Theory of Nonlocal Elasticity and Some Applications

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Theory of Nonlocal Elasticity and Some Applications

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ABSTRACT

Constitutive equations of finite nonlocal elasticity are obtained. Thermodynamic restriction are studied. The linear theory is given for anisotropic and isotropic solids. The physical and mathematical properties of the nonlocal elastic moduli are explored through lattice dynamics and dispersive wave propagations. The theory is applied to the problems of surface waves, screw dislocation and a crack. Excellent agreements with the results known in atomic lattice dynamics and experiments display the power and potential of the theory.

1. INTRODUCTION

All physical theories possess certain domain of applicability outside of which they fail to predict the physical phenomena with reasonable accuracy. While the boundaries of these domains are not known precisely, often the failure of a given mathematical model is indicated by its predictions that deviate considerably from experimental results or dramatically displayed by mathematical singularities that it leads to.

The domain of applicability of a theory is a function of some internal characteristic length and time scales of the media for which it is constructed. When these scales are sufficiently small as compared to the corresponding external scales, then the classical field theories give successful results. Otherwise, they fail.

Such is the situation with the classical elasticity theory which possesses no internal scales. Yet all elastic materials possess inner structures in the molecular and atomic scales. Consequently, when the external scales (such as wave length, period, the size of the area over which applied loads are continuous), becomes comparable with the inner scales (such as granular distance, relaxation time, lattice parameter), the theory fails to apply.

In classical elasticity, this situation is demonstrated dramatically by the singular stress field predicted at a sharp crack tip and the phase velocities that do not depend on wave lengths of propagating waves. As a result of the former a perfectly sensible physical criterion of fracture, based on maximum stress hypothesis, was replaced by various

ersatzs (e.g. Griffith energy, J-integral, etc). Clearly, the infinite stress is a sharp signal for the failure of the theory rather than the failure of the fracture criterion which must be based on the physical concept of cohesive stress. Regarding the phase velocity, at all wave lengths from infinite to the atomic distances, we have ample experimental measurements of dispersion curves. Only at the very large wave lengths is there an agreement with classically predicted constant phase velocity. Waves having short wave lengths have been observed to propagate with much smnaller phase velocities and in fact they cease to propagate near the boundaries of the Brillouin zone.

The question arises: "Should we altogether abandon classical field theories and appeal to atomic theories only?" The answer depends on the characteristic scale ratios. Indeed if the motion of each atom in a body is essential for the description of a physical phenomenon, then the lattice dynamics is the only answer. If, on the other hand, the collective behavior of large number of atoms is adequate for the description, then continuum theory offers much simpler and practical methodology. Between these two extremes, there lies a large domain full of rich physical phenomena.

Real materials possess a very complicated inner structure full of dislocations and impurities. Moreover the force law among the substructure is not known. Consequently, it is virtually impossible to carry out calculations on the basis of the atomic theories. Even if it were possible to accomplish such voluminous and difficult computations, results would be of no practical value. All experimental probes possess some characteristic lengths so that they can only measure statistical averages. Consequently

we need to calculate theoretically certain statistical averages so that comparison can be made with experimental observations. Hence we are back in the domain of continuum. Thus, continuum theory makes sense on its own grounds, provided it is properly constructed to predict these averages with sufficient accuracy.

Linear theory of nonlocal elasticity, which has been proposed independently by various authors [1]-[6], incorporates important features of lattice dynamics and yet it contains classical elasticity in the long wave length limit. It is capable of addressing small as well as large scale phenomena. Large number of references on the topic may be found in [7]-[9]. Interested readers may also consult [10], [11] for the nonlocal fluid dynamics and [12], [13] for nonlocal electromagnetic continua.

Here I present the theory of nonlocal nonlinear elasticity from a continuum point of view. (See also [14]-[17]). Constitutive equations are given in Section 3, where I employ the global entropy inequality rather than the local Clausius-Duhem inequality to place restrictions on the constitutive functionals. In Section 4, I derive a special class of stress-strain law for the additive functionals. Isotropic solids are studied in Section 5 and linear theory is presented in Section 6.

In nonlocal elasticity, the stress at a point is regarded as a functional of the strain tensor. For linear, homogeneous solids, this introduces material moduli which are functions of the distance. Physical and mathematical properties of these moduli are studied in Section 7. Section 8 gives the field equations.

Applications of the linear theory begins with Section 9 to wave

propagation. Dispersion curves are obtained for the plane harmonic waves in an infinite solid and for surface waves. Results are in excellent agreement with the corresponding ones obtained by means of lattice dynamics. In Section 10, I determine the stress distribution due to a screw-dislocation. Cohesive stress that holds the atomic bonds together in a perfect crystal is found to coincide with the so-called theoretical stress estimated on the basis of atomic theory or experiments. The last section (Section 11) treats the crack tip problem for anti-plane case (Mode III). Contrary to the classical result, the crack tip stress vanishes at the tip and possesses a finite maximum near the tip. The maximum stress hypothesis of fracture can now be restored. This enables us to calculate the fracture toughness which is shown to agree well with experimental results on several materials.

These few examples are sufficient to demonstrate the power and potential of the theory. There exist several other solutions in the literature, dealing with dislocations, cracks, wave propagations, defects, continuous distribution of dislocations. They also make successful predictions.

The purpose of this lecture is to share my enthusiasm with you and to draw your attention to the exploration of these new theories.

2. Balance Laws

Just as in classical field theories, the motion of a material point $\underset{\sim}{X}$ in a body B with volume V, enclosed by its surface ∂V, is described by the mapping

$$\underset{\sim}{x} = \underset{\sim}{x}(\underset{\sim}{X},t) \tag{2.1}$$

where $\underset{\sim}{x}$, at time t, is the spatial image of $\underset{\sim}{X}$, in the deformed configuration $\mathcal{B}$ having volume $\mathcal{V}$ enclosed within its surface $\partial\mathcal{V}$. We employ rectangular coordinates X_K and x_k to denote the position of $\underset{\sim}{X}$ and $\underset{\sim}{x}$ respectively, and assume that

$$J \equiv \det(\partial x_k/\partial X_K) > 0 \tag{2.2}$$

throughout B, so that the inverse of (2.1)

$$\underset{\sim}{X} = \underset{\sim}{X}(\underset{\sim}{x},t) \quad , \qquad \underset{\sim}{X} \in B \tag{2.3}$$

exists and is unique.

Under some mild assumptions, local balance laws of continuum mechanics are valid for the nonlocal theory. Thus, we assume that the body is made up of single nonpolar species and it is inert. Moreover, nonlocal gravitational effects can be neglected. Under these assumptions, the nonlocal residuals may be dropped and we have the usual balance laws

$$\dot{\rho} + \rho v_{k,k} = 0 , \tag{2.4}$$

$$t_{k\ell,k} + \rho(f_\ell - \dot{v}_\ell) = 0 , \tag{2.5}$$

$$t_{k\ell} = t_{\ell k} , \tag{2.6}$$

$$-\rho\dot{\varepsilon} + t_{k\ell} v_{\ell,k} + q_{k,k} + \rho h = 0 \tag{2.7}$$

and corresponding jump conditions which we do not list here (cf. Eringen [7],[15],[17]). Here, ρ, v_k, $t_{k\ell}$, f_ℓ, ε, q_k and h are, respectively, the mass density, velocity vector, stress tensor, body force density, internal energy density, heat vector and the energy source density. The usual summation convention is valid on repeated indices and a superposed dot ($\bullet$) denotes the material time rate and an index following a comma partial derivative, e.g.

$$\dot{v}_\ell = \frac{\partial v_\ell}{\partial t} + v_{\ell,m} v_m , \qquad v_{\ell,k} = \frac{\partial v_\ell}{\partial x_k}$$

$$x_{k,K} = \frac{\partial x_k}{\partial X_K}$$

In contrast to classical (local) field theories, the entropy inequality is a global statement covering the entire body. It is expressed as

$$\int_V [\rho\dot{\eta} - (q_k/\theta)_{,k} - (\rho h/\theta)] \, dv \geq 0 \tag{2.8}$$

where η is the entropy density and $\theta > 0$ is the absolute temperature.

By eliminating h between (2.7) and (2.8), we have

$$(2.9) \qquad \int_V \frac{1}{\theta}\left[-\rho(\dot{\psi}+\eta\dot{\theta}) + t_{k\ell}v_{\ell,k} + \frac{1}{\theta}q_k\theta_{,k}\right] dv \geq 0$$

where we also introduced the Helmholtz' free energy

$$(2.10) \qquad \psi = \varepsilon - \theta\eta$$

For some purpose, it is convenient to introduce the material tensors

$$(2.11) \qquad T_{KL} = J\, t_{k\ell}X_{K,k}X_{L,\ell}, \qquad Q_K = J\, q_k\, X_{K,k},$$
$$C_{KL} = x_{k,K}x_{k,L}, \qquad J = \rho_0/\rho$$

which enables us to write Eq. (2.9) in the form

$$(2.12) \qquad \int_V \frac{1}{\theta}\left[-\rho_0(\dot{\psi} + \eta\dot{\theta}) + \frac{1}{2}T_{KL}\dot{C}_{KL} + \frac{1}{\theta}Q_K\theta_{,K}\right] dV \geq 0$$

This is the material form of the global entropy inequality which is valid for the nonlocal field theories. Clearly, the Clausius-Duhem inequality which constitute the integrands of (2.9) and (2.12) are much too restricted. For example, it is not valid for the phase transformation where certain thermodynamic quantities exhibit discontinuities and/or singularities.

3. Constitutive Equations

According to the axiom of causality, [6,18] all physical processes are the result of motions (deformations) of all material points of the body in the past up to and including the present time. For thermomechanical phenomena, the independent variables are therefore

$$Y' \equiv \{\underset{\sim}{x}', \theta', \theta'_{,K}\} \tag{3.1}$$

where a prime is used to denote the values of these functions at $(X', t-\tau')$, e.g.

$$\underset{\sim}{x}' = \underset{\sim}{x}(\underset{\sim}{X}', t-\tau'), \qquad \theta' = \theta(\underset{\sim}{X}', t-\tau'),$$
$$\underset{\sim}{X}' \in B, \qquad 0 \leq \tau' < \infty \tag{3.2}$$

Constitutive equations express the functional dependence of the set

$$Z \equiv \{\psi, \eta, T_{KL}, Q_K\} \tag{3.3}$$

at $(\underset{\sim}{X}, t)$ on the set (3.1), e.g.,

$$\psi(\underset{\sim}{X}, t) = F[\underset{\sim}{x}', \theta', \theta'_{,K}] \tag{3.4}$$

For inhomogeneous materials, ψ also depends on $\underset{\sim}{X}'$ explicitly. Response functionals, such as F, must be form-invariant under

arbitrary spatial translations and rotations. This implies that ψ will depend on $\underset{\sim}{x}'$ and $\underset{\sim}{x}$ only through $|\underset{\sim}{x}'-\underset{\sim}{x}|$. Since the distance can be expressed as a functional of C_{KL}, it proves to be convenient to replace $\underset{\sim}{x}'$ in (3.4) by $C'_{KL} \equiv C_{KL}(\underset{\sim}{X}',t-\tau')$, i.e.

$$\psi(\underset{\sim}{X},t) = F[C'_{KL},\theta',\theta'_{,K},\underset{\sim}{X}'] \tag{3.5}$$

This functional equation constitutes a fundamental source from which constitutive equations of large classes of nonlocal memory-dependent thermomechanical materials (solids and fluids) can be derived (cf. [6]). Here we consider nonlocal elastic solids. In this case, θ' is uniform throughout B and the memory-dependence is not present. Consequently,

$$\psi(\underset{\sim}{X},t) = \Psi[C_{KL}(\underset{\sim}{X}'),\underset{\sim}{X}';\theta] \tag{3.6}$$

where in $C_{KL}(\underset{\sim}{X}')$ denotes $C_{KL}(\underset{\sim}{X}',t)$, i.e., t is suppressed for brevity.

