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Theory of Measurement in Quantum Mechanics

——*Mechanism of Reduction of Wave Packet. I*——

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New theory of measurement is presented to show that quantum mechanics can describe the whole processes of quantum-mechanical measurement leading to the so-called *reduction of wave packet* if applied properly to the total system of object and apparatus. We introduce a simple model, where momenta of particles are measured using a perfect rigid mirror, to yield the *reduction of wave packet* in a clear-cut way. Microscopic uncertainties of macroscopic quantities and the dynamical scattering theory play the essential roles. Before entering into the main part, we briefly summarize the controversial points in the theory of measurement.

§ 1. Introduction

One of the most fundamental topics in modern physics is undoubtedly the problem as to whether physical processes of measurement or observation for a quantum-mechanical system can be described in a self-contained way within the framework of quantum mechanics itself or not. Even though we have endless publications of so many papers after the famous work of von Neumann,[1] there still exist serious controversies among the authors conceiving affirmative or negative answers to the problem.[2] Needless to say, the central interest is to find what produces the so-called *reduction of wave packet* in the course of measurement. In this paper we shall discuss the problem by introducing simple models to elucidate the essential mechanism of *reduction of wave packet* in a clear-cut way. The aim of our theory is to show that quantum mechanics can describe the whole process of quantum-mechanical measurement or observation if measuring apparatus system and its interactions with the object system are properly formulated.

Here we briefly review predictions of quantum mechanics for measurement of an observable, say $\hat{F}$, in a quantum-mechanical object system. Let $\{\lambda_i\}$ and $\{|u_i\rangle\}$ be a complete set of eigenvalues and the corresponding eigenvectors belonging to λ_i, and then an arbitrary state of the object system is represented by

$$|\psi\rangle=\sum_i c_i|u_i\rangle \ ; \ c_i=\langle u_i|\psi\rangle \tag{1·1a}$$

following the principle of superposition, or equivalently

$$\hat{\rho}_0=|\psi\rangle\langle\psi|=\sum_i |c_i|^2\hat{\xi}(u_i)+\sum_{i\neq j} c_i c_j{}^*\hat{\eta}(u_i, u_j), \tag{1·1b}$$

where $\hat{\hat{\xi}}(u_i)=|u_i\rangle\langle u_i|$ is the projection operator into state $|u_i\rangle$ and $\hat{\eta}(u_i, u_j)=|u_i\rangle\langle u_j|$. Quantum mechanics predicts that $|c_i|^2=|\langle u_i|\psi\rangle|^2$ is equal to the probability of finding λ_i in a measurement of $\hat{F}$. If we obtained λ_i in a measurement of the first kind, then the object system should be in state $|u_i\rangle$ immediately after the measurement. Consequently, the object system suffers, by the measurement, a sudden change $|\psi\rangle\to|u_i\rangle$ which is often called *reduction of wave packet*. The *reduction of wave packet* must be rewritten more precisely in terms of statistical operator than in terms of state vector. After the measurement the object system should be described by the mixed-state statistical operator

$$\hat{\hat{\rho}}=\sum_i |c_i|^2\hat{\hat{\xi}}(u_i), \tag{1·2}$$

which represents an exclusive and probabilistic occurrence with probability $|c_i|^2$. Therefore, the *reduction of wave packet* is now to be recognized as a process described by

$$\hat{\rho}_0 \to \hat{\hat{\rho}}. \tag{1·3}$$

It is well known that this process is never a causal one to be described by the Schrödinger equation of the object system itself, but an acausal and probabilistic one. A serious question then arises as to whether quantum mechanics can completely describe in a self-contained way the whole process of measurement or observation within its own theoretical framework. However, the above argument does not take account of the presence of another important participant, that is, measuring apparatus. One can expect that interactions between a microscopic object system and a macroscopic apparatus system will produce the *reduction of wave packet*. The theory of measurement is therefore to give a definite answer to the above question, by applying quantum mechanics itself to the total system of object and apparatus. The present work also intends to approach the theory of measurement along the same line of thought.

In § 2 we summarize the famous controversial points in the theory of measurement for later convenience to develop our theory. In § 3 a simple model of a measuring apparatus using a perfect mirror is introduced to elucidate the essential mechanism of the *reduction of wave packet*. General formulation will be given in Part II.

§ 2. Summary of controversial points in the theory of measurement

Here we briefly summarize the controversial points in the theory of measurement which have been discussed among many authors for a long time. First we have to introduce von Neumann's view on the problem, because most of the authors have put their starting point on his account.

2.1. *von Neumann's view*

Following von Neumann, quantum-mechanical measurement of an observable

$\widehat{F}$ in a microscopic object system becomes possible if a measuring apparatus is set up so as to associate an eigenstate $|\Psi_i\rangle$ of a macroscopic dynamical variable $\widehat{G}$ belonging to its eigenvalue A_i, in one-to-one correspondence with each eigenstate $|u_i\rangle$ of the object system. In other words, it is supposed that if the object and the apparatus are, respectively, in $|u_i\rangle$ and $|\Psi\rangle$ before measurement, interactions between both the systems introduced by measurement (of the first kind) are to yield the following dynamical change:

$$|u_i\rangle\otimes|\Psi\rangle\rightarrow|u_i\rangle\otimes|\Psi_i\rangle\,. \tag{2·1}$$

Hence one can know that the object was in $|u_i\rangle$ before measurement, by observing that $\widehat{G}$ takes A_i in the apparatus after measurement. We call (2·1) the *first assumption* in von Neumann's view. If we accept the first assumption, then the *reduction of wave packet* should be written as a change from the initial state $\hat{\rho}_0=\hat{\rho}_0{}^Q\otimes\hat{\rho}_0{}^A$ (in which $\hat{\rho}_0{}^Q=|\psi\rangle\langle\psi|$ and $\hat{\rho}_0{}^A=|\Psi\rangle\langle\Psi|$, $|\psi\rangle$ being given by (1·1)) into the final mixed state

$$\hat{\hat{\rho}}=\sum_j|c_i|^2\hat{\Xi}_{ii};\ \ \hat{\Xi}_{ii}=\hat{\xi}^Q(u_i)\otimes\hat{\xi}^A(\Psi_i), \tag{2·2}$$

where $\hat{\xi}^Q(u_i)=|u_i\rangle\langle u_i|$ and $\hat{\xi}^A(\Psi_i)=|\Psi_i\rangle\langle\Psi_i|$.

