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# Theory of Polaron Mobility

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The polaron mobility is calculated by making use of the general theory of electrical conductivity. We take the states determined by Feynman's trial action as the unperturbed states and treat the difference between the true action and the trial action as a perturbation. Numerical values of the polaron mobility at very low temperatures are given and are discussed in comparison with the results obtained by Shultz and by Morita.

## § 1. Introduction

We have investigated the static properties of polaron at finite temperatures in the previous paper[1] (hereafter referred to as I). We shall investigate the polaron mobility in the present paper. In the same way as in I, we calculate the polaron mobility by using Feynman's path-integral method. In the polaron problem, the interaction between an electron and lattice vibrations is so large that the usual perturbation theoretic treatment fails. Therefore the calculation of the mobility in the present paper will be based on the general theory of electrical conductivity recently developed by many authors,[2] which is applicable to a system which does not allow us to set up the Boltzmann equation. We take the states determined by Feynman's trial action as unperturbed states and treat the difference between the trial action and the true action as a perturbation which yields a decay of electronic current correlation.

The following assumptions will be made in the present paper. i) The electrical conductivity is determined by the asymptotic form of a correlation function of electronic current. ii) This asymptotic form has the property of exponential decay in time. These assumptions are valid in the case of weak interaction. It is not sure whether or not they are applicable to the present problem which includes the case of strong interaction. In the present treatment, however, the states determined by Feynman's trial action instead of the states of a free electron are chosen as the unperturbed states. It is expected that the perturbation in the former case may be made weaker than in the latter case and that the above mentioned assumptions may be valid. In fact the self-energy of polaron at 0°K calculated by using these assumptions agrees with that obtained by Feynman[3] as will be shown in § 3. Therefore, the approximation used in the present calculation for the mobility may not be so bad even for strong interactions.

In § 2, the calculation of electrical conductivity, using the path-integral method, is discussed. Then, it is applied to the calculation of the polaron mobility in the case of a weak interaction. In § 3, we shall calculate the polaron mobility in the case of strong interaction based on the evaluation in § 2. In § 4, the result will be discussed in comparison with the results by Shultz and by Morita.

## § 2. Calculation of the polaron mobility
(in the case of weak interaction)

In this paper, we take the units so that $\hbar=1$, the electron mass $m=1$, and the optical phonon frequency $=1$.

The conductivity $\sigma$ is generally expressed by a current correlation function $\chi(t)$ in the following way:

$$\sigma=\beta\int_0^\infty \chi(t)dt, \tag{1}$$

$$\chi(t)=\langle j_x(t)j_x+j_xj_x(t)\rangle/2, \tag{2}$$

where $\beta=1/kT$, $j_x$ is the $x$ component of electronic current operator and

$$j_x(t)=e^{-iHt}j_xe^{iHt},$$

$H$ being the Hamiltonian of the system. In this equation and in what follows $\langle A\rangle$ denotes $\mathrm{Tr}\,\rho A/\mathrm{Tr}\,\rho$ where $\rho$ is the density matrix defined by $e^{-\beta H}$. The following equation is easily derived:

$$\begin{aligned}&\mathrm{Tr}(e^{-\beta H}e^{-iHt}j_xe^{iHt}j_x)\\&=-e^2\iint d\boldsymbol{X}_1d\boldsymbol{X}_2\,(\partial/\partial X_{2x}^{\beta-\lambda})\,(\partial/\partial X_{1x}^{\lambda})\left\{\iint d\boldsymbol{Q}_1d\boldsymbol{Q}_2\rho(\boldsymbol{X}_1\boldsymbol{Q}_1,\boldsymbol{X}_2\boldsymbol{Q}_2;\beta-\lambda)\right.\\&\left.\times\rho(\boldsymbol{X}_2\boldsymbol{Q}_2,\boldsymbol{X}_1\boldsymbol{Q}_1;\lambda)\right\},\end{aligned} \tag{3}$$

where $e$ is the electronic charge, $\lambda=-it$, and $\boldsymbol{X}$ is the electron coordinate, $\boldsymbol{Q}$ denotes the many-dimensional coordinates of phonons, and $\partial/\partial X_{2x}^{\beta-\lambda}$ operates only on $\boldsymbol{X}_2$ in $\rho(\boldsymbol{X}_1\boldsymbol{Q}_1,\boldsymbol{X}_2\boldsymbol{Q}_2;\beta-\lambda)$ and not on $\boldsymbol{X}_2$ in $\rho(\boldsymbol{X}_2\boldsymbol{Q}_2,\boldsymbol{X}_1\boldsymbol{Q}_1;\lambda)$. In what follows, we try to rewrite the density matrices by the use of the path-integral method.

Let us take the Hamiltonian as follows:

$$H=(p^2/2)+\sum_k M\omega_k q_k\gamma_k+\sum_k M\omega_k^2q_k^2/2+\sum_k p_k^2/2M, \tag{4}$$

where $\gamma_k(\boldsymbol{X})$ is some function of the electronic coordinate $\boldsymbol{X}$, and $p_k$ are the coordinate and its conjugate momentum of the $\boldsymbol{k}$-th normal mode of lattice vibrations, respectively, and $M$ is the reduced mass of the lattice ions. In the polaron problem, $\gamma_k=(2^{3/2}\alpha\pi/V)^{1/2}\cdot 1/k\cdot e^{ik\cdot X}$ and $\omega_k=1$ for all $k$, where $V$ is the volume of the crystal and $\alpha$ is the coupling constant used in I. The density matrix $\rho$ in the path-integral representation is given by

$$\rho(X_1Q_1, X_2Q_2; u)\{\prod_k (M\omega_k/2\pi\sinh\omega_k u)\}^{-1/2}$$

$$=\int\exp\Big[-\Big\{\int_0^u\frac{1}{2}\dot{X}^2(t)\,dt+\sum_k P(q_{1k}, q_{2k}, X(t))\Big\}\Big]\mathfrak{D}\Big(X(t):\frac{X_2(u)}{X_1(0)}\Big), \quad (5)$$

where

$$P(q_{1k}, q_{2k}, X(t))=[(q_{1k}^2+q_{2k}^2)\cosh\omega_k u-2q_{1k}\cdot q_{2k}$$
$$+2A_k\cdot q_{1k}+2B_k\cdot q_{2k}+2C_k]/2\sinh\omega_k u, \quad (6a)$$

and

$$A_k=\int_0^u\gamma_k(X(t))\sinh\omega_k t\,dt,\quad B_k=\int_0^u\gamma_k(X(t))\sinh\omega_k(u-t)\,dt,$$

$$C_k=-\int_0^u dt\int_0^t ds\,\gamma_k(X(t))\gamma_k(X(s))\sinh\omega_k(u-t)\sinh\omega_k s. \quad (6b)$$

$\mathfrak{D}\Big(X(t):\frac{X_2(u)}{X_1(0)}\Big)$ means that the path-integration must be carried out under the conditions $X(0)=X_1$ and $X(u)=X_2$.