We assume that $C_{KL}(\underset{\sim}{X}')$ is continuously differentiable and it belongs to a Hilbert space with an inner product defined by

$$(\underset{\sim}{C}_1,\underset{\sim}{C}_2)_H = \int_V H(\underset{\sim}{X}'-\underset{\sim}{X})\ \mathrm{tr}(\underset{\sim}{C}'_1\underset{\sim}{C}'_2)\ dV' \tag{3.7}$$

where the <u>influence function</u> H is a positive decreasing function of its argument such that the integral in (3.7) exists and

$$H(\underset{\sim}{0}) = 1 \tag{3.8}$$

In this case, the space of functions $\underset{\sim}{C}'$ is a Hilbert space H with a finite norm defined by

$$\|\underset{\sim}{C}'\| = (\underset{\sim}{C}', \underset{\sim}{C}')_H \tag{3.9}$$

In a Hilbert space, any continuous, linear, real-valued function $f(F)$ has the unique Riez-Fréchet representation

$$f(F) = (F, \underset{\sim}{C}')_H \tag{3.10}$$

There exists a great variety of influence functions. As an example, we mention

$$H(\underset{\sim}{X}) = \exp(-\alpha |\underset{\sim}{X}|), \qquad \alpha > 0 \tag{3.11}$$

It is now possible to calculate $\dot{\psi}$. Let

$$\rho_o \psi = F \tag{3.12}$$

then

$$\rho_0 \dot{\psi} = \delta F(C'_{KL} | \dot{C}'_{KL}, \underset{\sim}{X}') + \frac{\partial F}{\partial \theta} \dot{\theta} \tag{3.13}$$

where δF is the Fréchét derivative of F with the norm defined by (3.9). It is linear in $\dot{C}'_{KL}$. Consequently, it can be expressed in the form

$$\delta F = \int_V \frac{\delta F}{\delta C'_{KL}} \dot{C}'_{KL} \, dV' \tag{3.14}$$

where the operator $\delta(\ )/\delta(\ )$ represents Fréchet partial derivative.

Substituting (3.13) into (2.12), we organize it into

$$(3.15)\qquad -\int_V \frac{1}{\theta}\left(\eta + \frac{\partial F}{\partial \theta}\right)\dot{\theta}\, dV + \frac{1}{2}\int_V \left[\frac{1}{\theta} T_{KL} - 2\int_V \frac{1}{\theta'}\left(\frac{\delta F}{\delta C'_{KL}}\right)^{*} dV'\right]\dot{C}_{KL}\, dV$$

$$+\int_V \frac{1}{\theta^2} Q_K \theta_{,K}\, dV + \int_V\int_V \left[\frac{1}{\theta'}\left(\frac{\delta F}{\delta C'_{KL}}\right)^{*}\dot{C}_{KL} - \frac{1}{\theta}\frac{\delta F}{\delta C'_{KL}}\dot{C}'_{KL}\right]dV'\, dV \geq 0$$

where a superposed asterisk (*) indicates interchange of $\underset{\sim}{X}$ and $\underset{\sim}{X}'$, e.g.,

$$\overset{*}{A}(\underset{\sim}{X}',\underset{\sim}{X}) = A(\underset{\sim}{X},\underset{\sim}{X}')$$

It should be observed that the kernel of the last double integral in (3.15) is skew-symmetric in $\underset{\sim}{X}$ and $\underset{\sim}{X}'$. Hence it vanishes. Moreover, this inequality is linear in $\dot{\theta}$, $\dot{C}_{KL}$ and $\theta_{,K}$. For arbitrary and independent variations of these quantities throughout V, this inequality cannot be maintained unless

$$(3.16)\qquad \eta = -\frac{\partial F}{\partial \theta}\ ,$$

$$(3.17)\qquad T_{KL} = 2\int_V \left(\frac{\delta F}{\delta C'_{KL}}\right)^{*} dV'$$

$$(3.18)\qquad Q_K = 0$$

Note also that we set $\theta' = \theta$ since θ is assumed to be uniform throughout V. The spatial forms of the constitutive equations follow from (2.11) which gives for the stress tensor

$$(3.19)\qquad t_{k\ell} = \frac{\rho}{\rho_0} x_{k,K} x_{\ell,L} T_{KL}$$

4. Additive Functionals

For additive functions, in the sense of Friedman and Katz [19], a representation theorem exists, according to which

$$\rho_0\psi = F = \int_V G(C'_{KL}, C_{KL}, \underset{\sim}{X}', \underset{\sim}{X})\, dV' \tag{4.1}$$

Since the integral of F over the volume V must be symmetric in $\underset{\sim}{X}'$ and $\underset{\sim}{X}$, we set

$$G = \overset{*}{G} \tag{4.2}$$

Consequently,

$$T_{KL} = \int_V [(\frac{\partial G}{\partial C'_{KL}})^* + \frac{\partial G}{\partial C_{KL}}]\, dV' \tag{4.3}$$

and the Cauchy's stress tensor is given by (3.19).

Crystalline materials possess certain symmetry regulations. All thirty-two classes of perfect crystals can be characterized by a subgroup $\{S_{KL}\}$ of the full group of orthogonal transformations and translations $\{B_K\}$ of the material frame of reference, namely

$$\bar{X}_K = S_{KL}X_L + B_K \tag{4.4}$$

$$\underset{\sim}{S}\underset{\sim}{S}^T = \underset{\sim}{S}^T\underset{\sim}{S} = \underset{\sim}{1}, \qquad \det \underset{\sim}{S} = \pm 1$$

In the case of amorphous materials, S_{KL} and B_K may be different for different species. Here we consider that $\{S_{KL}\}$ and $\{B_K\}$ are constants. ψ is invariant under the transformations (4.4) and therefore:

$$\bar{G} = G, \tag{4.5}$$

for all members of $\{S_{KL}\}$ and $\{B_K\}$, where

$$\bar{G} = G(\underset{\sim}{S}\underset{\sim}{C}'\underset{\sim}{S}^T, \ \underset{\sim}{S}\underset{\sim}{C}\underset{\sim}{S}^T, \ \underset{\sim}{S}\underset{\sim}{X}' + \underset{\sim}{B}, \ \underset{\sim}{S}\underset{\sim}{X} + \underset{\sim}{B}) \tag{4.6}$$

For a given symmetry group, these expressions restricts the functional forms of the constitutive equations.

In the case of homogeneous materials, G will not depend on B, so that

$$G = G(\underset{\sim}{C}', \ \underset{\sim}{C}, \ \underset{\sim}{X}'-\underset{\sim}{X}) \tag{4.7}$$

For isotropic materials, $\{\underset{\sim}{S}\}$ is the full group of orthogonal transformations.

5. Isotropic Solids

For isotropic solids, $\{\underset{\sim}{S}\}$ is the full group of orthogonal transformations and Eq. (4.5) states that G is a function of the invariants of $\underset{\sim}{C}$, $\underset{\sim}{C}'$ and $\underset{\sim}{R} = \underset{\sim}{X}' - \underset{\sim}{X}$, i.e.

$$G = G(I_\alpha, \overset{*}{I}_\alpha; I_\beta) \qquad \begin{matrix}\alpha=1,2,\ldots,6\\ \beta=7,8,\ldots,10\end{matrix} \tag{5.1}$$

and a function of θ, where

$$\begin{aligned} &I_1 = \operatorname{tr} \underset{\sim}{C}, && I_2 = \operatorname{tr} \underset{\sim}{C}^2, && I_3 = \operatorname{tr} \underset{\sim}{C}^3, \\ &I_4 = \operatorname{tr} \underset{\sim}{C}^2\underset{\sim}{C}', && I_5 = \underset{\sim}{R}\cdot\underset{\sim}{C}\underset{\sim}{R}, && I_6 = \underset{\sim}{R}\cdot\underset{\sim}{C}^2\underset{\sim}{R}, \\ &I_2 = \operatorname{tr} \underset{\sim}{C}\underset{\sim}{C}', && I_8 = \operatorname{tr}(\underset{\sim}{C}^2\underset{\sim}{C}'^2), && I_9 = \underset{\sim}{R}\,\underset{\sim}{C}\underset{\sim}{C}'\,\underset{\sim}{R} \\ &I_{10} = \underset{\sim}{R}\cdot\underset{\sim}{R} \end{aligned} \tag{5.2}$$

From (4.2), it follows that

$$G(I_\alpha, \overset{*}{I}_\alpha; I_\beta) = G(\overset{*}{I}_\alpha, I_\alpha; I_\beta) \equiv \overset{*}{G} \tag{5.3}$$

since $\overset{*}{I}_\beta = I_\beta$.

Substituting (5.1) into (4.3), we obtain

$$(5.4)\qquad T_{KL} = 2\int_V \Big[\frac{\partial G}{\partial I_1}\,\delta_{KL} + 2\,\frac{\partial G}{\partial I_2}\,C_{KL} + 3\,\frac{\partial G}{\partial I_3}\,C_{KM}C_{ML}$$

$$+ \frac{\partial G}{\partial I_4}\,(C_{KM}C'_{ML} + C'_{KM}C_{ML}) + \frac{\partial G}{\partial I_5}\,R_K R_L$$

$$+ \frac{\partial G}{\partial I_6}\,(R_K C_{LM} R_M + R_M C_{MK} R_L) + \frac{\partial G}{\partial I_7}\,C'_{KL}$$

$$+ \frac{\partial G}{\partial I_8}\,(C_{KM}C'_{MN}C'_{NL} + C_{LM}C'_{MN}C'_{NK})$$

$$+ \frac{\partial G}{\partial I_9}\,(R_K C'_{LM} R_M + R_L C'_{KM} R_M)\Big]\,dV'$$

The spatial expression of the stress tensor follows from (3.19).

The exact constitutive equations (5.4) may be useful in dealing with finite deformation problems. In general, they are too complicated for practical applications. However, for certain simple geometries and loading and in those problems where the nonlinear region is localized (e.g., the crack tip), the solution may be tractable. Of course, there exists some important problems where the nonlinearity is the rule rather than the exception, e.g. phase transition, rubber elasticity.

