S-field

As the result of the preceding investigation, we find that the quantized S-field under the canonical formalism cannot have any familiar particle aspect. Therefore, it is impossible to grasp S-field in the same level as the usual elementary particle.

However, it may be a matter of interest for us to ask whether one can successfully confine the curious behavior of S-field only within an inner region without any appreciable effect in an asymptotic outer region, in a loose sense of the word.

In order to deal with such a problem, we have to take into consideration the interaction of S-field with other elementary particles. However, S-field quantized through the procedure in the previous section is no longer any simple local field, but rather depends on the time-like four vector ξ_μ's as if it were a bilocal field. At present, we have no legitimate means to deal with the interaction of such a field. Meanwhile, it is to be noted that the method of quantization of S-field given in the previous section does not mean the one and only way. For instance, in the familiar quantization method of a field in the Heisenberg representation, a different choice of the asymptotic field associated with the definition of the vacuum, in general, may lead to a unitary non-equivalent representation for the same system, as originally pointed out by Haag.[7)]

Anticipating such a possibility, let us consider an interaction system described by the following field equations and Schrödinger equation:

$$(\square-\kappa^2)\phi(x)=0, \qquad (4\cdot1)$$

$$(\gamma_\mu\partial_\mu+\kappa')\psi_{a,b}(x)=0, \qquad (4\cdot2)$$

$$i\frac{\delta\Psi(\sigma)}{\delta\sigma(x)}=(H_{\rm I}(x)+H_{\rm II}(x))\Psi(\sigma), \qquad (4\cdot3)$$

$$[\phi(x),\ \phi^*(x')]=i\Delta(x-x',\kappa), \qquad (4\cdot4)$$

$$\{\psi_a(x),\ \bar{\psi}_b(x')\}=(\gamma_\mu\partial_\mu-\kappa')\Delta(x-x',\kappa')\delta_{ab}, \qquad (4\cdot5)$$

where $\psi_{a,b}(x)$'s indicate usual spinor fields and

$$H_{\rm I}(x)=g\bar{\psi}_a(x)\psi_b(x)\phi(x)+\text{h. c.}, \qquad (4\cdot6)$$

$$H_{\rm II}(x)=-\delta\kappa^2\phi^*(x)\phi(x), \qquad (4\cdot7)$$

and further Δ-functions in the commutation relations (4·4) and (4·5) are the usual invariant Δ-functions with the real mass κ or κ'.

Now, if we take the formal "unitary" transformation $U(\sigma)$, for the above system, defined by

$$i\delta U(\sigma)/\delta\sigma(x)=H_{\rm II}(x)U(\sigma),$$

so as to eliminate $H_{\rm II}$ from the interaction terms, then in the above Eqs. (4·1)–(4·6) $\phi(x)$ is replaced by $\tilde{\phi}(x)$ in a new representation and κ^2 by $\kappa^2-\delta\kappa^2$. Therefore, if we assume

$$\kappa^2<\delta\kappa^2, \qquad (4\cdot8)$$

the original system may become equivalent to the interaction system of the quantized S-field, with a spinor field, $\psi_{a,b}(x)$.

However, as will be shown later, both systems are actually not unitary equivalent, so far as we assume the existence of the vacuum state for each of $\phi(x)$ and $\tilde{\phi}(x)$ in the same time. In spite of this non-equivalence of both systems, it is still interesting to investigate the original system, assuming the existence of the vacuum state with respect to $\phi(x)$.

The perturbation expansion of the S-matrix of the original system becomes as follows,

$$S=\sum_{l,m=0}^{\infty}(-i)^{l+m}\frac{1}{l!\,m!}\int dx_1\cdots dx_{l+m}$$
$$\times\langle f|P(j(x_1),\cdots,j(x_l),j^*(x_{l+1}),\cdots,j^*(x_{l+m}))|i\rangle\sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int dy_1\cdots dy_n$$
$$\times\langle f'|P(\phi(x_1),\cdots,\phi(x_l),\phi^*(x_{l+1}),\cdots,\phi^*(x_{l+m}),H_{\rm II}(y_1),\cdots,H_{\rm II}(y_n))|i'\rangle, \tag{4·9}$$

where $j(x)=g\bar{\psi}_a(x)\psi_b(x)$ and $|i\rangle$ ($|f\rangle$) and $|i'\rangle$ ($|f'\rangle$) denotes the initial (final) states associated with the spinor and the Bose field, respectively. Now, the above expression (4·9) indicates that the S-matrix of our system is equivalent to that of the system which involves only the interaction term $H_{\rm I}$, provided that we replace the free propagation function of the Bose field, $\Delta_F(x,\kappa^2)$, in the latter case by the function modified by the interaction $H_{\rm II}(x)$. This modified propagator, which we shall describe with $\Delta_F^s(x)$, can be expressed by the perturbation series:

$$\Delta_F^S(x-y)=2\sum_{n=0}^{\infty}(-i)^n\frac{1}{n!}\int dx_1\cdots dx_n\langle 0|P(\phi(x),\phi^*(y),H_{\rm II}(x_1),\cdots,H_{\rm II}(x_n)|0\rangle, \tag{4·10}$$