Using (5) and (6), we obtain, after some calculations, the following equation for the polaron problem:

$$(e^{\beta/2}/e^{\beta}-1)^{-3N}\iint dQ_1dQ_2\,\rho(X_1Q_1, X_2Q_2; \beta-\lambda)\rho(X_2Q_2, X_1Q_1: \lambda)$$

$$=\iint\exp\Big[-\int_0^{\beta-\lambda}\frac{1}{2}\dot{X}'^2(t)\,dt-\int_0^{\lambda}\frac{1}{2}\dot{X}''^2(t)\,dt+Q(X'X'')\Big]\mathfrak{D}\Big(X'(t):\frac{X_2(\beta-\lambda)}{X_1(0)}\Big)$$

$$\times\mathfrak{D}\Big(X''(t):\frac{X_1(\lambda)}{X_2(0)}\Big), \quad (7a)$$

where $N$ is the total number of unit cells in the crystal and

$$Q(X', X'')=Q_1(X'X''; \beta\lambda)+S_{cross}(X'X''; \beta\lambda).$$

$Q_1$ and $S_{cross}$ are given by

$$Q_1(X'X''; \beta, \lambda)\,V(e^{\beta}-1)/2^{3/2}\alpha\pi$$

$$=\sum_w\frac{1}{w^2}\Big[\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma\,I_w(X''(\tau), X''(\sigma); 1)+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma\,I_w(X''(\tau), X'(\sigma); 1)\Big], \quad (7b)$$

$$S_{cross}(\boldsymbol{X}'\boldsymbol{X}''\,;\,\beta\lambda)\,V(e^{\beta}-1)/2^{3/2}\alpha\pi=\Big[\sum_{w}\frac{1}{w^2}\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma\, I_w(\boldsymbol{X}'(\tau),\,\boldsymbol{X}''(\sigma)\,;\,1)\Big],\tag{7c}$$

where

$$I_w(\boldsymbol{x}(\tau),\boldsymbol{y}(\sigma)\,;\,\chi)=e^{\chi(\beta-\tau+\sigma)-i\boldsymbol{w}(\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma))}+e^{\chi(\tau-\sigma)+i\boldsymbol{w}(\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma))}.\tag{7d}$$

In this section, we hereafter confine ourselves to the case in which the interaction of the electron with phonons can be treated as a small perturbation. In order to calculate the mobility in such a case, we shall make the approximations that (1) the correlation function of electronic current is assumed to have the property of simple exponential decay in time and that (2) all averages over the canonical distribution are replaced by those of the unperturbed system. Substituting (7a) in (3) and omitting the unimportant numerical factor $(e^{\beta/2}/e^{\beta}-1)^{-3N}$ which is cancelled by the normalization factor for phonon, we get

$$\mathrm{Tr}(e^{-\beta H}e^{-iHt}j_x e^{iHt}j_x)=-e^2\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{2x}^{\beta-\lambda})\,(\partial/\partial X_{1x}^{\lambda})G(\boldsymbol{X}_1,\,\boldsymbol{X}_2),\tag{8a}$$

where

$$\begin{aligned}G(\boldsymbol{X}_1,\,\boldsymbol{X}_2)=&\iint\mathfrak{D}\Big(\boldsymbol{X}':\frac{\boldsymbol{X}_2(\beta-\lambda)}{\boldsymbol{X}_1(0)}\Big)\mathfrak{D}\Big(\boldsymbol{X}'':\frac{\boldsymbol{X}_1(\lambda)}{\boldsymbol{X}_2(0)}\Big)\\ &\times\exp\Big[-\int_0^{\beta-\lambda}\frac{1}{2}\dot{\boldsymbol{X}}'^2(t)dt-\int_0^{\lambda}\frac{1}{2}\dot{\boldsymbol{X}}''^2(t)dt\Big]\times\Big[1+(2^{3/2}\pi\alpha/(e^{\beta}-1)V)\\ &\times\Big\{\sum_{w}\frac{1}{w^2}\Big\{\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma\, I_w(\boldsymbol{X}''(\tau),\,\boldsymbol{X}''(\sigma):1)\\ &+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma\, I_w(\boldsymbol{X}'(\tau),\,\boldsymbol{X}'(\sigma):1)+\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma\, I_w(\boldsymbol{X}'(\tau),\,\boldsymbol{X}''(\sigma):1)\Big\}\Big\}\Big].\end{aligned}\tag{8b}$$

It should be noted here that the approximation

$$\exp[Q(\boldsymbol{X}',\,\boldsymbol{X}'')]\simeq 1+Q(\boldsymbol{X}',\,\boldsymbol{X}'')\tag{8c}$$

was used to obtain (8b). As is seen from (7b), $Q(\boldsymbol{X}'\boldsymbol{X}'')$ is proportional to the coupling constant $\alpha$ which is small in the present case.