Most of the authors have accepted the first assumption (2·1) on which they tried to develop their theories towards the final goal (2·2). However, the present authors do not agree with the assumption, because macroscopic variables such as pointer position can only indicate a macroscopic point to cover a wide microscopic region over atomic size. It seems to us (see § 3 and Part II, § 2) that macroscopic state variables can be defined only through a sort of ensemble average but never identified with quantum-mechanical observables. This idea is one of the most important points of the present paper different from von Neumann's view.

In von Neumann's view it is further emphasized that quantum mechanics, especially, the principle of superposition should be applied rigorously to the whole process of measurement. Consequently, if the object system is in a superposed state $|\psi\rangle$ given by (1·1), that is, the total system is in $|\psi\rangle\otimes|\Psi\rangle=\{\sum_i c_i|u_i\rangle\}\otimes|\Psi\rangle$, then the first assumption (2·1) and the principle of superposition immediately give

$$|\psi\rangle\otimes|\Psi\rangle\rightarrow\sum_i c_i|u_i\rangle\otimes|\Psi_i\rangle\,. \tag{2·3}$$

Let us call (2·3) the *second assumption* in von Neumann's view. The final state in (2·3) is rewritten as

$$\hat{\rho}=\sum_i|c_i|^2\hat{\Xi}_{ii}+\sum_{i\neq j}\sum c_ic_j{}^*\hat{\Xi}_{ij};$$
$$\hat{\Xi}_{ij}=\{|u_i\rangle\langle u_j|\}\otimes\{|\Psi_i\rangle\langle\Psi_j|\} \tag{2·4}$$

in terms of statistical operator. It is obvious that $\hat{\rho}$ differs from $\hat{\hat{\rho}}$ by the second phase-correlation term of $(2\cdot4)$—so that the measurement $(2\cdot3)$ in von Neumann's view never produces the *reduction of wave packet*. Since the second assumption is always valid, in von Neumann's view, at each step of a chain of succeeding measurements from the object system to a human observer (measuring apparatus→his eyes→his nerve system→his brain cell→⋯), it should be concluded that all the physical processes of measurement never give the *reduction of wave packet* but only the 'abstraktes Ich' or the 'consciousness' can do it. It is also concluded that a cut-point to divide the observer system from the object system can be arbitrarily shifted to the left or to the right.

It is well known that the above conclusions have evoked serious controversies among many authors with criticism or sympathy, as to whether quantum mechanics can be a self-contained theory or not. Most of those who never consider the 'abstraktes Ich' or the 'consciousness' to produce the *reduction of wave packet* went to search for its origin (to sweep the phase correlation term off from $\hat{\hat{\rho}}$) in irreversible processes taking place in the course of measurement. On the other hand Schrödinger[3] presented an interesting paradox coming from shifting the cut-point towards observer—famous for the story of 'Schrödinger's cat.' His paradox is so famous that we need not repeat it. Here we wish to mention another confusion brought from shifting the cut-point towards the object in formal application of the second assumption. A typical confusion is sometimes seen in arguments on the Stern-Gerlach experiment. Let $|\psi\rangle=\{c_+|+\rangle+c_-|-\rangle\}|\varphi\rangle$ be the wave function of a particle before measurement, in which c_+ and c_- are constants, $|+\rangle$ and $|-\rangle$ stand for spin up and down states, respectively, and $|\varphi\rangle$ for the position state. The process of passing through the magnet is written as

$$\begin{aligned}|\psi\rangle&=\{c_+|+\rangle+c_-|-\rangle\}|\varphi\rangle\\ &\to|\psi_0\rangle=c_+|+\rangle|\varphi_+\rangle+c_-|-\rangle|\varphi_-\rangle\,,\end{aligned}\tag{2·5}$$

where $|\varphi_+\rangle$ and $|\varphi_-\rangle$ are, respectively, the position wave functions going into detectors, D_+ and D_-, separated in space. If $(2\cdot5)$ were formally identified with $(2\cdot3)$, we might be led to the confusion that the position of the object particle itself is considered to be a measuring apparatus as was done by Wigner and others.[4),5)] It is, however, remarked that we have completely discarded the presence of macroscopic detectors to take main part in the *reduction of wave packet*. In the confusion, certainly, we have put too much confidence in the second assumption enough to shift the cut-point onto the object itself. We can never talk about any measuring procedure without resort to detection. The formality itself of the second assumption does not necessarily correspond to a practical measurement. Note that process $(2\cdot5)$ in the case of the Stern-Gerlach experiment is nothing but *preparation* process giving a sort of *spectral decomposition* for measurement.