though it is not a convergent series as will be seen from (4·12). However, there holds the well-known relation which connects both $\Delta_F(x,\kappa^2)$ and $\Delta_F^s(x)$ in terms of the proper self-energy part Π^* due to the interaction $H_{\rm II}(x)$, that is,

$$\Delta_F^S(p)=\Delta_F(p,\kappa^2)+\Delta_F^S(p)\Pi^*\Delta_F(p,\kappa^2), \tag{4·11}$$

or equivalently

$$\Delta_F^{S-1}(p)=\Delta_F^{-1}(p,\kappa^2)-\Pi^*, \tag{4·11'}$$

where

$$\Delta_F(p,\kappa^2)=1/2\pi i(p^2+\kappa^2-i\varepsilon),$$

and Π^* is given, after the simple consideration, by

$$\Pi^*=2\pi i\delta\kappa^2,$$

hence $\Delta_F^S(x)$ becomes as

$$\Delta_F^S(x)=\frac{-2i}{(2\pi)^4}\int\frac{e^{ipx}}{p^2+\kappa^2-\delta\kappa^2-i\varepsilon}d^4p. \tag{4·12}$$

The above function is, except a constant factor, nothing but the quasi-causal function $G_+(x, m^2)$ with $m^2=\delta\kappa^2-\kappa^2(>0)$ which was defined by (2·3) or (3·49), and then it was impossible to connect this function directly with the vacuum expectation value of the field operator, $\phi(x, \xi)$.

Now, the S-matrix of the Möller scattering in the lowest order with respect to the coupling constant g is given by

$$S^{(2)}=-i\iint dx\,dy\,j(x)G_+(x-y)j^*(y). \qquad (4\cdot13)$$

This can be written in terms of the quasi-retarded function according to the procedure given by Ma,[8] namely

$$S^{(2)}=-i\iint dx\,dy\,\{j^{(+)}(x)G_R(x-y, n)j^{*(-)}(y)+j^{(-)}(x)G_R(y-x, n)j^{*(+)}(y)\}, \qquad (4\cdot14)$$

where use is made of the relation (2·8) and $j^{(\pm)}(x)$ means the positive or negative time frequency part of the source function $j(x)$.

From the above expression, we just find that the quantized S-field propagates the energy with a super light velocity, because $G_R(x, \xi=n)$ is in general a non-vanishing function in a space-like region.

It should be worth-while to note that even though the original form of our system apparently belongs to a familiar theory of the local fields with a local interaction, hence each term of the perturbation expansion of the S-matrix with respect to $\delta\kappa^2$ and g, i.e. (4·9) is, of course, Lorentz-invariant and rigorously causal, the resultant expression after the summation of terms of the perturbation expansion with respect to $\delta\kappa^2$ shows Lorentz-invariant, but an explicitly acausal feature. This strange situation is, of course, caused mathematically by the non-convergence of the series of the perturbation expansion with respect to $\delta\kappa^2$. In terms of the physical word, however, we believe that this situation is due to the non-existence of the true vacuum state with regard to the interaction $H_{\text{II}}(x)$. In fact, if the latter statement is true, the appearance of the acausal property is reasonably conceived from the general framework according to Lehmann.[1] In order to support such a conjecture, let us consider the relation between the field $\phi(x, \xi)$ in the previous section and the field $\phi(x)$ in the present section, identifying both fields and their space-time derivatives at the hypersurface $\xi x=$const., more simply, $\xi x=0$:

$$\left.\begin{aligned}\phi(x, \xi)|_{\xi x=0}&=\phi(x)|_{\xi x=0},\\ \xi_\mu\partial_\mu\phi(x, \xi)|_{\xi x=0}&=\xi_\mu\partial_\mu\phi(x)|_{\xi x=0}.\end{aligned}\right\} \qquad (4\cdot15)$$

It is convenient to expand the field $\phi(x)$ in the following form in accordance with that of the field $\phi(x, \xi)$,

$$\phi(x)=\sum_{p\xi=0}\frac{1}{\sqrt{2\gamma\xi_0 V}}\{u(\boldsymbol{p}+\gamma\boldsymbol{\xi})e^{i(p+\gamma\xi)x}+v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})e^{i(p-\gamma\xi)x}\}, \qquad (4\cdot16)$$

where

$$\gamma=\sqrt{p^2+m^2}=\sqrt{\boldsymbol{p}^2-(\boldsymbol{p}\cdot\boldsymbol{\xi})^2/\xi_0^2+m^2}, \tag{4·17}$$

and $u(\boldsymbol{p})$ and $v(\boldsymbol{p})$ satisfy the usual canonical commutation relations. From (4·16), we obtain

$$\phi(x)|_{\xi x=0}=\sum\frac{1}{\sqrt{2\gamma\xi_0 V}}\{u(\boldsymbol{p}+\gamma\boldsymbol{\xi})+v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}e^{i(\boldsymbol{p}-(\boldsymbol{p}\cdot\boldsymbol{\xi})\boldsymbol{\xi}/\xi_0^2)\cdot\boldsymbol{x}},$$

$$\xi_\mu\partial_\mu\phi(x)|_{\xi x=0}=\sum(-i)\sqrt{\frac{\gamma}{2\xi_0 V}}\{u(\boldsymbol{p}+\gamma\boldsymbol{\xi})-v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}e^{i(\boldsymbol{p}-(\boldsymbol{p}\cdot\boldsymbol{\xi})\boldsymbol{\xi}/\xi_0^2)\cdot\boldsymbol{x}}. \tag{4·18}$$