6. Linear Theory

For linear theory, it is useful to employ the linear strain measure

$$E_{KL} = \frac{1}{2}(C_{KL} - \delta_{KL}) \simeq \frac{1}{2}(U_{K,L} + U_{L,K}) \tag{6.1}$$

where U_k is the displacement vector in B. A second degree polynomial for G satisfying Eq. (4.2) has the form

$$\begin{aligned} G = {} & \Sigma^1 + \Sigma^1_{KL}E_{KL} + \overset{*}{\Sigma}{}^1_{KL}E'_{KL} + \frac{1}{2}\Sigma^1_{KLMN}E_{KL}E'_{MN} \\ & + \frac{1}{2}\overset{*}{\Sigma}{}^1_{KLMN}E'_{KL}E_{MN} + \frac{1}{2}\Sigma^2_{KLMN}E_{KL}E_{MN} + \frac{1}{2}\overset{*}{\Sigma}{}^2_{KLMN}E'_{KL}E'_{MN} \end{aligned} \tag{6.2}$$

where constitutive coefficients $\Sigma^1, \ldots, \overset{*}{\Sigma}{}^2_{KLMN}$ are functions of $\underset{\sim}{X}' - \underset{\sim}{X}$ and θ and they have the symmetry regulations

$$\begin{aligned} & \Sigma^1_{KL} = \Sigma^1_{LK}, \qquad \Sigma^1_{KLMN} = \Sigma^1_{LKMN} = \Sigma^1_{KLNM}, \\ & \Sigma^2_{KLMN} = \Sigma^2_{LKMN} = \Sigma^2_{KLNM} = \Sigma^2_{MNKL} \end{aligned} \tag{6.3}$$

Substituting (6.2) into (4.3), we obtain

$$T_{KL} = \Sigma^0_{KL} + \Sigma^0_{KLMN}E_{MN} + \int_V \Sigma'_{KLMN}E'_{MN}\, dV' \tag{6.4}$$

where

(6.5)
$$\Sigma^0_{KL} = \int_V \Sigma^1_{KL}\, dV' ,$$
$$\Sigma^0_{KLMN} = \int_V \Sigma^2_{KLMN}\, dV' ,$$
$$\Sigma'_{KLMN} = \frac{1}{2}\,(\Sigma^1_{KLMN} + \overset{*}{\Sigma}{}^1_{MNKL})$$

from which and (6.3), we deduce the symmetry regulations

$$\Sigma^0_{KL} = \Sigma^0_{LK} , \quad \Sigma^0_{KLMN} = \Sigma^0_{LKMN} = \Sigma^0_{KLNM} = \Sigma^0_{MNKL} ,$$

(6.6)
$$\Sigma'_{KLMN} = \Sigma'_{LKMN} = \Sigma'_{KLNM} = \overset{*}{\Sigma}{}'_{MNKL}$$

Constitutive moduli Σ^0_{KL}, Σ^0_{KLMN} are functions of $\underset{\sim}{X}$ and θ, but Σ'_{KLMN} are functions of $\underset{\sim}{X}'-\underset{\sim}{X}$ and θ.

From Eq. (6.4), it is clear that the nonlocal effects are represented by the volume integral, i.e. when $\Sigma'_{KLMN} = 0$ we obtain the classical Hookes law of elasticity.

One may be tempted to incorporate Σ^0_{KLMN} into the nonlocal moduli Σ'_{KLMN} by writing

(6.7)
$$\Sigma_{KLMN} = \Sigma^0_{KLMN}\, \delta(\underset{\sim}{X}'-\underset{\sim}{X}) + \Sigma'_{KLMN}$$

where $\delta(\underset{\sim}{X}'-X)$ is the three-dimensional Dirac-delta measure. With this, Eq. (6.4) becomes

(6.8)
$$T_{KL} = \Sigma^0_{KL} + \int_V \Sigma_{KLMN}(\underset{\sim}{X}'-\underset{\sim}{X},\theta)\, E'_{MN}(\underset{\sim}{X}')\, dV(\underset{\sim}{X}')$$

In this form, the continuity requirement imposed on the stress functional is violated. As we shall see, this form is useful for mathematical treatment, but in some instances (for compact operators), it leads to unbounded inverses which may or may not be acceptable on physical grounds.

In Eq. (6.8), Σ^0_{KL} represents the initial stress present in the reference state, in the absence of which we set $\Sigma^0_{KL} = 0$.

The material symmetry group $\{S_{KL}\}$ place restrictions on Σ_{KLMN}. These are of the form

$$\Sigma_{KLMN}(\underset{\sim}{\underset{\sim}{S}}\underset{\sim}{R},\theta) = S_{KP}S_{LQ}S_{MR}S_{NT}\,\Sigma_{PQRT}(\underset{\sim}{R},\theta) \tag{6.9}$$

where $\underset{\sim}{R} = \underset{\sim}{X}' - \underset{\sim}{X}$.

The spatial form of the stress tensor is obtained by carrying (6.8) into (3.19) and using

$$\begin{aligned} &E_{KL} = e_{k\ell}\,x_{k,K}x_{\ell,L}, \qquad R_{KL} = r_{k\ell}\,x_{k,K}x_{\ell,L} \\ &x_{k,K} \simeq (\delta_{MK} + E_{MK} + R_{MK})\delta_{Mk}, \qquad \rho/\rho_0 \simeq 1 - e_{rr} \end{aligned} \tag{6.10}$$

where R_{KL} and $r_{k\ell}$ are the material and spatial rotation tensors, respectively. δ_{Mk} is the Kronecker delta when the spatial and material coordinates are coincident. In terms material and spatial displacement vectors U_k and u_k, we have

$$\begin{aligned} R_{KL} &= \frac{1}{2}(U_{K,L} - U_{L,K}), \\ r_{k\ell} &= \frac{1}{2}(u_{k,\ell} - u_{\ell,k}), \qquad e_{k\ell} = \frac{1}{2}(u_{k,\ell} + u_{\ell,k}) \end{aligned} \tag{6.11}$$

With these, Eq. (3.19) gives

$$t_{k\ell} = (1 - e_{rr})\,\delta_{ok\ell} + \sigma_{om\ell}(e_{km} + r_{km}) + \sigma_{okm}\,(e_{\ell m} + r_{\ell m}) + \int_V \sigma_{k\ell mn}(\underset{\sim}{x}'-\underset{\sim}{x},\theta)\; e_{mn}(x')\; dv(\underset{\sim}{x}') \tag{6.12}$$

where

$$\sigma_{ok\ell} = \Sigma_{OKL}\delta_{Kk}\delta_{L\ell}\,, \qquad \sigma_{k\ell mn} = \Sigma_{KLMN}\delta_{Kk}\delta_{L\ell}\delta_{Mm}\delta_{Nn} \tag{6.13}$$

For homogeneous and isotropic materials, we can show that, [7], [13],

$$\begin{aligned}\sigma_{ok\ell} &= \sigma_o\delta_{k\ell}\,,\\ \sigma_{k\ell mn} &= \lambda'\,\delta_{k\ell}\delta_{mn} + \mu'(\delta_{km}\delta_{\ell n} + \delta_{kn}\delta_{\ell m})\\ &\quad + \lambda_1\, r^{-2}(r_m r_n\delta_{k\ell} + r_k r_\ell\delta_{mn})\\ &\quad + \lambda_2\, r^{-2}(r_k r_m\delta_{\ell n} + r_k r_n\delta_{\ell m} + r_\ell r_m\delta_{kn} + r_\ell r_n\delta_{km})\\ &\quad + \lambda_3\, r^{-4}\, r_k r_\ell\, r_m r_n\end{aligned} \tag{6.14}$$

where

$$r^2 = r_k r_k \qquad r_k = x_k' - x_k \tag{6.15}$$

and σ_0 is a function of $\underset{\sim}{x}$ and θ but λ', μ' and λ_α are functions of $r=|\underset{\sim}{x}'-\underset{\sim}{x}|$ and θ, i.e.

$$\sigma_0 = \sigma_0(\underset{\sim}{x},\theta)\,, \quad \lambda' = \lambda'(|\underset{\sim}{x}'-\underset{\sim}{x}|,\theta)\,, \quad \mu' = \mu'(|\underset{\sim}{x}'-\underset{\sim}{x}|,\theta)\,, \quad \lambda_\alpha = \lambda_\alpha\,|\underset{\sim}{x}'-\underset{\sim}{x}|,\theta)\,, \quad \alpha = 1,2,3 \tag{6.16}$$

The appearance of λ_α in (6.14) indicates that the nonlocal effects can cause directional dependence, even in isotropic solids. These terms are expected to be small as compared to λ' and μ' because of the strong attenuation of the intermolecular forces with the distance. Henceforth, we shall drop these terms and also assume that the reference state is stress free, so that

$$(6.17) \qquad t_{k\ell} = \int_V \sigma_{k\ell mn}(\underset{\sim}{x}'-\underset{\sim}{x},\theta)\, e_{mn}(\underset{\sim}{x}')\, dv(\underset{\sim}{x}')$$

valid for linear anisotropic solids, but for isotropic solids

$$(6.18) \qquad \sigma_{k\ell mn} = \lambda'(|\underset{\sim}{x}'-\underset{\sim}{x}|,\theta)\,\delta_{k\ell}\delta_{mn} + \mu'(|\underset{\sim}{x}'-\underset{\sim}{x}|,\theta)(\delta_{km}\delta_{\ell n} + \delta_{kn}\delta_{\ell m})$$

7. Nonlocal Elastic Moduli

The nonlocal elastic moduli $\sigma_{k\ell mn}(\underset{\sim}{x}'-\underset{\sim}{x},\theta)$ are required to possess certain physically acceptable properties:

(i) From (6.17), it is clear that

$$\text{(7.1)} \qquad \text{Dimension } (\sigma_{k\ell mn}) = \text{stress}/(\text{length})^3$$

This indicates that elastic properties of materials depend on some internal characteristic lengths (in general, three), a property which is lacking from the classical (local) theory. If $\varepsilon(\theta)$ denotes this internal characteristic length for isotropic solids, then we can put

$$\text{(7.2)} \qquad \lambda' = \lambda(\theta)\,\alpha_1(|\underset{\sim}{x}'-\underset{\sim}{x}|/\varepsilon)\,, \qquad \mu' = \mu(\theta)\,\alpha_2(|\underset{\sim}{x}'-\underset{\sim}{x}|/\varepsilon)$$

where λ and μ are the classical Lamé constants, and α_1 and α_2 are the attenuation functions.

The internal characteristic length ε can be taken to be proportional to the lattice parameter, average granular distance, or some other internal length, depending on the inner structure of the material and the accuracy desired. We may put

$$\text{(7.3)} \qquad \varepsilon = e_0(\theta)a$$

where a is the internal characteristic length and $e_0(\theta)$ is a non-dimensional material function.

In the limit as $\varepsilon \to 0$, nonlocal theory should convert to classical (local) theory. Hence

$$\lim_{\varepsilon \to 0} (\alpha_1, \alpha_2) = \delta(\underset{\sim}{x}' - \underset{\sim}{x}) \tag{7.4}$$

It is also clear that α_1 and α_2 should assume their maxima at $\underset{\sim}{x}' = \underset{\sim}{x}$. Similar limits are valid for the anisotropic moduli $\sigma_{k\ell mn}(\underset{\sim}{x}' - \underset{\sim}{x}, \theta)$.

(ii) The nonlocal moduli must be continuous and bounded. This is the case, at least for $\varepsilon \neq 0$, because of the function space to which they belong. One may also be tempted to consider <u>compact</u> operators since they transform bounded sequences to sequences that have convergent subsequences. This however leads to unbounded inverse in an infinite dimensional space. In some physical situation, this could be acceptable (e.g. crack tip, where the strain may be infinite). Contrary to $\sigma_{k\ell mn}$, in the original Eq. (6.4) Σ'_{KLMN}, and the corresponding spatial moduli may define a compact operator.

(iii) The group symmetry of $\sigma_{k\ell mn}(\underset{\sim}{x}' - \underset{\sim}{x}, \theta)$ is identical to (6.9) as is clear from (6.13). For isotropic solids, it is expected that the attenuation of intermolecular attractions are the same in all directions. Consequently, we may also take

$$\alpha_1 = \alpha_2 = \alpha(|\underset{\sim}{x}' - \underset{\sim}{x}|/\varepsilon), \tag{7.5}$$

subject to

$$\alpha(0) = \max \alpha, \qquad \lim_{\varepsilon \to 0} \alpha = \delta(\underset{\sim}{x}' - \underset{\sim}{x}) \tag{7.6}$$

(iv) All materials, at some internal length scale, must be considered discrete. This characteristic of solids can be introduced by means of a cut-off length either through the so-called quasi-continuum approach [1],[4], or equivalently using a finite support for the Fourier transform of the nonlocal moduli $\sigma_{k\ell mn}$ and α, [4],[13].

