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Research Article

Theory of Nonlocal Point Transformations in General Relativity

Massimo Tessarotto[1,2] and Claudio Cremaschini[3]

[1] *Department of Mathematics and Geosciences, University of Trieste, 34128 Trieste, Italy*
[2] *Institute of Physics, Faculty of Philosophy and Science, Silesian University in Opava, Bezručovo Náměstí 13, 74601 Opava, Czech Republic*
[3] *Institute of Physics and Research Center for Theoretical Physics and Astrophysics, Faculty of Philosophy and Science, Silesian University in Opava, Bezručovo Náměstí 13, 74601 Opava, Czech Republic*

Correspondence should be addressed to Claudio Cremaschini; claudiocremaschini@gmail.com

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A discussion of the functional setting customarily adopted in General Relativity (GR) is proposed. This is based on the introduction of the notion of nonlocal point transformations (NLPTs). While allowing the extension of the traditional concept of GR-reference frame, NLPTs are important because they permit the explicit determination of the map between intrinsically different and generally curved space-times expressed in arbitrary coordinate systems. For this purpose in the paper the mathematical foundations of NLPT-theory are laid down and basic physical implications are considered. In particular, explicit applications of the theory are proposed, which concern (1) a solution to the so-called Einstein teleparallel problem in the framework of NLPT-theory; (2) the determination of the tensor transformation laws holding for the acceleration 4-tensor with respect to the group of NLPTs and the identification of NLPT-acceleration effects, namely, the relationship established via general NLPT between particle 4-acceleration tensors existing in different curved space-times; (3) the construction of the nonlocal transformation law connecting different diagonal metric tensors solution to the Einstein field equations; and (4) the diagonalization of nondiagonal metric tensors.

1. Introduction

The investigation carried out in this paper concerns basic theoretical issues and physical problems of critical importance in the classical field theory of gravity, that is, General Relativity (GR), as well as for both classical and quantum relativistic theories. Thus, while leaving the axiomatic framework of the *Standard Formulation to General Relativity* (SF-GR) unchanged which is based on the Einstein field equations, a new approach to SF-GR is proposed. This is obtained by introducing a family of *nonlocal point transformations* (NLPTs) which act between suitable sets of space-times and are referred to here as *NLPT-theory.* This concerns the extension of the customary functional setting which lies at the basis of SF-GR, which is realized by the notion of *local point transformations* (LPTs) P and their inverse P^{-1}:

$$\begin{aligned} P&: r^{\mu} \longrightarrow r'^{\mu} = r'^{\mu}(r), \\ P^{-1}&: r'^{\mu} \longrightarrow r^{\mu} = r^{\mu}(r'), \end{aligned} \tag{1}$$

which connect arbitrary GR-reference frames. In SF-GR the group $\{P\}$ (*LPT-group*) of these transformations is associated with in principle arbitrary possible parametrizations, that is, 4-dimensional curvilinear coordinate systems, of the physical space-time, the latter being identified with a 4-dimensional connected and time-oriented real metric space $D^4 \equiv (\mathbf{Q}^4, g)$, with $\mathbf{Q}^4 \equiv \mathbb{R}^4$. This determines for each parametrization a unique representation of the space-time metric tensor $g_{\mu\nu}(r)$ [1–7].

Hence, by definition, the group $\{P\}$ leaves invariant $(\mathbf{Q}^4, g)$, which must therefore be identified with a differential manifold. It is obvious that such a functional setting is intrinsic to SF-GR; that is, it is actually required for the validity of SF-GR itself. The same transformations defined by (1) are assumed also to warrant the global validity of the so-called Einstein *General Covariance Principle* (GCP) [8]; namely, they must be endowed with a suitable functional setting, referred to here as *LPT-functional setting* (see related discussion in Section 2), which permits in turn also the corresponding realization of GCP. Such a principle is therefore referred to as *LPT-GCP*. In particular, this means that LPT must be smoothly differentiable so as to uniquely and globally prescribe also the 4-*tensor transformation laws* of the displacement 4-vectors; namely,

$$dr^{\mu} = \mathcal{J}^{\mu}_{\nu} dr'^{\nu}, \qquad dr'^{\mu} = \left(\mathcal{J}^{-1}\right)^{\mu}_{\nu} dr^{\nu}. \tag{2}$$

Here, $\mathcal{J}^{\mu}_{\nu}$ and $(\mathcal{J}^{-1})^{\mu}_{\nu}$ denote the direct and inverse Jacobian matrices which take the so-called *gradient form*; that is,

$$\mathcal{J}^{\mu}_{\nu}\left(r'\right) \equiv \frac{\partial r^{\mu}\left(r'\right)}{\partial r'^{\nu}}, \tag{3}$$

$$\left(\mathcal{J}^{-1}\right)^{\mu}_{\nu}(r) \equiv \frac{\partial r'^{\mu}(r)}{\partial r^{\nu}}, \tag{4}$$

which uniquely globally prescribe also the corresponding 4-tensor transformation laws of *all tensor fields* which characterize SF-GR.

However, in this work we intend to show that—based on compelling physical considerations—an alternative approach to GR based on NLPT-theory actually exists, which involves a departure from the standard route adopted in SF-GR. This is founded on the introduction of an extended functional setting, referred to here as *NLPT-functional setting*, which maps in each other intrinsically different space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$, that is, space-times which cannot be otherwise connected by means of the group $\{P\}$.

Background and Physical Motivations. An ongoing subject of theoretical investigations in GR concerns its possible nonlocal modifications. Recent literature investigations of this type are several. Examples can be found, for instance, in [9–14], where nonlocal generalizations of the Einstein theory of gravitation have been proposed. Such a kind of nonlocal GR model leads typically to suitably modified forms of the Einstein equation [1] in which nonlocal field interactions are accounted for, by analogy with corresponding nonlocal features of the electromagnetic field occurring in classical electrodynamics.

It is well-know that the *LPT-functional setting* characteristic of the original Einstein formulation of GR is uniquely founded on the classical theory of tensor calculus on manifolds. The historical foundations of the latter, in turn, date back to the so-called absolute differential calculus developed at the end of the 19th century by Gregorio Ricci-Curbastro and later popularized by his former student and collaborator Tullio Levi-Civita [2, 4]. However, a basic issue that arises in GR and more generally in classical and quantum mechanics as well as in the theory of classical and quantum fields is whether these theories themselves might exhibit possible contradictions with the validity of the LPT-GCP and consequently a more general functional setting should be actually adopted for the treatment of these disciplines.

To better elucidate the scope and potential physical relevance of the topics indicated above, it is worth highlighting some of the main related physical issues which are relevant for the present investigation and whose solution, as explained below, appears of critical importance in GR. These include the following:

(1) *Problem #1: Teleparallel Approach to GR*. An example of violation of LPT-GCP occurs in the framework of the Einstein teleparallel approach to GR (see [15]) and possibly also in some of its recently proposed generalizations [16–18]. Indeed, such a theory is intended to map intrinsically different space-times. In the case of teleparallelism one of such space-times is identified, by construction, with the flat time-oriented Minkowski space-time. As discussed below (see Section 3), this is achieved by a suitable matrix transformation between the corresponding metric tensors, denoted as *teleparallel transformation problem* (TT-problem), which lies at the basis of such an approach (see (17) or equivalently (18)). A number of related issues arise which concern in particular the following:

   (i) *Problem #P1$_1$*. It is the realization and possible nonuniqueness feature of the mapping to be established between the two space-times occurring in the teleparallel transformation itself. This refers in particular to what might/should be the actual representation of the corresponding coordinate transformations, the prescription of possible nonlocal dependence, with particular reference to 4-velocity dependence, and the relationship between local and nonlocal coordinate transformations.

   (ii) *Problem #P1$_2$*. It is the fact that obviously such problems, and the TT-problem itself, cannot be solved in the framework of the validity of the LPT-GCP.

   (iii) *Problem #P1$_3$*. It is the physical implications of the theory, with particular reference to the explicit construction of special NLPT.

   (iv) *Problem #P1$_4$*. It is the possible existence/nonexistence of corresponding tensor transformation laws with respect to arbitrary NLPT and is referred to here as *NLPT 4-tensor laws*, for observable tensor fields and in particular for the metric tensors which are associated with a curved space-time $(\mathbf{Q}^4, g)$ and the corresponding Minkowski space-time $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}^4, \eta)$.

(2) *Problem #2: Diagonalization of Metric Tensors and Complex Transformation Approaches to GR*. A second notable example concerns the adoption in GR of complex-variable transformations, such as the so-called Newman-Janis algorithm [19–21]. This is used in the literature for the purpose of investigating a variety of standard or nonstandard GR black-hole solutions [22, 23], as well as alternative theories of gravitation, such as the one based on noncommutative geometry [24]. Its basic feature is that of

permitting one to transform, by means of a complex coordinate transformation, a diagonal metric tensor corresponding to a spherically symmetric and stationary configuration (like the Schwarzschild one) into a nondiagonal one corresponding to a rotating black-hole (like the Kerr solution). On the other hand, a number of issues arise concerning the Newman-Janis algorithm. These include the following:

(i) *Problem #P2$_1$*. First, it is complex, so that the transformed coordinates are complex too. This inhibits their objective physical interpretation in terms of physical observables.

(ii) *Problem #P2$_2$*. It is the fact that, as for the teleparallel transformation, the diagonalization problem at the basis of the same transformation cannot be solved in the framework of the validity of the LPT-GCP. Indeed, the Newman-Janis algorithm seems worth mentioning especially in view of the fact that it obviously represents a patent violation of the LPT-GCP.

(iii) *Problem #P2$_3$*. The physical meaning of the transformation: one cannot ignore that fact that there is no clear understanding regarding its physical interpretation and ultimately as to why the algorithm should actually work at all.

(iv) *Problem #P2$_4$*. Finally, despite the obvious fact that the teleparallel transformation provides in principle also a solution to the diagonalization problem, there is no clear connection emerging between the same transformation and the Newman-Janis algorithm.

(3) *Problem #3: Acceleration Effects in Relativistic Classical Electrodynamics*. A third issue worth pointing out for its potential relevance in the present discussion concerns the role of acceleration on GR-reference frames as discussed, for example, in [25, 26]. These papers deal with the necessity of taking into account, in the context of both GR and Maxwell's equations, possible acceleration-induced nonlocal effects. However, the precise mathematical formulation and physical mechanisms by which nonlocality should manifest itself must still be fully understood. In fact, a number of basic issues remain unanswered. These concern in particular the following ones:

(i) *Problem #P3$_1$*. First, the precise prescription of the mathematical setting of the theory and in particular the implementation and possible functional realization of the nonlocal acceleration effects in the context of GR remain unclear.

(ii) *Problem #P3$_2$*. Indeed, nonlocal acceleration effects are introduced by postulating directly "ad hoc" integral representations (or "transformation laws") for appropriate tensor fields.

(iii) *Problem #P3$_3$*. The validity of these transformation laws, namely, the reason why ultimately they should apply, and consequently their physical interpretation remain both unclear.

(4) *Problem #4: Nonlocal Effects in Classical Electrodynamics*. A further intriguing example which is by itself sufficient to demonstrate the role of nonlocality in physics can be found in the framework of a special-relativistic treatment of classical electrodynamics. This concerns the so-called electromagnetic radiation-reaction (EM-RR) problem, that is, the dynamics of an extended charge in the presence of its self-generated EM field. As shown in [27, 28] such a problem can be rigorously treated in the framework of a first-principle approach based on the Hamilton variational principle. In such a context the source of nonlocality appears at once as being due to the finite size of charged particles. Indeed, its physical origin is related to the retarded EM interaction of the extended particle with itself [29–33]. However, further fundamental physical issues emerge which should be answered:

(i) *Problem #P4$_1$*. First, the precise prescription of the transformation laws with respect to the group on NLPT should be achieved for the EM 4-potential A^μ and of the corresponding EM Faraday tensor $F^{\mu\nu}$.

(ii) *Problem #P4$_2$*. Second, it remains to be ascertained whether the transformations indicated above are realized by means of 4-tensor NLPT-transformation laws, that is, in particular for $F^{\mu\nu}$, transformation laws formally identical to those determined by the 4-position infinitesimal displacement dr^μ or the dyadic tensor $dr^\mu dr^\nu$.

The key question which needs to be ascertained in the context of GR is whether these problems do actually require, as anticipated above, the introduction of a more general class of GR-reference frames. In fact, despite previous solution attempts [25, 26], a basic issue which still remains unsolved nowadays concerns the construction of the explicit general form and physically admissible realizations which the transformations occurring among arbitrary GR-frames should take. The problem matter refers therefore to possible nonlocal generalization of the customary local tensor calculus and coordinate transformations to be adopted in GR. This is actually the task which we intend to undertake in the present investigation.

Under such premises it must be noted that the present work departs, while being at the same time also in some sense complementary, from the nonlocal GR theories indicated above. In fact it belongs to the class of studies aimed at introducing in the context of GR a new type of nonlocal phenomenon based on the coordinate transformations established between GR-reference frames and at the same time extending the functional setting customarily adopted in such a context.

Goals and Structure of the Paper. The work-plan of the investigation is to address the problem of the nonlocal generalization of GR achieved by a suitable extension of its functional setting. This task concerns basic theoretical issues and unsolved physical problems whose solution presented in this investigation for the first time appears of critical importance in General Relativity (GR). In detail these include the following:

(1) *Goal #1.* It is the identification of possible generalizations of the LPT-setting customarily adopted in GR, based on physical example-cases. A notable problem of this type is realized by Einstein's approach to the so-called *Einstein teleparallelism*. The issue arises whether such a theory can be recovered from SF-GR by means of a suitable mathematical, that is, purely conceptual, viewpoint. This involves the introduction of appropriate nonlocal point transformations (or NLPTs). It must be stressed that the possible prescription of NLPT is by no means "*a priori*" obvious since they remain—it must be stressed—largely arbitrary and intrinsically nonunique. For this purpose *Problems #P1*$_1$–*#P1*$_4$ are addressed in Sections 2–5.

Their solution is crucial for their identification. This goal can be reached based on the adoption of a suitable subset of NLPTs, referred to here as *special NLPT-group* $\{P_S\}$ acting on appropriate *extended GR-frames* which are defined with respect to prescribed space-times. For definiteness, in view of warranting the validity of suitable NLPT 4-tensor laws for the metric tensor which is associated with the teleparallel transformation (see (41) below), in the present treatment these transformations are assumed to preserve the line element (see Section 4 below); in other words they are required to map space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$ having the same line elements ds and ds'.

(2) *Goal #2.* In this context *Problems #P2*$_1$–*#P2*$_4$ are addressed. For such a purpose the determination is done of the group of *general nonlocal point transformations* (general NLPTs) connecting subsets of two generic curved space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$. This is referred to here as *general NLPT-group* $\{P_g\}$ (Section 6). The task posed here involves also their physical interpretation (Section 7). As an illustration of the theory, the explicit construction of possible physically relevant transformations of the group $\{P_g\}$ are addressed, with special reference to the problem of the NLPT between diagonal metric tensors (Section 8) and the diagonalization of metric tensors in GR (Section 9).

(3) *Goal #3.* It is the investigation of physical implications of the general NLPT-functional setting in reference to the identification of possible *acceleration effects* both in GR and in classical electrodynamics. The goal of Sections 10 and 11 is to look for a possible solution to *Problems #P3*$_1$–*#P3*$_3$ and *Problems #P4*$_1$-*#P4*$_2$ indicated above as well as to point out relevant possible realizations of general NLPT. This involves in particular the investigation of the role of acceleration on GR-reference frames and the search of NLPT 4-tensor laws occurring, respectively, for the acceleration 4-tensor and the EM 4-vector potential, with respect to the group of NLPT $\{P_g\}$ established between suitable subsets of two arbitrary curved space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$.

2. The LPT-Functional Setting

We first recall the functional setting which—as anticipated above—is usually adopted both in relativistic theories and in Einstein's 1915 theory of gravitation [1], that is, SF-GR itself. In both cases the goal is, in principle, to predict all physically relevant realizations of the observables. In the case of GR these concern the physical space-time itself $D^4 \equiv (\mathbf{Q}^4, g)$. As is well-known, in SF-GR this is identified with a 4-dimensional Lorentzian metric space on $\mathbf{Q}^4 \equiv \mathbb{R}^4$ which is endowed with a prescribed metric tensor $g_{\mu\nu}(r)$ when the same set $\mathbf{Q}^4$ is represented in terms of a given set of curvilinear coordinates $\{r^\mu\} \equiv r$. Nevertheless, validity of GR and in particular of the Einstein equation itself requires couching them in a suitable mathematical framework.

As recently pointed out in [6] in the context of a variational treatment of SF-GR, this involves, besides the fulfillment of a suitable property of gauge invariance, also the adoption of Classical Tensor Analysis on Manifolds. In other words both GR and the same Einstein equation should embody by construction the validity of LPT-GCP, namely, formulated consistent with the so-called LPT-functional setting. More precisely, this means explicitly that the following mathematical requirements (A–C) should apply:

(A) All physically observable tensor fields defined on space-time $(\mathbf{Q}^4, g)$ must be realized by means of 4-tensor fields with respect to a suitable ensemble of coordinate transformations connecting in principle arbitrary, but suitably related, 4-dimensional curvilinear coordinate systems, referred to as *GR-reference frames*, r^μ and r'^μ.

(B) The PDEs, together with their corresponding variational principles, which characterize all classical and quantum physical laws should satisfy *the criterion of manifest covariance*, *whereby it should be possible to cast them in all their realizations in manifest 4-tensor form*.

(C) The set of coordinate transformations indicated above is identified with the group of transformations that in Eulerian form are prescribed by means of the invertible maps (1) which identify the group $\{P\}$. For this purpose, suitable restrictions must be placed on the admissible GR-reference frames, that is, coordinate systems, prescribed by means of (1) which are realized by the following requirements:

(i) *LPT-Requirement #1.* For the validity of GCP, the two space-times must coincide and be transformed into one another by means of LPT; that is, $(\mathbf{Q}^4, g(r)) \equiv (\mathbf{Q}'^4, g'(r'))$, so as to define a single C^k*-differentiable Lorentzian manifold* with $k \geq 3$, that is, have either signature $(+,-,-,-)$ or analogous permutations.

(ii) *LPT-Requirement #2*. These transformations must be assumed as purely local, so that in (1) r'^{μ} and r^{μ} must depend only locally, respectively, on $r \equiv \{r^{\mu}\}$ and $r' \equiv \{r'^{\mu}\}$. In other words, the *local values* r^{μ} and r'^{μ} are required to be mutually mapped in each other by means of the same equations, with r'^{μ} (resp., r^{μ}) being a function of r^{μ} (and similarly r'^{μ}) *only*.

(iii) *LPT-Requirement #3*. The coordinates r^{μ} and r'^{μ} must realize *physical observables* and hence be prescribed in terms of real variables, while the functions relating them (P and P^{-1}) must be suitably smooth in the sense that they are of class $C^{(k)}$, with $k \geq 3$. This means that $(\mathbf{Q}^4, g(r))$ must realize a C^k*-differentiable Lorentzian manifold* with $k \geq 3$.

(iv) *LPT-Requirement #4*. Equations (1) generate the corresponding 4*-vector transformation equations* for the contravariant components of the displacement 4-vectors dr^{μ} and dr'^{μ} (see (2)). Analogous transformation laws follow, of course, for the covariant components of the displacements; namely, $dr_{\nu} = g_{\mu\nu}(r)dr^{\mu}$. In view of (1), by construction $\mathcal{J}^{\mu}_{\nu}$ and $(\mathcal{J}^{-1})^{\mu}_{\nu}$ are considered, respectively, local functions of $r' \equiv \{r'^{\mu}\}$ and $r \equiv \{r^{\mu}\}$ only and must necessarily coincide with the *gradient forms (3)-(4)*. Nevertheless, since $\mathcal{J}^{\mu}_{\nu}$ and $(\mathcal{J}^{-1})^{\mu}_{\nu}$ are mutually related being inverse matrices of each other and the point transformations are purely local, it follows that in view of (3) and (4) they can also both formally be regarded as functions, respectively, of the variables r' and r.

(v) *LPT-Requirement #5*. In terms of the Jacobian matrix $\mathcal{J}^{\mu}_{\nu}$ and its inverse $(\mathcal{J}^{-1})^{\mu}_{\nu}$ the fundamental *LPT 4-tensor transformation laws* for the group $\{P\}$ are set for arbitrary tensors. Consider, for example, the Riemann curvature tensor $R^{\rho}_{\sigma\mu\nu}(r)$. In terms of an arbitrary LPT it obeys the 4-tensor transformation law

$$R^{\rho}_{\sigma\mu\nu}(r) = \mathcal{J}^{\beta}_{\sigma}\left(\mathcal{J}^{-1}\right)^{\rho}_{\alpha}\mathcal{J}^{k}_{\mu}\mathcal{J}^{m}_{\nu}R'^{\alpha}_{\beta k m}\left(r'\right). \tag{5}$$

The same transformation law also requires that 4-scalars must be left unchanged under the action of the group $\{P\}$. Thus, by construction the 4-scalar proper-time element ds, that is, the Riemann-distance defined in terms of the equation $ds^2 = g_{\mu\nu}(r)dr^{\mu}dr^{\nu} \equiv g^{\mu\nu}(r)dr_{\mu}dr_{\nu}$, must satisfy the transformation law

$$ds^2 = g_{\mu\nu}(r)\,dr^{\mu}dr^{\nu} = g'_{\mu\nu}\left(r'\right)dr'^{\mu}dr'^{\nu}, \tag{6}$$

which can be equivalently expressed as

$$ds^2 = g^{\mu\nu}(r)\,dr_{\mu}dr_{\nu} = g'^{\mu\nu}\left(r'\right)dr'_{\mu}dr'_{\nu}. \tag{7}$$

Furthermore, the covariant and contravariant components of the metric tensor, that is, $g_{\mu\nu}(r)$ and $g^{\mu\nu}(r)$ and, respectively, $g'_{\mu\nu}(r')$ and $g'^{\mu\nu}(r')$, must satisfy, respectively, the *LPT 4-tensor transformation laws*

$$g'_{\mu\nu}\left(r'\right) = \mathcal{J}^{\alpha}_{\mu}\left(r'\right)\mathcal{J}^{\beta}_{\nu}\left(r'\right)g_{\alpha\beta}(r), \tag{8}$$

$$g'^{\mu\nu}\left(r'\right) = \left(\mathcal{J}^{-1}(r)\right)^{\mu}_{\alpha}\left(\mathcal{J}^{-1}(r)\right)^{\nu}_{\beta}g^{\alpha\beta}(r), \tag{9}$$

so that the validity of the scalar transformation laws (6) and (7) is warranted.