Substituting the above expressions and the corresponding ones for $\phi(x, \xi)$ derived from (3·14) and (3·15) into (4·15) and comparing the both sides, one finds the following relations,

$$\begin{pmatrix}a(p,\xi)\\ b(p,\xi)\end{pmatrix}=\frac{1}{2\sqrt{\eta\gamma}}\{(\eta\pm\gamma)u(\boldsymbol{p}+\gamma\boldsymbol{\xi})+(\eta\mp\gamma)v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}, \tag{4·19}$$

with the conditions $p^2\geq m^2$, $p\xi=0$, and

$$\begin{pmatrix}c(p,\xi)\\ d(p,\xi)\end{pmatrix}=\frac{1}{2\sqrt{\rho\gamma}}\{(\rho\pm i\gamma)(u(\boldsymbol{p}+\gamma\boldsymbol{\xi})+(\rho\mp i\gamma)v^*(\boldsymbol{p}-\gamma\boldsymbol{\xi})\}, \tag{4·20}$$

with the conditions $m^2>p^2\geq 0$, $p\xi=0$.

Now, as originally pointed out by Haag[7] for the relation between the two neutral Bose fields with different masses, we can easily verify the following fact, taking account of the relations (4·19) and (4·20). If there is the vacuum state associated with the field $\phi(x)$, namely

$$\left.\begin{array}{l}u(\boldsymbol{p})|0)=0\\ v(\boldsymbol{p})|0)=0\end{array}\right\}\quad\text{for all } p, \tag{4·21}$$

there is no normalizable vacuum state $|0\rangle$ associated with the field $\phi(x, \xi)$, which satisfies

$$\left.\begin{array}{l}a(p,\xi)|0\rangle=b(p,\xi)|0\rangle=0,\\ c(p,\xi)|0\rangle=d(p,\xi)|0\rangle=0,\end{array}\right\} \tag{4·22}$$

over all p's under the conditions in (4·19) and (4·20). This is, of course, due to the infinite degrees of freedom of the field operator.

Thus, the quantization of S-field in this section makes it possible to remove the eigenstates of the number operators of S-field from the Hilbert space. Speaking more physically, we can say that it is possible to quantize S-field so that S-field no longer exists in the *free* state with the modified Einstein relation (1·3), but takes a role only in the interactions with other particles, bringing the acausal effect into them.

§ 5. Summary and conclusion

We have investigated the matter with a super light velocity as one possible model in the course of looking for the physical concept which will essentially govern the future theory, but may be hidden behind the present field theory. In such an attempt, it is important to study to what extent this matter could be reconciled with the framework of the present theory or contradicts the present physical concept of elementary particles. With such a purpose, we investigated at first in § 2 the propagation character of the classical wave and explained the apparent contradiction between the group velocity over the light velocity and the hyperbolic propagation character. In § 3, it was shown that in the framework of the usual canonical quantization, this matter is no longer represented by a simple local field, but rather has an infinite number of unitary non-equivalent representations, each of which is characterized by the time-like unit four-vector as if being bilocal field and is transformed into each other by the Lorentz transformation. There is no vacuum state which satisfies the Lorentz-invariance and the lowest energy condition, while the displacement operator of the field does exist. Further, the abnormal component, which is inherent in S-field and characterized by its unbounded behavior in space and time, brings the indefinite metric into the Hilbert space. Therefore, we have to say as a whole that S-field, free from any interactions, has no physical reality. In § 4, we considered the interactions of the S-field with other fields attempting to confine the above curious behavior of S-field only within the structure of interaction with other particles. The result of this section shows that the suitable quantization of S-field provides the Lorentz-invariant, but quasi-causal S-matrix and further makes it possible to eliminate the free state of S-field from the Hilbert space by means of the infinite degrees of freedom of the field operators. These features of S-field seem to have similarity somewhat to those of B-matter referred to in the introduction of this paper.

In conclusion, the author would like to express his sincere thanks to Prof. T. Inoue for his kind interest in this work and to Prof. Y. Munakata and Dr. M. Ida for valuable discussions.