The idea is the same one that is used in lattice dynamics and information theory: Let $f(\underset{\sim}{n})$ be a scalar or tensor valued function of a vector $\underset{\sim}{n}$ which takes discrete values at lattice points through all displacements of the origin by vectors $\underset{\sim}{n}=n^k\underset{\sim}{e}_k$ (n^k and k are integers $\underset{\sim}{e}_k:(\underset{\sim}{e}_1,\underset{\sim}{e}_2,\underset{\sim}{e}_3)$). Let $\underset{\sim}{e}^k$ be the reciprocal triad to $\underset{\sim}{e}_k$, i.e.

$$\underset{\sim}{e}^k\cdot\underset{\sim}{e}_\ell = \delta^k_\ell \tag{7.7}$$

Define a sampling function

$$\delta_B(\underset{\sim}{x}) = (2\pi)^{-3}\int_B \exp(i\underset{\sim}{k}\cdot\underset{\sim}{x})\, dv(\underset{\sim}{k}) \tag{7.8}$$

where the domain of integration B is a parallelepiped $B\{-\pi\leq k_j\leq\pi\}$. Suppose that the Fourier transform $\bar{f}(\underset{\sim}{k})$ of a continuous function $f(\underset{\sim}{x})$ can be represented by a Fourier series

$$\bar{f}(\underset{\sim}{k}) = v_0\sum_{\underset{\sim}{n}} f(\underset{\sim}{n})\exp(i\underset{\sim}{n}\cdot\underset{\sim}{k}), \qquad \underset{\sim}{k}\in B \tag{7.9}$$

where v_0 is the volume of the cell B, and $f(\underset{\sim}{k})=0$ outside of B.

From the inversion theorem, it follows that

$$f(\underset{\sim}{x}) = v_0 \sum_{n} f(\underset{\sim}{n})\ \delta_B(\underset{\sim}{x}-\underset{\sim}{n}) \tag{7.10}$$

Since, according to (7.8), $\delta_B(0) = v_0$ and $\delta_B(\underset{\sim}{n}) = 0$ for all other points, the continuous function $f(\underset{\sim}{x})$ takes the discrete values $f(\underset{\sim}{n})$ at lattice points. This representation can be shown to be unique. Of course, between the lattice points $f(\underset{\sim}{x})$ can be quite arbitrary. Thus, by means of the sampling function δ_B, we can replace a function with discrete values by a continuous function. δ_B plays the role of the Dirac-delta distribution. The following nonlocal elastic moduli

$$\sigma_{k\ell mn}(\underset{\sim}{x}'-\underset{\sim}{x},\theta) = \sigma^0_{k\ell mn}(\theta)\ \delta_B(\underset{\sim}{x}'-\underset{\sim}{x}) \tag{7.11}$$

satisfies all the invariance requirements. in addition to being a continuous function.

Sampling functions for several crystals are known. For example, for a cubic crystal, it has the form

$$\delta_B(\underset{\sim}{x}'-\underset{\sim}{x}) = \pi^{-3} \prod_{j=1}^{3} (\underset{\sim}{x}_j'-\underset{\sim}{x}_j)^{-1} \sin[\pi(\underset{\sim}{x}_j'-\underset{\sim}{x}_j)/a] \tag{7.12}$$

where a is the lattice constant. For hexagonal, body and face-centered cubic lattices, see Kotowski [20]. For the so-called Debye continuum where the Brillouin zone is considered to be spherical, δ_B was calculated by Kunin and coworkers [21],[22].

$$\delta_\kappa(\underset{\sim}{r}) = (\kappa/2\pi^2 r^2)\left(\frac{\sin \kappa r}{\kappa r} - \cos \kappa r\right), \tag{7.13}$$

$$r \equiv |\underset{\sim}{x}'-\underset{\sim}{x}|, \qquad \kappa = \pi/a$$

where a is the lattice parameter. For the two-dimensional case, δ_B has the form

$$(7.14) \qquad \delta_\kappa(\underset{\sim}{x}'-\underset{\sim}{x}) = (\kappa/2\pi r)\, J_1(\kappa r), \qquad \kappa = \pi/a$$

where J_1 is the Bessel's function.

We note that δ_B and δ_κ given above, have no singularity for finite a, but they go into Dirac-delta distribution when $a \to 0$, as they should.

The Fourier transform of $\bar{\delta}_B = 1$ in B, and vanishes outside. Consequently, we have

$$(7.15) \qquad \begin{aligned} \bar{t}_{k\ell}(\underset{\sim}{k}) &= \bar{\sigma}_{k\ell}(\underset{\sim}{k}) && \text{when} && k_j \in B \\ &= 0 && \text{when} && k_j \notin B \end{aligned}$$

where

$$(7.16) \qquad \bar{\sigma}_{k\ell} = \bar{\sigma}_{k\ell mn}(\underset{\sim}{k})\ \bar{e}_{mn}(\underset{\sim}{k})$$

From this, by inversion, we have

$$(7.17) \qquad \bar{e}_{k\ell}(\underset{\sim}{k}) = \bar{s}_{k\ell mn}(\underset{\sim}{k})\ \bar{t}_{mn}(\underset{\sim}{k})$$

where $\bar{s}_{k\ell mn}$ is the Fourier transform of the compliance subject to

$$(7.18) \qquad \bar{\sigma}_{k\ell mn}\ \bar{s}_{mnrs} = \delta_{kr}\delta_{\ell s}$$

i.e., it is the inverse matrix to $\bar{\sigma}_{k\ell mn}$. For isotropic materials, it is given by

$$\bar{s}_{k\ell mn} = \frac{1}{4\bar{\mu}} (\delta_{km}\delta_{\ell n} + \delta_{kn}\delta_{\ell m}) - \frac{\bar{\lambda}}{2\bar{\mu}(3\bar{\lambda}+2\bar{\mu})} \delta_{k\ell}\delta_{mn} \tag{7.19}$$

For the case (7.5), it reads

$$\bar{s}_{k\ell mn} = s^0_{k\ell mn}(\theta)\, \bar{\beta}(\underset{\sim}{k}) \tag{7.20}$$

where $s^0_{k\ell mn}$ has the form (7.19) with $\bar{\lambda}$ and $\bar{\mu}$ replaced by Lamé constants λ and μ, and

$$\bar{\beta}(\underset{\sim}{k}) = 1/\bar{\alpha}(\underset{\sim}{k}) \tag{7.21}$$

(v) In calculations, the moduli $\bar{\underset{\sim}{\sigma}}(\underset{\sim}{k})$ or $\bar{\underset{\sim}{s}}(\underset{\sim}{k})$ in their full generality, cause major mathematical complexities. Often approximate expressions replacing them by power series in $\underset{\sim}{k}$ are used. We give here two examples involving second degree expansions in $\underset{\sim}{k}$ for isotropic solids

$$\bar{\alpha}(\underset{\sim}{k}) = 1 - \varepsilon^2 k^2 \tag{7.22}$$

$$\bar{\beta}(\underset{\sim}{k}) = 1 + \varepsilon^2 k^2 \tag{7.23}$$

where $k^2 = \underset{\sim}{k}\cdot\underset{\sim}{k}$ and ε given by (7.3) is independent of k. We note that the linear term in $\underset{\sim}{k}$ is missing because of the isotropy and center of symmetry.

In physical space, these approximations are equivalent to constitutive equations of the forms

$$t_{k\ell} = (1 + \varepsilon^2\nabla^2)\,\sigma_{k\ell} \tag{7.24}$$

for (7.22) and

$$\sigma_{k\ell} = (1 - \varepsilon^2\nabla^2)\,t_{k\ell} \tag{7.25}$$

for (7.23), where $\sigma_{k\ell}$ is the Hookean (local) stress tensor

$$\sigma_{k\ell} = \lambda\, e_{rr}\delta_{k\ell} + 2\mu\, e_{k\ell} \tag{7.26}$$

Applying the operator $(1-\varepsilon^2\nabla^2)$ to the exact constitutive equation

$$t_{k\ell} = \int_V \alpha(\underset{\sim}{x}'-\underset{\sim}{x},\theta)\,\sigma_{k\ell}(\underset{\sim}{x}')\,dv(\underset{\sim}{x}') \tag{7.27}$$

we obtain (7.25) if

$$(1 - \varepsilon^2\nabla^2)\alpha = \delta(\underset{\sim}{x}'-\underset{\sim}{x}) \tag{7.28}$$

This indicates that in this approximation, α is a Green's function for this operator. This approach was used extensively by Eringen and his coworkers (cf. [23]-[25]).

Similary, the application of the operator $1 + \varepsilon^2\nabla^2$ to

$$\sigma_{k\ell} = \int_V \beta(\underset{\sim}{x}'-\underset{\sim}{x},\theta)\, t_{k\ell}(\underset{\sim}{x}')\, dv(\underset{\sim}{x}') \tag{7.29}$$

gives (7.24) if

$$(1 + \varepsilon^2\nabla^2)\beta = \delta(\underset{\sim}{x}'-\underset{\sim}{x}) \tag{7.30}$$

Even though $\bar{\alpha}$ and $\bar{\beta}$ are given approximately by (7.22) and (7.23). We may employ either (7.28) or (7.30) for the determination of the attenuation function. For example, when α is determined from solving (7.28), we can then use (7.21) to determine $\beta(\underset{\sim}{x})$ exactly instead of solving (7.30) which is an approximate equation satisfied by β . In this sense, (7.27), with α given by (7.28), represents an exact nonlocality. However, $\bar{\alpha}$ being limited to a special class of function represented by Equation (7.22), is a limited or "short" nonlocality. Nevertheless, with this compppromise, we have achieved a major gain in that the integro-partial differential equations for the displacement field are now reduced to <u>singular</u> partial differential equations through the use of (7.24) or (7.25).

Below, we give solutions of (7.28) for solids of infinite extents. For these and other types of moduli, see [5], [17] and [25].

(i) <u>One Dimension</u>:

$$\alpha(|x'-x|) = (2\varepsilon)^{-1} \exp(-|x'-x|/\varepsilon) \tag{7.31}$$

ii) Two Dimensions

$$\alpha(|\underset{\sim}{x}'-\underset{\sim}{x}|) = (2\pi\,\varepsilon^2)^{-1}\, K_0(|\underset{\sim}{x}'-\underset{\sim}{x}|/\varepsilon) \tag{7.32}$$

iii) Three Dimensions

$$\alpha(|\underset{\sim}{x}'-\underset{\sim}{x}|) = (4\pi\,\varepsilon^2|\underset{\sim}{x}'-\underset{\sim}{x}|)^{-1} \exp(-\,|\underset{\sim}{x}'-\underset{\sim}{x}|/\varepsilon) \tag{7.33}$$

Excluding (7.31), these functions possess singularity at $\underset{\sim}{x}'=\underset{\sim}{x}$ and they go over the Dirac delta distribution when $\varepsilon \to 0$. From (7.31) we have

$$\bar{\alpha} = (1 + \varepsilon^2 k^2)^{-1} \exp(ikx') \tag{7.34}$$

and since $\bar{\beta} = 1/\bar{\alpha}$, we have formally,

$$\beta = \delta(\underset{\sim}{x}'-\underset{\sim}{x}) - \varepsilon^2\, \delta''(\underset{\sim}{x}'-\underset{\sim}{x}) \tag{7.35}$$

which upon substitution into (7.29), gives (7.25). Thus verifying our expectation.