(vi) *LPT-Requirement #6*. Introducing the corresponding *Lagrangian form* of the same equations, obtained by parametrizing both r^{μ} and r'^{μ} in terms of suitably smooth time-like worldlines $\{r^{\mu}(s),\ s \in I\}$ and $\{r'^{\mu}(s),\ s \in I\}$, (1) take the equivalent form

$$\begin{aligned} P&: r^{\mu}(s) \longrightarrow r'^{\mu}(s) = r'^{\mu}(r(s)), \\ P^{-1}&: r'^{\mu}(s) \longrightarrow r^{\mu}(s) = r^{\mu}\left(r'(s)\right), \end{aligned} \tag{10}$$

whereby the displacement 4-vectors $dr^{\mu} \equiv dr^{\mu}(s)$ and $dr'^{\mu} \equiv dr'^{\mu}(s)$ can be viewed as occurring during the proper-time ds. Then it follows that (10) imply also suitable transformation laws for the 4-velocities $u^{\mu}(s) = dr^{\mu}(s)/ds$ and $u'^{\mu}(s) = dr'^{\mu}(s)/ds$, which by definition span the tangent space $T\mathbb{D}^4$. The latter are provided by the equations

$$\begin{aligned} u^{\mu}(s) &= \mathcal{J}^{\mu}_{\nu}\left(r'\right)u'^{\nu}(s), \\ u'^{\mu}(s) &= \left(\mathcal{J}^{-1}\right)^{\mu}_{\nu}(r)\,u^{\nu}(s), \end{aligned} \tag{11}$$

implying the simultaneous validity of the mass-shell constraints

$$u^{\mu}(s)\,u_{\mu}(s) = 1, \tag{12}$$

$$u'^{\mu}(s)\,u'_{\mu}(s) = 1. \tag{13}$$

Notice that here also the Jacobian $\mathcal{J}^{\mu}_{\nu}$ and its inverse $(\mathcal{J}^{-1})^{\mu}_{\nu}$ must be considered as s-dependent (but just only through $r' = r'(s)$ and $r = r(s)$, resp.), that is, of the form

$$\begin{aligned} \mathcal{J}^{\mu}_{\nu}\left(r'\right) &= \mathcal{J}^{\mu}_{\nu}\left(r'(s)\right), \\ \left(\mathcal{J}^{-1}\right)^{\mu}_{\nu}(r) &= \left(\mathcal{J}^{-1}\right)^{\mu}_{\nu}(r(s)). \end{aligned} \tag{14}$$

(vii) *LPT-Requirement #7*. Finally, in terms of (10) and (11) one notices that LPT can be formally represented in terms of Lagrangian phase-space transformations of the type

$$\begin{gathered}
\left\{r^{\mu}(s), u^{\mu}(s)\right\} \longrightarrow \\
\left\{r^{\prime \mu}(s), u^{\prime \mu}(s)\right\}=\left\{r^{\prime \mu}(r(s)),\left(\mathscr{J}^{-1}\right)_{\nu}^{\mu}(r) u^{\nu}(s)\right\}, \\
\left\{r^{\prime \mu}(s), u^{\prime \mu}(s)\right\} \longrightarrow \\
\left\{r^{\mu}(s), u^{\mu}(s)\right\}=\left\{r^{\mu}\left(r^{\prime}(s)\right), \mathscr{J}_{\nu}^{\mu}\left(r^{\prime}\right) u^{\prime \nu}(s)\right\}
\end{gathered} \tag{15}$$

(*LPT-phase-space map*), with the vectors $\{r^{\mu}(s), u^{\mu}(s)\}$ and $\{r'^{\mu}(s), u'^{\mu}(s)\}$ to be viewed as representing the phase-space states, endowed by 4-positions $r^{\mu}(s)$ and $r'^{\mu}(s)$, respectively, and corresponding 4-velocities $u^{\mu}(s)$ and $u'^{\mu}(s)$. Hence, by construction transformation (15) warrants the scalar and tensor transformation laws (6) and (8) and preserves the structure of the space-time $(\mathbf{Q}^4, g)$.

This concludes the prescription of the LPT-functional setting required for the validity of GCP. The set of assumptions represented by *LPT-Requirements #1–#7* will be referred to here as *LPT-theory*.

It must be stressed that its adoption is of paramount importance in the context of GR and in particular for the subsequent considerations regarding the physical interpretations of Einstein teleparallelism. This happens at least for the following three main motivations. The first one is that, in validity of the *LPT-requirements #1–#6*, and in particular the gradient form requirement (3)-(4) for the Jacobian matrix, (11) are equivalent to the Eulerian equations (1) (and of course also to the corresponding Lagrangian equations (10)). Hence, both equations actually allow one to identify uniquely the group $\{P\}$ (*Proposition #1*).

The second one concerns the very notion of particular solution to be adopted in the context of GR for the Einstein equation. In fact, if $g_{\mu\nu}(r)$ denotes a *parametrized-solution* to the same equation obtained with respect to a GR-frame r^{μ}, the notion of particular solution for the same equation is actually peculiar. Indeed, it must necessarily coincide with the whole equivalence class of parametrized-solutions, represented symbolically as $\{g_{\mu\nu}(r)\}$, which are mapped in each other by means of an arbitrary LPT of the group $\{P\}$. Such a property, which is actually a consequence of GCP (and consequently of Classical Tensor Analysis on Manifolds), is usually being referred to in GR as the so-called *principle of frame's* (or observer's) *independence* (*Proposition #2*).

The third motivation concerns the very notion of curved space-time $(\mathbf{Q}^4, g(r))$, compared to that of the Minkowski flat space-time $(\mathbf{Q}^4, \eta)$, which when expressed in orthogonal Cartesian coordinates $r'^{\mu} \equiv (r^{0\prime}, (\mathbf{r}' \equiv x', y', z'))$ has the metric tensor $\eta_{\mu\nu} = \mathrm{diag}\{1, -1, -1, -1\}$. A generic space-time of this type is characterized, by definition, by a nonvanishing Riemann curvature 4-tensor $R^{\rho}_{\sigma\mu\nu}(r)$. As a consequence of the 4-tensor transformation laws (8)-(9) it follows that two generic space-times $(\mathbf{Q}^4, g(r))$ and $(\mathbf{Q}'^4, g'(r'))$ can be mapped in each other by means of LPTs and hence actually coincide, only provided the respective metric tensors, and hence also the corresponding Riemann curvature 4-tensors, are transformed into each other via the same equations (8)-(9). Hence, it is obvious that a generic curved space-time cannot be mapped into the said Minkowski space-time purely by means of LPT (*Proposition #3*).

3. Einstein's Teleparallel Transformation Problem

Most of the historical developments achieved so far in GR since its original appearance in 1915 have been obtained in the framework of the GCP-setting of GR [8]. Nonetheless for a long time the issue has been debated whether Relativistic Classical Mechanics and Relativistic Classical theory of fields might exhibit in each case (possibly different) nonlocal phenomena. In the literature there are several examples of studies aimed at extending in the context of GR the classical notions of local dynamics and local field interactions. A related question is, however, whether there actually exist additional nonlocal phenomena which might escape the validity of GCP and require the setup of a proper theoretical framework for their study.

As we intend to show, an instance of this type arises in the context of the so-called teleparallel approach to GR, also known as Einstein teleparallelism [15] (see also [16–18]). To state the issue in the appropriate physical context let us briefly highlight the basic ideas behind such an approach. This is based on the conjecture on Einstein part that at each point r^{μ} of the space-time manifold $(\mathbf{Q}^4, g(r))$ the corresponding tangent space $T\mathbb{D}^4$ can be "parallelized." This means, in other words, that at all 4-positions $r^{\mu} \in (\mathbf{Q}^4, g(r))$ it should be possible to cast each tangent 4-vector $u^{\mu}(s)$ in the form

$$\begin{aligned}
u^{\mu}(s) &= M^{\mu}_{\nu} u'^{\nu}(s), \\
u'^{\mu}(s) &= \left(M^{-1}\right)^{\mu}_{\nu} u^{\nu}(s),
\end{aligned} \tag{16}$$

with M^{μ}_{α} being an invertible matrix with inverse $(M^{-1})^{\alpha}_{\mu}$. More precisely, according to Einstein's approach the metric tensor of a generic curved space-time $(\mathbf{Q}^4, g(r))$ should satisfy an equation in the form

$$g_{\mu\nu}(r) = \left(M^{-1}\right)^{\alpha}_{\mu} \left(M^{-1}\right)^{\beta}_{\nu} \eta_{\alpha\beta} \tag{17}$$

or equivalently

$$M^{\mu}_{\alpha}(r) M^{\nu}_{\beta}(r) g_{\mu\nu}(r) = \eta_{\alpha\beta}, \tag{18}$$

with $\eta_{\alpha\beta}$ being here the metric tensor associated with the flat Minkowski space-time $(\mathbf{Q}'^4 \equiv \mathbf{M}^4, \eta)$ having the Lorentzian signature $(+,-,-,-)$. The goal is therefore to determine the map

$$\eta_{\alpha\beta} \longleftrightarrow g_{\mu\nu}(r), \tag{19}$$

known as the *teleparallel transformation* (TT), while (17) (or equivalently (18)) will be referred to as the *TT-problem*.

For definiteness, it must be stressed here what appears to be Einstein's key assumption underlying these equations: it is understood in fact that in (17) and (18) $\eta_{\alpha\beta}$ manifestly identifies the metric tensor of the Minkowski space-time $(\mathbf{M}^4, \eta)$ when expressed in terms of orthogonal Cartesian coordinates. On the other hand it is also understood that (17) and (18) should include the identity transformation among their possible solutions. This means that for consistency $g_{\mu\nu}(r)$ can always be identified with the metric tensor of the curved space-time $(\mathbf{Q}^4, g(r))$ when expressed as a local function of the *same Cartesian coordinates*. In the present paper such a viewpoint will be consistently adopted in the subsequent considerations to be developed below.

The following additional remarks must also be made regarding the TT-problem. The first one concerns the interpretation of (18) in the so-called tetrad formalism. It implies, in fact, that for $\mu = 0, 3$ the fields $M_0^\mu(r), M_1^\mu(r), M_2^\mu(r)$, and $M_3^\mu(r)$ can simply be interpreted as a *tetrad basis*, that is, a set of four independent real 4-vector fields that are mutually orthogonal, that is, such that for $\alpha \neq \beta$

$$e_\alpha^\mu(r)\, e_\beta^\nu(r)\, g_{\mu\nu}(r) = 0. \tag{20}$$

Also, all basis 4-vectors are unitary, in the sense that, for all $\alpha = 0, 3$, $|M_\alpha^\mu(r) M_{(\alpha)}^\nu(r) g_{\mu\nu}(r)| = 1$, one of them $(M_0^\mu(r))$ being time-like and the others being space-like; namely,

$$\begin{gathered} M_0^\mu(r)\, M_0^\nu(r)\, g_{\mu\nu}(r) = 1, \\ M_\alpha^\mu(r)\, M_{(\alpha)}^\nu(r)\, g_{\mu\nu}(r) = -1 \end{gathered} \tag{21}$$

together span the 4D tangent space at each point r^μ in the space-time $(\mathbf{Q}^4, g)$.

The second remark is about the choice of the curved space-time $(\mathbf{Q}^4, g(r))$ in the TT-problem. It must be stressed, in fact, that the space-time $(\mathbf{Q}^4, g(r))$ should remain in principle arbitrary. Therefore, it should always be possible to identify $(\mathbf{Q}^4, g(r))$ with the curved space-time having signature different from that of the Minkowski space-time. Therefore, the solution to the TT-problem should be possible also in the case in which $(\mathbf{Q}^4, g(r))$ and $(\mathbf{M}^4, \eta)$ have different signatures.

The third remark is about the ultimate goal of Einstein teleparallelism. This emerges perspicuously from (17) (or equivalently its inverse represented by (18)). The determination of the matrix $M_\alpha^\mu(r)$ solution to such an equation will be referred to here as *TT-problem*. In fact, (17) (i.e., if a solution exists to such an equation) should permit one to relate curved and flat space-time metric tensors, respectively, identified with $g_{\mu\nu}(r)$ and $\eta_{\alpha\beta}$.

From these premises, therefore, the fundamental problem of establishing a map between the generic curved space-time $(\mathbf{Q}^4, g)$ indicated above and the Minkowski space-time $(\mathbf{M}^4, \eta)$ emerges, which should have a global validity; namely, it should hold in the whole $(\mathbf{Q}^4, g)$ or at least in a finite subset of the same space-time. However, such a kind of transformation cannot be realized by means of LPT of type (1) in which $M_\alpha^\mu(r)$ is identified with the corresponding Jacobian (see (3) below). This happens because the teleparallel transformation cannot be realized by means of the group of LPTs $\{P\}$ (see also the related *Proposition #3* indicated above). The issue arises whether in the context of GR the teleparallel transformation (17) (and equivalently its inverse, i.e., (18)) might actually still apply in the case of a more general type of nonlocal point transformations, with the matrix $M_\alpha^\mu(r)$ to be identified with a corresponding suitably prescribed Jacobian matrix.

The existence of such a class of generalized GR-reference frames and coordinate systems is actually suggested by the Einstein equivalence principle (EEP) itself. This is expressed by two separate propositions, which in the form presently known must both be ascribed to Albert Einstein's 1907 original formulation [34] (see also [35]). The part of EEP which is mostly relevant for the current discussion is the one usually referred to as the so-called weak equivalence principle (WEP). This is related, in fact, to the fundamental notion of equivalence between gravitational and inertial mass as well as to Albert Einstein's observation that the gravitational "force" as experienced locally while standing on a massive body is actually the same as the pseudoforce experienced by an observer in a noninertial (accelerated) frame of reference. Apparently there is no unique formulation of WEP to be found in the literature. However, the form of WEP which is of key importance in the following consists in the two distinct claims by Einstein stating (a) the equivalence between accelerating frames and the occurrence of gravitational fields (see also [8]) and (b) the fact that "*local effects of motion in a curved space (gravitation)*" should be considered as "*indistinguishable from those of an accelerated observer in flat space*" [34, 35]. Incidentally, it must be stressed that statement (b) is the basis of Einstein's 1928 paper on teleparallelism.

From a historical perspective, the original introduction of WEP (and EEP) on the part of Albert Einstein was later instrumental for the development of GR. An interesting question concerns the conditions of validity of GCP and the choice of the class of LPTs to which WEP applies. In fact, based on the discussion above, the issue is whether it is possible to extend in such a framework the class of LPTs. In particular, here we intend to look for a more general group of point transformations, to be identified with NLPT. These are distinguished from the class $\{P\}$ introduced above and form a group of transformations denoted here as *special NLPT-group* $\{P_S\}$. This new type of transformation connects two accelerating frames, namely, curvilinear coordinate systems mutually related by means of suitable acceleration-dependent and necessarily nonlocal coordinate transformations. The latter should permit one to connect globally two suitable subsets of Lorentzian spaces which realize accessible domains (in the sense indicated below) and are endowed with different metric tensors having intrinsically different Riemann tensors. Therefore, these transformations should have the property of being globally defined and, together with the corresponding inverse transformations, be, respectively, endowed with Jacobians $M_\alpha^\mu(r)$ and $(M^{-1}(r))_\nu^\mu$.

We intend to show that provided suitable "ad hoc" restrictions are set on the class of manifolds among which NLPTs are going to be established, a nontrivial generalization of GR by means of the general NLPT-group $\{P_S\}$ can be achieved. These will be shown to be realized in terms of a suitably prescribed diffeomorphism between 4-dimensional Lorentzian space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ of the general form

$$P_g: r'^{\mu} \longrightarrow r^{\mu} = r^{\mu}\left\{r', \left[r', u'\right]\right\}, \quad (22)$$

with inverse transformation

$$P_g^{-1}: r^{\mu} \longrightarrow r'^{\mu} = r'^{\mu}\{r, [r, u]\}. \quad (23)$$

Here the squared brackets $[r', u']$ and $[r, u]$ denote possible suitable *nonlocal dependence* in terms of the 4-positions r'^{μ} and r^{μ} and corresponding 4-velocities $u^{\mu} \equiv dr^{\mu}/ds$ and $u'^{\mu} \equiv dr'^{\mu}/ds$, respectively. As a consequence, (22)-(23) identify a new kind of point transformation, which unlike LPTs (see (1)) is established between intrinsically different manifolds $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$, that is, which cannot be mapped in each other purely by means of LPTs.

4. Solution to the TT-Problem: The NLPT-Functional Setting

Let us now pose the problem of constructing explicitly the new type of point transformation, that is, the NLPT, which are involved in the representation problem of teleparallel gravity and identifying, in the process, the corresponding NLPT-functional setting.

For this purpose we introduce first the conjecture that, consistent with EEP, it should be possible to generate such a transformation introducing a suitable 4-velocity transformation $u^{\mu} \rightarrow u'^{\mu}$ which connects appropriate sets of GR-reference frames belonging to the two space-times indicated above. Indeed, the possibility of constructing "ad hoc" 4-velocity transformations which are not reducible to LPTs of type (1) is physically conceivable. To show how this task can be achieved in practice, we notice that the transformation laws for the 4-velocity which are realized, by assumption, by (16) necessarily imply the validity of corresponding transformation equations for the displacement 4-vectors $dr^{\mu}(s)$ and $dr'^{\mu}(s)$. These read manifestly

$$\begin{aligned} dr^{\mu}(s) &= M_{\nu}^{\mu} dr'^{\nu}(s), \\ dr'^{\mu}(s) &= \left(M^{-1}\right)_{\nu}^{\mu} dr^{\nu}(s), \end{aligned} \quad (24)$$

where for generality M_{ν}^{μ} and $(M^{-1})_{\nu}^{\mu}$ are considered of the forms $M_{\nu}^{\mu} = M_{\nu}^{\mu}(r', r)$ and $(M^{-1})_{\nu}^{\mu} = (M^{-1})_{\nu}^{\mu}(r, r')$. By analogy with (14), when evaluated along the corresponding world-lines, it follows that they take the general functional form

$$M_{\nu}^{\mu} = M_{\nu}^{\mu}\left(r'(s), r(s)\right), \quad (25)$$

$$\left(M^{-1}\right)_{\nu}^{\mu} = \left(M^{-1}\right)_{\nu}^{\mu}\left(r(s), r'(s)\right), \quad (26)$$

with M_{ν}^{μ} and $(M^{-1})_{\nu}^{\mu}$ being now smooth functions of s through the variables $r(s) \equiv \{r^{\mu}(s)\}$ and $r'(s) \equiv \{r'^{\mu}(s)\}$. More precisely, by analogy to the LPT-requirements recalled above, the following prescriptions can be invoked to determine the NLPT-functional setting:

(i) *NLPT-Requirement #1*. The coordinates r^{μ} and r'^{μ} realize by assumption physical observables and hence are prescribed in terms of real variables, while $(\mathbf{Q}^4, g(r))$ and $(\mathbf{M}^4, \eta)$ must both realize C^k-differentiable Lorentzian manifolds, with $k \geq 3$.

(ii) *NLPT-Requirement #2*. The matrices M_{ν}^{μ} and $(M^{-1})_{\nu}^{\mu}$ are assumed to be *locally* smoothly dependent only on 4-position, while admitting at the same time also possible nonlocal dependence. More precisely, in the case of the Jacobian $M_{\nu}^{\mu}(r', r)$ the second variable $r \equiv \{r^{\mu}\}$ which enters the same function can contain in general both local and nonlocal implicit dependence, the former one in terms of r'^{μ}. Similar considerations apply to the inverse matrix $(M^{-1})_{\nu}^{\mu}(r, r')$, which, besides local explicit and implicit dependence in terms of r^{μ}, may generally include additional nonlocal dependence through the variable $r' \equiv \{r'^{\mu}\}$.

(iii) *NLPT-Requirement #3*. The Jacobian matrix M_{ν}^{μ} and its inverse $(M^{-1})_{\nu}^{\mu}$ are assumed to be generally nongradient. In other words, at least in a subset of the two space-times $(\mathbf{M}^4, \eta) \equiv (\mathbf{Q}'^4, g')$ and $(\mathbf{Q}^4, g)$

$$\begin{aligned} M_{\nu}^{\mu}\left(r', r\right) &\neq \frac{\partial r^{\mu}\left(r', r\right)}{\partial r'^{\nu}}, \\ \left(M^{-1}\right)_{\nu}^{\mu}\left(r, r'\right) &\neq \frac{\partial r'^{\mu}\left(r, r'\right)}{\partial r^{\nu}}, \end{aligned} \quad (27)$$

while elsewhere they can still recover the gradient forms (3) and (4); namely,

$$\begin{aligned} M_{\nu}^{\mu}\left(r', r\right) &= \frac{\partial r^{\mu}\left(r', r\right)}{\partial r'^{\nu}}, \\ \left(M^{-1}\right)_{\nu}^{\mu}\left(r, r'\right) &= \frac{\partial r'^{\mu}\left(r, r'\right)}{\partial r^{\nu}}. \end{aligned} \quad (28)$$

In both cases the partial derivatives are performed with respect to the local dependence only.

(iv) *NLPT-Requirement #4*. Introducing the (proper-time) line elements ds and ds' in the two space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{M}^4, \eta) \equiv (\mathbf{Q}'^4, g')$ defined, respectively, according to (7) so that

$$\begin{aligned} ds^2 &= g_{\mu\nu}(r)\, dr^{\mu} dr^{\nu}, \\ ds'^2 &= g'_{\mu\nu}\left(r'\right) dr'^{\mu} dr'^{\nu} \equiv \eta_{\mu\nu} dr'^{\mu} dr'^{\nu}, \end{aligned} \quad (29)$$

the *Riemann-distance conservation law*

$$ds = ds' \quad (30)$$

is set. This implies that the equation

$$g_{\mu\nu}(r)\, dr^{\mu} dr^{\nu} = \eta_{\mu\nu} dr'^{\mu} dr'^{\nu} \quad (31)$$

must hold.

(v) *NLPT-Requirement #5*. Finally, we will assume that the 4-positions $r^{\mu}(s)$ and $r'^{\mu}(s)$ spanning the corresponding space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{M}^4, \eta)$ are represented in terms of the same Cartesian coordinates; that is,

$$r^{\mu} \equiv \{ct, (\mathbf{r} \equiv x, y, z)\}, \tag{32}$$

$$r'^{\mu} \equiv \{ct', (\mathbf{r}' \equiv x', y', z')\}. \tag{33}$$

Let us now briefly analyze the implications of these requirements. First, (24) (or equivalently (16)) can be integrated at once performing the integration along suitably smooth time- (or space-) like world-lines $r^{\mu}(s)$ and $r'^{\mu}(s)$:

$$\begin{aligned} P_S&: r^{\mu}(s) = r^{\mu}(s_o) + \int_{s_o}^{s} d\bar{s} M^{\mu}_{\nu}(r', r) u'^{\nu}(\bar{s}), \\ P_S^{-1}&: r'^{\mu}(s) = r'^{\mu}(s_o) + \int_{s_o}^{s} d\bar{s} (M^{-1})^{\mu}_{\nu}(r, r') u^{\nu}(\bar{s}), \end{aligned} \tag{34}$$

where the initial condition is set:

$$r^{\mu}(s_o) = r'^{\mu}(s_o). \tag{35}$$

Transformations (34) will be referred to as *special NLPT* in *Lagrangian form*, the family of such transformations identifying the special NLPT-group $\{P_S\}$, that is, a suitable subset of the group of general NLPT-group $\{P_g\}$. The subsets of two space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}^4, \eta)$ which are mapped in each other by a special NLPT, both assumed to have nonvanishing measure, will be referred to as *accessible subdomains*.