Appendix A

α) The field equation and the transformation property of spinor field

Let us consider the general spinor field equation which is invariant under the proper Lorentz-transformation:

$$(\gamma_\mu \partial_\mu + \alpha + \beta\gamma_5)\psi(x) = 0, \tag{A·1}$$

where α and β are complex numbers:

$$\alpha = \alpha_1 + i\alpha_2, \quad \beta = \beta_1 + i\beta_2. \tag{A·2}$$

The Lagrange function associated with the above equation takes the following form:

$$\mathcal{L}(x)=\psi^*\gamma_4(a+b\gamma_5)(\gamma_\mu\partial_\mu+\alpha+\beta\gamma_5)\psi(x). \quad (A\cdot3)$$

From the reality condition of $\mathcal{L}(x)$,* we obtain

$$a=a^*,\ b=b^*, \quad (A\cdot4)$$

$$\beta_1/\alpha_1=\alpha_2/\beta_2=-b/a. \quad (A\cdot5)$$

On the other hand, multiplying the operator $(\gamma_\mu\partial_\mu-\alpha+\beta\gamma_5)$ from the left to Eq. (A·1) gives

$$\{\Box-(\alpha_1^2+\beta_2^2-\beta_1^2-\alpha_2^2)\}\psi(x)=0. \quad (A\cdot6)$$

Now, applying the usual canonical quantization to the above field, we obtain the following commutation relation,

$$\{\psi(\boldsymbol{x},x_0),\ \psi^*(\boldsymbol{x}',x_0)\}=-\frac{a+b\gamma_5}{a^2-b^2}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \quad (A\cdot7)$$

Replacing $\psi(x)$ by $\tilde{\psi}(x)$:

$$\tilde{\psi}(x)=\exp(-f\gamma_5)\psi(x),$$

with a real constant, f, (A·7) is transformed into

$$\{\tilde{\psi}(\boldsymbol{x},x_0),\ \tilde{\psi}^*(\boldsymbol{x}',x_0)\}=-\frac{1}{a^2-b^2}$$

$$\times\{(a\cosh 2f+b\sinh 2f)+(b\cosh 2f+a\sinh 2f)\gamma_5\}\delta^3(\boldsymbol{x}-\boldsymbol{x}'). \quad (A\cdot8)$$

Therefore, choosing the magnitude of f as

$$\frac{b}{a}=-\tanh 2f,\quad \text{or}\quad \frac{a}{b}=-\tanh 2f, \quad (A\cdot9)$$

according to $|a|>|b|$ or $|a|<|b|$, makes it possible to drop the term involving γ_5 or the constant term in the right-hand side of (A·8), respectively.

Consequently, we can confine ourselves, without loss of generality, into the following two cases:

$$\text{i)}\quad a\neq 0,\ b=0,$$
$$\text{ii)}\quad a=0,\ b\neq 0. \quad (A\cdot10)$$

Let us now consider the above two cases, separately.

i) $a\neq 0$, $b=0$.

In this case, from (A·5)

$$\alpha_2=\beta_1=0,$$

thus the equation of motion becomes

* This condition implies that the Euler equations derived from the variation of ψ and ψ^*, respectively, are compatible with each other.

$$(\gamma_\mu \partial_\mu + \alpha_1 + i\beta_2\gamma_5)\psi = 0. \tag{A·11}$$

By making use of the transformation

$$\psi(x) = \exp i\theta\gamma_5 \tilde{\psi}(x), \tag{A·12}$$

with $\beta_2/\alpha_1 = -\tan 2\theta$, this equation is led to the usual Dirac field equation

$$(\gamma_\mu \partial_\mu + \sqrt{\alpha_1^2 + \beta_2^2})\tilde{\psi}(x) = 0, \tag{A·13}$$

and the commutation relation (A·8) yields

$$\{\tilde{\psi}(\boldsymbol{x}, x_0), \tilde{\psi}^*(\boldsymbol{x}', x_0)\} = -\frac{1}{a}\delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{A·14}$$

ii) $a=0,\ b \neq 0$

From (A·5), one has $\alpha_1 = \beta_2 = 0$, and the equation of motion becomes

$$(\gamma_\mu \partial_\mu + i\alpha_2 + \beta_1\gamma_5)\psi = 0. \tag{A·15}$$

Making use of the transformation of (A·12) with $\alpha_2/\beta_1 = -\tan 2\theta$ or $\beta_1/\alpha_2 = \tan 2\theta$, the equation of motion, respectively, becomes

$$\left.\begin{aligned} &(\gamma_\mu \partial_\mu + \sqrt{\alpha_2^2 + \beta_1^2}\gamma_5)\tilde{\psi} = 0, \\ \text{or}\quad &(\gamma_\mu \partial_\mu + i\sqrt{\alpha_2^2 + \beta_1^2})\tilde{\psi} = 0, \end{aligned}\right\} \tag{A·13$'$}$$

and the commutation relation takes the following form,

$$\{\tilde{\psi}(\boldsymbol{x}, x_0), \tilde{\psi}^*(\boldsymbol{x}', x_0)\} = \frac{\gamma_5}{b}\delta^3(\boldsymbol{x} - \boldsymbol{x}'). \tag{A·14$'$}$$

Summarizing the above result, we can say that the general spinor fields subject to (A·1) are classified into two cases, one of which is a well-known Dirac field and the other is the field with a super light velocity and described by

$$(\gamma_\mu \partial_\mu + m\gamma_5)\psi(x) = 0 \tag{A·16}$$

$$\{\psi(x), \bar{\psi}(x')\} = -i\gamma_5(\gamma_\mu \partial_\mu - m\gamma_5)D(x - x'), \tag{A·17}$$

making use of the invariant D-function defined by (2·11).