Singular kernels such as (7.32) and (7.33) may lead singular $\sigma_{k\ell}$ (hence singular strain) fields. For $t_{k\ell}$ may be a continuous function but $\nabla^2 t_{k\ell}$ may be singular at some point. The question arises whether this is allowable in a continuum theory. Recall that in the atomic scale there exists repulsive forces between ions when they are close

enough together. These forces become infinite when $\underset{\sim}{x}' = \underset{\sim}{x}$. Even in macroscopic levels at a sharp crack tip, classical elasticity leads to infinite strains. On these physical grounds, it seems that we are justified to employ such singular kernels. While these kernels do not possess an oscillatory character observed in crystal physics, it is possible to achieve such oscillations by adding a fourth degree term to the expansions (7.22) or (7.23), e.g.

$$\bar{\alpha} = 1 - \varepsilon^2 k^2 + b\, k^4 \tag{7.36}$$

where b is an appropriate constant.

In Section 9, we shall see that these kernels for the plane harmonic waves, lead to dispersion relations, which are nearly coincident to those obtained in lattice dynamics, in the entire Brillouin zone. Also, cohesive bond stresses in perfect crystals calculated by means of the nonlocal theory are in good agreement with those known in solid state physics (Section 10 and 11).

8. Field Equations

Upon substituting (7.27) into (2.5) we obtain

$$\int_V \alpha_{,k}\, \sigma_{k\ell}(\underset{\sim}{x}')\, dv(\underset{\sim}{x}') + \rho(f_\ell - \ddot{u}_\ell) = 0 , \tag{8.1}$$

where

$$\sigma_{k\ell} = \sigma^0_{k\ell mn} e_{mn} = \sigma^0_{k\ell mn}\, u_{m,n} \tag{8.2}$$

which assumes a single attenuation function for anisotropic solids. We put

$$\frac{\partial \alpha}{\partial x_k}\, \sigma_{k\ell}(\underset{\sim}{x}') = -\frac{\partial \alpha}{\partial x'_k}\, \sigma_{k\ell} = -\frac{\partial(\alpha\sigma_{k\ell})}{\partial x'_k} + \frac{\partial \sigma_{k\ell}}{\partial x'_k}$$

in (8.1) and convert the first term to a surface integral by means of the Green-Gauss theorem.

$$\int_V \alpha(\underset{\sim}{x}'-\underset{\sim}{x})\, \sigma_{k\ell,k}\,(\underset{\sim}{x}')\, dv(\underset{\sim}{x}') - \int_{\partial V} \alpha\, \sigma_{k\ell}\, da'_k + \rho(f_\ell - \ddot{u}_\ell) = 0 \tag{8.3}$$

Using (8.2) this gives the field equation of the nonlocal elasticity for the displacement field

$$\int_V \alpha(\underset{\sim}{x}'-\underset{\sim}{x})\, \sigma^0_{k\ell mn}\, u_{m,nk}(\underset{\sim}{x}')\, dv(\underset{\sim}{x}') - \int_{\partial V} \alpha(\underset{\sim}{x}'-\underset{\sim}{x})\, \sigma^0_{k\ell mn}\, u_{m,n}(\underset{\sim}{x}')\, da_k(\underset{\sim}{x}') + \rho(f_\ell - \ddot{u}_\ell) = 0 \tag{8.4}$$

Here, the surface integral represents the contributions of the surface stresses (e.g. surface tension). Consequently, *nonlocal theory accounts for the surface physics as well.*

For the isotropic solids we replace $\alpha\sigma^0_{k\ell mn} = \sigma_{k\ell mn}$ by (6.17) leading to

$$\int_V [(\lambda' + 2\mu')\, \nabla\nabla\cdot\underset{\sim}{u}' - \mu'\nabla\times\nabla\times\underset{\sim}{u}']dv' + \rho(\underset{\sim}{f} - \ddot{\underset{\sim}{u}}) = \underset{\sim}{0} \tag{8.6}$$

without the surface terms. If we assume (7.5), this gives

$$(8.7)\qquad \int_V \alpha(|\underset{\sim}{x}'-\underset{\sim}{x}|)\ [(\lambda+2\mu)\nabla\nabla\cdot\underset{\sim}{u}' - \mu\nabla\times\nabla\times\underset{\sim}{u}']dv' + \rho(\underset{\sim}{f} - \ddot{\underset{\sim}{u}}) = \underset{\sim}{0}$$

where λ and μ are the classical Lamé constraints.

For α we may choose various proposals made in section 7. If we employ the approximation leading to (7.25) we will have

$$(8.8)\qquad \sigma_{k\ell,k} + (1 - \varepsilon^2\nabla^2)(\rho f_\ell - \rho\ddot{u}_\ell) = 0$$

or in terms of the displacement field

$$(8.9)\qquad (\lambda + 2\mu)\ \nabla\nabla\cdot\underset{\sim}{u} - \mu\nabla\times\nabla\times\underset{\sim}{u} + (1 - \varepsilon^2\nabla^2)(\rho\underset{\sim}{f} - \rho\ddot{\underset{\sim}{u}}) = \underset{\sim}{0}$$

Here we see the advantage gained with the particular choice of α satisfying (7.28).

The field equations obtained above must be solved to determine the displacement field $\underset{\sim}{u}(\underset{\sim}{x},t)$ under appropriate boundary and initial conditions. Boundary and initial conditions involving the displacement and velocity fields are identical to those of the classical theory. Boundary condition on tractions is based on the stress field $t_{k\ell}$, <u>not</u> $\sigma_{k\ell}$, i.e.

$$(8.10)\qquad t_{k\ell}\, n_k = t_{(\underset{\sim}{n})\ell}$$

where $t_{(\underset{\sim}{n})\ell}$ are the prescribed boundary fractions.

9. Propagation Waves

1. Infinite Media

Using classical decomposition

$$\underset{\sim}{u} = \nabla\phi + \nabla \times \underset{\sim}{\psi} \qquad \nabla \cdot \underset{\sim}{\psi} = 0 \tag{9.1}$$

Eq. (8.9) with $\underset{\sim}{f} = \underset{\sim}{0}$ is satisfied if

$$c_1^2 \nabla^2\phi - (1 - \varepsilon^2\nabla^2)\ddot{\phi} = 0 , \tag{9.2}$$

$$c_2^2 \nabla^2\underset{\sim}{\psi} - (1 - \varepsilon^2\nabla^2)\ddot{\underset{\sim}{\psi}} = \underset{\sim}{0} , \tag{9.3}$$

where

$$c_1 = [(\lambda + 2\mu)/\rho]^{1/2} , \qquad c_2 = (\mu/\rho)^{1/2} \tag{9.4}$$

are the classical phase velocities of irrotational and equivoluminal waves. For plane harmonic waves, Eq. (9.2) leads to the dispersion relations

$$\omega/c_1 k = (1 + \varepsilon^2 k^2)^{-1/2} \tag{9.5}$$

For $\varepsilon = 0.39\, a$, where a is the lattice parameter, the plot of $\omega a/c$ versus ka is shown in Fig. 1, where the dispersion curve for the Born-Kármán model of lattice dynamics is also plotted for which

$$\omega a/c_1 = 2 \sin(ka/2) \tag{9.6}$$

The maximum deviation of this curve from the nonlocal result is less than 6%in the entire Brillouin zone. It should be remarked that Eq. (9.6) obeys the two natural conditions

$$(9.7) \qquad d\omega/dk\Big|_{k=0} = c_1 , \qquad d\omega/dk\Big|_{k=\pi/a} = 0$$

but the nonlocal result (9.5) obeys only the first one of these. Accordingly, the group velocity does not vanish at the end of the Brillouin zone. However, this situation can be remedied easily by taking

$$(9.8) \qquad \bar{\alpha}(k) = 1 - \varepsilon^2 k^2 + b\, k^4$$

instead of (7.22) where b is a constant (cf. Kunin [8,II], p. 38).

Similarly, the dispersion relations of the equivoluminal waves are obtained by means of (9.3).

(ii) Surface Waves

In the plane $x_3 = 0$, $\underset{\sim}{\psi} = \psi \underset{\sim}{i}_3$ has a single component $\psi(x_1, x_2, t)$ so that

$$(9.9) \qquad u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_2} , \qquad u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_1}$$

The general solution of (9.2) and (9.3) relevant to surface waves are of the form

$$(9.10) \qquad \begin{aligned} \phi &= A \exp[-k\nu_1 x_2 + ik(x_1 - ct)] , \\ \psi &= B \exp[-k\nu_2 x_2 + ik(x_2 - ct)] \end{aligned}$$

provided ν_α is given by

$$(9.11) \qquad \nu_\alpha^2 = 1 - (c/c_\alpha)^2 \left[1 - \varepsilon^2 k^2 (c/c_\alpha)^2\right]^{-1}$$

Using (9.9) and (7.27), the stress field can be calculated. If we set $t_{22} = t_{21} = 0$ on the surface $x_2 = 0$, we obtain the dispersion relations [25]:

$$(9.12) \qquad \left[(c_1/c_2)^2 (\nu_1^2 - 1) + 2\right](1 + \nu_2^2) - 4\,\nu_1\nu_2 = 0$$

This can be arranged into a quartic equation

$$(9.13) \qquad \gamma(a_1\gamma^3 + a_2\gamma^2 + a_3\gamma + a_4) = 0$$

where

$$(9.4) \qquad \begin{aligned} \gamma &= (c/c_2)^2 , \\ a_1 &= \frac{1}{16} + \frac{1}{4}(m+1)(\varepsilon k)^2 + \frac{1}{4}(1+4m-3m^2)(\varepsilon k)^4 + m(1-m)(\varepsilon k)^6 , \\ a_2 &= -\frac{1}{2} + \frac{1}{2}(2m^2 - m - 3)(\varepsilon k)^2 - (1+m-2m^2)(\varepsilon k)^4 , \\ a_3 &= \frac{3}{2} - m + (2 - m - m^2)(\varepsilon k)^2 \\ a_4 &= 1 - m , \qquad m = \frac{1-2\nu}{2(1-\nu)} \end{aligned}$$

A root of this equation for $\varepsilon = 0.39\,a$ and $\nu = 0.3$ is plotted against ka in Fig. 2 where the result of lattice dynamic calculations of Wallis and Gazs [26] for KCℓ are also shown. If we note that no free constant is available for matching, the agreement is beyond our expectations.

The nonlocal theory also gives new types of waves which are not present in local theory. However, these waves disappear in the long wave length regions (cf. Kaliski and Rymarz [27]).

It is important to note that the nonlocal moduli $\alpha(|\underset{\sim}{x}'-\underset{\sim}{x}|)$ is appropriate only to homogeneous and isotropic solids. Half-space ceases to be homogeneous in the vicinity of the surface $x_2 = 0$, where in a boundary layer of a few atomic distances, the material is inhomogeneous and therefore a perturbation is necessary in $\alpha(|\underset{\sim}{x}'-\underset{\sim}{x}|)$. The present results are however valid so long as the wave length is much larger than this boundary layer thickness.