The perturbational treatment in the present path-integral method consists of replacing $\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}''(\tau)-\boldsymbol{X}''(\sigma)))$, $\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}'(\sigma)))$, and $\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}''(\sigma)))$ by their average values over the path with a free action $-\int_0^{\lambda}1/2\cdot\dot{\boldsymbol{X}}''^2(t)\,dt$ and $-\int_0^{\beta-\lambda}1/2\cdot\dot{\boldsymbol{X}}'^2(t)\,dt$. In the same way as in I, these average values are given by

$$\langle e^{\pm(iw\cdot(X''(\tau)-X''(\sigma))}\rangle_{X_2X_1;\lambda}$$
$$=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_2-\boldsymbol{X}_1)(\tau-\sigma)/\lambda)\exp\left(-\frac{w^2}{2}(\tau-\sigma)\left(1-\frac{(\tau-\sigma)}{\lambda}\right)\right),$$
$$\langle e^{\pm(iw\cdot(X'(\tau)-X'(\sigma))}\rangle_{X_1X_2;\beta-\lambda}$$
$$=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_2-\boldsymbol{X}_1)(\tau-\sigma)/\beta-\lambda)\exp\left(-\frac{w^2}{2}(\tau-\sigma)\left(1-\frac{(\tau-\sigma)}{\beta-\lambda}\right)\right),$$
$$\langle e^{\pm iw\cdot X'(\tau)}\rangle_{X_1X_2;\beta-\lambda}=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_2-\boldsymbol{X}_1)\tau/\beta-\lambda)\exp\left(-\frac{w^2}{2}\tau\left(1-\frac{\tau}{\beta-\lambda}\right)\right),$$
$$\langle e^{\pm iw\cdot X''(\sigma)}\rangle_{X_2X_1;\lambda}=\exp(\pm i\boldsymbol{w}\cdot(\boldsymbol{X}_1-\boldsymbol{X}_2)\sigma/\lambda)\exp\left(\frac{w^2}{2}\sigma\left(1+\frac{\sigma}{\lambda}\right)\right). \qquad (9)$$

We shall give another method to calculate the average values of Eq. (9) in Appendix I, which will be used in the calculation in § 3. Noting that the identity

$$-\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{1x}^{\beta-\lambda})f_{\beta-\lambda}(\boldsymbol{X}_1,\boldsymbol{X}_2)(\partial/\partial X_{2x}^{\lambda})f_\lambda(\boldsymbol{X}_2,\boldsymbol{X}_1)$$
$$=\iint d\boldsymbol{K}_1 d\boldsymbol{K}_2 K_{1x}K_{2x}g_{\beta-\lambda}(\boldsymbol{K}_1,\boldsymbol{K}_2)g_\lambda(\boldsymbol{K}_2,\boldsymbol{K}_1) \qquad (10a)$$

holds with

$$f_\lambda(\boldsymbol{X}_1\boldsymbol{X}_2)=(2\pi)^{-3}\int g_\lambda(\boldsymbol{K}_1,\boldsymbol{K}_2)e^{i(\boldsymbol{K}_1\boldsymbol{X}_1-\boldsymbol{K}_2\boldsymbol{X}_2)}d\boldsymbol{K}_1 d\boldsymbol{K}_2$$

and that the density matrix $\rho_0(\boldsymbol{X}_1\boldsymbol{X}_2;u)$ for a free electron is $(\sqrt{2\pi/u})^3$ $\times\exp(-(\boldsymbol{X}_1-\boldsymbol{X}_2)^2/2u)$, and furthermore, using (8) and (9) one can obtain

$$\mathrm{Tr}(e^{-\beta H}e^{-iHt}j_x e^{iHt}j_x)=\sum_k e^2k_x^2e^{-k^2\beta/2}\Big[1+\sum_w\{2^{3/2}\alpha/Vw^2(e^\beta-1)\}$$
$$\times\Big\{\int_0^\lambda d\tau(\lambda-\tau)(e^{\beta-\tau(1-k\cdot w+w^2/2)}+e^{-\tau(-1+k\cdot w+w^2/2)})$$
$$+\int_0^{\beta-\lambda}d\tau(\beta-\lambda-\tau)(e^{\beta-\tau(1-k\cdot w+w^2/2)}+e^{-\tau(-1+k\cdot w+w^2/2)})$$
$$+\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma(e^{\beta-(\tau-\sigma)(1-k\cdot w+w^2/2)}+e^{(\tau-\sigma)(-1+k\cdot w+w^2/2)}\Big\}\Big]. \qquad (11)$$

The first four terms appearing in brackets in the right-hand side of Eq. (11) are simplified by using the formulae for a large time $t$

$$\sum_\omega F(\omega)\Big[\int_0^{\beta+it}ds(\beta+it-s)e^{-f(\omega)s}+\int_0^{-it}ds(-it-s)e^{-f(\omega)s}\Big]\simeq 2\pi t\sum_\omega F(\omega)\delta(f(\omega)),$$
$$(A\cdot 2\cdot 1)$$

which will be proved in Appendix II, Here $F(\omega)$ and $f(\omega)$ are some functions of $\omega$. The remaining terms which come from the terms $S_{cross}$ of Eq. (7c) become quantities being independent of $t$ for a large time $t$ as will be shown in Appendix II. Then, one obtains

$$\mathrm{Tr}(e^{-\beta H}e^{-iHt}j_x e^{iHt}j_x) \simeq \sum_k e^2 k_x^2 e^{-k^2\beta/2}(1-t/\tau_k) \tag{12a}$$

for a large time $t$, where

$$1/\tau_k = \{2\pi\cdot 2^{3/2}\alpha\pi/V(e^\beta-1)\} \sum_w \frac{1}{w^2}\left\{e^\beta\,\delta\left(1-\boldsymbol{k}\cdot\boldsymbol{w}+\frac{w^2}{2}\right)+\delta\left(-1+\boldsymbol{k}\cdot\boldsymbol{w}+\frac{w^2}{2}\right)\right\}. \tag{12b}$$

This expression of $\tau_k$ agrees with the collision time obtained by the ordinary perturbation theory with the electron-phonon interaction as a perturbing Hamiltonian. In the perturbation theory, the energy shift $\varDelta E_n$ and the collision time $\tau_n$ of an unperturbed state are given, respectively, by

$$\varDelta E_n = P\sum_m |\langle m|H'|n\rangle|^2/(E_n^0-E_m^0),$$

$$1/2\pi\tau_n = \sum_m \delta(E_n^0-E_m^0)\,|\langle m|H'|n\rangle|^2,$$

where $H'$ is the perturbing Hamiltonian and $E_m^0$ is the energy of the unperturbed state $m$. Comparing these two equations with one another, it is expected that the self energy $\varDelta E_k$ corresponding to $\tau_k$ given by Eq. (12) becomes

$$\varDelta E_k = \{2^{3/2}\alpha\pi/(e^\beta-1)V\} \sum_w \{e^\beta P/(1-\boldsymbol{k}\cdot\boldsymbol{w}+w^2/2)+P/(-1+\boldsymbol{k}\cdot\boldsymbol{w}+w^2/2)\}. \tag{13}$$

This result also agrees with that obtained by the ordinary perturbation treatment.