The equation of motion (A·16) has several invariances under the transformation such as space or time reflections, and charge conjugation. They are given by

$$\begin{aligned} \psi(x) &\to \gamma_5\gamma_4\psi(x), && \text{(space reflection, } P\text{)} \\ &\to \gamma_4\psi(x), && \text{(time reflection, } T\text{)} \\ &\to \gamma_5 C\bar{\psi}(x), && \text{(charge conjugation, } C\text{)*.} \end{aligned}$$

Under each of these transformation the Lagrangian is also invariant, but the commutation relation (A·18) is no longer invariant and changes the sign of the right-

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* The notation C follows Schwinger's:

$$C\gamma_\mu C^{-1} = -\gamma_\mu^T.$$

hand side. However, this is naturally invariant under the combined transformations CP or CT.

Thus, from the above argument we can say that the S-field with spin one half has the unique correspondence with the parity violation character.

β) *The quantization*

Let us divide the spinor field into the normal and the abnormal parts, in a similar way to (3·14):

$$\psi(x,\xi)=\psi^{(n)}(x,\xi)+\psi^{(ab)}(x,\xi), \tag{A·18}$$

where

$$\psi^{(n)}(x,\xi)=\sum_{\substack{p\xi=0\\ p^2\geq m^2}}\frac{1}{\sqrt{2\xi_0V}}\{a^r(p,\xi)\psi_r^+(p,\xi)e^{i(p+\eta\xi)x}+b^{r*}(p,\xi)\psi_r^-(p,\xi)e^{i(p-\eta\xi)x}\}, \tag{A·19}$$

$$\psi^{(ab)}(x,\xi)=\sum_{\substack{p\xi=0\\ m^2>p^2\geq 0}}\frac{1}{\sqrt{2\xi_0V}}\{c^r(p,\xi)\varphi_r^+(p,\xi)e^{i(p+i\rho\xi)x}+d^r(p,\xi)\varphi_r^-(p,\xi)e^{i(p-i\rho\xi)x}\}, \tag{A·20}$$

with the same notation η, ρ as (3·16) in the text. In the above expression the constant spinors $\psi_r^\pm$, $\varphi_r^\pm$ are subject to the following equation,

$$\{i(p_\mu\pm\eta\xi_\mu)\gamma_\mu+m\gamma_5\}\psi_r^\pm(p,\xi)=0, \tag{A·21}$$

$$\{i(p_\mu\pm i\rho\xi_\mu)\gamma_\mu+m\gamma_5\}\varphi_r^\pm(p,\xi)=0. \tag{A·22}$$

The solution for $\psi_r^\pm(p,\xi=n)$ is explicitly given by

$$\psi_{r=1,2}^+=\frac{m}{\sqrt{2(p\mp\epsilon)\epsilon}}\begin{pmatrix}u^\pm\\ (1/m)(\pm p-\epsilon)u^\pm\end{pmatrix}, \tag{A·23}$$*

$$\psi_{r=1,2}^-=\frac{m}{\sqrt{2(p\mp\epsilon)\epsilon}}\begin{pmatrix}u^\mp\\ (1/m)(\mp p+\epsilon)u^\mp\end{pmatrix}, \tag{A·24}$$

where

$$\epsilon=\sqrt{\boldsymbol{p}^2-m^2},\quad p=|\boldsymbol{p}|,$$

and

$$\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{p}u^\pm=\pm u^\pm,$$

with the obvious notation. The above solution satisfies the orthogonal conditions,

$$\left.\begin{aligned}\psi_r^{+*}(p,n)\gamma_5\psi_s^+(p,n)&=\epsilon_r\delta_{rs},\\ \psi_r^{-*}(p,n)\gamma_5\psi_s^-(p,n)&=\epsilon_r\delta_{rs},\\ \psi_r^{+*}(p,n)\gamma_5\psi_s^-(p,n)&=0,\end{aligned}\right\} \tag{A·25}$$

* Hereafter, we shall use the representation, where $\gamma_5=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$.

where

$$\epsilon_r = \pm 1 \quad \text{for} \quad r = \pm 1.$$

From (A·25), we obtain

$$\sum_r \epsilon_r \{\psi_r^+(p, n)\psi_r^{+*}(p, n) + \psi_r^-(p, n)\psi_r^{-*}(p, n)\} = \gamma_5. \tag{A·26}$$

Introducing the operators

$$\Lambda^\pm(p, n) = \{\pm(i\boldsymbol{p}\gamma_4\boldsymbol{\gamma} + m\gamma_4\gamma_5) + \epsilon\}/2\epsilon \tag{A·27}$$

and making use of (A·26), one finds

$$\sum_r \epsilon_r \psi_r^\pm(p, n)\psi_r^{\pm*}(p, n) = \Lambda^\pm(p, n)\gamma_5. \tag{A·28}$$