2.2. *Ergodic amplification point of view*

Such a particle detector as a counter, a bubble chamber, a spark chamber and so on is usually invented so as to amplify a microscopic input impulse up to a macroscopic output power through a thermal irreversible process. Let us call it 'ergodic amplification.' It would be natural to expect that the ergodic amplification inside detectors could produce the *reduction of wave packet.* In fact, it has already been known that the master equation or the phenomenological transport equations of irreversible processes can be derived from the fundamental many-body Schrödinger equation, by means of an asymptotic approach to a coarse-grained description in which the phase correlations vanish.[6),7)] Along the line of thought many authors have so far attempted to formulate the mechanism of *reduction of wave packet* within the framework of quantum mechanics.[8)] In particular, Green[9)] attempted to show that the *reduction of wave packet* really takes place in his special model of particle detector in the case of the Stern-Gerlach experiment. A general theory of the 'ergodic amplification' has been formulated by Daneri, Loinger and Prosperi[10)] on the basis of standard dynamical statistical theory. Green's model detector contains two sets of oscillators at different temperatures which become coupled by interaction with the object particle in the course of measurement, so that one can detect the particle by a temperature change. Practically, Green showed that the inequality

$$\mathrm{tr}\,\hat{\Xi}^{(\pm)} \gg \mathrm{tr}\,\hat{\Xi}^{(\pm\mp)} \tag{2·6}$$

holds after the interaction in the following statistical operator of the total system:

$$\begin{aligned}
\hat{\rho} &= |c_+|^2\hat{\Xi}^{(+)} + |c_-|^2\hat{\Xi}^{(-)} + c_+c_-{}^*\hat{\Xi}^{(+-)} + c_-c_+{}^*\hat{\Xi}^{(-+)}\,; \\
\hat{\Xi}^{(\pm)} &= [\{|\pm\rangle|\varphi_\pm\rangle\langle\pm|\langle\varphi_\pm|\}\otimes\hat{\rho}_\pm]\otimes\hat{\rho}_\mp\,, \\
\hat{\Xi}^{(\pm\mp)} &= [\{|\pm\rangle|\varphi_\pm\rangle\langle\mp|\langle\varphi_\mp|\}\otimes\hat{\rho}_\pm\otimes\hat{\rho}_\mp]\,,
\end{aligned} \tag{2·7}$$

where $c_\pm$, $|\pm\rangle$ and $|\varphi_\pm\rangle$ are the same ones as used in the preceding subsection, $\hat{\rho}_\pm$ stands for the statistical operator of detector $D_\pm$ and $[\cdots]$ means operators after the interaction. At first sight we might think as if inequality (2·6) meant the smallness of $\hat{\Xi}^{(\pm\mp)}$, namely, the *reduction of wave packet.* It has, however, been pointed out by Furry[11)] that the inequality does not necessarily mean the smallness of $\hat{\Xi}^{(\pm\mp)}$. He has proved that $\hat{\Xi}^{(+-)}\hat{\Xi}^{(-+)}$ cannot be regarded as small compared with $(\hat{\Xi}^{(\pm)})^2$ so that they are not small. The *reduction of wave packet,* therefore, does not occur in the Green model.

Furry's remark is actually a special application of a general theorem which states essentially that any superposed state does not evolve into a mixed state by a unitary transformation. It is obvious that this theorem is closely related to the heart of the problem of measurement. Wigner[4)] proved the theorem for measurements satisfying von Neumann's account and then claimed that the problem of measurement cannot be solved affirmatively in the realm of quantum mechanics. Fine[13] has generalized the theorem by allowing the measuring apparatus to be described as a mixed state in *one* Hilbert space, and then also claimed the same as Wigner did.

The Wigner-Fine criticism is also applied to the Daneri-Loinger-Prosperi theory. Especially, Jauch, Wigner and Yanase,[14] have severely criticized their theory on the basis of the Wigner theorem and the 'negative result measurements,' and excited the objections from Rosenfeld and Loinger.[15] Fine[13] has also critically discussed approximate theories that take into account the macroscopic nature of the apparatus including the ergodic amplification point of view. We do not enter into the details of their debate here.

In spite of the presence of the Wigner-Fine theorem, we will show in Part II that an affirmative solution to the problem of measurement *is indeed possible* in the quantum theory of measurement, by means of replacing their artificial assumption that the measuring apparatus be represented in *one* Hilbert space with a more realistic one.

Apart from the criticisms raised by Jauch, Wigner and Yanase and by Fine, we have to mention the following remarks on the ergodic amplification point of view: Even though it is true that the *reduction of wave packet*, to be described by $\hat{\rho}\rightarrow\hat{\bar{\rho}}$, is certainly a sort of irreversible process in the sense of yielding a mixed state from a superposed one, the problem is not so clear as to whether the *reduction of wave packet* in quantum-mechanical measurement is the same as thermal irreversible processes. Especially, we do not think that the amplification itself is inevitable for the *reduction of wave packet*. Indeed as will be shown in § 3, we can set up a simple model of measuring apparatus to give the *reduction of wave packet*, without resort to intervention of any amplification process or any thermal irreversible process.

2.3. *Negative result measurements*

Daneri, Loinger and Prosperi[10] viewed the microscopic interaction between object and apparatus as a triggering device of ergodic amplification in the macroscopic measuring apparatus. In other words, it is supposed there that the *reduction of wave packet* originates in actual operation of the particle detector, namely, actual occurrence of ergodic amplification in the apparatus. Against their view Jauch, Wigner and Yanase[14] pointed out that, in the so-called "negative result measurements" as discussed by Renninger,[16] quantum-mechanical measurement is performed without any microscopic triggering process. The Stern-Gerlach type version of the *negative result experiments* can be performed just by omitting one of the two detectors, say D_- in Fig. 1, and setting a new one D_0 in front of the magnet as in Fig. 2. Suppose the case of anti-coincidence between D_0 and D_+, that is to say, the case in which a wave packet of the object particle is certainly injected at time T_0 into the right passing through D_0 but we observe no signal

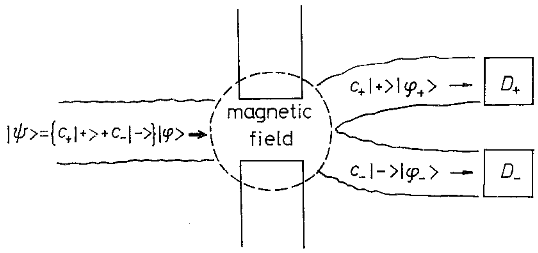

Fig. 1. The Stern-Gerlach experiment.