10. Screw Dislocation

Consider a screw dislocation located at $x_3 = 0$ of rectangular coordinates x_k (Fig. 3). The displacement field has only single component $u_3 = w(x_1,x_2)$ which satisfies the equation

$$\nabla^2 w = 0 \tag{10.1}$$

whose solution relevant to our problem may be written in plane polar coordinates (r,θ) as

$$w = \frac{b}{2\pi}\theta \tag{10.2}$$

where b is the Burger's vector. The non-zero components of $\sigma_{k\ell}$ are given by

$$\sigma_{31} = \mu_o \frac{\partial w}{\partial x_1} = -\frac{\mu b}{\varrho}\frac{}{2\pi r} \sin\theta ,$$

$$\sigma_{32} = \mu_o \frac{\partial w}{\partial x_2} = \frac{\mu b}{\varrho}\frac{}{2\pi r} \cos\theta \tag{10.3}$$

We now carry these into

$$(1 - \varepsilon^2\nabla^2)t_{k\ell} = \sigma_{k\ell} \tag{10.4}$$

and determine $t_{k\ell}$ by imposing the conditions that as $r\to\infty$ $t_{k\ell}=0$ and at $r=0$ (crack tip surface) $t_{zr}=0$. This leads to [25] (see also [28]).

$$(10.5) \qquad t_{z\theta} = \frac{\mu_0 b}{2\pi r}\left[1 - \frac{r}{\varepsilon} K_1(r/\varepsilon)\right] , \qquad t_{zr} = 0$$

where K_1 is the Bessel's function. The plot of

$$(10.6) \qquad T_{z\theta} = 2\pi \varepsilon t_{z\theta}/\mu_0 b = \rho^{-1}[1 - \rho K_1(\rho)] , \qquad \rho = r/\varepsilon$$

versus ρ is shown in Fig. 4, where is also plotted classical elasticity solution. We see that the stress at the center $\rho = 0$ is singular for the classical solution, but zero for the nonlocal solution. The maximum stress occurs at $\rho_c = 1.1$ and is given by

$$(10.7) \qquad t_{z\theta\,\max} = 0.3995\ \mu_0 b/2\pi e_0 a$$

If we equate this to the yield stress (theoretical) t_y and write $h = e_0 a/0.3995$, this agrees with the estimate of Frenkel, based on an atomic model. For $e_0 = 0.39$, $b/a = 1/\sqrt{2}$, we obtain $t_y/\mu = 0.12$, which compares with the known value 0.11 for Aℓ(fcc), W, α-Fe(bcc) and 0.12 for NaCℓ, MgO (cf. [29], p. 160).

The strain energy per unit length L in x_3-direction is given by

$$\rho_0 \psi/L = \Sigma/L = \frac{1}{2}\int_0^R r\, dr \int_0^{2\pi} t_{z\theta}\, e_{z\theta}\, d\theta$$

substituting from (10.5) and $e_{z\theta} = b/2\pi r$, we obtain

$$(10.8) \qquad \Sigma/L = (\mu_0 b^2/8\pi)[\ell n(R/2\varepsilon) + K_0(R/\varepsilon)]$$

where R is the outer radius of the solid. Again we see that for finite R the strain energy is finite. Eq. (10.8) shows how the energy grows with size.

The same problem can be treated by means of the nonlocal theory of continuous dislocations [30]. For the screw dislocations for a given Burger's vector b, we have

$$\frac{\partial e_{32}}{\partial x_1} - \frac{\partial e_{31}}{\partial x_2} = \frac{b}{2} \tag{10.9}$$

The equilibrium equations reduce to

$$\frac{\partial t_{31}}{\partial x_1} + \frac{\partial t_{32}}{\partial x_2} = 0 \tag{10.10}$$

For the stress tensor t_{31} and t_{32}, we have the constitutive equations

$$\{t_{31}, t_{32}\} = \int_V \mu(|\underset{\sim}{x}' - \underset{\sim}{x}|)\{e'_{31}, e'_{32}\}\, da' \tag{10.11}$$

where the anti-plane strains are given by

$$e_{31} = \frac{\partial w}{\partial x_1}, \qquad e_{32} = \frac{\partial w}{\partial x_2} \tag{10.12}$$

We introduce the stress potential ϕ by

$$t_{31} = -\frac{\partial \phi}{\partial x_2}, \qquad t_{32} = \frac{\partial \phi}{\partial x_1} \tag{10.13}$$

by means of which Eq. (10.10) is satisfied. The Fourier transforms of (10.9),

(10.11) to (10.13) may be combined to give

$$\bar{\phi} = \bar{\mu}\,\bar{b}/k^2 \tag{10.14}$$

where $k = (k_i k_i)^{1/2}$ is the magnitude of the wave vector.

Following Kunin [8,II], if we employ the Debye Model, then the inverse of (10.14) is found to be

$$\phi = \int_A G(|\underset{\sim}{x}'-\underset{\sim}{x}|)\; b(\underset{\sim}{x}')\; da(\underset{\sim}{x}') \tag{10.15}$$

where the integration is over the area in (x_1,x_2)-plane and

$$G(r) = \frac{\mu_o}{2\pi}\left[\int_0^{\rho} \frac{J_1(\tau)-1}{\tau}\,d\tau - \ell n(\kappa R)\right], \tag{10.16}$$

$$\rho = \kappa r, \qquad \kappa = \pi/a$$

For a single screw at the origin $r = 0$, this gives

$$t_{31} = \frac{\mu_o b\,\kappa^2 x_2}{2\pi}\,\frac{J_0(\rho)-1}{\rho^2}, \qquad t_{32} = -\frac{\mu_o b\,\kappa^2 x_1}{2\pi}\,\frac{J_0(\rho)-1}{\rho^2} \tag{10.17}$$

Alternatively,

$$\tau_\theta = 2\pi t_{z\theta}/\mu_o \kappa b = [1-J_0(\rho)]/\rho\,, \qquad t_{zr} = 0 \tag{10.18}$$

$t_{z\theta}$ vanishes at the origin $r = 0$. It has slightly oscillatory behavior for $r > 0$ (Fig. 5) and it acquires its first maximum at $\rho \simeq 2.76$ with an amplitude

$$t_{z\theta\,max} \simeq 0.423\;\mu_0 b\kappa/2\pi \tag{10.19}$$

This value is nearly identical to (10.7) where an entirely different (singular) kernel was used. Note however that the location of the maximum differ considerably.

11. Crack Tip Problem

An infinite plate with a line crack of length $2c$ subjected to loads at infinity is known as the Griffith crack. It is well-known that the classical elasticity solution of this problem gives stress singularity at the crack tip. Because of this, no fracture criterion based on maximum stress hypothesis could be employed. Various alternative criteria (Griffith energy criterion, J-integral, etc.) had to be established to circumvent this difficulty. With the development of nonlocal theory, this problem has been resolved by Eringen and his coworkers [31 - 34]. Here I discuss only the so-called Mode III problem, i.e. line crack subject to anti-plane shear load (Fig. 6).

The classical elasticity solution for a line crack in a plate under anti-plane loading at infinity is well known [35]

$$\sigma = \sigma_{23} - i\,\sigma_{13} = \sigma_0\,\bar{z}(\bar{z}^2 - c^2)^{-\frac{1}{2}} \tag{11.1}$$

where σ_0 is the applied shear, $2c$ is the crack length and $z = x_1 + i\,x_2$, $\bar{z} = x_1 - i\,x_2$.

Employing (11.1) in (7.25), we determine the nonlocal stress field

$$t_{23} - i\,t_{13} = (\pi\varepsilon/2r_1)^{\frac{1}{2}}\, e^{-r_1/\varepsilon}\,(C_1\, e^{i\theta_1/2} + C_2\, e^{-i\theta_1/2}) + \sigma \tag{11.2}$$

where (r_1, θ_1) are the polar coordinates with the origin at the right-hand crack tip(Fig. 6). The boundary condition at the crack tip is calculated by considering the tip as a small circular cylinder of radius r_1 so that

in the limit t_{rz} vanishes as $r_1 \to 0$, i.e.

$$\lim_{r_1 \to 0} t_{rz} = 0 \tag{11.3}$$

This gives $C_2 = 0$ and $C_1 = -(c/\pi\varepsilon)^{1/2}\ \sigma_0$ and we have [34]

$$t_{z\theta} - i\, t_{zr} = (t_{32} - i\, t_{31})\, e^{-i\theta_1} \tag{11.4}$$

$$= \sigma_0 (c/2r_1)^{1/2} \{(2r^2/cr_2)^{1/2} \exp[i\,(-\theta + \tfrac{\theta_2}{2})] - \exp(-r_1/\varepsilon)\} \exp(-i\theta_1/2)$$

(r_2, θ_2) are the plane polar coordinates attached to the left crack tip and (r, θ) have the origin $x_1 = x_2 = 0$.

Along the x_1-axis $(\theta = \theta_1 = \theta_2 = 0)$, $t_{z\theta}$ acquires its maximum near to the crack tip. From (11.4), we have

$$T_\theta(\rho) = \gamma^{1/2}\ t_{z\theta}/\sigma_0 = (\pi\varepsilon)^{1/2}\ t_{z\theta}/K_{III} \tag{11.5}$$

$$= (2\rho)^{-1/2}\ [(1 + \gamma\rho)(1 + \tfrac{\gamma\rho}{2})^{-1/2} - e^{-\rho}]$$

where

$$\rho = r_1/\varepsilon\ , \qquad \gamma = \varepsilon/c\ , \qquad K_{III} = (\pi c)^{1/2}\ \sigma_0 \tag{11.6}$$

It is clear that $t_{z\theta}$ vanishes at the crack tip $\rho = 0$ and acquires a maximum at $\rho = \rho_c = 1.2565$ since $\gamma << 1$ $(\gamma \leq 10^{-4})$.

$$t_{z\theta\,\max} = \sigma_0 (\varepsilon/c)^{-1/2}\ [(2\,\rho_c)^{1/2} + (2\rho_c)^{-1/2}]^{-1} \tag{11.7}$$

$T_\theta(\rho)$ given by (11.5) is plotted against ρ in Fig. 7. From this it is clear that

(a) The stress field vanishes at the crack tip instead of being singular as predicted by classical elasticity.

(b) Fracture begins when $t_{z\theta max} = t_y$, where t_y is the cohesive yield stress.

From (11.7), we have

$$K_c/t_y = 3.9278\,\varepsilon^{\frac{1}{2}} \tag{11.8}$$

where $K_c = (\pi c)^{\frac{1}{2}}\,\sigma_{oc}$ is the critical fracture toughness. For $e_0 = 0.39$, K_c/t_y is shown in Table 1 along with its values based on the classical fracture criterion, namely $K_c = (4\mu\,\gamma_s)^{\frac{1}{2}}$ where γ_s is the surface energy. This table also displays some experimental results. The present results are again in good agreement with experimental observations of Ohr and Chang [36], even though further considerations are necessary for the inhomogeneity of the material at the core region.

Material	*Classical*	*Present*	*Experiment*
Aℓ (fcc)	1.11	0.49	0.3
Cu (fcc)	3.86	0.47	0.66
Fe (bcc)	1.04	0.42	0.23

(c) The maximum stress acts at $\rho = \rho_c$ not at the tip of the crack $\rho = 0$. Although ρ_c is very small, this implies that fracture initiates ahead of the crack joining the tip. If the inhomogeneity and the presence of dislocations near the crack tip is taken into account, we expect ρ_c to become larger than atomic dimensions.

Interactions of dislocations with crack was treated by Eringen [34]. For the solution of the crack tip problems for modes I and II, see [32], [33] and for crack curving, see [24]. Point defects and elastic interactions were explored in the works of Gairola [37] and Kunin [8].