Adopting the assumption that the asymptotic form of a correlation function of electronic current has the property of simple exponential decay in time and using of Eqs. (1), (2) and (12a), we get, as the expression of the mobility $\mu_0$,

$$\mu_0 = e\beta\sum_k k_x^2 e^{-(k^2/2)\beta}\int_0^\infty e^{-t/\tau_k}dt\Big/\sum_k e^{-(k^2/2)\beta}.$$

Substituting the value $\tau_k$ of Eq. (12b) in this equation, the mobility $\mu_0$ at very low temperatures is given by

$$\mu_0 = e/2\bar{n}\alpha, \tag{14}$$

where

$$\bar{n} = 1/(e^\beta-1).$$

## § 3. Calculation of the polaron mobility

(in the case of strong interaction)

We shall calculate the polaron mobility in the case of strong interaction by using the evaluation in § 2. We take states determined by Feynman's trial action as the unperturbed states and treat the difference between it and the true action as a perturbation.

We shall start with the approximate expression

$$\mathrm{Tr}(e^{-(\beta+it)H}j_x e^{iHt}j_x) = -e^2\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{2x}^{\beta-\lambda})(\partial/\partial X_{1x}^{\lambda})G(\boldsymbol{X}_1, \boldsymbol{X}_2), \quad (15a)$$

where

$$G(\boldsymbol{X}_1, \boldsymbol{X}_2) = \iint \mathfrak{D}\left(\boldsymbol{X}'(t) : \frac{\boldsymbol{X}_2(\beta-\lambda)}{\boldsymbol{X}_1(0)}\right)\mathfrak{D}\left(\boldsymbol{X}''(t) : \frac{\boldsymbol{X}_1(\lambda)}{\boldsymbol{X}_2(0)}\right)$$
$$\times \exp[S_{0,\beta-\lambda}(\boldsymbol{X}') + S_{0,\lambda}(\boldsymbol{X}'') + S_{0,\lambda,\beta-\lambda}(\boldsymbol{X}', \boldsymbol{X}'')](1+F(\boldsymbol{X}'\boldsymbol{X}''; \beta, \lambda)), \quad (15b)$$

and

$$F(\boldsymbol{X}'\boldsymbol{X}''; \beta, \lambda) = 2^{-3/2}\alpha(e^\beta-1)^{-1}\left\{\int_0^\lambda d\tau\int_0^\tau d\sigma F_1(\boldsymbol{X}''(\tau), \boldsymbol{X}''(\sigma))\right.$$
$$\left. + \int_0^{\beta-\lambda} d\tau\int_0^\tau d\sigma F_1(\boldsymbol{X}'(\tau), \boldsymbol{X}'(\sigma)) + \int_0^{\beta-\lambda} d\tau\int_0^\lambda d\sigma F_1(\boldsymbol{X}'(\tau), \boldsymbol{X}''(\sigma))\right\}$$
$$+C(e^{\beta\omega}-1)^{-1}/2\left\{\int_0^\lambda d\tau\int_0^\tau d\sigma F_2(\boldsymbol{X}''(\tau), \boldsymbol{X}''(\sigma))\right.$$
$$\left. + \int_0^{\beta-\lambda} d\tau\int_0^\tau d\sigma F_2(\boldsymbol{X}'(\tau), \boldsymbol{X}'(\sigma)) + \int_0^{\beta-\lambda} d\tau\int_0^\lambda d\sigma F_2(\boldsymbol{X}'(\tau), \boldsymbol{X}''(\sigma)\right\}, \quad (15c)$$

where

$$S_{0u}(\boldsymbol{X}) = -\frac{1}{2}\int_0^u \dot{\boldsymbol{X}}^2(t)dt - C(e^{\beta\omega}-1)^{-1}/2\int_0^u d\tau\int_0^\tau d\sigma F_2(\boldsymbol{X}(\tau), \boldsymbol{X}(\sigma)), \quad (15d)$$

$$S_{0,\lambda,\beta-\lambda}(\boldsymbol{X}'\boldsymbol{X}'') = C(e^{\beta\omega}-1)^{-1}/2\int_0^{\beta-\lambda} d\tau\int_0^\lambda d\sigma F_2(\boldsymbol{X}(\tau), \boldsymbol{X}(\sigma)), \quad (15e)$$

$$F_1(\boldsymbol{x}(\tau), \boldsymbol{y}(\sigma)) = |\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma)|^{-1}(e^{(\beta-\tau+\sigma)}+e^{(\tau-\sigma)}), \quad (15f)$$

$$F_2(\boldsymbol{x}(\tau), \boldsymbol{y}(\sigma)) = |\boldsymbol{x}(\tau)-\boldsymbol{y}(\sigma)|^2(e^{\omega(\beta-\tau+\sigma)}+e^{\omega(\tau-\sigma)}). \quad (15g)$$

Here $S_{0u}$ denotes Feynman's trial action. To derive (15) from (3), we considered $F(\boldsymbol{X}'\boldsymbol{X}''; \beta\lambda)$ small, since it corresponds to the difference between the true action and the trial action, and we used an approximation similar to (8c) in the case of weak interaction. $C$ and $\omega$ are variational parameters which should be determined so as to minimize the free energy of the system.