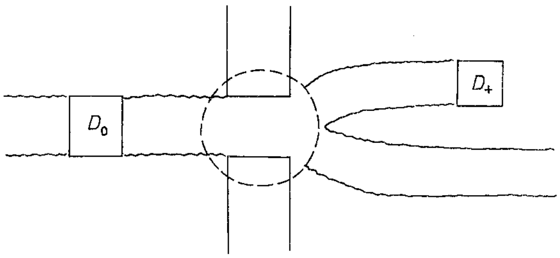

Fig. 2. The 'negative result measurement' in the Stern-Gerlach experiment.

on detector D_+ at an appropriate time $T>T_0$. We then obtain the information at time T that the object system is in state $|-\rangle$ and has gone downwards, assuming the detector efficiency to be 100%. This is an example of quantum-mechnical measurement without actual occurrence of thermal irreversible processes.

Since no event occurred in the detector, it is evident that thermal irreversible processes themselves in macroscopic apparatus cannot be the origin of the *reduction of wave packet*. Through the arguments of the 'negative result measurements,' however, one might be led to a misleading confusion that the *reduction of wave packet* does take place without any *interaction* between the microscopic object and the macroscopic apparatus.[14] Note that 'no event' does not mean 'no interaction'! An important logical point to overcome this confusing situation is that the *interaction* does not have one-to-one correspondence to the *event*. That is to say, the event 'no signal' on D_+ in the above case never means that the particle wave function did not interact with detector D_+. Quantum-mechanical wave function does not give a definite prediction for each measurement performed on a system in a superposed state, but only a probabilistic prediction over a large ensemble.

Summarizing, though the arguments of the 'negative result measurements' exclude the viewpoint that the *reduction of wave packet* occurs as a consequence of *event* as discharges, they cannot exclude the viewpoint that the reduction occurs as a consequence of *interaction*. In § 3 of Part II we show that the *reduction of wave packet* does occur as a consequence of interaction even in the case of the negative result measurements.

§ 3. Measurement of particle momentum by a perfect mirror

In this section we discuss a simple model of measuring apparatus which was originally introduced by one of the present authors to show that ergodic

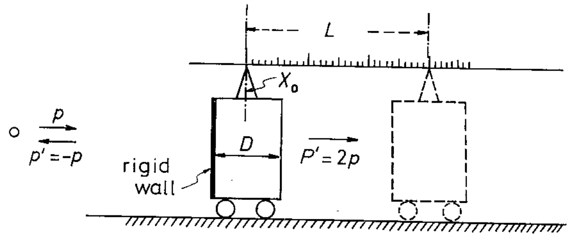

Fig. 3. Model apparatus using a perfect mirror for measurement of particle momentum.

amplification is not always necessary for quantum mechanical observation.[17]

A perfect mirror with a rigid wall can be used to measure momentum of a microscopic particle with very high energy ($\gtrsim 10^{19}$ eV$\simeq$1 Joule, for example) such as cosmic ray protons. See Fig. 3 in which a small wagon with a perfect mirror is put on a horizontal railway. The particle energy is quite enough to move a macroscopic body, that is, the wagon by a macroscopic distance after a collision of the object particle with the mirror at rest. If the friction between the wagon and the railway is very small, then the collision can be regarded as an elastic one between two free particles. Assume that the friction force is proportional to velocity, then the running distance of the wagon is given by $L=(P'/f)$, f being the friction coefficient and P' the wagon momentum after the collision. Because of large wagon mass $M(\gg$ particle mass $m)$, we can safely put that $Mc\gg p\gg mc$ and then $p'=-p+(2p^2/Mc)$ and $P'=2p-(2p^2/Mc)$ where p and p' are, respectively, particle momenta before and after the collision, and c the light velocity. For simplicity we put $p'=-p$ and $P'=2p$ in what follows. And note that the relative momentum is equal to the particle momentum because of $M\gg m$. We can then get the particle momentum before the collision through the formula $p=(Lf/2)$ by reading L from the pointer position. It is noted here that these approximations in kinematical relations are not essential to our final conclusion but only introduced to make the following expressions clear and simple. The essential fact is that the relative momentum becomes very large proportionally to the incident particle momentum p as $p\to\infty$. If the particle energy is not so high as to move a macroscopic body by a macroscopic distance, we have only to make the apparatus run with very high speed towards the object particle. We can give an arbitrary high speed to the apparatus. On the rest frame of the apparatus we can develop the theory just in the same way as in the above case. Speaking in principle, it is very important that we can develop our theory assuming $p\to\infty$.

Now let us develop quantum-mechanical description for the total system of the object particle and the mirror-wagon system. First we formulate a free motion of the mirror-wagon system. Divide the motion of the system into two parts: One is its center-of-mass motion and the other its inner motion.