Notice that the Jacobians $M^{\mu}_{\nu}(r', r)$ and $(M^{-1})^{\mu}_{\nu}(r, r')$ remain still in principle arbitrary. In particular, in case they take the gradient forms (28) the Lagrangian LPT defined by (10) is manifestly recovered. Furthermore, (16) or equivalently (34) can be also represented in terms of the equations for the infinitesimal 4-displacements, given by (24). In particular, assuming the matrix M^{μ}_{ν} to be continuously connected to the identity δ^{μ}_{ν} implies that the Jacobian matrix M^{μ}_{ν} and its inverse $(M^{-1})^{\mu}_{\nu}$ can always be represented in the form

$$M^{\mu}_{\nu}(r', r) = \delta^{\mu}_{\nu} + \mathscr{A}^{\mu}_{\nu}(r', r), \tag{36}$$

$$(M^{-1})^{\mu}_{\nu}(r, r') = \delta^{\mu}_{\nu} + \mathscr{B}^{\mu}_{\nu}(r, r'), \tag{37}$$

with $\mathscr{A}^{\mu}_{\nu}$ and $\mathscr{B}^{\mu}_{\nu}$ being suitable *transformation matrices*, which are mutually related by matrix inversion. Hence, in terms of (36)-(37), the special NLPT in Lagrangian form (34) yields then the corresponding Lagrangian and Eulerian forms:

$$\begin{aligned} r^{\mu}(s) &= r'^{\mu}(s) + \int_{s_o}^{s} d\bar{s} \mathscr{A}^{\mu}_{\nu}(r', r) u'^{\nu}(\bar{s}) \\ r'^{\mu}(s) &= r^{\mu}(s) + \int_{s_o}^{s} d\bar{s} \mathscr{B}^{\mu}_{\nu}(r, r') u^{\nu}(\bar{s}), \end{aligned} \tag{38}$$

$$\begin{aligned} r^{\mu} &= r'^{\mu} + \int_{r'^{\nu}(s_o)}^{r'^{\nu}} dr'^{\nu} \mathscr{A}^{\mu}_{\nu}(r', r) \\ r'^{\mu} &= r^{\mu} + \int_{r^{\nu}(s_o)}^{r^{\nu}} dr^{\nu} \mathscr{B}^{\mu}_{\nu}(r, r'). \end{aligned} \tag{39}$$

We stress that, in difference with the treatment of LPT, in the proper-time integral on the rhs of (34) and (38) the tangent-space curve $u'^{\nu}(\bar{s})$ (resp., $u^{\nu}(\bar{s})$) must be considered as an independent variable. This is a peculiar feature of (34) which cannot be avoided. The reason lies in the fact that there is no way by which $u'^{\nu}(\bar{s})$ (and $u^{\nu}(\bar{s})$) can be uniquely prescribed by means of the same equations. Indeed, (34) (or equivalently (38) and (39)) together with (16) truly establish a phase-space transformation of the following form:

$$\begin{aligned} &\{r^{\mu}(s), u^{\mu}(s)\} \longrightarrow \\ &\{r'^{\mu}(s), u'^{\mu}(s)\} = \{r'^{\mu}\{r(s), [r, u]\}, (\mathscr{M}^{-1})^{\mu}_{\nu} u^{\nu}(s)\} \\ &\{r'^{\mu}(s), u'^{\mu}(s)\} \longrightarrow \\ &\{r^{\mu}(s), u^{\mu}(s)\} = \{r^{\mu}\{r'(s), [r', u']\}, \mathscr{M}^{\mu}_{\nu} u'^{\nu}(s)\}. \end{aligned} \tag{40}$$

This will be referred to as *NLPT-phase space map*. The latter applies to a new type of reference frame, denoted as *extended GR-frames*, which are represented by the vectors $\{r^{\mu}(s), u^{\mu}(s)\}$ and $\{r'^{\mu}(s), u'^{\mu}(s)\}$, respectively. These can be viewed as phase-space states (of the corresponding extended GR-frames) having, respectively, 4-positions $r^{\mu}(s)$ and $r'^{\mu}(s)$ and 4-velocities $u^{\mu}(s)$ and $u'^{\mu}(s)$. Finally, let us mention that transformation (40), in contrast with (15), obviously does not preserve the structure of the space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{M}, \eta)$. Nevertheless the scalar transformation law (6) is still by construction warranted, while at the same time the metric tensor satisfies by construction the TT-problem, that is, (17).

Let us now show how the matrices A^{μ}_{ν} and B^{μ}_{ν} can be explicitly determined in terms of the teleparallel transformation (17). The relevant results, which actually prescribe the general form of related NLPT, are summarized by the following proposition.

Theorem 1 (realization of the special NLPT-group $\{P_S\}$ for the TT-problem). *Let one assumes that $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$ identify, respectively, a generic curved space-time and the Minkowski space-time both parametrized in terms of orthogonal Cartesian coordinates (32) and (33).*

Then, given validity of the NLPT-Requirements #1–#5, the following propositions hold:

(P_1) *In the accessible subdomain of $(\mathbf{Q}^4, g)$ the teleparallel transformation (17) (or equivalently its inverse, i.e., (18)), relating $(\mathbf{Q}^4, g)$ with the Minkowski space-time $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$, is realized by a nonlocal point transformation of type (34) or equivalently (38) and (39), with a Jacobian M^{ν}_{μ} and its inverse $(M^{-1})^{\alpha}_{\mu}$ being of forms (25) and (26), respectively. This is required to* *satisfy the NLPT 4-tensor laws prescribed by the matrix equation*

$$g_{\mu\nu}(r) = \left(M^{-1}\right)^{\alpha}_{\mu}\left(r, r'\right)\left(M^{-1}\right)^{\beta}_{\nu}\left(r, r'\right)\eta_{\alpha\beta}, \tag{41}$$

and similarly its inverse (see (18)) where $g_{\mu\nu}(r)$ identifies a prescribed symmetric metric tensor associated with the space-time $(\mathbf{Q}^4, g)$, by assumption expressed in the Cartesian coordinates (32). Hence, $(M^{-1})^{\alpha}_{\mu}(r, r')$ necessarily coincides with the Jacobian matrix of the TT-problem (see (17)).

(P_2) *The set of special NLPTs has the structure of a group.*

Proof. Let us prove proposition (P_1). For this purpose it is sufficient to construct explicitly a possible, that is, nonunique, realization of the NLPT and the corresponding set $\{P_S\}$, satisfying (41). In fact, let us consider the equation for the infinitesimal 4-displacement dr'^{μ} (see (24)), which in validity of (37) becomes

$$dr'^{\mu} = \left[\delta^{\mu}_{\nu} + \mathscr{B}^{\mu}_{\nu}\left(r, r'\right)\right] dr^{\nu} \tag{42}$$

and similarly

$$dr^{\mu} = \left[\delta^{\mu}_{\nu} + \mathscr{A}^{\mu}_{\nu}\left(r', r\right)\right] dr'^{\nu}, \tag{43}$$

where the matrices $\mathscr{B}^{\mu}_{\nu}(r, r')$ and $\mathscr{A}^{\mu}_{\nu}(r', r)$ are suitably related. Substituting dr'^{μ} on the rhs of the last equation and invoking the independence of the components of the infinitesimal displacement dr^{μ}, this means for consistency that the covariant components of the metric tensor, that is, $g_{\mu\nu}(r)$ and, respectively, $g'_{\mu\nu}(r') \equiv \eta_{\mu\nu}$, must satisfy the NLPT 4-tensor law (41). Such a tensor equation delivers, therefore, a set of 10 algebraic equations. Their solution can be determined in a straightforward way for the 16 components of the matrix $\mathscr{B}^{\mu}_{\nu}(r, r')$. For example, one of these equations reads

$$\begin{aligned} g_{00}(r) = {} & \left[1 + \mathscr{B}^{0}_{0}\left(r, r'\right)\right]^2 - \left[\mathscr{B}^{1}_{0}\left(r, r'\right)\right]^2 \\ & - \left[\mathscr{B}^{2}_{0}\left(r, r'\right)\right]^2 - \left[\mathscr{B}^{3}_{0}\left(r, r'\right)\right]^2. \end{aligned} \tag{44}$$

The remaining equations following from (41) are not reported here for brevity.

One can nevertheless show that the solution to this set is nonunique. In fact, due to the freedom in the choice of the matrix elements of $\mathscr{B}^{\mu}_{\nu}(r, r')$, the latter can in principle be chosen arbitrarily by suitably prescribing appropriate components of the same matrix. A particular solution is obtained, for example, by requiring validity of the constraint equations

$$\begin{aligned} & \mathscr{B}^{0}_{1}\left(r, r'\right) = \mathscr{B}^{0}_{3}\left(r, r'\right) = \mathscr{B}^{1}_{0}\left(r, r'\right) = \mathscr{B}^{1}_{3}\left(r, r'\right) \\ & \quad = 0, \\ & \mathscr{B}^{2}_{0}\left(r, r'\right) = \mathscr{B}^{2}_{1}\left(r, r'\right) = \mathscr{B}^{2}_{3}\left(r, r'\right) = \mathscr{B}^{3}_{0}\left(r, r'\right) \\ & \quad = \mathscr{B}^{3}_{1}\left(\mathrm{r}, r'\right) = 0. \end{aligned} \tag{45}$$

The surviving components of $\mathscr{B}^{\mu}_{\nu}$ are then determined by the same algebraic equations of the set (41). From these considerations it follows that necessarily it must be $\mathscr{B}^{\mu}_{\nu} = \mathscr{B}^{\mu}_{\nu}(r)$. In particular, here we notice that all diagonal components $\mathscr{B}^{i}_{i}(r)$ for $i = 0, 3$ can be viewed as determined, up to an arbitrary sign, by the diagonal components of the metric tensor $g_{\mu\mu}(r)$. Instead, the remaining nondiagonal matrix elements are then prescribed in terms of the nondiagonal components of the metric tensor, which follow analogously from the corresponding 6 equations of the set. Then, both the 4-displacement transformations (42) and their inverse ones (43) exist and can be nonuniquely prescribed. An example of possible realization is given by

$$\begin{aligned} dr'^{0} &= \left[1 + \mathscr{B}^{0}_{0}\right] dr^{0} + \mathscr{B}^{0}_{2} dr^{2}, \\ dr'^{1} &= \left[1 + \mathscr{B}^{1}_{1}\right] dr^{1} + \mathscr{B}^{1}_{0} dr^{0} + \mathscr{B}^{1}_{2} dr^{2}, \\ dr'^{2} &= \left[1 + \mathscr{B}^{2}_{2}\right] dr^{2}, \\ dr'^{3} &= \left[1 + \mathscr{B}^{3}_{3}\right] dr^{3} + \mathscr{B}^{3}_{2} dr^{2}, \end{aligned} \tag{46}$$

with determinant

$$\begin{aligned} & \begin{vmatrix} 1 + \mathscr{B}^{0}_{0} & 0 & \mathscr{B}^{0}_{2} & 0 \\ \mathscr{B}^{1}_{0} & 1 + \mathscr{B}^{1}_{1} & \mathscr{B}^{1}_{2} & 0 \\ 0 & 0 & \left[1 + \mathscr{B}^{2}_{2}\right] & 0 \\ 0 & 0 & \mathscr{B}^{3}_{2} & \left[1 + \mathscr{B}^{3}_{3}\right] \end{vmatrix} \\ & \quad = \prod_{i=0,3}\left(1 + \mathscr{B}^{i}_{i}\right), \end{aligned} \tag{47}$$

to be assumed as nonvanishing, and with inverse transformation

$$\begin{aligned} dr^{0} &= \frac{1}{1 + \mathscr{B}^{0}_{0}}\left[dr'^{0} + \frac{\mathscr{B}^{0}_{2}}{1/\left(1 + \mathscr{B}^{2}_{2}\right)} dr'^{2}\right], \\ dr^{1} &= \frac{1}{1 + \mathscr{B}^{1}_{1}}\left[dr'^{1} - \frac{\mathscr{B}^{1}_{0}}{1 + \mathscr{B}^{0}_{0}} dr'^{0}\right. \\ & \quad \left. - \frac{1}{1 + \mathscr{B}^{2}_{2}} \frac{\mathscr{B}^{1}_{0}\mathscr{B}^{0}_{2}}{1 + \mathscr{B}^{0}_{0}} dr'^{2}\right], \\ dr^{2} &= \frac{1}{1 + \mathscr{B}^{2}_{2}} dr'^{2}, \\ dr^{3} &= \frac{1}{1 + \mathscr{B}^{3}_{3}}\left[dr'^{3} - \frac{\mathscr{B}^{3}_{2}}{1 + \mathscr{B}^{2}_{2}} dr'^{2}\right]. \end{aligned} \tag{48}$$

In particular, from (48) one can easily evaluate in terms of $B^{\mu}_{\nu}(r)$ the precise expression taken by the matrix A^{μ}_{ν}. Hence one finds that necessarily $\mathscr{A}^{\mu}_{\nu} = \mathscr{A}^{\mu}_{\nu}(r)$, with $r \equiv \{r^{\mu}\}$ being now considered as prescribed by means of the NLPT (38). Finally, the corresponding finite NLPT generated by (55) and (48) can always be equivalently represented in terms of (34).

Next, dropping the assumption of validity of (45), let us prove proposition (P_2). For this purpose let us consider the two special NLPTs

$$\mathcal{J}^{\mu}_{(i)\nu} \equiv \delta^{\mu}_{\nu} + \mathcal{A}^{\mu}_{(i)\nu}\left(r', r_{(i)}\right) \tag{49}$$

which map the space-times $(\mathbf{Q}^4_{(i)}, g)$ (for $i = 1, 2$) onto $(\mathbf{M}^4, \eta)$ and where by construction the Jacobians $J^{\mu}_{(i)\nu}$ for $i = 1, 2$ admit the inverse matrices $(\mathcal{J}^{-1})^{\mu}_{(i)\rho} \equiv \delta^{\mu}_{\nu} + \mathcal{B}^{\mu}_{(i)\nu}(r, r')$. Requiring that both the corresponding admissible subsets of $(\mathbf{M}^4, \eta)$ and their intersection have a nonvanishing measure the product of two special NLPTs is defined on such a set. Its Jacobian is

$$\begin{aligned}\mathcal{J}^{\mu}_{\nu} &= \left(\delta^{\mu}_{\alpha} + \mathcal{A}^{\mu}_{(1)\alpha}\left(r', r_{(1)}\right)\right)\left(\delta^{\alpha}_{\nu} + \mathcal{B}^{\alpha}_{(2)\nu}\left(r_{(2)}, r'\right)\right) \\ &= \left(\delta^{\mu}_{\nu} + \mathcal{C}^{\mu}_{\nu}\left(r_{(1)}, r', r_{(2)}\right)\right),\end{aligned} \tag{50}$$

with $\mathcal{C}^{\mu}_{\nu}(r_{(1)}, r', r_{(2)}) \equiv \mathcal{A}^{\mu}_{(1)\nu}(r', r_{(1)}) + \mathcal{B}^{\mu}_{(2)\nu}(r_{(2)}, r') + \mathcal{A}^{\mu}_{(1)\alpha}(r', r_{(1)})\mathcal{B}^{\alpha}_{(2)\nu}(r_{(2)}, r')$. It follows that in such a circumstance the product of the two special NLPTs belongs necessarily to the same set $\{P_S\}$, which is therefore a group. □

Theorem 1 provides the formal solution to Einstein's TT-problem in the framework of the theory of NLPT. This is achieved by means of the introduction of a nonlocal phase-space transformation of type (15), which is realized by means of a special NLPT (34) and the corresponding 4-velocity transformation law (16). In this reference the following comments must be mentioned:

(i) First, the NLPT-functional setting has been prescribed in terms of the special NLPT-group $\{P_S\}$, determined here by (34) together with the *NLPT-Requirements #1–#5*.

(ii) Due to the nonuniqueness of the matrix $\mathcal{B}^{\mu}_{\nu}(r)$ solution to the TT-problem (see (41)) and of the related matrix $\mathcal{A}^{\mu}_{\nu}$, the realization of the NLPT-transformation (55) [and hence (48)] yielding the solution to the TT-problem is manifestly nonunique too. For a prescribed curved space-time $(\mathbf{Q}^4, g)$ which is parametrized in terms of the Cartesian coordinates, the ensemble of NLPT which provide particular solutions to the TT-problem will be denoted as $\{P_g\}_{\mathrm{TT}}$.

(iii) Both for (46) and for (48) the corresponding Jacobians determined by means of (36) and (37) take by construction and consistently with (27) a manifest *nongradient form*. This follows immediately from *Proposition #1* thanks to the validity of (41) and the requirement that $(\mathbf{Q}^4, g)$ is a curved space-time.

(iv) In terms of the Jacobian matrix $M^{\mu}_{\nu}(r', r)$ (and its inverse $(M^{-1})^{\mu}_{\nu}(r, r')$) (41) means that $g_{\mu\nu}(r)$ should actually satisfy the original Einstein equations (17) and (18). The latter can be interpreted as NLPT 4-tensor laws for the metric tensor $g_{\mu\nu}(r)$.

(v) Similarly and by analogy with (6) holding in the case of LPT, the validity of the scalar transformation law (7) is warranted also in the case of NLPT, thanks to the transformation law (41).

(vi) The transformation law (41) for the metric tensor can be interpreted as tensor transformation law with respect to the special NLPT-group $\{P_S\}$. This will be referred to as *NLPL 4-tensor transformation law*. In terms of the same Jacobian matrix $M^{\mu}_{\nu}(r', r)$ and its inverse $(M^{-1})^{\mu}_{\nu}(r, r')$, analogous NLPT 4-tensor laws can be set in principle for tensors of arbitrary order. Nevertheless, it must be noted that—specifically because of the validity of the same transformation law (41)—such a type of tensor transformation law cannot be fulfilled by the Riemann curvature tensor $R^{\rho}_{\sigma\mu\nu}(r)$, the reason being that it manifestly vanishes identically in the case of the Minkowski space-time.

A further issue concerns the identification of the physical domain of existence and the actual possible realization of NLPT which are implied by Theorem 1. In this regard it is obvious that NLPT, just like LPT, can actually be defined only in the accessible subdomains of $(\mathbf{Q}^4, g)$, namely, the connected subsets which in the curved space-time can be covered by time- (or space-) like world-lines $r^{\mu}(s)$ which are endowed with a finite 4-velocity. Nevertheless, the components of the same 4-velocity can still be in principle arbitrarily large, so that the corresponding world-line can be arbitrarily close to light trajectories (and therefore to the light cones).

Another aspect of the existence problem for NLPT is related to the solubility conditions of the algebraic equations arising in Theorem 1, which follow from the requirement that all components of the matrix $\mathcal{B}^{\mu}_{\nu}(r, r')$ should be real. For example, in the case of (44) the corresponding condition is determined by the inequality

$$\begin{aligned}&g_{00}(r) + \left[\mathcal{B}^{1}_{0}\left(r, r'\right)\right]^2 + \left[\mathcal{B}^{2}_{0}\left(r, r'\right)\right]^2 + \left[\mathcal{B}^{3}_{0}\left(r, r'\right)\right]^2 \\ &\quad \geq 0.\end{aligned} \tag{51}$$

It must be stressed that the validity of inequalities of this type for the remaining equations in general cannot be warranted in the whole admissible subset of the space-times $(\mathbf{Q}^4, g)$, that is, in particular in the subset in which $ds^2 > 0$. On the other hand, "*a priori*" the symmetric metric tensor $g_{\mu\nu}(r)$ must be regarded in principle as completely arbitrary. Hence it is obvious that such inequalities following from Theorem 1 cannot place any "unreasonable" physical constraint on the same tensor $g_{\mu\nu}(r)$.

In fact, consider the case in which the metric tensor $g_{\mu\nu}(r)$ has the signature $(+, -, -, -)$ and is also diagonal; namely, $g_{\mu\nu}(r) = \mathrm{diag}\{g_{00}(r), g_{11}(r), g_{22}(r), g_{33}(r)\}$. Then, necessarily the metric tensor must be such that everywhere in the same admissible subset $g_{00}(r) > 0$, while $g_{11}(r), g_{22}(r), g_{33}(r) < 0$. As a consequence the functional class $\{P_g\}_{\mathrm{TT}}$ contains transformations which may not exist everywhere in the same set. In fact, some of the inequalities of the group (51) which involve the spatial components, that is, $g_{ii}(r)$ (with $i = 1, 2, 3$), must be considered as local, that is, are subject to the condition of local validity of the same inequalities. Although NLPTs of this kind are physically admissible, the question arises whether particular solutions actually exist which are not required to fulfill the same inequalities (51). These solutions, if they actually exist, have therefore necessarily a global character; that is, they are defined everywhere in the same admissible subset of $(\mathbf{Q}^4, g)$. In view of these considerations, since the only acceptable physical restriction on $g_{\mu\nu}(r)$ concerns its signature, it can be shown that global validity is warranted everywhere in $(\mathbf{Q}^4, g)$ provided the following two sets of constraints are required to hold:

$$\left[\mathscr{B}_0^1\left(r,r'\right)\right]^2+\left[\mathscr{B}_0^2\left(r,r'\right)\right]^2+\left[\mathscr{B}_0^3\left(r,r'\right)\right]^2=0, \tag{52}$$

and in validity of the signature indicated above

$$\begin{aligned}
&\left[\mathscr{B}_1^0\left(r,r'\right)\right]^2-\left[\mathscr{B}_1^2\left(r,r'\right)\right]^2-\left[\mathscr{B}_1^3\left(r,r'\right)\right]^2 \\
&\quad\geq -\inf\left\{g_{11}\left(r\right)\right\}, \\
&\left[\mathscr{B}_2^0\left(r,r'\right)\right]^2-\left[\mathscr{B}_2^1\left(r,r'\right)\right]^2-\left[\mathscr{B}_2^3\left(r,r'\right)\right]^2 \\
&\quad\geq -\inf\left\{g_{22}\left(r\right)\right\}, \\
&\left[\mathscr{B}_3^0\left(r,r'\right)\right]^2-\left[\mathscr{B}_3^1\left(r,r'\right)\right]^2-\left[\mathscr{B}_3^2\left(r,r'\right)\right]^2 \\
&\quad\geq -\inf\left\{g_{33}\left(r\right)\right\}.
\end{aligned} \tag{53}$$

The first equations actually require these 3 independent equations

$$\mathscr{B}_0^1\left(r,r'\right)=\mathscr{B}_0^2\left(r,r'\right)=\mathscr{B}_0^3\left(r,r'\right)=0 \tag{54}$$

to apply separately. Particular solutions to the components of $\mathscr{B}_\nu^\mu$ satisfying the 3 constraint equations (54) and either the 3 inequalities (53) or corresponding equations obtained replacing the inequality symbol with = will be denoted, respectively, as *partially unconditional* or *unconditional* solutions. In both cases it is immediately shown that these solutions are nonunique, even if in all cases the transformation matrix is again a local function of r; that is, $\mathscr{B}_\nu^\mu = \mathscr{B}_\nu^\mu(r)$. In particular, here we notice that all the diagonal components $B_i^i(r)$ for $0 = 1, 3$ can be viewed as determined, up to an arbitrary sign, by the diagonal components of the metric tensor $g_{\mu\nu}(r)$. Instead, the remaining nondiagonal matrix elements are then prescribed in terms of the nondiagonal components of the metric tensor, which follow analogously from the set of equations mentioned in Theorem 1. In validity of the constraints given above, that is, both for partially unconditional or for unconditional particular solutions, the 4-displacement transformations (42) become

$$\begin{aligned}
dr'^0&=\left[1+\mathscr{B}_0^0\right]dr^0+\mathscr{B}_1^0dr^1+\mathscr{B}_2^0dr^2+\mathscr{B}_3^0dr^3, \\
dr'^1&=\left[1+\mathscr{B}_1^1\right]dr^1+\mathscr{B}_2^1dr^2+\mathscr{B}_3^1dr^3, \\
dr'^2&=\left[1+\mathscr{B}_2^2\right]dr^2+dr^1\mathscr{B}_1^2+\mathscr{B}_3^2dr^3, \\
dr'^3&=\left[1+\mathscr{B}_3^3\right]dr^3+\mathscr{B}_1^3dr^1+\mathscr{B}_2^3dr^2.
\end{aligned} \tag{55}$$

Similarly, one can show that also the corresponding inverse NLPTs exist.

5. Application of Special NLPT: Diagonal Metric Tensors

As pointed out above the theory of special NLPT must in principle hold also when the space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$ have different signatures. In particular, if $(\mathbf{Q}^4, g)$ coincides with a flat space-time, then it might still have in principle an arbitrary signature. To clarify this important point we present in this section a sample application. For definiteness, let us consider here a curved space-time $(\mathbf{Q}^4, g)$ which is diagonal when expressed in terms of Cartesian coordinate. The following two possible realizations are considered:

(A) $\operatorname{diag}(g_{\mu\nu}) \equiv \operatorname{diag}(S_0(r), -S_1(r), -S_2(r), -S_3(r))$,

(B) $\operatorname{diag}(g_{\mu\nu}) \equiv \operatorname{diag}(-S_0(r), S_1(r), -S_2(r), -S_3(r))$.