Let us first consider the term which does not include the terms $F(\boldsymbol{X}'\boldsymbol{X}''\,;\beta\lambda)$ in Eq. (15b), that is,

$$-e^2\iint d\boldsymbol{X}_1 d\boldsymbol{X}_2(\partial/\partial X_{2x}^{\beta-\lambda})(\partial/\partial X_{1x}^{\lambda})\iint \mathfrak{D}\left((\boldsymbol{X}'(t):\begin{matrix}\boldsymbol{X}_2(\beta-\lambda)\\ \boldsymbol{X}_1(0)\end{matrix}\right)$$
$$\times\mathfrak{D}\left(\boldsymbol{X}''(t):\begin{matrix}\boldsymbol{X}_1(\lambda)\\ \boldsymbol{X}_2(0)\end{matrix}\right)e^{\bar{S}_0(\boldsymbol{X}'\boldsymbol{X}'')}, \tag{16}$$

where

$$\bar{S}_0(\boldsymbol{X}'\boldsymbol{X}'')=S_{0,\beta-\lambda}(\boldsymbol{X}')+S_{0,\lambda}(\boldsymbol{X}'')+S_{0,\lambda,\beta-\lambda}(\boldsymbol{X}'\boldsymbol{X}'').$$

As was shown in I, the states determined by the trial action $S_0$ are equivalent to the states of a system with the Lagrangian

$$L_0=(\dot{\boldsymbol{X}}^2+M\dot{\boldsymbol{q}}^2-\kappa(\boldsymbol{q}-\boldsymbol{X})^2)/2, \tag{17}$$

where

$$\kappa=M\omega^2 \text{ and } M=4C/\omega^2.$$

The Hamiltonian $H_0$ corresponding to the Lagrangian $L_0$ is expressed by

$$H_0=-\frac{1}{2m_0}\partial^2/\partial \boldsymbol{x}_0{}^2+\left(-\frac{1}{2\eta}\partial^2/\partial \boldsymbol{r}^2+\frac{\kappa}{2}\boldsymbol{r}^2\right), \tag{18}$$

where the new variables $\boldsymbol{x}_0$ and $\boldsymbol{r}$ defined by

$$\boldsymbol{x}_0=(M\boldsymbol{q}+\boldsymbol{X})/(M+1),\quad \boldsymbol{r}=\boldsymbol{X}-\boldsymbol{q} \tag{19}$$

are used instead of $X$ and $q$, and also

$$m_0=\nu^2/\omega^2,\quad \eta=4C/\nu^2.$$

The energy eigenvalues of this Hamiltonian are given by

$$E_0(\boldsymbol{k},\, n_1n_2n_3)=\boldsymbol{k}^2/2m_0+(n_1+n_2+n_3+3/2)\nu, \tag{20}$$

where $\boldsymbol{k}$ is the wave number specifying a free motion for $\boldsymbol{x}_0$ and $n_1$, $n_2$ and $n_3$ are the quantum numbers of the harmonic oscillator for $\boldsymbol{r}$.

It is not so difficult to show, by performing calculations similar to that used in § 2, that

$$\text{the quantity (16)}=\sum_{mm'}e^{-(\beta-\lambda)E_m^0}e^{-\lambda E_{m'}^0}(m|j_x|m')(m'|j_x|m)$$
$$=\mathrm{Tr}(e^{-\beta H_0}e^{-iH_0t}j_xe^{iH_0t}j_x), \tag{21}$$

where we simply express the eigenstates $|\boldsymbol{k}\,n_1n_2n_3)$ and their eigenvalues $E_0(\boldsymbol{k}n_1n_2n_3)$ by $|m)$ and $E_m{}^0$, respectively. Using Eq. (21) and noting that

$$\partial/\partial\boldsymbol{X}=(m_0)^{-1}\,\partial/\partial\boldsymbol{x}_0+\partial/\partial\boldsymbol{r}, \tag{22}$$

we obtain that

$$\text{the quantity (16)}=\sum_{kn_1n_2n_3}e^2\exp\{-\beta E_0(\boldsymbol{k}n_1n_2n_3)\}[kx^2/m_0^2$$
$$+(\kappa\eta)^{1/2}(n_1+1)e^{-\lambda\nu}/2+(\kappa\eta)^{1/2}n_1e^{\lambda\nu}/2]. \tag{23}$$

Next, we consider the term including $F(X'X'';\beta\lambda)$ in Eq. (15). This quantity is transformed into

$$-e^2\iint dX_1dX_2(\partial/\partial X_{2x}^{\beta-\lambda})(\partial/\partial X_{1x}^{\lambda})\Big[\iint\mathfrak{D}\Big(X'(t):\frac{X_2(\beta-\lambda)}{X_1(0)}\Big)\mathfrak{D}\Big(X''(t):\frac{X_1(\lambda)}{X_2(0)}\Big)\Big]e^{\bar{S}_0(X'X'')}$$

$$\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma I_u(X''(\tau),X''(\sigma):1)+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma I_u(X'(\tau),X'(\sigma):1)$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma I_u(X'(\tau),X''(\sigma);1)\Big\}+\sum_u D_u^2(e^{\beta\omega}-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma I_u(X''(\tau),X''(\sigma);\omega)$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma I_u(X'(\tau),X'(\sigma);\omega)+\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma I_u(X'(\tau),X''(\sigma);\omega)\Big\}\Big],\qquad(24)$$

where we used the notation of Eq. (7d) and the identities

$$|X_1-X_2|^{-1}=4\pi\sum_u e^{iu\cdot(X_1-X_2)}/Vu^2,$$

$$|X_1-X_2|^2=-8\pi^3\sum_u\delta(u)\nabla_u^2e^{iu\cdot(X_1-X_2)}/V$$

and the definitions

$$C_u=\sqrt{2^{3/2}\pi\alpha/Vu},\quad D_u=-i\sqrt{2^2\pi^3C\delta(u)\nabla_u^2/V}.\qquad(25)$$

Now we replace $e^{\pm iu\cdot(X''(\tau)-X''(\sigma))}$, $e^{\pm iu\cdot(X'(\tau)-X'(\sigma))}$ and $e^{\pm iu\cdot(X'(\tau)-X''(\sigma))}$ by their average values $\overline{e^{\pm iu\cdot(X''(\tau)-X''(\sigma))}}$, $\overline{e^{\pm iu\cdot(X'(\tau)-X'(\sigma))}}$ and $\overline{e^{\pm iu\cdot(X'(\tau)-X''(\sigma))}}$ taken along the path with the action $S_{0,\beta-\lambda}(X')+S_{0,\lambda}(X'')+S_{0,\lambda,\beta-\lambda}(X'X'')$ as was done in § 2. These average values are calculated as follows. Noting that

$$\iint\mathfrak{D}\Big(X'(t):\frac{X_2(\beta-\lambda)}{X_1(0)}\Big)\mathfrak{D}\Big(X''(t):\frac{X_1(\lambda)}{X_2(0)}\Big)e^{\bar{S}_0(X',X'')}$$