(i) Center-of-mass motion of the mirror-wagon system

The mirror-wagon system is a macroscopic body, so that its center-of-mass state can be expressed by a well-localized wave packet $|\tilde{P}, X_0\rangle$ with uncertainties δP and δX_0 around mean values P and X_0, respectively, of momentum and position—for example,

$$|\tilde{P}, X_0\rangle = \int |P'\rangle dP' \langle P'|\tilde{P}, X_0\rangle;$$

$$\langle P'|\tilde{P}, X_0\rangle = [2\pi(\delta P)^2]^{-1/4} \exp\left[-(P'-P)^2/4(\delta P)^2 - \frac{i}{\hbar}P'X_0\right], \qquad (3\cdot1)$$

where both δP and $\delta X_0 = (\hbar/2\delta P)$ are very small on a macroscopic scale. The tilde on P is put to stress wave packet states. Since we can neglect spreading of the wave packet— $(\delta P/M) \ll 1$—during the measurement, the free center-of-mass motion can be described by the time evolution

$$|\tilde{P}, X_0\rangle_t = \exp\left[-\frac{i}{\hbar}\hat{H}^{AX}t\right]|\tilde{P}, X_0\rangle = \exp\left[-\frac{i}{\hbar}\frac{P^2}{2M}t\right]|\tilde{P}, X_0 + Pt/M\rangle, \qquad (3\cdot2)$$

where $\hat{H}^{AX}$ is the Hamiltonian operator of the center-of-mass motion. Note that the neglection of spreading of the wave packet is not essential to our final conclusion but used only to make the following expression simple.

Here it should be remarked that the center-of-mass is to be identified with the pointer position, and that we cannot determine a macroscopic point such as the pointer position on a microscopic scale, but can only do it on a macroscopic scale with a certain allowance at least larger than the order of atomic size. Consequently, the macroscopic center-of-mass state should be represented by the statistical operator

$$\hat{\rho}_0^{AX}(P, X) = \int |\tilde{P}, X_0\rangle W^{(X)}(X_0 - X)\, dX_0 \langle \tilde{P}, X_0|, \qquad (3\cdot3)$$

where $W^{(X)}(X_0 - X)$ is the normalized weight function representing the X_0-distribution with a width δX around X, for example, given by

$$W^{(X)}(X_0 - X) = [2\pi(\delta X)^2]^{-1/2} \exp[-(X_0 - X)^2/2(\delta X)^2].$$

Thus, the free center-of-mass motion of the mirror-wagon system is described by the statistical operator

$$\hat{\rho}_t^{AX}(P, X) = \exp\left[-\frac{i}{\hbar}\hat{H}^{AX}t\right]\hat{\rho}_0^{AX}(P, X) \exp\left[\frac{i}{\hbar}\hat{H}^{AX}t\right]$$

$$= \hat{\rho}_0^{AX}\left(P, X + \frac{P}{M}t\right) \qquad (3\cdot4)$$

which means that the pointer position runs with velocity (P/M).

(ii) Inner state of the mirror-wagon system

Suppose that quantum-mechanical inner states of the mirror-wagon system are governed by the Hamiltonian $\hat{H}^{AI}$ having the eigenvalue equation

$$\hat{H}^{AI}|n, D\rangle = E_n^{AI}|n, D\rangle, \tag{3·5}$$

where $|n, D\rangle$ is the eigenvector of $\hat{H}^{AI}$ belonging to eigenvalue E_n^{AI} and D denotes thickness of the mirror body. It is noted that the eigenvectors are to be determined by solving the boundary-value problem in a spatial domain of the mirror body with thickness D. The eigenvalues and eigenvectors depend on D. Similarly as for the pointer position, we cannot determine the thickness of a macroscopic body on a microscopic scale, but can only measure it on a macroscopic scale with a certain allowance at least larger than the order of atomic size.*) Therefore, the inner state of the mirror-wagon system should be represented by the statistical operator

$$\hat{\sigma}_0^{AI}(d) = \int dD W^{(I)}(D-d)\hat{\rho}_0^{AI}(D), \tag{3·6}$$

where

$$\hat{\rho}_0^{AI}(D) = \sum_n w_n \hat{\xi}_n^I(D); \ \hat{\xi}_n^I(D) = |n, D\rangle\langle n, D|. \tag{3·7}$$

Here $W^{(I)}(D-d)$ is the normalized statistical weight for the D-distribution with a width δD around its center d, for example, given by

$$W^I(D-d) = [2\pi(\delta D)^2]^{-1/2}\exp[-(D-d)^2/2\pi(\delta D)^2],$$

and it is natural to put w_n equal to the Boltzmann weight in thermal equilibrium. Parameter d is nothing other than the thickness of the mirror body in a macroscopic sense. Now, we obtain

$$\exp\left[-\frac{i}{\hbar}\hat{H}^{AI}t\right]\hat{\sigma}_0^{AI}(d)\exp\left[\frac{i}{\hbar}\hat{H}^{AI}t\right] = \hat{\sigma}_0^{AI}(d) \tag{3·8}$$

because of (3·5), so that the inner state does not change.

Consequently we can describe a macroscopic state of the total mirror-wagon system by the statistical operator

$$\begin{aligned}\hat{\rho}_0^A(P, X, d) &= \hat{\rho}_0^{AX}(P, X) \otimes \hat{\sigma}_0^{AI}(d)\\ &= \int dD W^I(D-d)\hat{\rho}_0^A(P, X, D),\end{aligned} \tag{3·9}$$

where

$$\begin{aligned}\hat{\rho}_0^A(P, X, D) &= \hat{\rho}_0^{AX}(P, X) \otimes \hat{\rho}_0^{AI}(D)\\ &= \sum_n w_n \hat{\rho}_0^{AX}(P, X) \otimes \hat{\xi}_n^I(D),\end{aligned} \tag{3·10}$$

*) This is apparently related with the uncertainty of the number of atoms in the detector and the openness of macroscopic body discussed in general in §2, Part II.

and its free motion by the time dependent statistical operator

$$\hat{\rho}_t{}^A(P,X,d)=\exp\left[-\frac{i}{\hbar}(\hat{H}^{AX}+\hat{H}^{AI})\,t\right]\hat{\rho}_0{}^A(P,X,d)\exp\left[\frac{i}{\hbar}(\hat{H}^{AX}+\hat{H}^{AI})\,t\right]$$
$$=\hat{\rho}_0{}^A\left(P,X+\frac{P}{M}t,d\right). \tag{3·11}$$

The mirror-wagon system is now represented by the macroscopic state variables P, X and d. If the mirror-wagon system is put on the pointer position $X=0$ at rest $(P=0)$ before measurement, then its initial state is represented by $\hat{\rho}_0{}^A(0,0,d)$.