In both cases here the functions $S_\mu(r)$ are assumed to be *prescribed* real functions which are strictly positive for all $r \equiv r^\mu \in (\mathbf{Q}^4, g)$. Since by construction the Riemannian distance ds is left invariant by arbitrary NLPT, it follows that in the two cases either the differential identity

$$\begin{aligned}
ds^2&=S_0\left(dr^0\right)^2-S_1\left(dr^1\right)^2-S_2\left(dr^2\right)^2-S_3\left(dr^3\right)^2 \\
&=\left(dr'^0\right)^2-\left(dr'^1\right)^2-\left(dr'^2\right)^2-\left(dr'^3\right)^2
\end{aligned} \tag{56}$$

or

$$\begin{aligned}
ds^2&=S_0\left(dr^0\right)^2-S_1\left(dr^1\right)^2-S_2\left(dr^2\right)^2-S_3\left(dr^3\right)^2 \\
&=-\left(dr'^0\right)^2+\left(dr'^1\right)^2-\left(dr'^2\right)^2-\left(dr'^3\right)^2,
\end{aligned} \tag{57}$$

respectively, must hold. Let us point out the solutions to the TT-problem, that is, (17) or equivalently (18), in the two cases.

5.1. Solution to Case A. In validity of (56), if one adopts a special NLPT of the form

$$dr^\mu=\left(1+A_{(\mu)}^{(\mu)}\left(r',r\right)\right)dr'^{(\mu)}, \tag{58}$$

in terms of (18) this delivers for diagonal matrix elements $A_{(\mu)}^\mu(r', r)$ for all $\mu = 0, 3$ the equations

$$1=S_\mu\left(r\right)\left(1+A_{(\mu)}^{(\mu)}\left(r',r\right)\right)^2, \tag{59}$$

with the formal solutions

$$A_{(\mu)}^\mu\left(r',r\right)=\sqrt{\frac{1}{S_{(\mu)}\left(r\right)}}-1. \tag{60}$$

Notice that here only the positive algebraic roots have been retained in order to recover from (60) the identity transformation when letting $S_\mu(r) = 1$. From (38) one obtains therefore the special NLPT

$$r^\mu\left(s\right)=r^\mu\left(s_o\right)+\int_{s_o}^s ds\frac{dr'^{(\mu)}\left(s\right)}{ds}\sqrt{\frac{1}{S_{(\mu)}\left(r\right)}}, \tag{61}$$

where in the integrand r is to be considered as an implicit function of r' and, as indicated above, $dr'^{(\mu)}(s)/ds$ remains still arbitrary. Thus, explicit solution to (61) can be obtained by suitably prescribing $dr'^{(\mu)}(s)/ds$.

5.2. Solution to Case B. Let us now consider the solution to the TT-problem when (57) applies. For definiteness, let us look for a special NLPT of the type

$$\begin{aligned} dr^0 &= M_{(1)}^{(0)}\left(r',r\right)dr'^{(1)}, \\ dr^1 &= M_{(0)}^{(1)}\left(r',r\right)dr'^{(0)}, \\ dr^2 &= M_{(2)}^{(2)}\left(r',r\right)dr'^{(2)}, \\ dr^3 &= M_{(3)}^{(3)}\left(r',r\right)dr'^{(3)}. \end{aligned} \tag{62}$$

In terms of (18) this delivers for diagonal matrix elements $M_{(\mu)}^{(\mu)}$ the equations

$$\begin{aligned} 1 &= S_1(r)\,M_{(0)}^{(1)}\left(r',r\right)^2, \\ 1 &= S_0(r)\,M_{(1)}^{(0)}\left(r',r\right)^2, \\ 1 &= S_2(r)\,M_{(2)}^{(2)}\left(r',r\right)^2, \\ 1 &= S_3(r)\,M_{(3)}^{(3)}\left(r',r\right)^2, \end{aligned} \tag{63}$$

with the formal solutions

$$\begin{aligned} M_{(0)}^{(1)}\left(r',r\right) &= \sqrt{\frac{1}{S_{(1)}(r)}}, \\ M_{(1)}^{(0)}\left(r',r\right)^2 &= \sqrt{\frac{1}{S_{(0)}(r)}}, \\ M_{(2)}^{(2)}\left(r',r\right)^2 &= \sqrt{\frac{1}{S_{(2)}(r)}}, \\ M_{(3)}^{(3)}\left(r',r\right)^2 &= \sqrt{\frac{1}{S_{(3)}(r)}}. \end{aligned} \tag{64}$$

Hence, the corresponding NLPTs in integral form are found to be in this case

$$\begin{aligned} r^0(s) &= r^0\left(s_o\right) + \int_{s_o}^{s} ds \frac{dr'^{(1)}}{ds}\sqrt{\frac{1}{S_{(0)}(r)}}, \\ r^1(s) &= r^1\left(s_o\right) + \int_{s_o}^{s} ds \frac{dr'^{(0)}}{ds}\sqrt{\frac{1}{S_{(1)}(r)}}, \\ r^2(s) &= r^2\left(s_o\right) + \int_{s_o}^{s} ds \frac{dr'^{(2)}}{ds}\sqrt{\frac{1}{S_{(2)}(r)}}, \\ r^2(s) &= r^2\left(s_o\right) + \int_{s_o}^{s} ds \frac{dr'^{(3)}}{ds}\sqrt{\frac{1}{S_{(2)}(r)}}, \end{aligned} \tag{65}$$

where, again, in the integrands r is to be considered as an implicit function of r' while $dr'^{(i)}/ds$ has to be suitably prescribed.

Cases A and B correspond, respectively, to curved space-times having the same or different signatures with respect to the Minkowski flat space-time. Therefore, based on the discussion displayed above, it is immediately concluded that NLPT which maps mutually the two space-times indicated above must necessarily exist in all cases considered here.

Physical insight on the class of special NLPTs $\{P_S\}$ emerges from the following two statements, represented, respectively, by the following:

(i) Proposition (P_2) of Theorem 1.

(ii) The explicit realization obtained by the 4-velocity transformation laws (16) which follows in turn from (24).

Let us briefly analyze the first one, that is, in particular the fact that the set $\{P_S\}$ is endowed with the structure of a group. For this purpose, consider two arbitrary connected and time-oriented curved space-times $(\mathbf{Q}_{(i)}^4, g_{(i)})$ for $i = 1, 2$ and assume that the corresponding admissible subsets of $(\mathbf{M}^4, \eta)$, on which the same space-times are mapped by means of special NLPT, have a nonempty intersection with nonvanishing measure. The corresponding Jacobian matrices are by assumption of type (36) so that their product must necessarily belong to $\{P_S\}$ (Proposition (P_2)). The conclusion is of outmost importance from the physical standpoint. Indeed, it implies that by means of two special NLPTs it is possible to mutually map in each other *two*, in principle arbitrary, *curved space-times*. Therefore, the same theory can be applied in principle to the treatment of arbitrary curved space-times in terms of products of suitable special NLPT.

The validity of the second statement indicated above is also perspicuous. In fact, the prescription of the "geometry" of the transformed space-time $(\mathbf{Q}^4, g)$, namely, its metric tensor $g_{\mu\nu}(r)$ and the corresponding Riemann curvature tensor $R_{\sigma\mu\nu}^{\rho}(r)$, is obtained by means of a suitable nonlocal point transformation mapping the two space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}^4, \eta)$. This involves, in turn, the prescription of suitable nonuniform (i.e., position dependent) 4-velocity transformations between the same space-times. In particular, in the case of the solution indicated above for the transformation matrix $\mathscr{B}_{\nu}^{\mu}(r)$, the transformed 4-velocity has the following qualitative properties. First, its time-component, besides depending on the corresponding time-component of the Minkowski space-time, in general may carry also finite contributions which are linearly dependent on all spatial components of the Minkowskian 4-velocity. Second, the spatial components of the same 4-velocity depend linearly only on the corresponding spatial components of the Minkowskian 4-velocity and hence remain unaffected by its time-component, that is, its energy content in the Minkowski space-time.

6. Theory of General NLPT

In this section the problem is posed of the search of possible generalizations of the nonlocal point transformations (22) and (23). In the following these will be referred to as *general NLPT* and *general NLPT-theory,* respectively. More precisely, besides *NLPT-Requirements #1–#5*, the new transformations should embody the following additional optional features:

(i) *NLPT-Requirement #6*. They should realize a mapping between two in principle arbitrary connected and time-oriented 4-dimensional curved space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$.

(ii) *NLPT-Requirement #7*. The space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ should be possibly referring to arbitrary curvilinear coordinate systems which may differ in the two space-times. In addition, as for special NLPT we will require again that also general NLPTs establish between $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ suitably prescribed real diffeomorphisms of forms (22) and (23), the square brackets denoting appropriate nonlocal dependence. In particular, here $r' \equiv \{r'^\mu\}$, $r \equiv \{r^\mu\}$, $u' \equiv \{u'^\mu\}$, and $u \equiv \{u^\mu\}$, while $D'u'/Ds \equiv D'u'^\mu/Ds$ and $Du/Ds \equiv Du^\mu/Ds$ identify as usual the covariant derivatives defined in the two space-times $(\mathbf{Q}'^4, g')$ and $(\mathbf{Q}^4, g)$, respectively.

For definiteness, we will also assume that (22) and (23) are also consistent with the *NLPT-Requirements #1–#4*. It is then immediately noticed that an obvious particular realization of these transformations can be obtained simply by considering explicitly s-dependent smooth real direct and inverse transformations of the type

$$\begin{aligned} P_g &: r^\mu \longrightarrow r'^\mu = r'^\mu (r, s), \\ P_g^{-1} &: r'^\mu \longrightarrow r^\mu = r^\mu \left(r', s\right), \end{aligned} \tag{66}$$

defined for all $s \in I$. Again, for dr^μ and $u^\mu \equiv dr^\mu/ds$ transformations of types (24) and (16) are implied. However, the Jacobians are of the types $\mathcal{J}^\mu_\nu(r', s)$ and $(\mathcal{J}^{-1})^\mu_\nu(r, s)$ and read, respectively,

$$\mathcal{J}^\mu_\nu \left(r', s\right) \equiv \frac{\partial r^\mu \left(r', s\right)}{\partial r'^\nu} + \frac{\partial r^\mu \left(r', s\right)}{\partial s} g'_{\alpha\nu} u'^\alpha, \tag{67}$$

$$\left(\mathcal{J}^{-1}\right)^\mu_\nu (r, s) \equiv \frac{\partial r'^\mu (r, s)}{\partial r^\nu} + \frac{\partial r'^\mu (r, s)}{\partial s} g_{\alpha\nu} u^\alpha, \tag{68}$$

thus losing their gradient form (see (39) and (50) above). Nevertheless, it is obvious that transformations of the type indicated above generally imply the violation of the Riemann-distance constraint (30) (see *NLPT-Requirement #4*).

On the other hand, once the implications of the same equation are properly taken into account the representation problem posed here can be readily solved. Consider in fact again (30). Due to the arbitrariness of $r \equiv \{r^\mu\}$ and of s and dr^μ it follows that the same equation requires simultaneously that

$$\begin{aligned} dr^\mu &= M^\mu_{(g)\nu} dr'^\nu, \\ dr'^\mu &= \left(M^{-1}_{(g)}\right)^\mu_\nu dr^\nu, \end{aligned} \tag{69}$$

$$\begin{aligned} g_{\mu\nu} (r) &= \left(M^{-1}_{(g)}\right)^\alpha_\mu \left(M^{-1}_{(g)}\right)^\beta_\nu g'_{\alpha\beta} \left(r'\right), \\ g'_{\mu\nu} \left(r'\right) &= M^\alpha_{(g)\mu} M^\beta_{(g)\nu} g_{\alpha\beta} (r) \end{aligned} \tag{70}$$

must hold, with $M^\mu_{(g)\nu}$ denoting a suitable and still undetermined real Jacobian matrix and $(M^{-1}_{(g)})^\mu_\nu$ being its inverse. Therefore, (69) imply that

$$\begin{aligned} M^\mu_{(g)\nu} (s) &= M^\mu_{(g)\nu} \left(r' (s), r (s)\right), \\ \left(M^{-1}_{(g)}\right)^\mu_\nu (s) &= \left(M^{-1}_{(g)}\right)^\mu_\nu \left(r (s), r' (s)\right); \end{aligned} \tag{71}$$

that is, the Jacobian matrices can only be functions of $r'(s)$ or, respectively, $r(s)$. More precisely, on the rhs of the first (second) equation $r(s)$ $(r'(s))$ must be considered as a function of $r'(s)$ (resp., of $r(s)$) determined by means of an equation analogous to that holding for special NLPT. Hence (66) must recover the form

$$\begin{aligned} P_g &: r^\mu (s) = r'^\mu (s_o) + \int_{s_o}^{s} d\bar{s} M^\mu_{(g)\nu} (s) u'^\nu (\bar{s}), \\ P_g^{-1} &: r'^\mu (s) = r^\mu (s_o) + \int_{s_o}^{s} d\bar{s} \left(M^{-1}_{(g)}\right)^\mu_\nu (s) u^\nu (\bar{s}), \end{aligned} \tag{72}$$

with $M^\mu_{(g)\nu}$ being a suitable Jacobian matrix and $(M^{-1}_{(g)})^\mu_\nu$ being its inverse. Such transformations will be referred to as *general NLPT*. The corresponding phase-space map analogous to (40), namely,

$$\begin{aligned} &\left\{r^\mu (s), u^\mu (s)\right\} \longrightarrow \\ &\left\{r'^\mu (s), u'^\mu (s)\right\} \\ &\quad = \left\{r'^\mu \left\{r (s), [r, u]\right\}, \left(M^{-1}_{(g)}\right)^\mu_\nu (s) u^\nu (s)\right\}, \\ &\left\{r'^\mu (s), u'^\mu (s)\right\} \longrightarrow \\ &\left\{r^\mu (s), u^\mu (s)\right\} \\ &\quad = \left\{r^\mu \left\{r' (s), \left[r', u'\right]\right\}, M^\mu_{(g)\nu} (s) u'^\nu (s)\right\}, \end{aligned} \tag{73}$$

will be denoted as *general NLPT-phase space map*. Then the following result holds.

Theorem 2 (realization of the general NLPT-group $\{P_g\}$). *The group $\{P_g\}$ of general NLPTs of type (71) can always be realized by means of Jacobians $M^\mu_{(g)\nu}$ and $(M^{-1}_{(g)})^\mu_\nu$ of the form*

$$\begin{aligned} M^\mu_{(g)\nu} &= \frac{\partial g^\mu_A \left(r'\right)}{\partial r'^\nu} + A^\mu_{(g)\nu} \left(r', r\right), \\ \left(M^{-1}_{(g)}\right)^\mu_\nu &= \frac{\partial f^\mu_A (r)}{\partial r^\nu} + B^\mu_{(g)\nu} \left(r, r'\right), \end{aligned} \tag{74}$$

with $A^\mu_{(g)\nu}(r', r)$ and $B^\mu_{(g)\nu}(r, r')$ being suitable transformation matrices. As a consequence, an arbitrary general NLPT can be represented as

$$\begin{aligned} P_g &: r^\mu (s) = g^\mu_A \left(r' (s)\right) + \int_{s_o}^{s} d\bar{s} A^\mu_{(g)\nu} (s) u'^\nu (\bar{s}), \\ P_g^{-1} &: r'^\mu (s) = f^\mu_A (r (s)) + \int_{s_o}^{s} d\bar{s} B^{-1\mu}_{(g)\nu} (s) u^\nu (\bar{s}). \end{aligned} \tag{75}$$

Proof. In fact, given validity of (74) it follows, for example, that

$$r^{\mu}(s)=r'^{\mu}(s_o)+\int_{s_o}^{s}d\bar{s}\left[\frac{\partial g_A^{\mu}(r')}{\partial r'^{\nu}}+A^{\mu}_{(g)\nu}(r',r)\right]u'^{\nu}(\bar{s}),\tag{76}$$

where manifestly $\int_{s_o}^{s}d\bar{s}(\partial g_A^{\mu}(r')/\partial r'^{\nu})u'^{\nu}(\bar{s}) = g_A^{\mu}(r'(s)) - g_A^{\mu}(r'(s_o))$. Now we notice that it is always possible to set the initial condition so that $r'^{\mu}(s_o)=g_A^{\mu}(r'(s_o))$. This implies the validity of the first of (75). The proof of the second one is analogous. □

Notice that, in difference with the special NLPT defined by (34), transformations (75) (or equivalently (72)) now establish a diffeomorphism between two *different*, *connected*, and *time-oriented* space-times $(\mathbf{Q}^4,g)$ and $(\mathbf{Q}'^4,g')$. For definiteness let us consider the possible optional choices:

(A1) $(\mathbf{Q}^4,g)$ is an *arbitrary curved space-time*.

(A2) $(\mathbf{Q}'^4,g')$ is an *arbitrary curved space-time*.

(B1) the space-times $(\mathbf{Q}^4,g)$ and $(\mathbf{Q}'^4,g')$ are referred to as *arbitrary* GR-frames.

(B2) the same space-times $(\mathbf{Q}^4,g)$ and $(\mathbf{Q}'^4,g')$ are referred to as *different* GR-frames.

Let us consider possible particular realizations of the general NLPT given above.

The first one is obtained dropping assumption (B2), that is, requiring that the GR-frames of the two space-times $(\mathbf{Q}^4,g)$ and $(\mathbf{Q}'^4,g')$ coincide. In fact, if the coordinate systems for $(\mathbf{Q}^4,g)$ and $(\mathbf{Q}'^4,g')$ are the same ones while still remaining arbitrary, then one obtains that the constraint equations

$$\begin{aligned}g_A^{\mu}(r')&=r'^{\mu},\\ f_A^{\mu}(r)&=r^{\mu}\end{aligned}\tag{77}$$

must hold identically. In such a case, denoting the transformations matrices as

$$\begin{aligned}A^{\mu}_{(g)\nu}&=A^{\mu}_{\nu},\\ B^{\mu}_{(g)\nu}&=B^{\mu}_{\nu},\end{aligned}\tag{78}$$

transformations (75) recover the same form given by (38) and (39) above. These can be conveniently written as

$$\begin{aligned}P_g&: r^{\mu}(s)=r'^{\mu}(s)+\Delta r'^{\mu}(s),\\ P_g^{-1}&: r'^{\mu}(s)=r^{\mu}(s)+\Delta r^{\mu}(s),\end{aligned}\tag{79}$$

with $\Delta r'^{\mu}(s)$ and $\Delta r^{\mu}(s)$ identifying the nonlocal displacements

$$\begin{aligned}\Delta r'^{\mu}(s)&=\int_{s_o}^{s}d\bar{s}A^{\mu}_{\nu}(s)u'^{\nu}(\bar{s}),\\ \Delta r^{\mu}(s)&=\int_{s_o}^{s}d\bar{s}B^{\mu}_{\nu}(s)u^{\nu}(\bar{s}).\end{aligned}\tag{80}$$

Therefore (75) in validity of (77) identify again a special NLPT belonging to the group $\{P_S\}$ (see also Theorem 1). From this conclusion the relationship between general and special NLPT is immediately inferred. In fact, it is obvious that for an arbitrary general NLPT the relationship existing between the Jacobians $M^{\mu}_{(g)\nu}$ and M^{μ}_{ν}, as well as the corresponding transformation matrices $A^{\mu}_{(g)\nu}(r',r)$ and $A^{\mu}_{\nu}(r',r)$, is simply provided by the matrix equation

$$M^{\mu}_{(g)\nu}=M^{\mu}_{\alpha}J^{\alpha}_{\nu},\tag{81}$$

with $J^{\alpha}_{\nu}\equiv\partial g_A^{\alpha}(r')/\partial r^{\nu}$ being the Jacobian of a suitable LPT.

Another interesting realization occurs when the space-time $(\mathbf{Q}'^4,g')$ is identified with the Minkowski space-time $(\mathbf{Q}'^4,g')\equiv(\mathbf{M}'^4,\eta'(r'))$ represented in terms of general curvilinear coordinates $r'\equiv\{r'^{\mu}\}$. In such a case its metric tensor is of the form

$$\eta'_{\mu\nu}(r')=J^{\alpha}_{\mu}(r')J^{\beta}_{\nu}(r')\eta_{\alpha\beta},\tag{82}$$

with $\eta_{\alpha\beta}$ being the corresponding Minkowski metric tensor in orthogonal Cartesian coordinates. The corresponding NLPT 4-tensor laws (70) become now

$$\begin{aligned}g_{\mu\nu}(r)&=\left(M^{-1}_{(g)}\right)^{\alpha}_{\mu}\left(M^{-1}_{(g)}\right)^{\beta}_{\nu}\eta'_{\alpha\beta}(r'),\\ \eta'_{\mu\nu}(r')&=M^{\alpha}_{(g)\mu}M^{\beta}_{(g)\nu}g_{\alpha\beta}(r),\end{aligned}\tag{83}$$

which are analogous to (41) (see Theorem 1). However, remarkably, the corresponding coordinate transformations become now—in difference with the special NLPT introduced above—of the general NLPT type (75).

It is interesting to stress that the same conclusions, that is, in particular equations (72), can actually be recovered following an alternative route. This is obtained by introducing suitable prescriptions on transformations (22) and (23). Consider in fact the following possible realization of the said maps:

$$\begin{aligned}r^{\mu}&=g_A^{\mu}(r')+\int_{s_o}^{s}d\bar{s}g_B^{\mu}\left(r'(\bar{s}),u'(\bar{s}),\frac{D'u'(\bar{s})}{D\bar{s}}\right),\\ r'^{\mu}&=f_A^{\mu}(r)+\int_{s_o}^{s}d\bar{s}f_B^{\mu}\left(r(\bar{s}),u(\bar{s}),\frac{Du(\bar{s})}{D\bar{s}}\right),\end{aligned}\tag{84}$$

where the functions $g_A^{\mu}(r')$ and $g_B^{\mu}(r'(\bar{s}),u'(\bar{s}),D'u'(\bar{s})/D\bar{s})$ and $f_A^{\mu}(r)$ and $f_B^{\mu}(r(\bar{s}),u(\bar{s}),Du(\bar{s})/D\bar{s})$ are suitably defined real and smooth 4-vector functions. Notice that by construction equations (84) are understood as being evaluated along the corresponding world-lines $r^{\mu}(s)$ and $r'^{\mu}(s)$, and therefore they realize a Lagrangian representation of the NLPT. In particular, let us assume that the acceleration 4-tensor enters most linearly; namely,

$$g_B^{\mu}\left(r'(\bar{s}),u'(\bar{s}),\frac{D'u'(\bar{s})}{D\bar{s}}\right)\equiv G^{\mu}_{k}\frac{D'u'^{k}(\bar{s})}{D\bar{s}},\tag{85}$$

$$f_B^{\mu}\left(r(\bar{s}),u(\bar{s}),\frac{Du(\bar{s})}{D\bar{s}}\right)\equiv F^{\mu}_{k}\frac{Du^{k}(\bar{s})}{D\bar{s}}.\tag{86}$$

Here, G_k^{μ} and F_k^{μ} are real functions of the forms $G_k^{\mu}(r', [r', u'])$ and $F_k^{\mu}(r, [r, u])$, respectively. Next, one notices that, thanks to the validity of the kinematic constraints (13), the acceleration 4-tensor $D'u'^k(\bar{s})/D\bar{s}$ and acceleration 4-tensor$Du^k(\bar{s})/D\bar{s}$ must necessarily satisfy constraint equations of the type

$$\begin{aligned} D'u'^{\mu} &= u'^{\nu} H'^{\mu}_{\nu} ds \equiv H'^{\mu}_{\nu} dr'^{\nu}, \\ Du^{\mu} &= u^{\nu} H^{\mu}_{\nu} ds \equiv H^{\mu}_{\nu} dr^{\nu}, \end{aligned} \tag{87}$$

with H'^{μ}_{ν} and H^{μ}_{ν} denoting suitable antisymmetric tensors, yet to be determined. As a consequence, the functional form of g_B^{μ} and $f_B'^{\mu}$ becomes of the type

$$\begin{aligned} g_B^{\mu} &= A^{\mu}_{(g)\nu} \frac{dr'^{\nu}}{ds}, \\ f_B'^{\mu} &= B^{\mu}_{(g)\nu} \frac{dr^{\nu}}{ds}, \end{aligned} \tag{88}$$

where the real matrices A^{μ}_{ν} and B^{μ}_{ν} are defined as

$$\begin{aligned} A^{\mu}_{(g)\nu} &= G_k^{\mu} H'^{k}_{\nu}, \\ B^{\mu}_{(g)\nu} &= F_k^{\mu} H^{k}_{\nu}. \end{aligned} \tag{89}$$

We remark that, despite the matrices H'^{μ}_{ν} and H^{μ}_{ν} being antisymmetric in the upper and lower indices, $A^{\mu}_{(g)\nu}$ and $B^{\mu}_{(g)\nu}$ remain in principle arbitrary, that is, without definite symmetry (or antisymmetry) index properties. In addition, both matrices $A^{\mu}_{(g)\nu}$ and $B^{\mu}_{(g)\nu}$ may still retain both local and nonlocal functional dependence. Therefore, (84) manifestly recover form (75), that is, once (74) are invoked in (72).