$$=\sum_{mm'}\iint dq_1dq_2e^{-(\beta-\lambda)E_m^0}e^{\lambda E_{m'}^0}\psi_m^{0*}(X_1q_1)\psi_m^0(X_2q_2)\psi_{m'}^{0*}(X_2q_2)\psi_{m'}^0(X_1q_1),$$

where $\psi_m^0(X_1q_1)$ is the eigen-function of $H_0$, and making use of Eqs. (A·1·1) and (A·1·2), it is shown that

$$\Big[\iint\mathfrak{D}\Big(X'(t):\frac{X_2(\beta-\lambda)}{X_1(0)}\Big)\mathfrak{D}\Big(X''(t):\frac{X_1(\lambda)}{X_2(0)}\Big)e^{\bar{S}_0(X',X'')}\Big]\overline{e^{\pm iu\cdot(X''(\tau)-X''(\sigma))}}$$

$$=\sum_{mm'}\iint dq_1dq_2e^{-(\beta-\lambda)E_m^0+\lambda E_{m'}^0}\psi_m^{0*}(X_1q_1)\psi_m^0(X_2q_2)\psi_{m'}^{0*}(X_2q_2)\psi_{m'}^0(X_1q_1)$$

$$\times\langle m'|e^{\pm iu\cdot X(\tau)}e^{\mp iu\cdot X(\sigma)}|m'\rangle,\qquad(26a)$$

$$\Big[\iint\mathfrak{D}\Big(X'(t):\frac{X_2(\beta-\lambda)}{X_1(0)}\Big)\mathfrak{D}\Big(X''(t):\frac{X_1(\lambda)}{X_2(0)}\Big)e^{\bar{S}_0(X',X'')}\Big]\overline{e^{\pm iu\cdot(X'(\tau)-X'(\sigma))}}$$

$$=\sum_{mm'}e^{-(\beta-\lambda)E_m^0+\lambda E_{m'}^0}\psi_m^{0*}(X_1q_1)\psi_m^0(X_2q_2)\psi_{m'}^{0*}(X_2q_2)\psi_{m'}^0(X_1q_1)$$

$$\times\langle m|e^{\pm iu\cdot X(\tau)}e^{\mp iu\cdot X(\sigma)}|m\rangle,\qquad(26b)$$

and

$$\left[\iint \mathfrak{D}\left(\boldsymbol{X}'(t):\begin{matrix}\boldsymbol{X}_2(\beta-\lambda)\\ \boldsymbol{X}_1(0)\end{matrix}\right)\mathfrak{D}\left(\boldsymbol{X}''(t):\begin{matrix}\boldsymbol{X}_1(\lambda)\\ \boldsymbol{X}_2(0)\end{matrix}\right)e^{\bar{S}_0(X',X'')}\right]\overline{e^{\pm i\boldsymbol{u}\cdot(\boldsymbol{X}'(\tau)-\boldsymbol{X}''(\sigma))}}$$

$$=\lim_{\sigma'\to 0,\tau'\to 0}\sum_{mm'}\iint d\boldsymbol{q}_1 d\boldsymbol{q}_2 e^{-(\beta-\lambda)E_m^0+\lambda E_{m'}^0}\psi_m^{0*}(\boldsymbol{X}_1\boldsymbol{q}_1)\psi_m{}^0(\boldsymbol{X}_2\boldsymbol{q}_2)$$

$$\times\psi_{m'}^{0*}(\boldsymbol{X}_2\boldsymbol{q}_2)\psi_{m'}^0(\boldsymbol{X}_1\boldsymbol{q}_1)\langle m|e^{\pm i\boldsymbol{u}\cdot\boldsymbol{X}(\tau)}e^{\mp i\boldsymbol{u}\cdot\boldsymbol{X}(\tau')}|m\rangle\langle m'|e^{\pm i\boldsymbol{u}\cdot\boldsymbol{X}(\sigma')}e^{\mp i\boldsymbol{u}\cdot\boldsymbol{X}(\sigma)}|m'\rangle, \tag{26c}$$

where

$$e^{\pm i\boldsymbol{u}\cdot\boldsymbol{X}(\tau)}=e^{H_0\tau}e^{\pm i\boldsymbol{u}\cdot\boldsymbol{X}}e^{-H_0\tau}.$$

The quantity (24) can be replaced, using of Eq. (26), by

$$\sum_{\boldsymbol{u}}C_{\boldsymbol{u}}{}^2(e^{\beta}-1)^{-1}\Big\{\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma(e^{(\beta-\tau+\sigma)}f_{\boldsymbol{u}}^{\lambda,+}(\tau,\sigma)+e^{(\tau-\sigma)}f_{\boldsymbol{u}}^{\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma(e^{(\beta-\tau+\sigma)}f_{\boldsymbol{u}}^{\beta-\lambda,+}(\tau,\sigma)+e^{(\tau-\sigma)}f_{\boldsymbol{u}}^{\beta-\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma(e^{(\beta-\tau+\sigma)}g_{\boldsymbol{u}}^{\lambda,\beta-\lambda,+}(\tau,\sigma)+e^{(\tau-\sigma)}g_{\boldsymbol{u}}^{\lambda,\beta-\lambda,-}(\tau,\sigma))\Big\}$$

$$+\sum_{\boldsymbol{u}}D_{\boldsymbol{u}}{}^2(e^{\beta\omega}-1)^{-1}\Big\{\int_0^{\lambda}d\tau\int_0^{\tau}d\sigma(e^{\omega(\beta-\tau+\sigma)}f_{\boldsymbol{u}}^{\lambda,+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}f_{\boldsymbol{u}}^{\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\tau}d\sigma(e^{\omega(\beta-\tau+\sigma)}f_{\boldsymbol{u}}^{\beta-\lambda,+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}f_{\boldsymbol{u}}^{\beta-\lambda,-}(\tau,\sigma))$$

$$+\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma(e^{\omega(\beta-\tau+\sigma)}g_{\boldsymbol{u}}^{\lambda,\beta-\lambda,+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}g_{\boldsymbol{u}}^{\lambda,\beta-\lambda,-}(\tau,\sigma))\Big\}, \tag{27a}$$