We can easily understand that δD, δX and δX_0 are of the order of magnitude larger than the atomic size but much smaller than the macroscopic scale unit. For example, we may estimate

$$\delta D\sim 10^{-8}\,\mathrm{cm} \quad \text{or} \quad \delta P_D=\hbar/\delta D\sim 1\,\mathrm{keV}/c \tag{3·12}$$

as a measure, even though (3·12) is an underestimate for δD and an overestimate for δP_D.

(iii) Object particle state

Let us prepare a wave packet state $|\tilde{p},x_0\rangle$ very close to a plane wave with momentum p for the object particle, for example, given by

$$|\tilde{p},x_0\rangle=\int|p''\rangle dp''\langle p''|\tilde{p},x_0\rangle;$$
$$\langle p''|\tilde{p},x_0\rangle=[2\pi(\delta p)^2]^{-1/4}\exp\left[-(p''-p)^2/4(\delta p)^2-\frac{i}{\hbar}p''x_0\right], \tag{3·13}$$

δp being the momentum uncertainty. The wave packet distributes over a spatial domain with breadth $\delta x=\hbar/2\delta p$ around its center x_0. Note that the wave packet before measurement is located in a remote region $(-x_0\gg\delta x)$ on the left side of the mirror-wagon system. Speaking in principle, we can arbitrarily improve colimation of the wave packet state $|\tilde{p},x_0\rangle$, so that we can assume $\delta p\to 0$. Consequently we have

$$\delta P\gg\delta p, \text{ or equivalently } \delta x\gg\delta D\,, \tag{3·14}$$

but δx is still very small on a macroscopic scale (D or $L\gg\delta x$). In this case we can discard spreading of the wave packet for propagation in our instrument, —even though this discarding is not essential to our final conclusion,— so that the wave packet state can be described by the statistical operator

$$\hat{\xi}_t{}^Q(p,x_0)=\exp\left[-\frac{i}{\hbar}\hat{H}^Q t\right]\hat{\xi}_0{}^Q(p,x_0)\exp\left[\frac{i}{\hbar}\hat{H}^Q t\right]$$
$$=\hat{\xi}_t{}^Q\left(p,x_0+\frac{p}{m}t\right), \tag{3·15}$$

where $\hat{\xi}_0^Q(p, x_0) = |\tilde{p}, x_0\rangle\langle\tilde{p}, x_0|$ and $\hat{H}^Q$ is the free Hamiltonian of the object particle.

Under the above preparation we can now proceed to discuss our measuring processes using the mirror-wagon system. For simplicity, suppose that the object system before the measurement is in a superposed state of two momentum states $|\tilde{p}_a, x_0\rangle$ and $|\tilde{p}_b, x_0\rangle$, given by

$$|\psi\rangle = c_a|\tilde{p}_a, x_0\rangle + c_b|\tilde{p}_b, x_0\rangle \tag{3·16a}$$

or equivalently

$$\begin{aligned}\hat{\rho}_0^Q(\psi) &= |\psi\rangle\langle\psi| \\ &= |c_a|^2\hat{\xi}_0^Q(p_a, x_0) + |c_b|^2\hat{\xi}_0^Q(p_b, x_0) \\ &\quad + c_a c_b^*\hat{\eta}_0^Q(p_a, p_b; x_0) + c_a^* c_b\hat{\eta}_0^Q(p_b, p_a; x_0)\end{aligned} \tag{3·16b}$$

in terms of the statistical operator, where $\hat{\eta}_0^Q(p_a, p_b; x_0) = |\tilde{p}_a, x_0\rangle\langle\tilde{p}_b, x_0|$ and $\hat{\eta}_0^Q(p_b, p_a; x_0) = |\tilde{p}_b, x_0\rangle\langle\tilde{p}_a, x_0|$. Needless to say, it is easy to extend $|\psi\rangle$ to a general superposed state. The initial state of the total system before the measurement is now represented by the statistical operator

$$\hat{\rho}_0 = \hat{\rho}_0^Q(\psi) \otimes \hat{\rho}_0^A(0, 0, d). \tag{3·17}$$

The measuring process is therefore described by the time-dependent statistical operator

$$\hat{\rho}_t = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\rho}_0 \exp\left[\frac{i}{\hbar}\hat{H}t\right], \tag{3·18}$$

where

$$\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}};\ \hat{H}_0 = \hat{H}^Q + \hat{H}^{AX} + \hat{H}^{AI}, \tag{3·19}$$