Concerning the realization of the general NLPT introduced here the following comments are in order.

(1) First, it must be stressed that the two involved metric tensors $g_{\mu\nu}$ and $g'_{\mu\nu}$ remain arbitrary. For example, one can always require that both metric tensors are particular solutions to the Einstein equation. In this case (70) can be interpreted as equations for the still unknown Jacobian matrix, to be determined accordingly. This includes as a particular case the one in which, for example, the transformed metric tensor $g'_{\alpha\beta}(r')$ coincides with the Minkowski metric tensor. If $g_{\mu\nu}(r)$ and $g'_{\mu\nu}(r')$ are realizations holding for the two different space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ when they are referring, respectively, to the coordinate systems r^{μ} and r'^{μ}, the tensor transformation laws (70) must hold. If the vector functions $g_A^{\mu}(r')$ and $f_A^{\mu}(r)$ are considered prescribed, then the first of these equations becomes

$$\begin{aligned} g_{\mu\nu}(r) = {} & \left[\frac{\partial g_A^{\alpha}(r')}{\partial r'^{\mu}} + A^{\alpha}_{(g)\mu}(r', r) \right] \\ & \cdot \left[\frac{\partial g_A^{\beta}(r')}{\partial r'^{\nu}} + A^{\beta}_{(g)\nu}(r', r) \right] g'_{\alpha\beta}(r'), \end{aligned} \tag{90}$$

which, for special NLPT (see, e.g., (79)), reduces simply to

$$\begin{aligned} & g_{\mu\nu}(r) \\ & = \left[\delta^{\alpha}_{\mu} + A^{\alpha}_{(g)\mu}(r', r) \right] \left[\delta^{\beta}_{\nu} + A^{\beta}_{(g)\nu}(r', r) \right] g'_{\alpha\beta}(r'). \end{aligned} \tag{91}$$

Equation (90) or alternatively (91) yields actually a set of implicit, that is, integral, equations for the components of the same matrix. The explicit construction of the solution for $A^{\beta}_{(g)\nu}$ actually requires representing it in Eulerian form. This involves as before (see related discussion in the previous section) representing the proper-time s in terms of the instantaneous 4-position $r' \equiv \{r'^{\mu}\}$, so that $s = s(r')$.

(2) Second, an alternative interpretation is the one in which one of the two metric tensors, say, $g'_{\alpha\beta}(r')$, is prescribed together with the Jacobian $M^{\alpha}_{\mu}(r')$ so that (70) provides an explicit representation for the transformed metric tensor $g_{\mu\nu}(r)$. In this case an interesting remaining issue concerns its possible identification as an admissible particular solution to the Einstein equation corresponding to prescribed physical sources.

(3) Finally, the problem of the construction of the NLPT—or, better, given validity of the representation (72), the corresponding special NLPT to which in principle it should always be possible to refer—amounts therefore to looking for the still unknown matrix $A^{\beta}_{(g)\nu}(r', r)$.

7. Physical Implications of NLPT-Theory

In this section we analyze certain physical/mathematical implications of the general NLPT determined by (75) (see Theorem 1) and the related NLPT phase-space transformations equation (40).

The first one concerns the physical interpretation of the NLPT-phase-space map (40), which concerns the existence of a classical dynamical system (CDS) which is generated by it. The existence of such a CDS is actually immediate. The conclusion follows in a straightforward way, being in fact analogous to the one realized in the context of Special Relativity by means of an s-dependent Lorentz boost (see Appendix A). For this purpose, let us notice that the NLPT-phase transformation (40) does indeed generate a CDS. In fact, consider the states $\{r^{\mu}(s), u^{\mu}(s)\}$ and $\{r'^{\mu}(s), u'^{\mu}(s)\}$ involved in the same transformation (40).

The two maps (73) are immediately determined (they are again not independent), both being prescribed for all $s_o, s \in I$. More precisely, (a) the first one is obtained by considering the state $\{r'^{\mu}(s), u'^{\mu}(s)\}$ as a prescribed function of s in a suitable interval I, so that, at all s in the same interval, $\{r^{\mu}(s), u^{\mu}(s)\}$ is uniquely determined by the same NLPT; (b) the second one is realized by the inverse transformation; namely, it is obtained instead by considering the state $\{r^{\mu}(s), u^{\mu}(s)\}$ as a prescribed function of s, while $\{r'^{\mu}(s), u'^{\mu}(s)\}$ is then determined by the corresponding NLPT. The two cases (a) and (b) identify, respectively, the *active* and *passive viewpoints for the same transformation*. More precisely, for an arbitrary NLPT phase-space transformation (40), the active viewpoint is realized by first assuming that the transformed phase-state (i.e., the transformed extended GR-frame) $\{r'^{\mu}(s), u'^{\mu}(s)\}$ is prescribed. This means that $\{r'^{\mu}(s), u'^{\mu}(s)\}$ remains in principle an arbitrary, but suitably predetermined, function of s. Thus, for example, $u'^{\mu}(s)$ can always be assumed to be constant for all s in a prescribed interval I. This permits one to uniquely ideally "measure" the time-evolution of the state $\{r^{\mu}(s), u^{\mu}(s)\}$ of the current space-time $(\mathbf{Q}^4, g)$. In the passive viewpoint, instead, the current state (i.e., the current extended GR-frame) $\{r^{\mu}(s), u^{\mu}(s)\}$ is regarded as prescribed. This point of view permits one to "measure" the behavior of the transformed state $\{r'^{\mu}(s), u'^{\mu}(s)\}$ for the same prescribed NLPT phase-space transformation (40).

Further interesting physical implications of the NLPT-theory should be mentioned.

The first one is about the physical domain of existence of NLPTs. In this regard we stress that, just as in the case of LPT, NLPTs must be defined in the accessible subdomains of $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$, namely, the connected subsets which in each space-time can be covered by time-like world-lines or their limit functions to be suitably defined. In fact, for example, in the case of light cones, NLPTs can be defined for time-like world-lines $r^{\mu}(s)$ which are endowed with a 4-velocity having arbitrarily large spatial components and/or time-components and therefore arbitrarily close to the same light trajectories. In addition, we stress that the structure of the two space-times themselves remains "*a priori*" arbitrary. Thus, for example, each of them may be characterized by different ensembles of event-horizons, while NLPTs remain defined in the subsets internal or external to the same event-horizons such that the mapped subsets have the same signature.

The second aspect concerns the role of NLPT-tensor transformation laws (69)-(70). These can be intended as prototypes of tensor transformations laws applicable to virtually arbitrary higher-rank tensors. Thus, as an illustration, let us consider the case of a 4-scalar field $\Phi(r)$, that is, a function which remains invariant under the action of an arbitrary transformation of the group $\{P_g\}$, for example, identified with the special NLPT

$$r^{\mu} \equiv r^{\mu}\left(r'^{\mu}(s), s\right) = r'^{\mu}(s) + \Delta r'^{\mu}(s), \tag{92}$$

with $\Delta r'^{\mu}(s)$ being defined by (80). Then, denoting as $\Phi'(r')$ (resp., $\Phi(r)$) the realization of the same scalar field in the GR-reference frame r'^{μ} (resp., r^{μ}), it follows that the Eulerian equation

$$\Phi'\left(r'\right) = \Phi(r) \tag{93}$$

must hold identically. On the other hand, on the rhs of the same equation $r \equiv \{r^{\mu}\}$ is to be considered a function of $r' \equiv \{r'^{\mu}\}$ when represented via the special NLPT given above. It follows that $\Phi(r(s)) \equiv \Phi(r'^{\mu}(s) + \Delta r'^{\mu}(s))$ when cast in Lagrangian form; that is, it is parametrized in terms of the world-line $r^{\mu}(s)$ or $r'^{\mu}(s)$, respectively, and the corresponding proper-time s. As a result, (93) yields also the relationship expressed in Lagrangian form, that is, in terms of the world-lines $r(s)$ and $r'(s)$. Since by construction $r(s)$ is a nonlocal function of $r'(s)$ and the initial and transformed fields $\Phi(r(s))$ must still coincide identically, that is,

$$\Phi'\left(r'(s)\right) = \Phi(r(s)) \equiv \Phi\left(r'^{\mu}(s) + \Delta r'^{\mu}(s)\right), \tag{94}$$

it follows that $\Phi'(r'(s))$ becomes necessarily a nonlocal function of $r'^{\mu}(s)$. To determine the corresponding Eulerian fields in terms of (93) it is sufficient to represent the proper-time s in terms of the instantaneous 4-position $r' \equiv \{r'^{\mu}\}$, so that $s = s(r')$. The way how this can be done, once the world-line $r'^{\mu}(s)$ is considered prescribed, is discussed in Appendix B. Once the representation $s = s(r')$ is introduced, it follows that the rhs of (94) determines actually a function of $r' \equiv \{r'^{\mu}\}$ only; namely,

$$\Phi\left(r'^{\mu} + \Delta r'^{\mu}(s)\right) \equiv \widehat{\Phi}\left(r'\right), \tag{95}$$

so that (93) implies

$$\Phi'\left(r'\right) \equiv \widehat{\Phi}\left(r'\right) \tag{96}$$

too. In other words, the scalar fields $\Phi(r)$ and hence $\Phi'(r')$ become formally a composite and nonlocal function of $r' \equiv \{r'^{\mu}\}$. However, the existence of further NLPT-tensor transformation laws must be mentioned.

7.1. NLPT Properties of the Acceleration 4-Tensor. The first one concerns the transformation properties of the acceleration 4-tensor defined in two Riemannian manifolds $\{\mathbf{Q}^4, g\}$ and $\{\mathbf{Q}'^4, g'\}$ connected by means of a general NLPT. Theorem 1 and in particular equation (C.8) can be used to determine the relationships holding between them. In fact, let us identify the acceleration 4-tensor with the covariant derivatives of u^{ν} and u'^{ν} defined in $\mathbf{Q}^4$ and $\mathbf{Q}'^4$ as

$$\begin{aligned} a^{\mu} &\equiv \frac{D}{Ds}u^{\mu}, \\ a'^{\mu} &\equiv \frac{D'}{Ds}u'^{\mu}, \end{aligned} \tag{97}$$

where D/Ds and D'/Ds are identified with the ordinary differential operators (C.6). This means that in the two space-times they must be identified, respectively, as

$$\frac{D}{Ds}u^{\nu} = \frac{d}{ds}u^{\nu} + u^{\alpha}u^{\beta}\Gamma^{\nu}_{\alpha\beta}, \tag{98}$$

$$\frac{D'}{Ds}u'^{\nu} = \frac{d}{ds}u'^{\nu} + u'^{\alpha}u'^{\beta}\Gamma'^{\nu}_{\alpha\beta}, \tag{99}$$

where $\Gamma^{\nu}_{\alpha\beta}$ and $\Gamma'^{\nu}_{\alpha\beta}$ denote the corresponding standard connections defined in the same space-times. Let us consider for definiteness (98), the other one being uniquely dependent on it (as will be obvious from the subsequent considerations). Invoking the NLPT 4-tensor laws for the 4-velocity (16) then (98) implies that

$$\begin{aligned} \frac{d}{ds}u^{\mu} = {} & M^{\mu}_{(g)\nu}\frac{D}{Ds}u'^{\nu} - M'^{\mu}_{(g)\nu}u'^{\alpha}u'^{\beta}\Gamma'^{\nu}_{\alpha\beta} \\ & + u'^{\nu}\frac{d}{ds}M'^{\mu}_{(g)\nu}, \end{aligned} \tag{100}$$

where $M^{\mu}_{(g)\nu}$ is a suitable Jacobian matrix and $(M^{-1}_{(g)})^{\mu}_{\nu}$ is its inverse which are defined according to (72). Then, based on Lemmas B.1, B.2, and C.1 reported, respectively, in Appendices A and B, it is immediately proven that, thanks to the validity of Theorem 1, the acceleration 4-tensor $(D/Ds)u^{\nu}$ and acceleration 4-tensor$(D'/Ds)u'^{\nu}$ are linearly related. In particular the following result holds.

Theorem 3 (NLPT law for the acceleration 4-tensor). *If M'^{μ}_{ν} is the Jacobian of an arbitrary NLPT, defined according to (71), and $(D/Ds)u^{\nu}$ and $(D'/Ds)u'^{\nu}$ are the acceleration 4-tensors defined according to (98) and (99), then with respect to an arbitrary NLPT of the group $\{P_g\}$ it follows that they are related by means of the NLPT 4-tensor laws:*

$$\frac{D}{Ds}u^{\mu} = M^{\mu}_{(g)\nu}\left(r',r\right)\frac{D'}{Ds}u'^{\nu}, \tag{101}$$

$$\frac{D'}{Ds}u'^{\mu} = \left(M^{-1}_{(g)}\right)^{\mu}_{\nu}\left(r,r'\right)\frac{D}{Ds}u^{\nu}. \tag{102}$$

The result is analogous to that holding for arbitrary LPTs belonging to the group $\{P_g\}$.

Proof. First it is obvious that (101) and (102) mutually imply each other so that it is sufficient to prove that one of the two actually holds. Consider then the proof of (101). First, let us invoke the transformation law for the 4-velocity (16) and invoke (98) to give

$$\begin{aligned}\frac{D}{Ds}u^{\mu} &= \frac{D}{Ds}\left[M^{\mu}_{(g)\nu}\left(r',r\right)u'^{\nu}\right]\\ &= \frac{d}{ds}\left[M^{\mu}_{(g)\nu}\left(r',r\right)u'^{\nu}\right]\\ &\quad + u'^{h}u'^{k}M^{\alpha}_{(g)h}\left(r',r\right)M^{\beta}_{(g)k}\left(r',r\right)\Gamma^{\nu}_{\alpha\beta},\end{aligned} \tag{103}$$

where the chain rule delivers

$$\begin{aligned}&\frac{d}{ds}\left[M^{\mu}_{(g)\nu}\left(r',r\right)u'^{\nu}\right]\\ &\quad = M^{\mu}_{(g)\nu}\left(r',r\right)\frac{d}{ds}u'^{\nu} + u'^{\nu}\frac{d}{ds}\left[M^{\mu}_{(g)\nu}\left(r',r\right)\right].\end{aligned} \tag{104}$$

It is immediately shown that Lemma C.1 given in Appendix C and (C.8) then imply the identity

$$\begin{aligned}&dM^{\mu}_{(g)\nu}\left(r',r\right) + \Gamma^{\mu}_{\alpha\beta}M^{\alpha}_{(g)\nu}\left(r',r\right)M^{\beta}_{(g)\gamma}\left(r',r\right)dr'^{\gamma}\\ &\quad = M^{\mu}_{(g)\gamma}\left(r',r\right)\Gamma'^{\gamma}_{\nu\beta}dr'^{\beta}.\end{aligned} \tag{105}$$

Hence the thesis is proved. Incidentally, thanks to Lemma B.1 in Appendix B, it is obvious that the same conclusion holds in the case of arbitrary LPTs belonging to group $\{P\}$. □

The following comments are in order regarding Theorem 3:

(1) Equations (101) and (102) determine the tensor transformation laws for the acceleration 4-tensor occurring in the two space-times $\{\mathbf{Q}^4, g\}$ and $\{\mathbf{Q}'^4, g'\}$.

(2) Again the Jacobian matrices $M^{\mu}_{(g)\nu}$ and $(M^{-1}_{(g)})^{\mu}_{\nu}$ can be represented in terms of the transformation matrices $A^{\mu}_{(g)\nu}(r',r)$ and $B^{\mu}_{(g)\nu}(r,r')$. The latter ones identify therefore in the Jacobian matrices the acceleration-dependent contributions arising specifically due to nonlocal dependence.

7.2. NLPT Laws of the EM Faraday Tensor. A further notable transformation law to be pointed out here concerns the EM Faraday tensor, again defined with respect to the same Riemannian manifolds $\{\mathbf{Q}^4, g\}$ and $\{\mathbf{Q}'^4, g'\}$ indicated above. As shown below this follows in a direct way from the analogous NLPT laws for the acceleration 4-tensor. Consider, in fact, for this purpose the dynamics of a charged point-particle of rest-mass m_o and electric charge q immersed in an external EM field. As is well-known, in the curved space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ this is determined by the relativistic equation of motion which in the same space-times takes, respectively, the forms

$$m_o\frac{Du^{\mu}}{Ds} = qF^{(\mathrm{ext})\mu}_{\nu}(r)\,u^{\nu}, \tag{106}$$

$$m_o\frac{D'u'^{\mu}}{Ds} = qF'^{(\mathrm{ext})\mu}_{\nu}\left(r'\right)u'^{\nu}. \tag{107}$$

Here u^{μ} and u'^{μ} and, respectively, $F^{(\mathrm{ext})\mu}_{\nu}(r)$ and $F'^{(\mathrm{ext})\mu}_{\nu}(r')$ denote in the same space-times the 4-velocities and the Faraday tensors generated by an externally produced EM field. Assuming that a general NLPT maps in each other $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$, since

$$\frac{D}{Ds}u^{\mu} = M^{\mu}_{(g)\nu}\left(r',r\right)\frac{D'}{Ds}u'^{\nu} \tag{108}$$

it must be identically that

$$\begin{aligned}&qF^{(\mathrm{ext})\mu}_{\alpha}(r)\,u^{\alpha}\\ &\quad = M^{\mu}_{(g)\beta}\left(r',r\right)qF'^{(\mathrm{ext})\beta}_{\nu}\left(M^{-1}\right)^{\nu}_{(g)\alpha}\left(r,r'\right)u^{\alpha}.\end{aligned} \tag{109}$$

Therefore, due to the arbitrariness of the 4-vector u'^{α}, the quantity $F^{(\mathrm{ext})\mu}_{\alpha}(r)$ necessarily satisfies the NLPT 4-tensor law

$$\begin{aligned}&F^{(\mathrm{ext})\mu}_{\alpha}(r)\\ &\quad = M^{\mu}_{(g)\beta}\left(r',r\right)F'^{(\mathrm{ext})\beta}_{\nu}\left(r'\right)\left(M^{-1}\right)^{\nu}_{(g)\alpha}\left(r,r'\right).\end{aligned} \tag{110}$$

Hence by construction it follows that

$$\begin{aligned}&\left(M^{-1}\right)^{k}_{(g)\mu}F^{(\mathrm{ext})\mu}_{\alpha}M^{\alpha}_{(g)j}\\ &\quad = \left(M^{-1}\right)^{k}_{(g)\mu}M^{\mu}_{(g)\beta}F'^{(\mathrm{ext})\beta}_{\nu}\left(M^{-1}\right)^{\nu}_{(g)\alpha}M^{\alpha}_{(g)j},\end{aligned} \tag{111}$$

which yields the corresponding inverse transformation law too:

$$F_{j}^{\prime(\mathrm{ext})k}\left(r'\right) = \left(M^{-1}\right)_{(g)\mu}^{k}\left(r,r'\right)F_{\alpha}^{(\mathrm{ext})\mu}\left(r\right)M_{(g)j}^{\alpha}\left(r',r\right). \tag{112}$$

Equations (110) and (112) provide the transformation equations connecting the Faraday tensors $F_{\alpha}^{(\mathrm{ext})\mu}(r)$ and $F_{\alpha}^{\prime(\mathrm{ext})\mu}(r')$ which are defined, respectively, in the two space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$. In particular we stress that on the rhs of the first equation $r \equiv \{r^{\mu}\}$ must be regarded as a nonlocal function of $r' \equiv \{r'^{\mu}\}$ whose form is determined by the same NLPT. This means that $F_{\nu}^{(\mathrm{ext})\mu}(r)$ (and conversely $F_{\alpha}^{\prime(\mathrm{ext})\beta}(r')$ when represented via the inverse transformation (112)) must be regarded in turn as a nonlocal function of r' too.

There remains an important question to answer, that is, whether the transformed Faraday tensor $F_{\alpha}^{\prime(\mathrm{ext})\beta}$ can be identified or not with an exact solution to the Maxwell equations defined in the space-time $(\mathbf{Q}^4, g)$, $F_{\nu}^{(\mathrm{ext})\mu}(r)$ being an exact solution to the same equations in $(\mathbf{Q}'^4, g')$. The answer to this question requires proving that $F_{\nu}^{(\mathrm{ext})\mu}(r)$ and $F_{\nu}^{\prime(\mathrm{ext})\mu}(r')$ are, respectively, solutions to the Maxwell equations in the two space-times, that is, that these equations are endowed with a tensor transformation law with respect to the group of general NLPTs $\{P_g\}$. The proof of this statement will be reported elsewhere.

Finally, one notices that the validity of NLPT-transformation laws for the EM Faraday tensor, represented by (110) and (112), is not completely unexpected. Indeed they appear in qualitative consistency with the famous Einstein equivalence principle (EEP, [8]) and, more precisely, with Einstein's key related conjecture which actually lies at the basis of GR, namely, that "*local effects of motion in a curved space (produced by gravitation)*" should be considered as "*indistinguishable from those of an accelerated observer in flat space*" [34, 35].

8. Application of General NLPT-Theory #1: NLPT between Diagonal Metric Tensors

The first application to be considered concerns the construction of NLPT mapping two connected and time-oriented space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ both having diagonal form with respect to suitable sets of coordinates. More precisely we will require the following:

(i) When $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ are referring to the same coordinate systems, both are realized by diagonal metric tensors

$$g_{\mu\nu}(r) \equiv \operatorname{diag}\left(S_0(r), -S_1(r), -S_2(r), -S_3(r)\right)$$
$$g'_{\mu\nu}\left(r'\right) \equiv \operatorname{diag}\left(S'_0\left(r'\right), -S'_1\left(r'\right), -S'_2\left(r'\right), -S'_3\left(r'\right)\right), \tag{113}$$

respectively. The accessible subsets are as follows: (a) for $(\mathbf{Q}'^4, g')$ it is that in which, for all $\mu = 0, 3$, $S'_{\mu}(r') > 0$; (b) for $(\mathbf{Q}^4, g)$ it is either the set in which, for all $\mu = 0, 3$, $S_{\mu}(r) > 0$ or the other one in which $S_0(r) < 0$, $S_1(r) < 0$, $S_2(r) > 0$, and $S_3(r) > 0$.

(ii) $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ are intrinsically *different*; that is, the corresponding Riemann curvature tensors $R_{\mu\nu}(r)$ and $R'_{\mu\nu}(r')$ cannot be globally mapped in each other by means of any LPT. This means that a mapping between the accessible subsets of the said space-times can only possibly be established by means of a suitable NLPT.

(iii) Two occurrences are considered: (a) the *same signature* case in which both $(\mathbf{Q}'^4, g')$ and $(\mathbf{Q}^4, g)$ have the same Lorentzian signature $(+,-,-,-)$; (b) the *opposite-signature* case in which $(\mathbf{Q}'^4, g')$ and $(\mathbf{Q}^4, g)$ have signatures $(+,-,-,-)$ and $(-,+,+,+)$, respectively.