where

$$f_{\boldsymbol{u}}^{\lambda,\pm}(\tau,\sigma)=\sum_{kn_1n_2n_3}e^{-\beta E_0(kn_1n_2n_3)}e^2\{k_x{}^2/m_0{}^2+\sqrt{\kappa\eta}(n_1+1)e^{-\lambda\nu}/2$$

$$+\sqrt{\kappa\eta}\,n_1e^{\lambda\nu}/2\}h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\sigma), \tag{27b}$$

$$f_{\boldsymbol{u}}^{\beta-\lambda,\pm}(\tau,\sigma)=\sum_{kn_1n_2n_3}e^{-\beta E_0(kn_1n_2n_3)}e^2\{k_x{}^2h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k}n_1n_2n_3;\tau\sigma)/m_0{}^2$$

$$+(n_1+1)\sqrt{\kappa\eta}\,e^{-\lambda\nu}h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k},n_1+1,n_2n_3;\tau\sigma)/2$$

$$+n_1\sqrt{\kappa\eta}\,e^{\lambda\nu}h_{\boldsymbol{u}}^{\pm}(\boldsymbol{k},n_1-1,n_2n_3;\tau\sigma)/2\}, \tag{27c}$$

and

$$g_u^{\lambda,\beta-\lambda,\pm}(\tau,\sigma)=\lim_{\tau'\to 0,\sigma'\to 0}\sum_{kn_1n_2n_3}e^{-\beta E_0(kn_1n_2n_3)}e^2$$
$$\times\{k_x^2h_u^\pm(kn_1n_2n_3\,;\,\tau\tau')h_u^\pm(kn_1n_2n_3\,;\,\sigma'\sigma)/m_0^2+(n_1+1)\sqrt{\kappa\eta}e^{-\lambda\nu}$$
$$\times h_u^\pm(kn_1n_2n_3\,;\,\tau\tau')h_u^\pm(k,\,n_1+1,\,n_2n_3\,;\,\sigma'\sigma)/2+n_1\sqrt{\kappa\eta}e^{\lambda\nu}$$
$$\times h_u^\pm(kn_1n_2n_3\,;\,\tau\tau')h_u^\pm(k,\,n_1-1,\,n_2n_3\,;\,\sigma'\sigma)/2\},\tag{27d}$$

and notation

$$h_u^\pm(kn_1n_2n_3\,;\,\tau\sigma)=(kn_1n_2n_3|e^{\pm iu\cdot X(\tau)}e^{\mp iu\cdot X(\sigma)}|kn_1n_2n_3)\tag{27e}$$

is employed, for simplicity.

For convenience of calculations in what follows, let us consider operators $X_s(\tau)$ and $e^{iX_0(\tau)\cdot u}$ defined below. $X_s(\tau)$ is defined by

$$X_s(\tau)=x_{0s}(\tau)+\eta r_s(\tau)=x_{0s}(\tau)+\sqrt{\eta/2\nu}(b_s{}^*e^{-\nu\tau}+b_se^{\nu\tau}),\tag{28}$$

where the suffix $s$ stands for one of $x$, $y$ and $z$-components, $b_s{}^*$ and $b_s$ denote the creation and annihilation operators of the harmonic oscillator for $r_s$. In Eq. (28), the first equality comes from Eq. (18) and the second equality from the definition

$$r_s(\tau)=(b_s{}^*(\tau)+b_s(\tau))/\sqrt{2\nu\eta}.$$

$e^{ix_0(\tau)\cdot u}$ is defined by

$$e^{ix_0(\tau)\cdot u}=e^{H_0\tau}e^{ix_0\cdot u}e^{-H_0\tau}=\exp\left(\frac{p_0{}^2-(p_0-u)^2}{2m_0}\right)e^{ix_0\cdot u},\tag{29}$$

where $p_0$ is the momentum conjugate to the coordinate $x_0$. It is shown, according to Eqs. (28) and (29), that

$$h_u^\pm(kn_1n_2n_3\,;\,\tau\sigma)=\exp\{(k^2-(k\mp u)^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\langle n_1n_2n_3|\prod_s\exp(\mp i\sqrt{\eta}\,u_sb_s{}^*e^{\nu\tau}/\sqrt{2\nu})\exp(\pm i\sqrt{\eta}\,u_sb_se^{-\nu\tau}/\sqrt{2\nu})$$
$$\times\prod_{s'}\exp(\pm i\sqrt{\eta}\,u_{s'}b_{s'}^*e^{\nu\sigma}/\sqrt{2\nu})\exp(\mp i\sqrt{\eta}\,u_{s'}b_{s'}e^{-\nu\sigma}/\sqrt{2\nu})|n_1n_2n_3\rangle.\tag{30}$$

If we are concerned only with the phenomena at low temperatures, it is sufficient to carry out the evaluation of Eq. (30) in the case of $n_1=n_2=n_3=0$. In such a case, we have

$$h_u^\pm(kn_1n_2n_3\,;\,\tau\sigma)_{n_1=n_2=n_3=0}=\exp\{(k^2-(k\pm u)^2)(\tau-\sigma)/2m_0\}$$
$$\times\exp(-\eta u^2/2\nu)\sum_{l_s=0}^{\infty}\prod_s(\eta u_s^2)^{l_s}\exp(-\nu l_s(\tau-\sigma))/l_s!\,(2\nu)^{l_s},\tag{31a}$$

$$h_u^\pm(k,\,n_1+1,\,n_2n_3\,;\,\tau\sigma)_{n_1=n_2=n_3=0}=\exp\{(k^2-(k\pm u)^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times[\eta u_x{}^2\sum_{l_s=0}^{\infty}\prod_s(\eta u_s^2)^{l_s}e^{-\nu(l_s-1)(\tau-\sigma)}/2\nu l_s!$$

$$+\sum_{l_1=0,\,l_2=0,\,l_3=0}^{\infty}(l_1+1)^2(\eta)^{l_1+l_2+l_3}(u_x)^{2l_1}(u_y)^{2l_2}(u_z)^{2l_3}$$
$$\times e^{-\nu(l_1+l_2+l_3)(\tau-\sigma)}/(l_1+1)!\,l_2!\,l_3!\,(2\nu)^{l_1+l_2+l_3}]. \tag{31b}$$