Now, let us consider the Lorentz-transformation which transforms n_μ into a given ξ_μ, namely

$$\xi_\mu = \Lambda_{\mu\nu} n_\nu \tag{A·29}$$

and take the matrix S,

$$S\gamma_\mu S^{-1} = \Lambda_{\nu\mu}\gamma_\nu. \tag{A·30}$$

Then, since the functions $S\psi_r^\pm(\Lambda^{-1}p, n)$ satisfy Eq. (A·21), we can identify them with $\psi_r^\pm(p, \xi)$, namely

$$\psi_r^\pm(p, \xi) = S\psi_r^\pm(\Lambda^{-1}p, n). \tag{A·31}$$

In this case, one can easily verify that

$$\mathfrak{S}(p, \xi)\psi_r^\pm(p, \xi) = \pm\epsilon_r\psi_r^\pm(p, \xi), \tag{A·32}$$

where $\mathfrak{S}(p, \xi)$ is the generalized spin operator,

$$\mathfrak{S}(p, \xi) = -i\gamma_5\gamma_{[\mu}\gamma_{\nu]}p_\mu\xi_\nu/\sqrt{p_\sigma^2} \tag{A·33}$$

and especially $\mathfrak{S}(p, n) = \boldsymbol{\sigma}\cdot\boldsymbol{p}/p$.

Further, corresponding to (A·28),

$$\sum_r \epsilon_r \psi_r^\pm(p, \xi)\bar{\psi}_r^\pm(p, \xi) = \mp\{i(p_\mu \pm \eta\xi_\mu)\gamma_5\gamma_\mu - m\}/2\eta. \tag{A·34}$$

Also, concerning the abnormal part $\varphi_r^\pm(p, \xi)$, the similar treatment is applied. The solutions $\varphi_r^\pm(p, \xi = n)$'s are given by

$$\left.\begin{aligned} \varphi_r^+(p, n) &= \frac{1}{2\sqrt{\lambda}}\begin{pmatrix}(i\sqrt{m+\lambda} \pm \sqrt{m-\lambda})u^\pm \\ (\sqrt{m+\lambda} \pm i\sqrt{m-\lambda})u^\pm\end{pmatrix}, \\ \varphi_r^-(p, n) &= \frac{1}{2\sqrt{\lambda}}\begin{pmatrix}(\sqrt{m-\lambda} \mp i\sqrt{m+\lambda})u^\pm \\ (-i\sqrt{m-\lambda} \pm \sqrt{m+\lambda})u^\pm\end{pmatrix}, \end{aligned}\right\} \tag{A·35}$$

where

$$\lambda = \sqrt{m^2 - \boldsymbol{p}^2}.$$

Taking

$$\varphi_r^{\pm}(p, \xi) = S\varphi_r^{\pm}(\Lambda^{-1}p, n),$$

we obtain

$$\mathfrak{S}(p, \xi)\varphi_r^{\pm}(p, \xi) = \epsilon_r \varphi_r^{\pm}(p, \xi). \tag{A·36}$$

The final relations corresponding to (A·34) are

$$\left.\begin{aligned} \sum_r \epsilon_r \varphi_r^{+}(p, \xi)\bar{\varphi}_r^{-}(p, \xi) &= \{i(p_\mu + i\rho\xi_\mu)\gamma_5\gamma_\mu - m\}/2i\rho, \\ \sum_r \epsilon_r \varphi_r^{-}(p, \xi)\bar{\varphi}_r^{+}(p, \xi) &= -\{i(p_\mu - i\rho\xi_\mu)\gamma_5\gamma_\mu - m\}/2i\rho. \end{aligned}\right\} \tag{A·37}$$

On the other hand, the right-hand side of the commutation relation (A·17) is also divided into the normal and the abnormal parts. Making use of the expressions (2·13), (2·15), they are given by

$$\begin{aligned} &-i\gamma_5(\gamma_\mu\partial_\mu - m\gamma_5)D^{(n)}(x, \xi) \\ &= \frac{-1}{(2\pi)^3}\int_{\substack{p\xi=0 \\ p^2\geq m^2}} \frac{d^3p}{2\eta\xi_0}[\{i(p_\mu + \eta\xi_\mu)\gamma_5\gamma_\mu - m\}\exp\{i(p+\eta\xi)x\} \\ &\quad - \{i(p_\mu - \eta\xi_\mu)\gamma_5\gamma_\mu - m\}\exp\{i(p-\eta\xi)x\}], \end{aligned} \tag{A·38}$$

and

$$\begin{aligned} &-i\gamma_5(\gamma_\mu\partial_\mu - m\gamma_5)D^{(ab)}(x, \xi) \\ &= \frac{-1}{(2\pi)^3}\int_{\substack{p\xi=0 \\ m^2>p^2\geq 0}} \frac{d^3p}{2i\rho\xi_0}[\{i(p_\mu - i\rho\xi_\mu)\gamma_5\gamma_\mu - m\}\exp\{i(p-i\rho\xi)x\} \\ &\quad - \{i(p_\mu + i\rho\xi_\mu)\gamma_5\gamma_\mu - m\}\exp\{i(p+i\rho\xi)x\}]. \end{aligned} \tag{A·39}$$

Substituting the expressions (A·19), (A·20) and (A·38), (A·39) into (A·17) and comparing both sides, we obtain

$$\{a^r(p, \xi), a^{s*}(p', \xi)\} = \{b^r(p, \xi), b^{s*}(p', \xi)\} = \epsilon^r\delta_{rs}\delta_{pp'}, \tag{A·40}$$

under the condition

$$p^2 \geq m^2, \quad p\xi = 0,$$

$$\{c^r(p, \xi), d^{s*}(p', \xi)\} = \epsilon_r\delta_{rs}\delta_{pp'}, \tag{A·41}$$

under the condition

$$m^2 > p^2 \geq 0, \quad p\xi = 0,$$

and all other commutators vanish.