In the case of diagonal metric tensors the tensor transformation equation (70) takes obviously the general form:

$$S_{\mu}(r) = \left(M_{(g)}^{-1}\right)_{\mu}^{\alpha}\left(r,r'\right)\left(M_{(g)}^{-1}\right)_{(\mu)}^{\alpha}\left(r,r'\right)S'_{\alpha}\left(r'\right),$$
$$S'_{\mu}\left(r'\right) = M_{(g)\mu}^{\alpha}\left(r',r\right)M_{(g)(\mu)}^{\alpha}\left(r',r\right)S_{\alpha}(r), \tag{114}$$

where manifestly $M_{(g)\mu}^{\alpha}(r',r) \equiv M_{\mu}^{\alpha}(r',r)$ and $(M_{(g)}^{-1})_{\mu}^{\alpha}(r, r') = (M^{-1})_{\mu}^{\alpha}(r,r')$ correspondingly to the case of a special NLPT. For such a type of space-time in the following we intend to display a number of explicit particular solutions to (114) for the Jacobian M_{μ}^{α} and its inverse $(M^{-1})_{\mu}^{\alpha}$ and to construct also the corresponding NLPT-phase-space maps.

8.1. The Same Signature Diagonal NLPT. In the case in which $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ have the same signatures, it is immediately shown that a particular solution to (114) in the accessible subsets of $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ is provided by a diagonal Jacobian matrix, that is, of the form

$$M_{\mu}^{\alpha}\left(r',r\right) = M_{\mu}^{\mu}\left(r',r\right)\delta_{\mu}^{\alpha} \equiv \left[\delta_{\mu}^{\mu} + A_{\mu}^{\mu}\left(r',r\right)\right]\delta_{\mu}^{\alpha}. \tag{115}$$

Indeed from (114) one finds

$$M_{(\mu)}^{\mu}\left(r',r\right) = \frac{1}{\left(M^{-1}\right)_{(\mu)}^{\mu}} = \sqrt{\frac{S'_{\mu}\left(r'\right)}{S_{(\mu)}(r)}}, \tag{116}$$

where $S'_{\mu}(r')/S_{(\mu)}(r) > 0$ in the accessible subsets. In terms of (72) one then determines the corresponding special NLPT; namely,

$$r^{\mu}(s) = r'^{\mu}\left(s_o\right) + \int_{s_o}^{s} d\bar{s}\sqrt{\frac{S'_{\mu}\left(r'\right)}{S_{(\mu)}(r)}}u'^{\mu}(\bar{s}),$$
$$r'^{\mu}(s) = r^{\mu}\left(s_o\right) + \int_{s_o}^{s} d\bar{s}\sqrt{\frac{S_{(\mu)}\left(r'\right)}{S'_{\mu}\left(r'\right)}}u^{\mu}(\bar{s}), \tag{117}$$

as well as the corresponding 4-velocity transformation.

Let us now consider a possible physical realization for the space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ and the corresponding metric tensors $g_{\mu\nu}(r)$ and $g'_{\mu\nu}(r')$, respectively. Examples are provided by the Schwarzschild or alternatively the Reissner-Nordström space-times, both being characterized by a single event-horizon. In terms of the spherical coordinates (r, ϑ, φ) an analogous (*Schwarzschild-analogue*) representation holds of the form $g_{\mu\nu}(r) \equiv \operatorname{diag}((S_0(r), -S_1(r), -S_2(r), -S_3(r)))$ with

$$\begin{aligned} S_0(r) &= f(r), \\ S_1(r) &= \frac{1}{f(r)}, \\ S_2(r) &= r^2, \\ S_3(r) &= r^2 \sin^2\vartheta, \end{aligned} \tag{118}$$

where in the two cases $f(r)$ is identified, respectively, with

$$\begin{aligned} f(r) &= \left(1 - \frac{r_s}{r}\right), \\ f(r) &= \left(1 - \frac{r_s}{r} + \frac{r_Q^2}{r^2}\right). \end{aligned} \tag{119}$$

Here, $r_s = 2GM/c^2$ is the Schwarzschild radius and $r_Q = \sqrt{Q^2 G/4\pi\varepsilon_0 c^4}$ is a characteristic length scale, with Q being the electric charge and $1/4\pi\varepsilon_0$ being the Coulomb coupling constant. Introducing the curvilinear coordinates

$$\left(r^0,\ r^1 \equiv r,\ r^2 \equiv r\vartheta,\ r^3 \equiv \varphi r \sin\vartheta\right), \tag{120}$$

here referred to as pseudospherical coordinates, one obtains $r^2 d\Omega^2 = (dr^2)^2 + (dr^3)^2$. It follows that, in (119), $S_2(r)$ and $S_3(r)$ are replaced with

$$\begin{aligned} S_2(r) &= 1, \\ S_3(r) &= 1. \end{aligned} \tag{121}$$

In both cases, the transformed space-time $(\mathbf{Q}'^4, g')$ is assumed to be again Schwarzschild-analogue, namely, of type (118). Expressed in the pseudospherical coordinates this is prescribed to be

$$\begin{aligned} S'_0(r') &= f'(r'), \\ S'_1(r') &= \frac{1}{f'(r')}, \\ S'_2(r') &= 1, \\ S'_3(r') &= 1. \end{aligned} \tag{122}$$

Here $f'(r')$ is assumed to be an analytic function having $n > 1$ positive simple root $r'_1 < r'_2 < \cdots < r'_n$ in the positive real axis $[0, +\infty]$ such that $f'(r') > 0$ for $r' > r'_n$. In particular, we will require that the Schwarzschild radius occurs in the interval

$$r'_1 < r_s < r'_n. \tag{123}$$

The admissible subdomains of $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$, where NLPT can possibly be established between the two space-times, are therefore defined, respectively, by the inequalities $r > r_s$ and $r' > r'_n$. In these subsets the transformation matrix $A_\mu^\nu(r', r)$ becomes

$$A_0^0(r', r) = \sqrt{\frac{f'(r')}{f(r)}} - 1, \tag{124}$$

$$A_1^1(r', r) = \sqrt{\frac{f(r)}{f'(r')}} - 1, \tag{125}$$

$$A_2^2(r', r) = \sqrt{\frac{1}{1}} - 1 = 0, \tag{126}$$

$$A_3^3(r', r) = \sqrt{\frac{1}{1}} - 1 = 0, \tag{127}$$

where in the first terms on the rhs of the previous equations the positive values of the square roots have been taken. Therefore, the NLPT corresponding to (124)–(127) is the identity transformation as far as the coordinates r^2 and r^3 are concerned. The nontrivial contributions giving rise to nonlocal terms in (72) are produced therefore only by the time and radial components of the 4-velocity, that is, u'^0 and u'^1 only. The following physical interpretation is proposed:

(i) The special NLPT corresponding to (124)–(127) is only defined in the accessible subset of the space-time, namely, when $r' > r'_n$ and $r > r_s$, respectively, occur.

(ii) The effect of the special NLPT produced by (124)–(127) is that of mapping the accessible subsets of Schwarzschild or Reissner-Nordström space-time in the corresponding accessible subset of a Schwarzschild-analogue space-time. The basic feature of the transformed space-time is that of exhibiting $n > 1$ event-horizon instead of a single one as in the initial space-time.

(iii) The physical origin for the generation of such an effect is the special NLPT introduced here, which in turn arises when nonlocal effects are included in (72) which are carried out only by the time and radial components of the 4-velocity. In particular, assuming that the NLPT is of the form determined according to requirements (85) it follows that (124)–(127) correspond to the case in which only a tangential acceleration 4-tensor $a'^\mu = D'u'^\mu/Ds$ can occur, namely, in which its only nonvanishing components correspond to $\mu = 2, 3$.

A final remark must be made concerning the limit $\lim_{r' \to r'^{(+)}_*}$ in (125) and, respectively, $\lim_{r \to r^{(+)}_*}$ in (124), where r'_* and r_* are the largest roots of the equations $f'(r') = 0$

and $f(r) = 0$. In terms of the pseudospherical coordinates the previous limits do not exist and therefore the limit NLPT is not defined on the event-horizons. Nevertheless, these divergences can be cured by preliminarily recurring to a suitable coordinate system, which in the case of the Schwarzschild metric can be identified with the Kruskal-Szekeres coordinates [4].

8.2. Opposite-Signature NLPT. Let us now consider the case in which $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ have opposite signatures, namely, respectively, $(-,+,+,+)$ and $(+,-,-,-)$, while the metric tensors are still diagonal when expressed with respect to the same coordinate systems, that is, are in diagonal form. It follows that in the accessible subset of $(\mathbf{Q}'^4, g')$ it occurs, respectively, that

$$\begin{gathered} S_0(r) < 0, \\ S_1(r) < 0. \end{gathered} \tag{128}$$

In this case it is immediately shown that in the accessible subsets of $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ a particular realization is provided by a Jacobian matrix of the form

$$\begin{aligned} M_1^0(r', r) &= \frac{1}{(M^{-1})_0^1} = \sqrt{-\frac{S_1'(r')}{S_0(r)}}, \\ M_0^1(r', r) &= \frac{1}{(M^{-1})_1^0} = \sqrt{-\frac{S_0'(r')}{S_1(r)}}, \\ M_2^2(r', r) &= \frac{1}{(M^{-1})_3^3} = \sqrt{\frac{S_2'(r')}{S_2(r)}}, \\ M_3^3(r', r) &= \frac{1}{(M^{-1})_3^3} = \sqrt{\frac{S_3'(r')}{S_3(r)}}, \end{aligned} \tag{129}$$

where $-S_1(r)/S_0'(r') > 0$ in the accessible subsets. The corresponding special NLPT follows immediately from (72). Once again a possible application is provided by Schwarzschild-analogue space-times. More precisely let us consider the case in which the following occurs:

(A) The space-time $(\mathbf{Q}'^4, g')$ is assumed to be again Schwarzschild-analogue of type (118), so that in pseudospherical coordinates it is given again by (122). In particular in the accessible subset of $(\mathbf{Q}'^4, g')$ we will require

$$\begin{gathered} S_0'(r') = f'(r') > 0, \\ S_1'(r') = \frac{1}{f'(r')} > 0. \end{gathered} \tag{130}$$

(B) The space-time $(\mathbf{Q}^4, g)$ is the Schwarzschild one, the accessible subset being such that

$$\begin{gathered} S_0(r) = 1 - \frac{r_s}{r} < 0, \\ S_1(r) = \frac{1}{1 - r_s/r} < 0. \end{gathered} \tag{131}$$

As a consequence, the Jacobian becomes

$$\begin{aligned} M_1^0(r', r) &= \frac{1}{(M^{-1})_0^1} = \sqrt{-\frac{1}{(1 - r_s/r) f'(r')}}, \\ M_0^1(r', r) &= \frac{1}{(M^{-1})_1^0} = \sqrt{-\left(1 - \frac{r_s}{r}\right) f'(r')}, \\ M_2^2(r', r) &= \frac{1}{(M^{-1})_3^3} = 1, \\ M_3^3(r', r) &= \frac{1}{(M^{-1})_3^3} = 1. \end{aligned} \tag{132}$$

Therefore, in this case the resulting special NLPT maps the interior domain of the Schwarzschild space-time, namely, its *black-hole domain*, onto the exterior domain of a Schwarzschild-analogue space-time. As a final comment, it must be stressed that the starting equations adopted in this section, namely, (115), can be in principle easily reformulated when arbitrary different coordinate systems are adopted for representing the two space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$. Although details are here omitted for brevity, it is worth mentioning that this extension can easily be accomplished adopting the general NLPT-theory developed here.

9. Application of General NLPT-Theory #2: Diagonalization of Metric Tensors

As a second example, the problem of diagonalization of a nondiagonal metric tensor is posed in the framework of NLPT-theory. More precisely, this concerns the construction of NLPT mapping two connected and time-oriented space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$. Here we will require that when $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ are referring to the same coordinate systems they are realized by the metric tensors

$$g_{\mu\nu}(r) \equiv \operatorname{diag}(S_0(r), -S_1(r), -S_2(r), -S_3(r)), \tag{133}$$

$$g'_{\mu\nu}(r') = \begin{vmatrix} S_0'(r') & & & S_{03}'(r') \\ & -S_1'(r') & & \\ & & -S_2'(r') & \\ S_{03}'(r') & & & -S_3'(r') \end{vmatrix}, \tag{134}$$

respectively. The accessible subsets are assumed to be both for $(\mathbf{Q}'^4, g')$ and for $(\mathbf{Q}^4, g)$ as follows: for all $\mu = 0, 3$, $S_\mu'(r') > 0$ and $S_\mu(r) > 0$.

As before, the realization of the NLPT which maps the two metric tensors is not unique. A possible choice is provided by a special NLPT of the form

$$\begin{aligned} dr^0 &= \left(1 + A^0_{(g)0}\right) dr'^0 + A^0_{(g)3} dr'^3, \\ dr^i &= \left(1 + A^i_{(g)(i)}\right) dr'^{(i)}, \end{aligned} \tag{135}$$

for $i = 1, 2, 3$, namely, such that

$$\begin{aligned} r^0(s) = r'^0(s) &+ \int_{s_o}^{s} d\bar{s} \left[A^0_{(g)0}\left(r'(\bar{s}), r(\bar{s})\right) u'^0(\bar{s}) \right. \\ &\left. + A^0_{(g)3}\left(r'(\bar{s}), r(\bar{s})\right) u'^3(\bar{s}) \right], \\ r^i(s) = r'^i(s) &+ \int_{s_o}^{s} d\bar{s} A^i_{(g)(i)}\left(r'(\bar{s}), r(\bar{s})\right) u'^{(i)}(\bar{s}), \end{aligned} \tag{136}$$

where again the indices in brackets are not subject to the summation rule. The transformation bringing r'^{μ} in r^{μ} will be referred to as *diagonalizing NLPT*. The transformation equations for the matrix elements $A^0_{(g)0}$, $A^0_{(g)3}$, and $A^i_{(g)(i)}$, for $i = 1, 2, 3$, are therefore

$$S'_j\left(r'\right) = \left(1 + A^{(j)}_{(g)(j)}\left(r', r\right)\right)^2 S_j(r), \tag{137}$$

$$\begin{aligned} S'_3\left(r'\right) &= \left(1 + A^3_{(g)3}\left(r', r\right)\right)^2 S_3(r) \\ &\quad - \left(A^0_{(g)3}\left(r', r\right)\right)^2 S_0(r), \end{aligned} \tag{138}$$

$$S'_{03}\left(r'\right) = A^0_{(g)3}\left(r', r\right) A^0_{(g)0}\left(r', r\right) S_0(r), \tag{139}$$

for $j = 0, 1, 2$. The first set of (137) has a formal solution of the type

$$A^j_{(g)(j)}\left(r', r\right) = \sqrt{\frac{S'_j\left(r'\right)}{S_{(j)}(r)}} - 1. \tag{140}$$

The third equation (139) gives then

$$A^0_{(g)3}\left(r', r\right) = \frac{S'_{03}\left(r'\right)}{S_0(r)} \left[\sqrt{\frac{S'_0\left(r'\right)}{S_0(r)}} - 1 \right]^{-1}. \tag{141}$$

Finally, (138) delivers

$$A^3_{(g)3}\left(r', r\right) = \sqrt{\frac{S'_3\left(r'\right) + \left(A^0_3\left(r', r\right)\right)^2 S_0(r)}{S_3(r)}} - 1. \tag{142}$$

The signs of the square roots in the previous equations have been chosen in such a way to recover the correct result for identity transformations.

A number of remarks must be made.

(1) Also the present application can be in principle reformulated adopting arbitrary different coordinate systems for the representation of the space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$. This ultimately involves adopting the general NLPT-theory developed here.

(2) Transformation (136) is defined provided the inequality

$$\sqrt{\frac{S'_0\left(r'\right)}{S_0(r)}} - 1 \neq 0 \tag{143}$$

holds. In this case in fact all the matrix elements A^{μ}_{ν} determined above are real and smooth functions.

(3) A solution satisfying inequality (143) can always be found by suitably prescribing $S_0(r)$ once $S'_0(r')$ is considered fixed.

(4) An alternate possibility, in case condition (143) is not satisfied, is to look for another possible realization of transformation (136). The general solution can be cast in the form

$$\begin{aligned} dr^0 &= \left(1 + A^0_{(g)0}\right) dr'^0 + A^0_{(g)3} dr'^3, \\ dr^i &= \left(1 + A^i_{(g)(i)}\right) dr'^{(i)}, \\ dr^3 &= \left(1 + A^3_{(g)3}\right) dr'^3 + A^3_{(g)0} dr'^0, \end{aligned} \tag{144}$$

for $i = 1, 2$, namely, such that

$$\begin{aligned} r^0(s) = r'^0(s) &+ \int_{s_o}^{s} d\bar{s} \left[A^0_{(g)0}\left(r'(\bar{s}), r(\bar{s})\right) u'^0(\bar{s}) \right. \\ &\left. + A^0_{(g)3}\left(r'(\bar{s}), r(\bar{s})\right) u'^3(\bar{s}) \right], \\ r^i(s) = r'^i(s) &+ \int_{s_o}^{s} d\bar{s} A^i_{(g)(i)}\left(r'(\bar{s}), r(\bar{s})\right) u'^{(i)}(\bar{s}), \\ r^3(s) = r'^3(s) &+ \int_{s_o}^{s} d\bar{s} \left[A^3_{(g)3}\left(r'(\bar{s}), r(\bar{s})\right) u'^3(\bar{s}) \right. \\ &\left. + A^3_{(g)0}\left(r'(\bar{s}), r(\bar{s})\right) u'^0(\bar{s}) \right]. \end{aligned} \tag{145}$$

The resulting equations can be immediately solved.

(5) The diagonalization of the Kerr metric tensor expressed in spherical coordinates, as well as the Kerr-Newman and analogous Kerr-like solutions, can be carried out in terms of a transformation of either type (136) or type (145).

(6) Regarding the physical interpretation of the differential equations (135) we notice that the first equation implies that the time-component of the 4-velocity in the initial frame is modified by the combined effects of time-components and 3-components of the 4-velocity in the transformed frame. In the case of the Kerr metric, in particular, the latter corresponds to an azimuthal component of the 4-velocity. Therefore, the corresponding nonlocal coordinate transformation (136) produces a modification of the coordinate time $r^0(s)$ taking into account also the contribution of the azimuthal velocity.

(7) Also for the diagonalizing NLPT a teleparallel realization can be given. This follows by identifying now the space-time $(\mathbf{Q}^4, g)$ with the Minkowski space-time. The solution

for the Jacobian of such a transformation is obtained from (140)–(142) by setting $S_\mu(r) = 1$ identically. This means that it is always possible to transform a nondiagonal metric tensor into the Minkowski one by means of the inverse diagonalizing NLPT.

(8) Finally, an interesting comparison is possible with the so-called Newman-Janis algorithm [19–21]. As is well-known (see also related discussion in Section 4) this algorithm can be used to diagonalize nondiagonal metric tensors and is frequently used in the literature for the purpose of investigating a variety of standard or nonstandard GR black-hole solutions [22, 23]. Its basic feature involves adopting a complex coordinate transformation, a feature which effectively inhibits its physical interpretation and puts in doubt its very validity. In contrast, within the present NLPT-approach, the physical consistency of the transformation approach is preserved. Hence, the present conclusions seem particularly rewarding. Indeed, based on the NLPT-approach indicated above, the difficulties and physical limitations of the complex Newman-Janis algorithm are effectively avoided by adopting the NLPT-theory. This is of paramount importance for theoretical and astrophysical applications, such as the physics around rotating black-holes and gravitational waves.

10. Application of General NLPT-Theory #3: Acceleration Effects in Schwarzschild, Reissner-Nordström, and Schwarzschild-Analogue Space-Times

The next application to be considered concerns the construction of NLPT mapping two connected and time-oriented space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ both having diagonal form with respect to suitable sets of coordinates. More precisely we will require the following:

(i) When $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ are referring to the same coordinate systems both are realized by diagonal metric tensors. The accessible subsets are as follows: (a) for $(\mathbf{Q}'^4, g')$ this is that in which, for all $\mu = 0, 3$, $S'_\mu(r') > 0$; (b) for $(\mathbf{Q}^4, g)$ it is either the set in which, for all $\mu = 0, 3$, $S_\mu(r) > 0$ or the other one in which $S_0(r) < 0$, $S_1(r) < 0$, $S_2(r) > 0$, and $S_3(r) > 0$.

(ii) $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ are intrinsically *different*, that is, that the corresponding Riemann curvature tensors $R_{\mu\nu}(r)$ and $R'_{\mu\nu}(r')$ cannot be globally mapped in each other by means of any LPT. This means that a mapping between the accessible subsets of the said space-times can only possibly be established by means of a suitable NLPT.

(iii) We will consider for definiteness only the case in which both $(\mathbf{Q}'^4, g')$ and $(\mathbf{Q}^4, g)$ have the same Lorentzian signature $(+,-,-,-)$.

Provided the metric tensors are diagonal the tensor transformation equation for the metric tensor takes the general form given again by (114), where manifestly $M^\alpha_{(g)\mu}(r', r) \equiv M^\alpha_\mu(r', r)$ and $(M^{-1}_{(g)})^\alpha_\mu(r, r') = (M^{-1})^\alpha_\mu(r, r')$ correspondingly to the case of a special NLPT. For such a type of space-time in the following we intend to display a number of explicit particular solutions to (114) for the Jacobian M^α_μ and its inverse $(M^{-1})^\alpha_\mu$ and to construct also the corresponding NLPT-phase-space maps.

In the case in which $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ have the same signatures a particular solution in the accessible subsets of $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ is provided by a diagonal Jacobian matrix of form (115). This in turn corresponds to a *diagonal NLPT* of the form

$$dr^i = M^i_{(i)}(r', r)\, dr'^{(i)} \quad (i = 0, 1, 2, 3). \tag{146}$$

Indeed, from (114) one finds

$$M^\mu_{(g)(\mu)}(r', r) = \frac{1}{\left(M^{-1}_{(g)}\right)^\mu_{(\mu)}} = \sqrt{\frac{S'_\mu(r)}{S_{(\mu)}(r')}}, \tag{147}$$

where $S_\mu(r)/S'_{(\mu)}(r') > 0$ in the accessible subsets. In terms of (101) and (102) one obtains the acceleration transformation laws

$$\begin{aligned} \frac{D}{Ds}u^\mu &= M^\mu_{(g)(\mu)}(r', r)\frac{D'}{Ds}u'^\mu, \\ \frac{D'}{Ds}u'^\mu &= \frac{1}{\left(M^{-1}_{(g)}\right)^\mu_{(\mu)}}\frac{D}{Ds}u^\mu. \end{aligned} \tag{148}$$

Let us now consider a possible physical realization for the space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$ and the corresponding metric tensors $g_{\mu\nu}(r)$ and $g'_{\mu\nu}(r')$, respectively. Here we consider examples analogous to those pointed out in Section 5. In pseudospherical coordinates (r, r^2, r^3) (see (120)) the following generic representation is assumed to hold for all of them of the form $g_{\mu\nu}(r) \equiv \operatorname{diag}((S_0(r), -S_1(r), -S_2(r), -S_3(r)))$,

$$\begin{aligned} S_0(r) &= a(r), \\ S_1(r) &= b(r), \\ S_2(r) &= g(r), \\ S_3(r) &= g(r). \end{aligned} \tag{149}$$

In particular the Schwarzschild, Reissner-Nordström, and Schwarzchild-analogue cases are obtained letting

$$\begin{aligned} a(r) &= f(r), \\ b(r) &= \frac{1}{f(r)}, \\ g(r) &= 1, \end{aligned} \tag{150}$$

where, respectively,

$$f(r)=\left(1-\frac{r_s}{r}\right)\quad \text{Case A},$$
$$f(r)=\left(1-\frac{r_s}{r}+\frac{r_Q^2}{r^2}\right)\quad \text{Case B}, \tag{151}$$
$$f(r)=\prod_{i=1,n}\left(1-\frac{r_i}{r}\right)\quad \text{Case C}.$$

Here, r_s, r_Q, and r_i (for $i = 1, n$) are suitably prescribed characteristic scale lengths; in particular

$$r_s=\frac{2GM}{c^2}, \tag{152}$$

$$r_Q=\sqrt{\frac{Q^2G}{4\pi\varepsilon_0 c^4}} \tag{153}$$

are, respectively, the Schwarzschild and Reissner-Nordström radii, with Q being the electric charge and $1/4\pi\varepsilon_0$ being the Coulomb coupling constant. In all Cases A, B, and C we will require that the function $f(r)$ defined according to (151) is strictly positive; that is, $r > r_s$ and $r > r_n$, with r_n denoting the largest root of the equation $f(r) = (1 - r_s/r + r_Q^2/r^2) = 0$ or $\prod_{i=1,n}(1 - r_i/r) = 0$. In all cases, the transformed space-time $(\mathbf{Q}'^4, g')$ when expressed in the same pseudospherical coordinates is identified either with the Minkowski space-time or with the Schwarzschild-analogue space-time, so that, respectively, either for all $\mu = 0, 3$

$$S'_\mu(r')=1, \tag{154}$$

or where (149), (150), and Case C of (151) applies. In the subsets where $f(r) > 0$ and for Case C $f'(r') > 0$ the transformation matrix $M_\mu^\nu(r', r)$ becomes

$$\begin{aligned} M_0^0(r',r)&=\sqrt{\frac{a'(r')}{a(r)}},\\ M_1^1(r',r)&=\sqrt{\frac{b'(r')}{b(r)}},\\ M_2^2(r',r)&=M_3^3(r',r)=1, \end{aligned} \tag{155}$$

where in the first terms on the rhs of the previous equations the positive values of the square roots have been taken.