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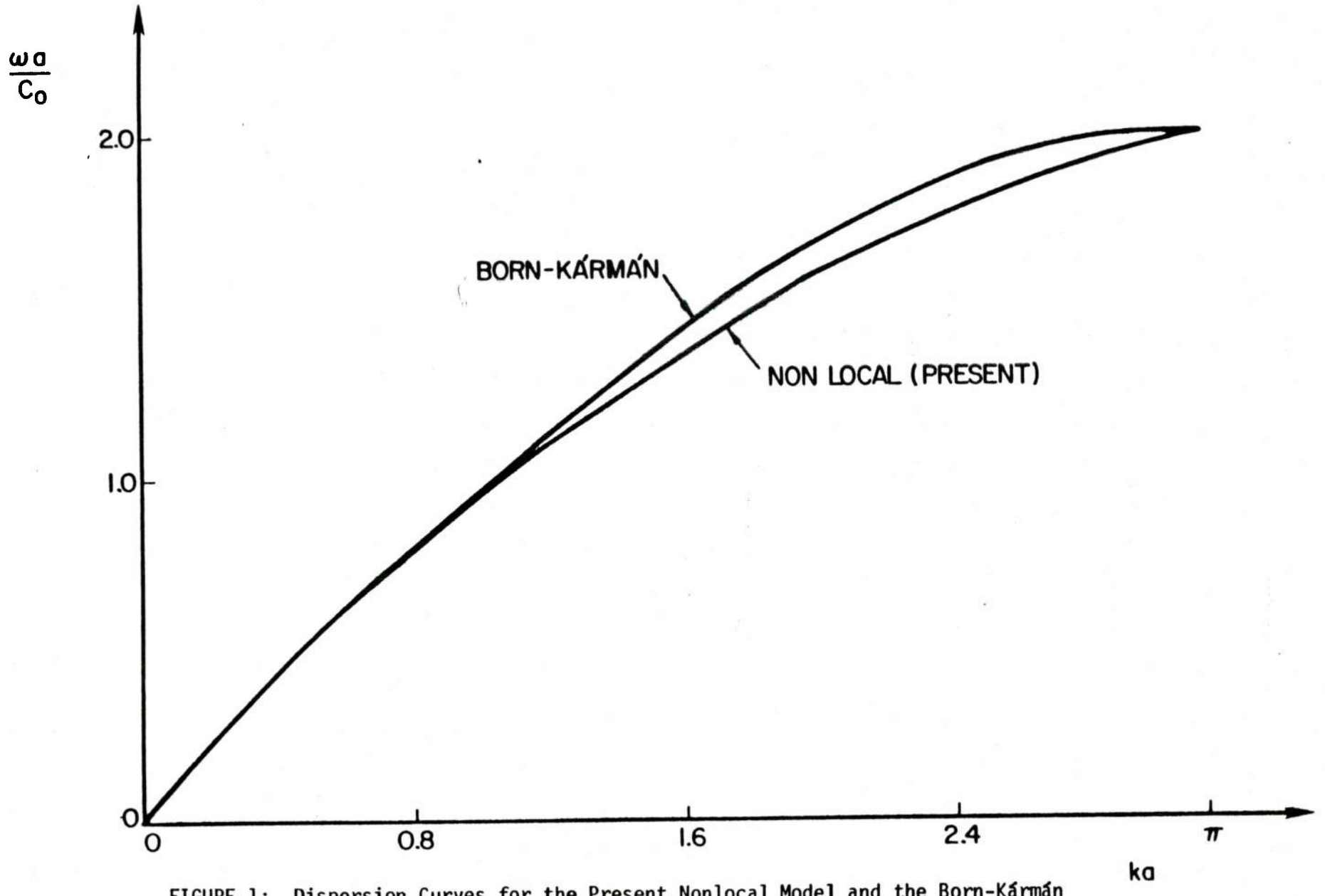


FIGURE 1: Dispersion Curves for the Present Nonlocal Model and the Born-Kármán Model of Lattice Dynamics

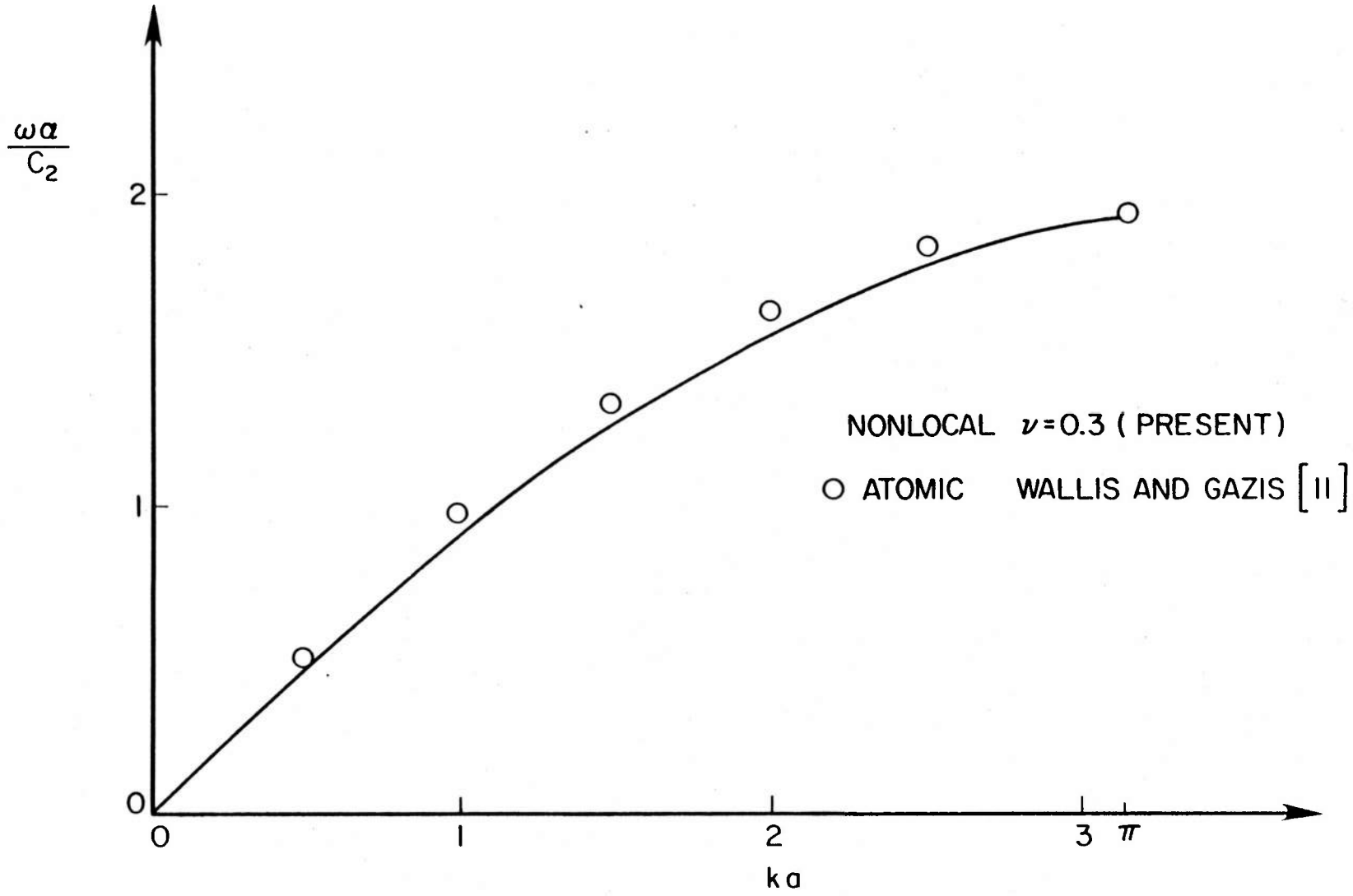


Figure 2: Dispersion Relations for Raleigh Surface Waves

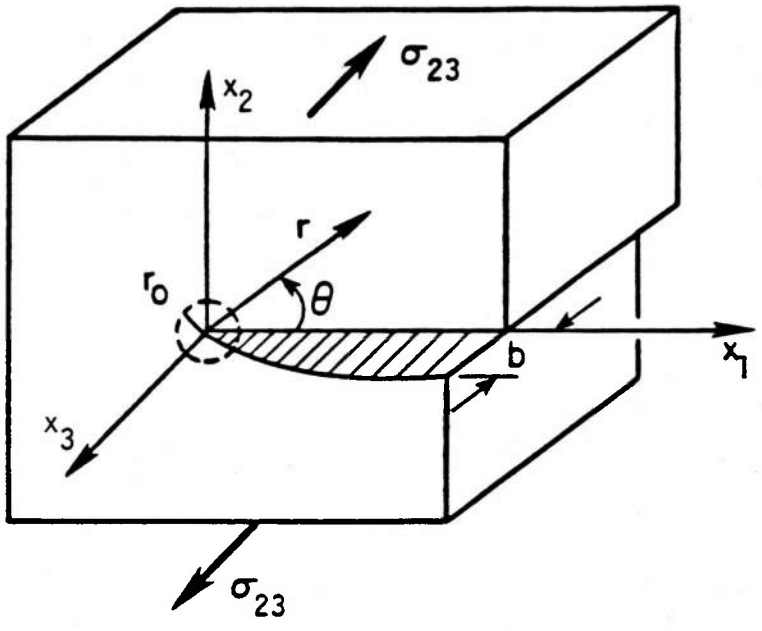
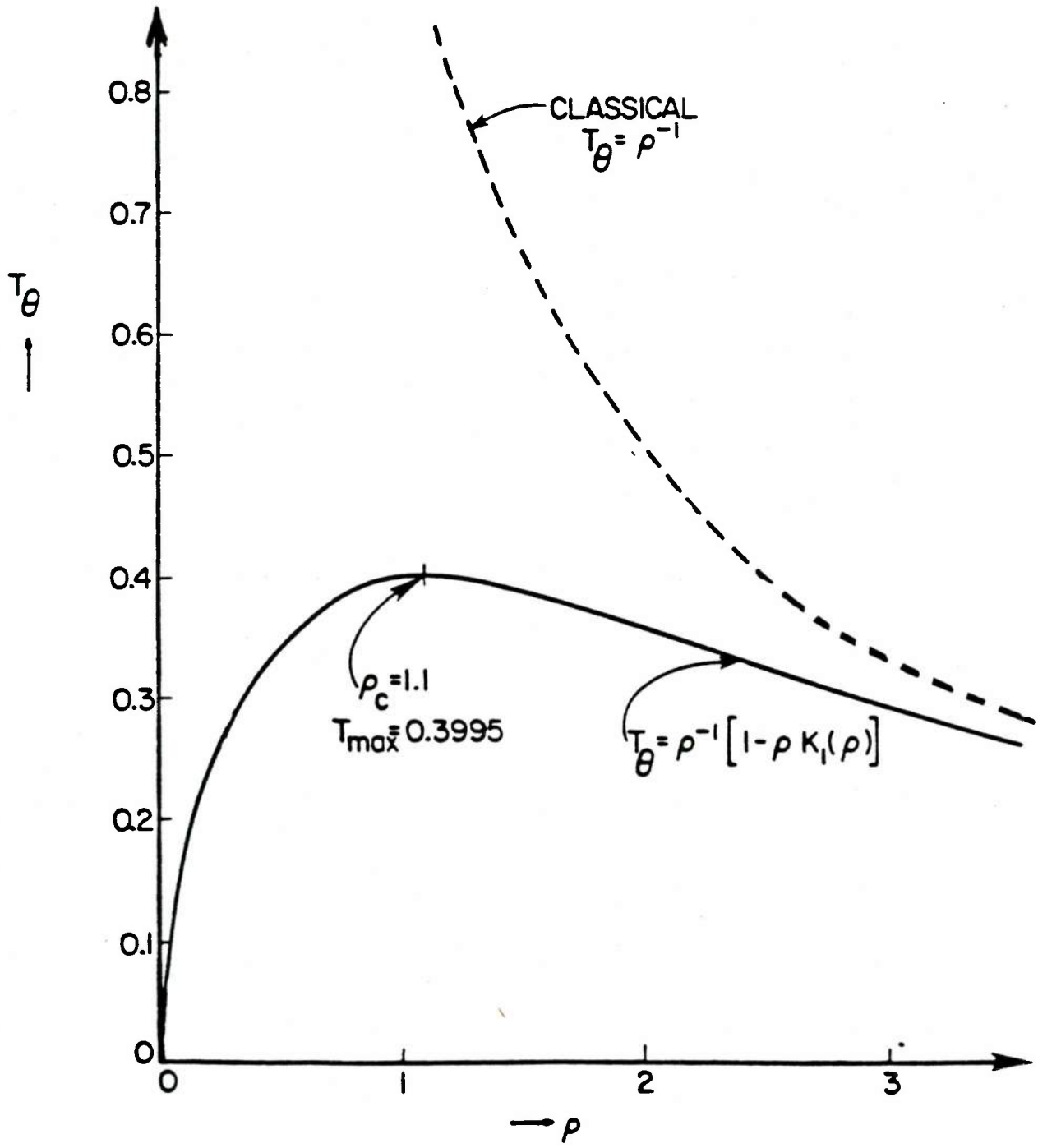