The several problems concerning the transformation property under the Lorentz-transformation, the displacement operator or the vacuum state can be carried through in essentially the same way as in § 3 for the Bose field, hence we do not enter further in detail.

However, it is to be noted that, as seen from (A·40), there appears a new indefinite metric in the Hilbert space with a different origin than that due to the abnormal part. In fact, the appearance of the sign factor ϵ_r in the right-hand side of (A·40) implies that the states with a different spin direction must have an inverse sign of metric in a Hilbert space. Further, this situation cannot be avoided by the modification of the vacuum state, because of the anticommutator. Rather, this is a reasonable result since *S*-field with spin one half becomes necessarily parity indefinite mentioned early.

Appendix B

The group theoretical structure of S-field under the Lorentz-transformation

In the usual Klein-Gordon field, there exist for any pair of annihilation operators, $a(\boldsymbol{p})$ and $a(\boldsymbol{p}')$, a large number of the Lorentz-transformations, which transform the pair in each other and are ∞^3 in number for each pair. Further, since the total number of operators $a(\boldsymbol{p})$ is obviously ∞^3, this field is complete to represent the Lorentz-group which has ∞^6 elements. On the contrary, the situation is different for the field under consideration. There do not necessarily exist the Lorentz-transformations for some pair of operators $a(p, \xi)$ and $a(p', \xi')$.* Let us consider the pair, $a(p, n)$ and $a(p', n)$. Then, the fourth components of p_μ and p_μ' are vanishing because of the condition in (3·19), namely $p_\mu = (\boldsymbol{p}, 0)$ and $p_\mu' = (\boldsymbol{p}', 0)$. Lorentz-transformations which connect both operators must, at least, belong to the spatial rotation, so they exist when and only when $\boldsymbol{p}^2 = \boldsymbol{p}'^2$ and are ∞^1 in number. Further, one finds that the number of operators $a(p', n)$'s which can be transformed by the Lorentz-transformation into the specified operator $a(p, n)$ is ∞^2, taking account of the condition $\boldsymbol{p}^2 = \boldsymbol{p}'^2$. In order to clarify the above situation more generally, it is convenient to define the " invariant class " of operators $a(p, \xi)$'s in such a way that operators which are connected by a Lorentz-transformation to each other belong to the same invariant class.** (Of course, the operators in the usual Klein-Gordon field has one and only one invariant class.) Taking into consideration the above argument confined to $\xi_\mu = n_\mu$, we can say, without rigorous proof here, the following properties about the structure of classes.

i) The number of operators with the common ξ_μ in one invariant class is ∞^2, and thus the total number of operators in one invariant class becomes $\infty^{2+3} = \infty^5$, taking account of the degrees of freedom of ξ_μ, namely ∞^3.

ii) The number of Lorentz-transformations which connect a given pair of operators in one invariant class is ∞^1. Thus, we find that each invariant class

* For simplicity, we confine ourselves only to the discussion about operators $a(p, \xi)$'s. Of course, the situation is just same for operators b's, c's and d's.

** The author is indebted to Dr. M. Ida for the following group theoretical consideration.

is complete to represent the Lorentz-group, taking into consideration the number of operators, ∞^5, in each invariant class as stated in i) and the degeneracy in number of ∞^1 above-mentioned.

iii) The number of class is ∞^1, which yields the total number of operators, $a(p, \xi)$'s over all classes, ∞^6, in connection with the latter statement in i).

Appendix C

Some remarks on the invariant functions in § 2

In § 2, we defined the retarded and the advanced functions as follows,

$$G_{R,A}(x) = -\frac{1}{(2\pi)^4}\int \frac{e^{i(k\pm i\eta\xi)x}}{(k\pm i\eta\xi)^2 - m^2} d^4k, \tag{2·4}$$

under the condition $\eta > m$. The latter condition guarantees that the integrand in the above integration has no poles on the real k_μ's axes in the complex k_μ-planes and thus the integration has a definite meaning. Now we shall show that $G_{R,A}(x)$'s have no dependence on the direction of the time-like unit vector, ξ_μ. For this purpose, it is enough to show the invariance of $G_{R,A}(x)$'s under the infinitesimal Lorentz-transformation for ξ_μ:

$$\xi_\mu' = \xi_\mu + \omega_{\mu\nu}\xi_\nu, \tag{C·1}$$

where $\omega_{\mu\nu}$'s are infinitesimals satisfying

$$\omega_{\mu\nu} = -\omega_{\nu\mu}. \tag{C·2}$$

Making use of the following relations,

$$\begin{aligned} \xi_\mu' x_\mu &= \xi_\mu (x_\mu - \omega_{\mu\nu} x_\nu) \\ &= \xi_\mu x_\mu', \quad (x_\mu' \equiv x_\mu - \omega_{\mu\nu} x_\nu), \end{aligned} \tag{C·3}$$