Let us briefly analyze the physical implications of (155):

(i) The first one is that (155) generate a *diagonal special NLPT* in which nonlocal effects are carried only by the time and radial components of the 4-displacement, that is, of 4-velocity and correspondingly of the acceleration 4-tensor.

(ii) The corresponding NLPT which map, respectively, either the Schwarzschild (A) or the Reissner-Nordström (B) space-times onto the Minkowski (C) or Schwarzschild-analogue (D) space-times are provided in all cases by (146). In particular, the acceleration transformation (148) implies that a point-particle endowed with acceleration 4-tensor $(D'/Ds)u'^\mu$ with respect to the space-time C or D, in the space-time A or B mapped via (146), is necessarily endowed with acceleration $(D/Ds)u^\mu$ given by the same equations (i.e., (148)).

11. Application of NLPT-Theory #4: Acceleration Effects in Kerr-Newman and Kerr Space-Times

As a final example, let us consider the case of Kerr-Newman and Kerr space-times, identified here for definiteness with the primed space-time $(\mathbf{Q}'^4, g'(r'))$. In both cases, when cast in spherical coordinates (r', θ', φ') the corresponding metric tensor is of the generic nondiagonal form

$$g'_{\mu\nu}(r')=\begin{vmatrix} S'_0(r') & & & S'_{03}(r')\\ & -S'_1(r') & & \\ & & -S'_2(r') & \\ S'_{03}(r') & & & -S'_3(r')\end{vmatrix}. \tag{156}$$

For definiteness, let us first introduce the standard notations

$$\begin{aligned} \alpha&=\frac{J}{Mc},\\ \rho^2&=r^2+\alpha^2\cos^2\theta,\\ \Delta&=r^2-rr_s+\alpha^2+r_Q^2, \end{aligned} \tag{157}$$

where α identifies a constant scale length and r_s and r_Q are the Schwarzschild and Reissner-Nordström radii (see (152) and (153)). Then, the Kerr-Newman metric is defined:

$$\begin{aligned} S'_0(r')&=\frac{\Delta+\alpha^2\sin^2\theta}{\rho^2},\\ S'_1(r')&=\frac{\rho^2}{\Delta},\\ S'_2(r')&=\rho^2,\\ S'_3(r')&=\frac{\Delta}{\rho^2}\alpha^2\sin^4\theta'+\left(r'^2+\alpha^2\right)^2\frac{\sin^2\theta'}{\rho^2},\\ S'_{03}(r')&=S'_{30}(r')=\alpha\left(r'^2+\alpha^2\right)\frac{\sin^2\theta'}{\rho^2}. \end{aligned} \tag{158}$$

Instead the Kerr metric is prescribed requiring

$$\begin{aligned} S'_0(r')&=1-\frac{r_sr'}{\rho^2},\\ S'_1(r')&=\frac{\rho^2}{\Delta},\\ S'_2(r')&=\rho^2, \end{aligned}$$

$$S'_3(r') = \left(r'^2 + \alpha^2 + \frac{r_s r' \alpha^2}{\rho^2} - \frac{r_s r'}{\rho^2}\sin^2\theta'\right)\sin^2\theta',$$
$$S'_{03}(r') = S'_{30}(r') = -\frac{r_s r' \alpha}{\rho^2}\sin^2\theta'. \tag{159}$$

Let us now pose the problem of mapping either the Kerr-Newman or the Kerr space-times onto the Minkowski space-time $(\mathbf{M}^4, \eta)$. For convenience let us represent also the latter space-time in spherical coordinates. This gives for the Minkowski metric the customary diagonal representation

$$\eta_{\mu\nu}(r) \equiv \operatorname{diag}\big(S_0(r) = 1,\ -S_1(r) = -1,\ -S_2(r) = -r^2,\ -S_3(r) = -r^2\sin^2\theta\big). \tag{160}$$

A possible nonunique realization of the NLPT between the two space-times $(\mathbf{Q}'^4, g')$ and $(\mathbf{M}^4.\eta)$ indicated above, in some sense analogous to the one developed here in Section 10, is proposed here. This is provided by the *nondiagonal special NLPT* of the form

$$\begin{aligned} dr^0 &= M^0_0 dr'^0 + M^0_1 dr'^1 + M^0_3 dr'^3, \\ dr^1 &= M^1_1 dr'^1 + M^1_0 dr'^0, \\ dr^2 &= M^2_2 dr'^2, \\ dr^3 &= M^3_1 dr'^1 + M^3_3 dr'^3, \end{aligned} \tag{161}$$

subject to the validity of the constraints

$$\begin{aligned} S_0(r) M^0_0 M^0_1 - S_1(r) M^1_1 M^1_0 &= 0, \\ S_0(r) M^0_1 M^0_3 - S_3(r) M^3_3 M^3_0 &= 0. \end{aligned} \tag{162}$$

One can readily show that (161) indeed realize NLPT which mutually maps in each other the two space-times $(\mathbf{Q}'^4, g')$ and $(\mathbf{M}^4.\eta)$. For this purpose, in validity of (160), let us require that the Jacobian matrix M^μ_ν satisfies the further tensor equations

$$\begin{aligned} \left(M^0_0\right)^2 - \left(M^1_0\right)^2 &= S'_0(r'), \\ \left(M^0_3\right)^2 - \left(M^1_1\right)^2 - r^2\sin^2\theta\left(M^3_1\right)^2 &= -S'_1(r'), \\ r^2\left(M^2_2\right)^2 &= S'_2(r'), \\ \left(M^0_1\right)^2 - r^2\sin^2\theta\left(M^3_3\right)^2 &= -S'_3(r'). \end{aligned} \tag{163}$$

From (163) and (162) elementary algebra gives the general solution

$$\begin{aligned} M^0_1 &= \frac{M^1_1}{M^0_0} M^1_0, \\ M^0_3 &= \frac{r^2\sin^2\theta M^0_0 M^3_3}{M^1_1 M^1_0} M^3_0, \\ M^0_0 &= \sqrt{S'_0(r') + \left(M^1_0\right)^2}, \\ M^1_1 &= \sqrt{\frac{S'_1(r') - r^2\left(M^3_1\right)^2}{S'_0(r')}}\sqrt{S'_0(r') + \left(M^1_0\right)^2}, \\ M^3_3 &= \sqrt{S'_0(r') + \left(M^1_0\right)^2}, \end{aligned} \tag{164}$$

where the matrix elements M^1_0 and M^3_1 still remain in principle arbitrary. Notice that the ratios $S_1(r)M^1_1/S_0(r)M^0_0$ and $S_3(r)M^0_0/S_1(r)M^1_1$ read then

$$\begin{aligned} \frac{M^1_1}{M^0_0} &= \sqrt{\frac{S'_1(r') - r^2\left(M^3_1\right)^2}{S'_0(r')}}, \\ \frac{r^2\sin^2\theta M^0_0}{M^1_1} &= r^2\sin^2\theta\sqrt{\frac{S'_1(r') - r^2\left(M^3_1\right)^2}{S'_0(r')}}. \end{aligned} \tag{165}$$

Hence it follows that

$$\begin{aligned} M^0_1 &= \sqrt{\frac{S'_1(r') - r^2\left(M^3_1\right)^2}{S'_0(r')}} M^1_0, \\ M^0_3 &= r^2\sin^2\theta\sqrt{\frac{S'_1(r') - r^2\left(M^3_1\right)^2}{S'_0(r')}} \frac{M^3_3}{M^1_0} M^3_1. \end{aligned} \tag{166}$$

Notice, however, that the existence of solution (164) demands manifestly

$$S'_1(r') - r^2\left(M^3_1\right)^2 > 0 \tag{167}$$

to be interpreted as solubility condition. A number of remarks can be made.

(1) Notice that when letting in particular $M^0_1 = M^3_1$, (161) reduce to the nondiagonal special NLPT considered above in Section 9.

(2) Equations (161) imply that the time-component of the acceleration 4-tensor in the Minkowski space-time is generated by time-components and radial and tangential components in the Kerr-Newman and Kerr space-times, respectively, namely, $(D'/Ds)u'^0$, $(D'/Ds)u'^1$, and $(D'/Ds)u'^3$.

(3) Similarly, the tangential component $(D/Ds)u^3$ depends also on the radial component $(D'/Ds)u'^1$ arising in the Kerr-Newman or Kerr space-times, besides $(D'/Ds)u'^3$.

(4) Let us now consider the inverse transformations following from (161). By analogy with Application #2 (see Section 5) also in this case both the time and radial components arising in the Kerr-Newman or Kerr space-times, namely, respectively, $(D'/Ds)u'^0$ and $(D'/Ds)u'^1$, generally depend on the analogous components of the acceleration 4-tensor arising in the Minkowski space-time, namely, $(D/Ds)u^0$ and $(D/Ds)u^1$, as well as the tangential component $(D/Ds)u^3$. Thus, for example, one obtains that

$$\frac{D'}{Ds}u'^{0} = \frac{\left[M_3^3 M_1^1 (D/Ds)\, u^0 + \left(M_3^0 M_1^3 - M_3^3 M_1^0\right)(D/Ds)\, u^1 - M_1^1 M_3^0 (D/Ds)\, u^3\right]}{\left(M_0^0 M_1^1 - M_1^0 M_0^1\right) M_3^3 - M_3^0 M_1^3 M_0^1}, \tag{168}$$

and similarly the radial component reads

$$\frac{D'}{Ds}u'^{1} = \frac{1}{M_1^1}\left[\frac{D}{Ds}u^1 - M_0^1 \frac{D'}{Ds}u'^{0}\right]. \tag{169}$$

(5) In the previous equations the matrix elements M_0^1 and M_1^3 remain still in principle arbitrary, with the second one required to fulfill the inequality (167) indicated above. A further solubility condition is provided, however, by the equation for M_3^0 in (164). In fact, this is only defined provided also

$$M_0^1 \neq 0. \tag{170}$$

(6) Since due to (162) the matrix elements M_1^0 and M_3^0 become linear functions of M_0^1 and M_1^3, respectively, it follows that both the time and radial acceleration (168) and (169) strongly depend on the choices of the same parameters. Notice that, in particular, the coupling of $(D'/Ds)u'^{1}$ with nonradial components occurs *always* due to the solubility condition (170).

12. Concluding Remarks

In view of these considerations, we are now in position to draw the main conclusions.

The investigation carried out in this paper concerns a new approach to GR, here denoted as NLPT-theory, which involves the extension of the customary functional setting usually adopted in SF-GR, namely, LPT-theory. This goal is achieved by means of the introduction of a suitable family of nonlocal point transformations.

The adoption of NLPT-theory involves a departure from the standard route customarily followed in the literature for SF-GR. Indeed, the validity of SF-GR relies in particular on the principle of general covariance with respect to the group of local point transformations. The latter by construction map a given space-time in itself only. Instead, in contrast to such a limitation, NLPT-theory allows one—by means of appropriate NLPT—to map in each other two intrinsically different and virtually arbitrary curved or flat space-times $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g')$. These are characterized by intrinsically different Riemann curvature tensors, so that in particular one of the two space-times can, for example, be identified with the flat Minkowski space-time $(\mathbf{M}^4, \eta')$.

As shown in this paper the adoption of NLPT-theory permits reaching an answer to physical issues which are of critical importance in GR. These include the following:

(1) The first point is the solution to the teleparallel transformation problem (TT-problem) arising in the context of Einstein's teleparallel approach to GR. This concerns the determination of the transformation matrix connecting the metric tensors of curved and flat space-times when expressed in terms of the same orthogonal Cartesians coordinates (see related discussion in Theorem 1). As shown in Section 4 this involves the construction of a suitable special NLPT (nonlocal point transformation). In particular it is found that by means of suitable assumptions (see *NLPT-Requirements #1–#5*) the TT-problem can be solved in terms of a suitable class of nonlocal point transformations, referred to as special NLPT (see Theorem 1), connecting the 4-positions in the two space-times. Such a transformation is necessarily a real one and involves both local and nonlocal dependence in terms of both 4-position and 4-velocity. The transformation laws for the corresponding infinitesimal 4-displacements have a tensor character, referred to here as NLPT 4-tensor laws. The latter transformations imply the validity of analogous NLPT 4-tensor laws for the corresponding 4-velocities.

(2) Second one is the construction, based on two optional additional requirements (*NLPT-Requirements #6* and *#7*), of the general form of NLPT, denoted here as general NLPT (see Theorem 2) and its application to the determination of the mappings between different curved space-times (see Sections 8 and 9), with particular reference to space-times which are represented in different coordinate systems and also possibly exhibit nondiagonal metric tensors. Such types of transformations are applied first to the diagonalization problem of metric tensors associated with curved space-times. Its basic feature is permitting one to transform mutually, by means of real nonlocal point transformations, nondiagonal metric tensors—such as those associated with rotating black-holes, like the Kerr solution—with a diagonal metric tensor corresponding to a spherically symmetric and stationary configuration. This approach avoids the adoption of complex-variable transformations, like the so-called Newman-Janis algorithm [19–21].

(3) Third point is the investigation of the NLPT-transformation laws for the acceleration 4-tensor and the EM Faraday tensor. As shown here both the acceleration 4-tensor and the Faraday tensor, when defined with respect to different curved space-times which are mapped in each other by means of a general NLPT, are shown to satisfy analogous 4-tensor NLPT-transformation laws (see Theorem 3 in Section 7 and the discussion reported in Sections 7.1 and 7.2).

(4) Selected applications involving the determination of the NLPT connecting a variety of curved space-times

and the related acceleration transformation laws have been pointed out. These include the following examples of NLPT:

(i) Realization of special NLPT for the TT-problem in the case of diagonal metric tensors (Section 5).
(ii) Realization of general NLPT in the case of diagonal metric tensors (Section 8).
(iii) Diagonalization of metric tensors (Section 9).
(iv) Acceleration effects in the Schwarzschild, Reissner-Nordström, and the Minkowski or Schwarzschild-analogue space-times (see Section 10).
(v) Acceleration effects in the Kerr-Newman or Kerr space-times and the Minkowski space-time (see Section 11).

As shown here NLPT-theory rests purely on first principles. In this regard in the present paper the following remarks have turned out to be crucial. The first one is realized by Proposition #1, namely, the fact that two different space-times, such as those occurring in Einstein's TT-problem, namely, $(\mathbf{Q}^4, g)$ and $(\mathbf{Q}'^4, g') \equiv (\mathbf{M}'^4, \eta)$, cannot be directly mapped in each other just by means of LPT. The second one is that general 4-velocity transformations of the form given by (16) manifestly can always be introduced in which the Jacobian of the transformation is not of the gradient form indicated by (3) and (4). The third fundamental remark concerns the existence of NLPT. This is actually suggested by the Einstein equivalence principle itself, a principle which also lies at the heart of his approach to the TT-problem. Such a feature appears of critical importance. In fact, as shown here, it directly leads to the identification of the precise form of the NLPT which provides an explicit solution to the same TT-problem. Finally, two characteristic aspects of the new NLPTs proposed here must be stressed. The first one is their nonlocality, which appears in their both Lagrangian and Eulerian forms. This arises because of their nonlocal dependence with respect to 4-velocity. The second, and in turn related, one is due to the form of their Jacobians. In fact, in difference with the treatment of LPT, for NLPT the same ones are not identified with gradient operators. Nevertheless, since the Jacobians still are by assumption locally velocity-independent, NLPT 4-tensor laws can actually be recovered once again. These follow from the corresponding transformation equations which hold for the infinitesimal 4-position displacements and the corresponding 4-velocities.

These conclusions strongly support the crucial importance of nonlocality effects in GR arising as a consequence of nonlocal point transformations. As pointed out in this paper a convenient framework is provided by NLPT-theory. The investigation carried out in this paper concerns basic theoretical issues and physical problems of critical importance in General Relativity. However, it appears promising for its potential implications and susceptible of a plethora of potential applications ranging from classical relativistic mechanics and electrodynamics [27–32], General Relativity, and cosmology to quantum theory of extended particle dynamics [32, 36–39], relativistic kinetic theory [30], and relativistic quantum mechanics [33] and quantum gravity.

Appendix

A. An Example from Special Relativity

In this appendix a possible realization is considered of the classical dynamical system (CDS) discussed at the beginning of Section 7. This is achieved by performing a suitable GR-frame transformation—determined by means of an s-dependent Lorentz boost—in the context of the Special Relativity (SR) setting, that is, in the time-oriented Minkowski space-time. We will distinguish—in such a process—the so-called active and passive viewpoints of the transformation, that is, in which either a point-particle evolves in time ("moves") or the reference frame itself changes, respectively. In order to define properly the two viewpoints let us introduce the 4-displacement and corresponding 4-velocity transformation of the type

$$\begin{aligned} dr^{\mu} &= \mathcal{J}^{\mu}_{\nu} dr'^{\nu}, \\ dr'^{\mu} &= \left(\mathcal{J}^{-1}\right)^{\mu}_{\nu} dr^{\nu}, \end{aligned} \tag{A.1}$$

$$\begin{aligned} u^{\mu} &= \mathcal{J}^{\mu}_{\nu} u'^{\nu}, \\ u'^{\mu} &= \left(\mathcal{J}^{-1}\right)^{\mu}_{\nu} u^{\nu}. \end{aligned} \tag{A.2}$$

Equations (A.2) can be viewed as a Gedanken experiment (GDE) advancing in time separately the states $\{r^{\mu}(s), u^{\mu}(s)\}$ and $\{r'^{\mu}(s), u'^{\mu}(s)\}$. To elucidate this point consider the following two CDSs:

$$\{r^{\mu}(s_o), u^{\mu}(s_o)\} \longleftrightarrow \{r^{\mu}(s), u^{\mu}(s)\}, \tag{A.3}$$

$$\left\{r'^{\mu}(s_o), u'^{\mu}(s_o)\right\} \longleftrightarrow \left\{r'^{\mu}(s), u'^{\mu}(s)\right\}, \tag{A.4}$$

which are assumed to be prescribed for all s_o, $s \in I$. Assuming validity of (A.1) and (A.2) it follows that the two states $\{r^{\mu}(s), u^{\mu}(s)\}$ and $\{r'^{\mu}(s), u'^{\mu}(s)\}$ are manifestly not independent. Indeed, the same CDSs are not independent, as it follows at once by direct inspection of (A.2) and (A.1). In particular, the first one (A.3) and, respectively, the second one (A.4) are obtained by considering the state $\{r'^{\mu}(s), u'^{\mu}(s)\}$ (or correspondingly $\{r^{\mu}(s), u^{\mu}(s)\}$) as prescribed. The two choices will be referred to as the *active* and *passive viewpoints* in which the GDE can be considered, more precisely: (A) in the *active viewpoint* the state $\{r^{\mu}(s), u^{\mu}(s)\}$ acting on the curved ("transformed") space-time evolves in time with $\{r'^{\mu}(s), u'^{\mu}(s)\}$, the "background" state defined in the Minkowski space-time being considered a prescribed smooth s-function and generating the phase-space flow. (B) In the *passive viewpoint* the state $\{r^{\mu}(s), u^{\mu}(s)\}$ is considered a prescribed smooth function of s, so that the background state $\{r'^{\mu}(s), u'^{\mu}(s)\}$ must evolve in time accordingly.

For definiteness, let us consider for the Jacobian matrix $\mathcal{J}^{\mu}_{\nu}$, with $(\mathcal{J}^{-1})^{\mu}_{\nu}$ being its inverse, a realization which corresponds to a boost transformation, that is, a space-time rotation for which $\mathcal{J}^{\mu}_{\nu} \equiv \mathcal{J}^{\mu}_{\nu}(s)$, where

$$\mathcal{J}^{\mu}_{\nu}(s) = \begin{vmatrix} \gamma(s) & -\beta(s)\gamma(s) & 0 & 0 \\ -\beta(s)\gamma(s) & \gamma(s) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}, \tag{A.5}$$

while $\gamma(s)$ and $\beta(s)$ are the Lorentz and relativistic factors $\gamma(s) = 1/\sqrt{1-\beta^2(s)}$ and $\beta(s) \equiv |\mathbf{v}(s)|/c$ and $\mathbf{v}(s)$ denotes the spatial components of a local and nonuniform reference velocity. In particular, let us require that $\mathbf{v}(s)$ is parametrized in terms of the arc length s, to be established on a suitable time-like word-line r^{μ} (see below). It follows that by construction u^{μ} and u'^{μ} belong to different tangent spaces defined with respect to the same Minkowskian space-time, since by construction the identity

$$\eta_{\alpha\beta}\frac{dr^{\alpha}}{ds}\frac{dr^{\beta}}{ds} = \eta_{\nu\mu}\frac{dr'^{\nu}}{ds}\frac{dr'^{\mu}}{ds} \tag{A.6}$$

manifestly holds. The corresponding coordinate transformation $r^{\alpha}(s) \to r'^{\alpha}(s)$ and its inverse, both defined in (M^4,η) and generated by integrating the 4-velocity transformations (A.2) along arbitrary time-like world-lines of (M^4,η), are manifestly of the type indicated above (see (34)) and therefore identify a particular possible realization of NLPT. It follows that (34) can be interpreted as being performed as a result of the said Gedanken experiment. More precisely, (A) in the *active viewpoint* a point-particle endowed with a 4-position r'^{ν} (or r^{ν}) acquires a displacement which carries it to the transformed 4-position r^{μ} (or r'^{μ}, resp.), by means of a suitable dynamical flow of some kind producing also such a change in the particle 4-position. (B) In the *passive viewpoint* the point-particle 4-position remains invariant, while the reference frame changes in such a way that the 4-position r'^{ν} (resp., r^{ν}) is transformed into r^{μ} (r'^{μ}).