Figure 3: SCREW DISLOCATION



NON-DIMESIONAL HOOP STRESS (Screw Dislocation)

FIGURE 4

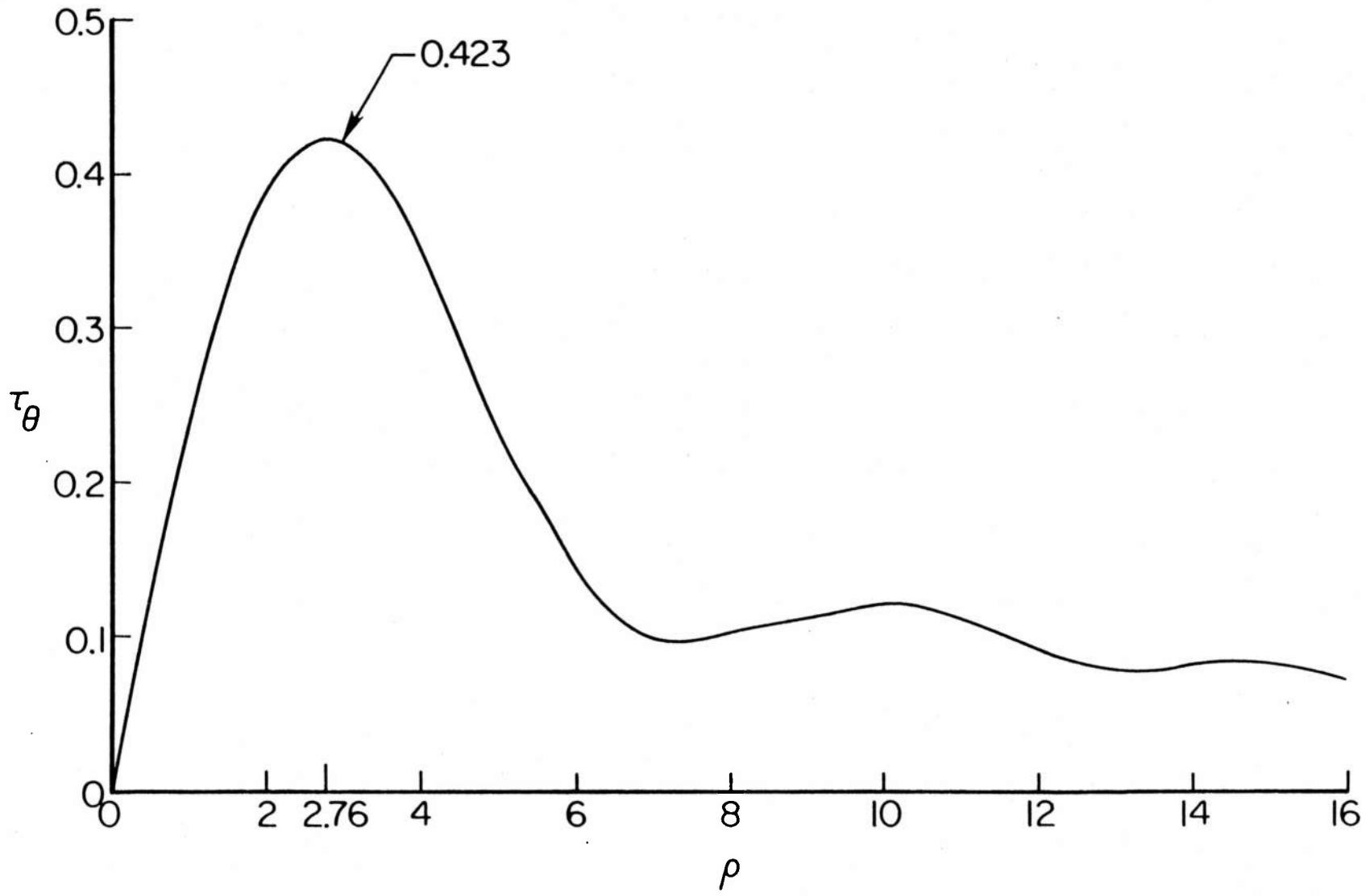
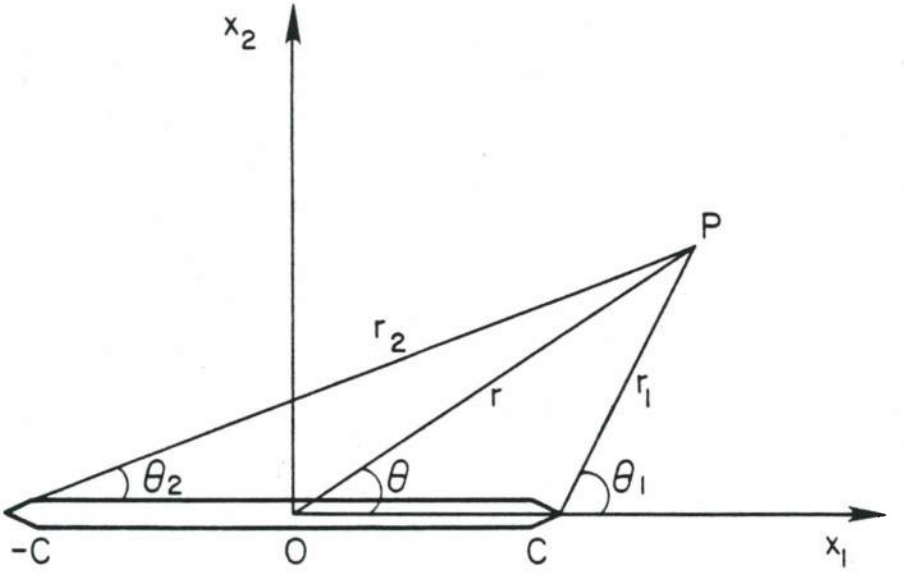


FIGURE 5. SHEAR STRESS IN SCREW DISLOCATION (Eq. 10.18)



CRACK SUBJECT ANTI-PLANE SHEAR (MODE III)

FIGURE 6

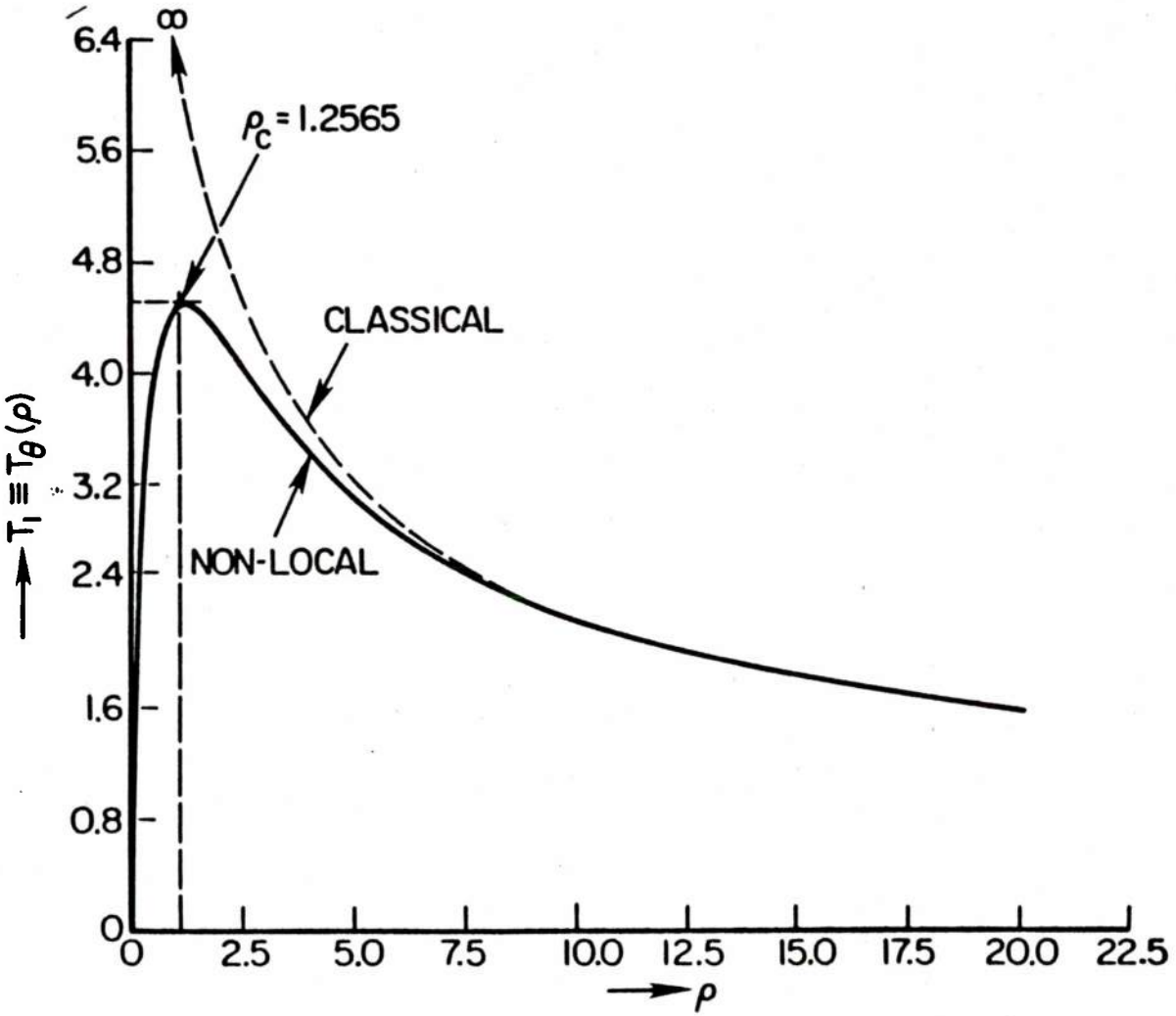


Figure 7: Nondimensional Shear in Mode III

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