and

$$k_\mu x_\mu = k_\mu' x_\mu', \quad (k_\mu' \equiv k_\mu - \omega_{\mu\nu} k_\nu), \tag{C·4}$$

we obtain

$$\begin{aligned} \Delta G_{R,A}(x, \xi) &\equiv G_{R,A}(x, \xi') - G_{R,A}(x, \xi) \\ &= \frac{1}{(2\pi)^4}\int \left\{ \frac{e^{(ik\mp\eta\xi')x}}{(ik\mp\eta\xi')^2 + m^2} - \frac{e^{(ik\mp\eta\xi)x}}{(ik\mp\eta\xi)^2 + m^2} \right\} d^4k \\ &= \frac{1}{(2\pi)^4}\int \frac{e^{(ik\mp\eta\xi)x'} - e^{(ik\mp\eta\xi)x}}{(ik\mp\eta\xi)^2 + m^2} d^4k \\ &= \frac{-i}{(2\pi)^4}\int \frac{(ik\mp\eta\xi)_\mu \omega_{\mu\nu}}{(ik\mp\eta\xi)^2 + m^2} \frac{\partial}{\partial k_\nu} e^{(ik\mp\eta\xi)x} d^4k. \end{aligned} \tag{C·5}$$

Further, performing the partial integration with respect to variables k_μ's which is permissible by virtue of the condition $\eta > m$, we get

$$\begin{aligned}&\Delta G_{R,A}(x,\xi)\\&=\frac{-i}{(2\pi)^4}\sum_{\nu=1}^{4}\left\{\left[\int\frac{(ik\mp\eta\xi)_\mu\omega_{\mu\nu}}{(ik\mp\eta\xi)^2+m^2}e^{(ik\mp\eta\xi)x}(dk)_\nu\right|_{k_\nu=-\infty}^{k_\nu=+\infty}\right\}\\&\quad-i\int e^{(ik\mp\eta\xi)x}\frac{\partial}{\partial k_\mu}\left\{\frac{\omega_{\mu\nu}(ik\mp\eta\xi)_\nu}{(ik\mp\eta\xi)^2+m^2}\right\},\end{aligned} \quad (C\cdot 6)$$

where

$$(dk)_\nu(=d^4k/dk_\nu)=(dk_2dk_3dk_0,\ dk_1dk_3dk_0,\ dk_1dk_2dk_0,\ dk_1dk_2dk_3/i).$$

The first term in the left side of Eq. (C·6) obviously vanishes. Further, one easily finds that the second term also vanishes, because of the fact:

$$\frac{\partial}{\partial k_\mu}\left\{\frac{\omega_{\mu\nu}(ik\mp\eta\xi)_\nu}{(ik\mp\eta\xi)^2+m^2}\right\}=-\frac{2i(ik\mp\eta\xi)_\mu\omega_{\mu\nu}(ik\mp\eta\xi)_\nu}{\{(ik\mp\eta\xi)^2+m^2\}^2}+\frac{i\omega_{\mu\mu}}{(ik\mp\eta\xi)^2+m^2}=0,$$

on account of (C·2). Thus, $G_{R,A}$'s have no dependence on ξ_μ.

Consequently, the $D(x)$-function, which was defined by (2·10) as the difference between $G_R(x)$ and $G_A(x)$, is also independent of ξ_μ. Therefore, in order to show the properties of the $D(x)$-function, (2·16) and (2·17), it is enough to investigate the $D(x)$-function expressed in terms of the special direction $\xi_\mu=n_\mu$, namely

$$D(x)=D^{(n)}(x,n)+D^{(ab)}(x,n).$$

Taking into consideration that both $D^{(n)}(x,n)$ and $D^{(ab)}(x,n)$, as seen from (2·13′) and (2·14′), vanish at $x_0=0$ and that $D(x)$ is a form-invariant function with respect to the Lorentz-transformation, we easily find (2·16), that is,

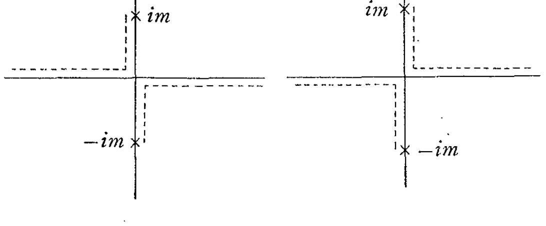
$$D(x)=0,\quad \text{for}\quad x^2>0. \quad (2\cdot 16)$$

Further, the relations in (2·17) are also seen from the expression, (2·13′) and (2·14′).

Finally, let us remark on the integral expression of the quasi-causal function defined by (2·3):

$$G_\pm(x)=-\frac{1}{(2\pi)^4}\int\frac{e^{ikx}}{k^2-m^2\mp i\epsilon}d^4k, \quad (2\cdot 3)$$

where ϵ is an infinitesimal positive number. The poles of the integrand in the complex k_0-plane are shown by the dotted lines in Fig. 5 *a* and *b*, corresponding to $G_+(x)$ and $G_-(x)$. Thus, the integration in (2·3) on the real k_0-axis has a definite meaning.



a. *b.*

Fig. 5.

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