$\hat{H}_{\text{int}}$ being the interaction Hamiltonian between the object particle and the mirror-wagon system. Corresponding to decomposition (3·16b), $\hat{\rho}_t$ is rewritten as

$$\hat{\rho}_t = |c_a|^2\hat{\Xi}_t^{aa} + |c_b|^2\hat{\Xi}_t^{bb} + c_a c_b^*\hat{\Xi}_t^{ab} + c_a^* c_b\hat{\Xi}_t^{ba}, \tag{3·20}$$

where

$$\hat{\Xi}_t^{aa} = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\Xi}_0^{aa}\exp\left[\frac{i}{\hbar}\hat{H}t\right];\ \hat{\Xi}_0^{aa} = \hat{\xi}_0^Q(p_a, x_0) \otimes \hat{\rho}_0^A(0, 0, d), \tag{3·21a}$$

$$\hat{\Xi}_t^{bb} = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\Xi}_0^{bb}\exp\left[\frac{i}{\hbar}\hat{H}t\right];\ \hat{\Xi}_0^{bb} = \hat{\xi}_0^Q(p_b, x_0) \otimes \hat{\rho}_0^A(0, 0, d), \tag{3·21b}$$

$$\hat{\Xi}_t^{ab} = \exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\Xi}_0^{ab}\exp\left[\frac{i}{\hbar}\hat{H}t\right];\ \hat{\Xi}_0^{ab} = \hat{\eta}_0^Q(p_a p_b; x_0) \otimes \hat{\rho}_0^A(0, 0, p), \tag{3·21c}$$

$$\hat{\Xi}_t{}^{ba}=\exp\left[-\frac{i}{\hbar}\hat{H}t\right]\hat{\Xi}_0{}^{ba}\exp\left[\frac{i}{\hbar}\hat{H}t\right];\ \hat{\Xi}_0{}^{ba}=\hat{\eta}_0{}^Q(p_b,p_a;x_0)\otimes\hat{\rho}_0{}^A(0,0,d). \tag{3·21d}$$

Introducing the time evolution operator

$$\hat{U}_I(t)=\exp\left[\frac{i}{\hbar}\hat{H}_0t\right]\exp\left[-\frac{i}{\hbar}\hat{H}t\right] \tag{3·22}$$

in the interaction representation, we can extract the elementary process from (3·18) by the equation

$$\exp\left[-\frac{i}{\hbar}\hat{H}t\right]|\tilde{p},x_0;\tilde{P},X_0,D\rangle=\exp\left[-\frac{i}{\hbar}\hat{H}_0t\right]\hat{U}_{\pm}(t)|\tilde{p},x_0;\tilde{P},X_0,D\rangle$$

$$\xrightarrow[t\to\infty]{}\exp\left[-\frac{i}{\hbar}\hat{H}_0t\right]\hat{S}|\tilde{p},x_0;\tilde{P},X_0,D\rangle\,, \tag{3·23}$$

where $\hat{S}$ is the S-matrix and $|\tilde{p},x_0;\tilde{P},X_0,D\rangle=|\tilde{p},x_0\rangle\otimes|\tilde{P},X_0\rangle\otimes|n,D\rangle$. Following the theory of scattering, we can obtain the S-matrix element from the asymptotic form of the relative coordinate part $\phi(r)$ of the wave function $\Psi(x',X')=\langle x',X'|U_I(0,-\infty)|p,P\rangle$, where x',X' and $r=x'-X'$ are, respectively, the object particle coordinate, the center-of-mass coordinate (of the mirror-wagon system) and the relative coordinate. In our case the object particle collides with the rigid body of thickness D, so that $\phi(r)$ has the same form as the wave function in the case of collision with a rigid wall located at $r=-(D/2)$. Then we have

$$\phi(r)=\text{const}\left\{\exp\left[\frac{i}{\hbar}pr\right]-\exp\left[-\frac{i}{\hbar}pD\right]\exp\left[-\frac{i}{\hbar}pr\right]\right\} \tag{3·24}$$

which certainly vanishes at $r=-(D/2)$. Equation (3·24) gives us the S-matrix element

$$\langle p',P'|\hat{S}|p,P\rangle=\exp\left[-\frac{i}{\hbar}pD\right], \tag{3·25}$$

where $P=0$, $P'=2p$ and $p'=-p$ in our collision—see kinematics given at the beginning and note that the relative momentum is equal to the particle momentum. Rigorously speaking, we have only to remark that the phase shift increases proportionally to p and D as $p\to\infty$. Hence the particle state of momentum p_a changes into

$$\int\left(-\exp\left[-\frac{i}{\hbar}p''D\right]\right)|-p''\rangle dp''\langle p''||\tilde{p}_a,x_0\rangle$$

by the collision. Because of $\delta p\delta D\ll\hbar$ obtained from (3·14) or the possible limit $\delta p\to0$, we can neglect the last term of $p''D=p''d+p_a(D-d)+(p''-p_a)(D-d)$ in the phase shift, so that the above state is equal to

$$\left[-\exp\left\{-\frac{i}{\hbar}p_a(D-d)\right\}\right]\int|-p''\rangle dp''\exp\left\{-\frac{i}{\hbar}p''d\right\}\langle p''|\tilde{p}_a, x_0\rangle$$
$$=\left[-\exp\left\{-\frac{i}{\hbar}p_a(D-d)\right\}\right]\int|p''\rangle dp''\langle p''|-\tilde{p}_a, -x_0-d\rangle$$
$$=\left[-\exp\left\{-\frac{i}{\hbar}p_a(D-d)\right\}\right]|-\tilde{p}_a, -x_0-d\rangle\,.$$