This simple example further supports the discussion reported above regarding the asserted physical inadequacy of the traditional concept of reference frame (the so-called GR-frame) adopted in particular in the context of GR, that is, of a coordinate system based on the 4-position $r \equiv \{r^{\mu}\}$ only, which is founded—in turn—on the adoption of purely *local* coordinate transformations. The rationale behind the issue considered here lies on the Einstein equivalence principle (EEP, [8]) itself. This is actually realized by two separate propositions, which in the form presently known must both be ascribed to Albert Einstein's 1907 original formulation [34] (see also [35]). In Einstein's original approach this actually is realized by the following two distinct claims stating (a) the equivalence between accelerating frames and the occurrence of gravitational fields (see also [8]) and (b) the fact that "local effects of motion in a curved space (gravitation)" should be considered as "*indistinguishable from those of an accelerated observer in flat space*" [34, 35].

B. Mathematical Preliminaries: Differential Properties of $M^{\mu}_{(g)\nu}$

In this section the relevant properties of the Jacobian $M^{\mu}_{(g)\nu}$ which characterizes general NLPT (see (72)) which is associated with a generic transformation of the group $\{P_g\}$ are summarized.

For the sake of reference, let us consider first the case in which transformations (72) reduce to the customary form of local point transformations. This case occurs manifestly if the matrices $A^{\mu}_{(g)\nu}$ and $B^{\mu}_{(g)\nu}$ vanish identically so that the transformations reduce to $C^{(k)}$-diffeomorphism (with $k \geq 3$):

$$\begin{aligned} r^{\mu}(s) &= g^{\mu}_{A}\left(r'(s)\right), \\ r'^{\mu}(s) &= f'^{\mu}_{A}(r(s)), \end{aligned} \tag{B.1}$$

while the corresponding Jacobian M'^{μ}_{ν} becomes a local $C^{(k-1)}$-function of the form $M'^{\mu}_{\nu} = M'^{\mu}_{\nu}(r')$. Hence, the differential of M'^{μ}_{ν} takes the form prescribed by the (Leibnitz) chain rule of differentiation. The following proposition holds.

Lemma B.1 (differential identity for LPT). *Given validity of (B.1) the Jacobian $M^{\mu}_{(g)\nu} = M^{\mu}_{\nu}(r')$ is identified with $C^{(k-1)}$-function:*

$$M^{\mu}_{\nu}\left(r'\right) = \frac{\partial g^{\mu}_{A}\left(r'\right)}{\partial r'^{\nu}} \tag{B.2}$$

so that the differential of the Jacobian $M^{\mu}_{\nu}(r')$ reads

$$dM^{\mu}_{\nu}\left(r'\right) = dr'^{\alpha}\frac{\partial M^{\mu}_{\nu}\left(r'\right)}{\partial r'^{\alpha}}, \tag{B.3}$$

where on the rhs $\partial M^{\mu}_{\nu}(r', r(r'))/\partial r'^{\alpha}$ denotes the partial derivative with respect to r'^{α}.

Next, let us consider the case of an arbitrary NLPT, for which the Jacobian $M^{\mu}_{(g)\nu}$ is instead of the form $M^{\mu}_{(g)\nu}(r', r)$ (see (71)), where $r \equiv r(r') \equiv \{r^{\mu}(r')\}$ and the implicit (and nonlocal) dependence in terms of $r' \equiv \{r'^{\mu}\}$ occurring via $r \equiv \{r^{\mu}\}$ is considered as prescribed via the NLPT. As an example, in the case of a special NLPT (see Section 4) it follows that

$$M^{\mu}_{(g)\nu}\left(r', r\right) \equiv M^{\mu}_{(g)\nu}\left(r', r' + \Delta r'(s)\right), \tag{B.4}$$

where $\Delta r'^{\mu}(s) \equiv \Delta r'^{\mu}$ takes the form

$$\Delta r'^{\mu} \equiv \int_{r'^{\mu}_{o}}^{r'^{\mu}} dr'^{\nu} A^{\mu}_{(g)\nu}\left(r', r' + \Delta r'\right). \tag{B.5}$$

As a consequence, invoking again (72), one obtains, respectively,

$$\begin{aligned} \frac{\partial r^{\beta}}{\partial r'^{\alpha}} &= \frac{\partial}{\partial r'^{\alpha}}\left[r'^{\beta} + \Delta r'^{\beta}(s)\right] \\ &= \delta^{\beta}_{\alpha} + A^{\beta}_{\alpha}\left(r', r' + \Delta r'\right) \equiv M^{\beta}_{(g)\alpha}\left(r', r\right), \end{aligned} \tag{B.6}$$

$$\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\left(r'\right)\right)}{\partial r^{\beta}}\right|_{r'} = \frac{\partial r'^{\beta}}{\partial r^{\alpha}}\left.\frac{\partial M^{\mu}_{(g)\nu}\left(\left(r'\right),r\left(r'\right)\right)}{\partial r'^{\beta}}\right|_{(r')} = \left(M^{-1}\right)^{\beta}_{\alpha}\left.\frac{\partial M^{\mu}_{(g)\nu}\left(\left(r'\right),r\left(r'\right)\right)}{\partial r'^{\beta}}\right|_{(r')}, \tag{B.7}$$

where we have denoted symbolically $r(r') \equiv r' + \Delta r'$. As a consequence, the following proposition, analogous to that warranted by Lemma B.1 in the case of LPTs, holds.

Lemma B.2 (differential identity for NLPT). *Given validity of Theorem 1 the differential of the Jacobian $M^{\mu}_{(g)\nu}(r',r) = M^{\mu}_{(g)\nu}(r',r(r'))$ reads*

$$dM^{\mu}_{(g)\nu}\left(r',r\left(r'\right)\right) = dr'^{\alpha}\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\left(r'\right)\right)}{\partial r'^{\alpha}}, \tag{B.8}$$

where on the rhs $\partial M^{\mu}_{(g)\nu}(r',r(r'))/\partial r'^{\alpha}$ denotes the "total" partial derivative with respect to r'^{α}, namely, defined such that the differential $dM^{\mu}_{(g)\nu}(r',r(r'))$ is written explicitly as

$$dM^{\mu}_{(g)\nu}\left(r',r\right) = dr'^{\alpha}\left[\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\left(r'\right)\right)}{\partial r'^{\alpha}}\right|_{r(r')} + \left.\frac{\partial M^{\mu}_{(g)\nu}\left(\left(r'\right),r\left(r'\right)\right)}{\partial r'^{\alpha}}\right|_{r'}\right], \tag{B.9}$$

where the partial derivatives on the rhs of the previous equation are performed, respectively, at constant $r(r')$ for the first one and at constant r' for the other one.

Proof. In fact, thanks to (B.7), the partial derivative of the Jacobian $M^{\mu}_{(g)\nu}(r',r)$ with respect to r'^{α} becomes

$$\frac{\partial}{\partial r'^{\alpha}}M^{\mu}_{(g)\nu}\left(r',r\right) = \left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r'^{\alpha}}\right|_{r} + \frac{\partial r^{\beta}}{\partial r'^{\alpha}}\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r^{\beta}}\right|_{r'}, \tag{B.10}$$

while its differential is just

$$dM^{\mu}_{(g)\nu}\left(r',r\right) = dr'^{\alpha}\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r'^{\alpha}}\right|_{r} + dr^{\beta}\left.\frac{\partial M^{\mu}_{\nu}\left(r',r\right)}{\partial r^{\beta}}\right|_{r'} = dr'^{\alpha}\left[\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r'^{\alpha}}\right|_{r} + M^{\beta}_{(g)\alpha}\left(r',r\right)\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r^{\beta}}\right|_{r'}\right]. \tag{B.11}$$

Hence due to (B.7) it follows that

$$\begin{aligned} dM^{\mu}_{(g)\nu}\left(r',r\right) &= dr'^{\alpha}\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r'^{\alpha}}\right|_{r} + dr^{\beta}\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r^{\beta}}\right|_{r'} \\ &= dr'^{\alpha}\left[\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\left(r'\right)\right)}{\partial r'^{\alpha}}\right|_{r(r')} + M^{\beta}_{(g)\alpha}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{\beta}_{\alpha}\left.\frac{\partial M^{\mu}_{(g)\nu}\left(r',r\left(r'\right)\right)}{\partial r'^{\beta}}\right|_{r'}\right], \end{aligned} \tag{B.12}$$

which manifestly implies (B.9). The rhs of the same equation coincides then with the rhs of (B.8) so that the thesis is proved. □

C. NLPT Properties of the Christoffel Symbols

Let us now inspect further mathematical implications, in part based on the lemmas presented in Appendix B, which concern the construction of the NLPT here reported. In this appendix we intend to determine in particular the transformations properties of the Christoffel symbols with respect to the general NLPT-group $\{P_g\}$. For definiteness, let us denote as $\Gamma^{\nu}_{\alpha\beta}$ and $\Gamma'^{\nu}_{\alpha\beta}$ the (initial and transformed) Christoffel symbols when referring to the two GR-reference frames r^{μ} ("initial frame") and r'^{μ} ("transformed frame"), respectively. The issue is the determination of the relationship between $\Gamma^{\nu}_{\alpha\beta}$ and $\Gamma'^{\nu}_{\alpha\beta}$ when the coordinate transformation relating the coordinates r^{μ} and r'^{μ} is suitably prescribed.

As we intend to show here, the solution to such a problem is closely related to the requirement, already included in the prescription of the group of NLPTs $\{P_g\}$, that the (initial and transformed) metric tensors $g^{ij}(r)$ and $g'^{ij}(r')$ defined, respectively, for the two Riemannian manifolds $\{\mathbf{Q}^4, g\}$ and $\{\mathbf{Q}'^4, g'\}$ are extremal; namely, they satisfy identically the *extremal conditions*

$$\nabla_k g^{ij}(r) = 0, \tag{C.1}$$

$$\nabla'_k g'^{ij}\left(r'\right) = 0. \tag{C.2}$$

Here ∇_j and ∇'_j denote as usual the covariant derivatives, defined as

$$\nabla_k g^{ij}(r) = \frac{\partial g^{ij}(r)}{\partial r^k} + \Gamma^{i}_{kl}g^{lj}(r) + \Gamma^{j}_{kl}g^{il}(r), \tag{C.3}$$

$$\nabla'_k g'^{ij}\left(r'\right) = \frac{\partial g'^{ij}\left(r'\right)}{\partial r'^k} + \Gamma'^{i}_{kl}g'^{lj}\left(r'\right) + \Gamma'^{j}_{kl}g'^{il}\left(r'\right), \tag{C.4}$$

where Γ^{i}_{kl} and Γ'^{i}_{kl} denote the initial and transformed Christoffel symbols defined on $\{\mathbf{Q}^4, g\}$ and $\{\mathbf{Q}'^4, g'\}$, respectively. We notice that (C.1) and (C.2) are manifestly equivalent to the ODEs

$$\frac{D}{Ds}g^{ij}(r(s)) = 0, \quad \frac{D'}{Ds}g'^{ij}(r'(s)) = 0, \tag{C.5}$$

once D/Ds and D'/Ds are identified with the *covariant derivatives* of $g^{ij}(r(s))$ and $g'^{ij}(r'(s))$. These are defined, respectively, in $\mathbf{Q}^4$ and $\mathbf{Q}'^4$ in terms of the differential operators acting on the covariants components $g^{ij}(r(s))$ and $g'^{ij}(r'(s))$ as

$$\frac{D}{Ds}g^{ij}(r) = u^k\nabla_k g^{ij}(r), \quad \frac{D'}{Ds}g'^{ij}(r') = u'^k\nabla'_k g'^{ij}(r'), \tag{C.6}$$

with u^k and $u'^k = (M^{-1})^k_j u'$ denoting the 4-velocities in the corresponding tangent spaces. Then the following proposition holds.

Lemma C.1 (NLPT laws for the Christoffel symbols). *Within the group $\{P_g\}$ the following two propositions hold:*

(P_1) *The extremal conditions (C.1) and (C.2) holding for the metric tensors $g^{ij}(r)$ and $g'^{ij}(r')$ are equivalent to require that the initial and transformed Christoffel symbols $\Gamma^k_{\nu\gamma}$ and $\Gamma'^k_{\nu\gamma}$ defined, respectively, on $\{\mathbf{Q}^4, g\}$ and $\{\mathbf{Q}'^4, g'\}$ satisfy the constraint differential equation*

$$dr'^s\frac{\partial}{\partial r'^s}M^\mu_{(g)\nu}(r',r) + \Gamma^\mu_{\alpha\beta}M^\alpha_{(g)\nu}(r',r)M^\beta_{(g)s}(r',r)dr'^s = M^\mu_{(g)\gamma}(r',r)\Gamma'^\gamma_{\nu s}dr'^s. \tag{C.7}$$

(P_2) *The previous equation in turn is equivalent to the equation*

$$\Gamma'^k_{\nu\gamma} = \left(M^{-1}_{(g)}\right)^k_\mu(r,r')\frac{\partial M^\mu_{(g)\nu}(r',r)}{\partial r'^\gamma} + \left(M^{-1}_{(g)}\right)^k_\mu(r,r')\Gamma^\mu_{\alpha\beta}M^\alpha_{(g)\nu}(r',r)M^\beta_{(g)\gamma}(r',r), \tag{C.8}$$

which determines the transformation laws for the transformed Christoffel symbol $\Gamma'^k_{\nu\gamma}$.

Proof. We first prove Proposition (P_2). Notice for this purpose that (C.7), due to the arbitrariness of the differential displacement dr'^s, implies also that

$$\frac{\partial}{\partial r'^s}M^\mu_{(g)\nu}(r',r) + \Gamma^\mu_{\alpha\beta}M^\alpha_\nu(r',r)M^\beta_{(g)s}(r',r) = M^\mu_{(g)\gamma}(r',r)\Gamma'^\gamma_{\nu s}, \tag{C.9}$$

which, after multiplying it term by term by $(M^{-1}_{(g)})^k_\mu(r,r')$, exchanging the indexes $s \leftrightarrow \gamma$, and recalling that $M^\mu_{(g)s}(r,r')(M^{-1}_{(g)})^k_\mu(r,r') = \delta^k_s$, reduces to (C.8).

Next we address the proof of Proposition (P_1), that is, that (C.1) is equivalent to (C.8). To start with, let us consider the definition of the covariant derivative recalled above (see (C.3)). Then, (C.1) delivers necessarily

$$\begin{aligned}\nabla_k g^{ij}(r) &= \frac{\partial g^{ij}(r)}{\partial r^k} + \Gamma^i_{kl}g^{lj}(r) + \Gamma^j_{kl}g^{il}(r) \\ &= \frac{\partial}{\partial r^k}\left[M^i_{(g)\alpha}(r',r)M^j_{(g)\beta}(r',r)g'^{\alpha\beta}(r')\right] \\ &\quad + \Gamma^i_{kl}M^l_{(g)\alpha}(r',r)M^j_{(g)\beta}(r',r)g'^{\alpha\beta}(r') \\ &\quad + \Gamma^j_{kl}M^i_{(g)\alpha}(r',r)M^l_{(g)\beta}(r',r)g'^{\alpha\beta}(r') = 0.\end{aligned} \tag{C.10}$$

Invoking now the identity

$$\frac{\partial}{\partial r^k}g'^{\alpha\beta}(r') = \left(M^{-1}_{(g)}\right)^s_k(r,r')\frac{\partial}{\partial r'^s}g'^{\alpha\beta}(r'), \tag{C.11}$$

thanks to the chain rule, it follows that the first term on the rhs of (C.10) becomes

$$\begin{aligned}&\frac{\partial}{\partial r^k}\left[M^i_{(g)\alpha}(r',r)M^j_{(g)\beta}(r',r)g'^{\alpha\beta}(r')\right] \\ &= g'^{\alpha\beta}(r')\frac{\partial}{\partial r^k}\left[M^i_{(g)\alpha}(r',r)M^j_{(g)\beta}(r',r)\right] \\ &\quad + M^i_{(g)\alpha}(r',r)M^j_{(g)\beta}(r',r)\left(M^{-1}_{(g)}\right)^s_k \\ &\quad \cdot (r,r')\frac{\partial}{\partial r'^s}g'^{\alpha\beta}(r').\end{aligned} \tag{C.12}$$

Therefore, noting that thanks to (C.2) it must be

$$\frac{\partial}{\partial r'^s}g'^{\alpha\beta}(r') = \nabla'_s g'^{\alpha\beta}(r') - \Gamma'^\alpha_{sl}g'^{l\beta}(r') + \Gamma'^\beta_{sl}g'^{\alpha l}(r'), \tag{C.13}$$

it follows that (C.10) requires necessarily the validity of the following constraint equation, obtained also upon exchanging summations indexes; namely

$$\begin{aligned}&g'^{\alpha\beta}(r')M^i_{(g)\alpha}(r',r)\frac{\partial}{\partial r^k}M^j_{(g)\beta}(r',r) + g'^{\alpha\beta}(r') \\ &\cdot M^j_{(g)\beta}(r',r)\frac{\partial}{\partial r^k}M^i_{(g)\alpha}(r',r) - M^i_{(g)q}(r',r) \\ &\cdot M^j_{(g)\beta}(r',r)\left(M^{-1}_{(g)}\right)^s_k(r,r')\Gamma'^q_{s\alpha}g'^{\alpha\beta}(r') \\ &- M^i_{(g)q}(r',r)M^j_{(g)p}(r',r)\left(M^{-1}_{(g)}\right)^s_k(r,r') \\ &\cdot \Gamma'^p_{s\beta}g'^{\alpha\beta}(r') + \Gamma^i_{kl}M^l_{(g)\alpha}(r',r)M^j_{(g)\beta}(r',r) \\ &\cdot g'^{\alpha\beta}(r') + \Gamma^j_{kl}M^i_{(g)\alpha}(r',r)M^l_{(g)\beta}(r',r) \\ &\cdot g'^{\alpha\beta}(r') = 0.\end{aligned} \tag{C.14}$$

Considering now $g'^{\alpha\beta}(r')$ as independent of the Jacobian matrix of the transformation and then thanks to the symmetry of the indexes α and β the previous equation delivers

$$M^{j}_{(g)\beta}\left(r',r\right)\left[\frac{\partial}{\partial r^{k}}M^{i}_{(g)\alpha}\left(r',r\right) - M^{i}_{q}\left(r',r\right)\left(M^{-1}\right)^{s}_{k}\left(r,r'\right)\Gamma'^{q}_{s\alpha}\right] + \Gamma^{i}_{kl}M^{l}_{\alpha}\left(r',r\right) \cdot M^{j}_{\beta}\left(r',r\right) = 0. \quad \text{(C.15)}$$

Namely, multiplying term by term by $(M^{-1})^{s}_{j}(r,r')$

$$\delta^{s}_{\beta}\left[\frac{\partial}{\partial r^{k}}M^{i}_{\alpha}\left(r',r\right) - M^{i}_{(g)q}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{s}_{k}\left(r,r'\right)\Gamma'^{q}_{s\alpha}\right] + \Gamma^{i}_{kl}M\left(g\right)^{l}_{\alpha} \cdot \left(r',r\right)M^{j}_{(g)\beta}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{s}_{j}\left(r,r'\right) = 0, \quad \text{(C.16)}$$

which yields

$$\delta^{s}_{\beta}\left[\frac{\partial}{\partial r^{k}}M^{i}_{(g)\alpha}\left(r',r\right) - M^{i}_{(g)q}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{r}_{k}\left(r,r'\right)\Gamma'^{q}_{r\alpha} + \Gamma^{i}_{kl}M^{l}_{(g)\alpha}\left(r',r\right)\right] = 0. \quad \text{(C.17)}$$

Therefore, one has that

$$\frac{\partial}{\partial r^{k}}M^{i}_{(g)\alpha}\left(r',r\right) - M^{i}_{(g)q}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{r}_{k}\left(r,r'\right)\Gamma'^{q}_{r\alpha} + \Gamma^{i}_{kl}M^{l}_{(g)\alpha}\left(r',r\right) = 0, \quad \text{(C.18)}$$

implying also

$$\left(M^{-1}_{(g)}\right)^{r}_{k}\left(r,r'\right)\frac{\partial}{\partial r'^{r}}M^{i}_{(g)\alpha}\left(r',r\right) - M^{i}_{(g)q}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{r}_{k}\left(r,r'\right)\Gamma'^{q}_{r\alpha} + \Gamma^{i}_{kl}M^{l}_{(g)\alpha}\left(r',r\right) = 0. \quad \text{(C.19)}$$

Hence it follows that

$$M^{k}_{(g)x}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{r}_{k}\left(r,r'\right)\frac{\partial}{\partial r'^{r}}M^{i}_{(g)\alpha}\left(r',r\right) - M^{i}_{(g)q}\left(r',r\right)M^{k}_{(g)x}\left(r',r\right)\left(M^{-1}_{(g)}\right)^{r}_{k}\left(r,r'\right)\Gamma'^{q}_{r\alpha} + M^{k}_{(g)x}\left(r',r\right)\Gamma^{i}_{kl}M^{l}_{(g)\alpha}\left(r',r\right) = 0, \quad \text{(C.20)}$$

so that

$$\delta^{r}_{x}\frac{\partial}{\partial r'^{r}}M^{i}_{(g)\alpha}\left(r',r\right) - M^{i}_{(g)q}\left(r',r\right)\delta^{r}_{x}\Gamma'^{q}_{r\alpha} + M^{k}_{(g)x}\left(r',r\right)\Gamma^{i}_{kl}M^{l}_{(g)\alpha}\left(r',r\right) = 0. \quad \text{(C.21)}$$

Finally, replacing the index x with k one gets

$$\frac{\partial}{\partial r'^{k}}M^{i}_{(g)\alpha}\left(r',r\right) - M^{i}_{(g)q}\left(r',r\right)\Gamma'^{q}_{k\alpha} + M^{p}_{(g)k}\left(r',r\right)M^{q}_{(g)\alpha}\left(r',r\right)\Gamma^{i}_{pq} = 0. \quad \text{(C.22)}$$

Straightforward algebra shows that this equation coincides with (C.9) and hence (C.7) too. Finally, one can show that in a similar way (C.8) implies (C.1) too. In view of the equivalence between (C.8), (C.7), and (C.8) the thesis is reached. □

The implication of Lemma C.1 is therefore that the requirements that both the initial and transformed metric tensors are extremal, that is, in the sense that the corresponding covariant derivatives vanish identically in both cases (see (C.5)), is necessarily equivalent to impose between the initial and transformed Christoffel symbols—that is, $\Gamma^{k}_{\nu\gamma}$ *and* $\Gamma'^{k}_{\nu\gamma}$ *which are defined, respectively, on* $\{\mathbf{Q}^4, g\}$ *and* $\{\mathbf{Q}'^4, g'\}$—the transformation law (C.8). The conclusion, as shown in Section 7.1, is important to establish the tensor transformation laws which hold for the acceleration 4-tensor for arbitrary NLPT belonging to the group $\{P_g\}$.

As a final comment, we remark that if the space-time $\{\mathbf{Q}'^4, g'\}$ is identified with the flat Lorentzian Minkowski space-time $\{\mathbf{M}^4, \eta\}$ then the following proposition holds.

Corollary to Lemma C.1 (case of Minkowski space-time). *Within the group* $\{P_g\}$ *if the space-time* $\{\mathbf{Q}'^4, g'\}$ *coincides with the flat Lorentzian Minkowski space-time* $\{\mathbf{M}^4, \eta\}$ *expressed in orthogonal Cartesian coordinates, then (C.8) reduces to*

$$0 = \frac{\partial M^{\mu}_{(g)\nu}\left(r',r\right)}{\partial r'^{\gamma}} + \Gamma^{\mu}_{\alpha\beta}M^{\alpha}_{(g)\nu}\left(r',r\right)M^{\beta}_{(g)\gamma}\left(r',r\right), \quad \text{(C.23)}$$

which provides a representation for the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$.

Proof. Assume in fact that the space-time $\{\mathbf{Q}'^4, g'\}$ coincides with the flat Minkowski space-time $\{\mathbf{M}, \eta\}$. Then by construction in (C.8) the transformed Christoffel symbols $\Gamma'^{\mu}_{\alpha\beta}$ in such a space-time necessarily vanish identically. Then the same equation reduces to (C.23). □

Competing Interests

The authors declare that they have no competing interests.

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