Therefore, we obtain

$$\hat{\Xi}_t^{aa}\xrightarrow[t\to\infty]{}\hat{\xi}_t^Q(-p_a, -x_0-d)\otimes\sum_n\int\int dX_0dD\left(\exp\left[-\frac{i}{\hbar}p_a(D-d)\right]\right)$$
$$\times|2\tilde{p}_a, X_0\rangle_t\otimes|n, D\rangle\cdot W^{(X)}(X_0-0)\,w_nW^{(I)}(D-d)$$
$$\times{}_t\langle 2\tilde{p}_a, X_0|\otimes\langle n, D|\left(-\exp\left[\frac{i}{\hbar}p_a(D-d)\right]\right)$$
$$=\hat{\xi}_t^Q(-p_a; -x_0-d)\otimes\hat{\rho}_0^A\left(2p_a, \frac{2p_a}{M}t, d\right) \tag{3·26a}$$

for (3·21a) and also

$$\hat{\Xi}_t^{bb}\xrightarrow[t\to\infty]{}\hat{\xi}_t^Q(-p_a, -x_0-d)\otimes\hat{\rho}_0^A\left(2p_b, \frac{2p_b}{M}t, d\right) \tag{3·26b}$$

for (3·21b). From (3·21c), however, we have

$$\hat{\Xi}_t^{ab}\xrightarrow[t\to\infty]{}\hat{\eta}_t^Q(-p_a, -p_b; -x_0-d)\otimes\sum_n\int\int dX_0dD\left(-\exp\left[-\frac{i}{\hbar}p_a(D-a)\right]\right)$$
$$\times|2\tilde{p}_a, X_0\rangle_t\otimes|n, D\rangle W^{(X)}(X_0-D)\,w_nW^{(I)}(D-d)$$
$$\times{}_t\langle 2\tilde{p}_b, X_0|\otimes\langle n, D|\left(-\exp\left[\frac{i}{\hbar}p_b(D-d)\right]\right),$$

in which the left side phase shift $e^{-(i/\hbar)p_a(D-d)}$ is never cancelled out by the right side one $e^{(i/\hbar)p_b(D-d)}$ if $p_a\neq p_b$. Here we meet with the following integral:

$$\int dDW^I(D-d)\exp\left[-\frac{i}{\hbar}(p_a-p_b)\,(D-d)\right]\hat{\rho}^{AI}(D)\,. \tag{3·27}$$

Considering that $\hat{\rho}^{AI}(D)$ is a slowly varying function of D because $d\gg\delta D$, it is easy to show that the integral vanishes if

$$|p_a-p_b|\gg\delta P_D\,, \tag{3·28}$$

δP_D is given by (3·12). Hence we get

$$\hat{\Xi}_t^{ab}\xrightarrow[t\to\infty]{}0 \tag{3·26c}$$

and also

$$\hat{\Xi}_t^{ba} \xrightarrow[t\to\infty]{} 0\,. \tag{3·26d}$$

As a result we are led to

$$\hat{\rho}_t \to \hat{\bar{\rho}}_t = |c_a|^2 \hat{\xi}_t^Q(-p_a, -x_0-d) \otimes \hat{\rho}_0^A\left(2p_a, \frac{2p_a}{M}t, d\right)$$
$$+|c_b|^2 \hat{\xi}_t^Q(-p_b, -x_0-d) \otimes \hat{\rho}_0^A\left(2p_b, \frac{2p_b}{M}t, d\right), \tag{3·29}$$

which exactly corresponds to the *reduction of wave packet.* We can now understand that our model has every property to be required to a measuring apparatus for quantum-mechanical observation.

Finally we have to remark on the condition (3·28) to have (3·26c, d). The condition is apparantly inherent to our model apparatus, but it does actually work as the limited accuracy only when used as a measuring apparatus without any momentum analyzer for the measurement of particle momentum. Indeed, the limited accuracy (3·28) can easily been removed if we use our model apparatus as a particle detector combined with a momentum analyzer. Suppose that the magnet in Fig. 1 is a momentum analyzer to deflect particles having p_b towards D_-, and that each of D_+ and D_- is a perfect mirror apparatus just discussed here. The wave packets $|\tilde{p}_a, x_0\rangle$ and $|\tilde{p}_b, x_0\rangle$, after separated from each other, come into D_+ and D_-, respectively. In this case it is easy to show that the off-diagonal part $\hat{\Xi}_t^{ab}$ of the total statistical operator is asymptotically proportional to

$$F(p_a, p_b) = \int dD_+' \exp\left[-\frac{i}{\hbar}p_a(D_+'-d_+)\right] W_+^{(I)}(D_+'-d_+)$$
$$\times \int dD_-' \exp\left[\frac{i}{\hbar}p_b(D_-'-d_-) W_-^{(I)}(D_-'-d_-)\right] \tag{3·30}$$

and $\hat{\Xi}_t^{ba}$ is proportional to $F^*(p_a, p_b)$, where $D_\pm'$ and $d_\pm$ are the size variables of the detectors and $W_\pm^{(I)}$ the weight functions just like $W^{(I)}$ given in (3·6). Consequently we have $F(p_a, p_b) \simeq 0$ and then the *reduction of wave packet* under the condition

$$|p_a| \gg \delta P_D \quad \text{or} \quad |p_b| \gg \delta P_D \tag{3·31}$$

instead of (3·28). Furthermore we have to recall here the important remark that we can develop the theory of measurement assuming p_a or $p_b \to \infty$. Following this remark we can safely apply the Riemann-Lebesque theorem

$$\lim_{p\to\infty} \int dD \exp\left[-\frac{i}{\hbar}p(D-d)\right] W^{(I)}(D-d) = 0 \tag{3·32}$$

which yields $F(p_a, p_b) = 0$ and then exactly the *reduction of wave packet.* It may be worth while to note that we have exactly obtained the *reduction of wave packet* but not approximately. The above arguments can also be applied to the general theory of measurement which will be discussed in Part II.

We can also discuss the negative result measurements using our simple model. In this case the off-diagonal part of the total statistical operator is asymptotically proportional to

$$\int dD_+ \exp\left[-\frac{i}{\hbar} p_a (D_+ - d_+)\right] W_+^{(I)}(D_+ - d_+)$$

and its complex conjugate, which vanishes because of (3·32) and the above remark. Consequently we can exactly obtain the *reduction of wave packet* even in the case of the negative result measurements.

General theory of the *reduction of wave packet* in the quantum mechanical measurement will be given in Part II.

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