According to Eqs. (23), (27), (28), (29), (31a) and (31b), Eq. (15a) is rewritten, in the case of $n_1=n_2=n_3=0$, as

$$\begin{aligned}
&\mathrm{Tr}(e^{-(\beta+it)H}j_x e^{iHt}j_x)\\
&=\sum_k e^{-k^2\beta/2m_0}e^2\{k_x^2/m_0^2+\sqrt{\kappa\eta}\,e^{-\lambda\nu}/2\}+\sum_k e^{-k^2\beta/2m_0}e^2k_x^2/m_0^2\\
&\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\\
&+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\Big\}\\
&+\sum_u D_u^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\\
&+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\Big\}\Big]\\
&+\sum_k e^{-k^2\beta/2m_0}e^2\sqrt{\kappa\eta}\,e^{-\lambda\nu}/2\\
&\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\\
&+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{(\beta-\tau+\sigma)}C_{uk}^+(\tau-\sigma)+e^{(\tau-\sigma)}C_{uk}^-(\tau-\sigma))\Big\}\\
&+\sum_u D_u^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^\lambda d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}B_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}B_{uk}^-(\tau-\sigma))\\
&+\int_0^{\beta-\lambda}d\tau\int_0^\tau d\sigma(e^{\omega(\beta-\tau+\sigma)}C_{uk}^+(\tau-\sigma)+e^{\omega(\tau-\sigma)}C_{uk}^-(\tau-\sigma))\Big\}\Big]\\
&+\sum_k e^{-k^2\beta/2m_0}e^2k_x^2/m_0^2\\
&\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma(e^{(\beta-\tau+\sigma)}\overline{B_{uk}^+}(\tau,\sigma)+e^{(\tau-\sigma)}\overline{B_{uk}^-}(\tau,\sigma))\Big\}\\
&+\sum_u D_u^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^{\beta-\lambda}d\tau\int_0^\lambda d\sigma(e^{\omega(\beta-\tau+\sigma)}\overline{B_{uk}^+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}\overline{B_{uk}^-}(\tau,\sigma))\Big\}\Big]\\
&+\sum_k e^{-k^2\beta/2m_0}\sqrt{\kappa\eta}\,e^{-\lambda\nu}/2
\end{aligned}$$

$$\times\Big[\sum_u C_u^2(e^\beta-1)^{-1}\Big\{\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma(e^{(\beta-\tau+\sigma)}\overline{C_{uk}^+}(\tau,\sigma)+e^{(\tau-\sigma)}\overline{C_{uk}^-}(\tau,\sigma))\Big\}$$

$$+\sum_u D_u^2(e^{\omega\beta}-1)^{-1}\Big\{\int_0^{\beta-\lambda}d\tau\int_0^{\lambda}d\sigma(e^{\omega(\beta-\tau+\sigma)}\overline{C_{uk}^+}(\tau,\sigma)+e^{\omega(\tau-\sigma)}\overline{C_{uk}^-}(\tau,\sigma))\Big\}\Big],\tag{32a}$$

where

$$B_{uk}^\pm(\tau-\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\sum_{l_1,l_2,l_3=0}^{\infty}f_u(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)(\tau-\sigma)},\tag{32b}$$

$$\overline{B_{uk}^\pm}(\tau,\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\Big(\sum_{l_1,l_2,l_3=0}^{\infty}f_u(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)\tau}\Big)\Big(\sum_{l_1',l_2',l_3'=0}^{\infty}f_u(l_1'l_2'l_3')\,e^{\nu(l_1'+l_2'+l_3')\sigma}\Big),\tag{32c}$$

$$C_{uk}^\pm(\tau-\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\Big\{(\eta u_x^2/2\nu)\sum_{l_1,l_2,l_3=0}^{\infty}f_u(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3-1)(\tau-\sigma)}+\sum_{l_1,l_2,l_3=0}^{\infty}g_u(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)(\tau-\sigma)}\Big\},\tag{32d}$$

$$\overline{C_{uk}^\pm}(\tau,\sigma)=\exp\{(k^2-(\boldsymbol{k}\mp\boldsymbol{u})^2)(\tau-\sigma)/2m_0\}\exp(-\eta u^2/2\nu)$$
$$\times\Big(\sum_{l_1,l_2,l_3=0}^{\infty}f_u(l_1l_2l_3)\,e^{-\nu(l_1+l_2+l_3)\tau}\Big)\Big\{(\eta u_x^2/2\nu)$$
$$\times\sum_{l_1',l_2',l_3'=0}^{\infty}f_u(l_1'l_2'l_3')\,e^{\nu(l_1'+l_2'+l_3'-1)\sigma}+\sum_{l_1',l_2',l_3'=0}^{\infty}g_u(l_1'l_2'l_3')\,e^{\nu(l_1'+l_2'+l_3')\sigma}\Big\},\tag{32e}$$

and

$$f_u(l_1l_2l_3)=(\eta/2\nu)^{l_1+l_2+l_3}(u_x)^{2l_1}(u_y)^{2l_2}(u_z)^{2l_3}/l_1!\,l_2!\,l_3!,\tag{32f}$$

$$g_u(l_1l_2l_3)=(\eta/2\nu)^{l_1+l_2+l_3}(u_x)^{2l_1}(u_y)^{2l_2}(u_z)^{2l_3}(l_1+1)^2/(l_1+1)!\,l_2!\,l_3!.\tag{32g}$$

The second and third term in the right-hand of Eq. (32a) are simplified by using Eqs. (A·2·1), (A·2·2) and (A·2·3), and is proportional to $t$. The fourth and fifth term become a constant quantities independent of $t$ for a large time $t$, as will be shown in Appendix II, and can be omitted.

According to the discussion mentioned just above, we can obtain

$$\mathrm{Tr}(e^{-(\beta+it)H}j_xe^{iHt}j_x)=\sum_k e^{-k^2\beta/2m_0}e^2\Big[(1-|t|/\tau(\boldsymbol{k},0))k_x^2/m_0^2$$
$$+\sqrt{\kappa\eta}e^{i\nu t}\{1+it(\Delta E(k,1)-\Delta E(k,0))-|t|(1/\tau(\boldsymbol{k},0)+1/\tau(\boldsymbol{k},1))/2\}\Big]$$
$$\simeq\sum_k e^{-k^2\beta/2m_0}e^2\Big[e^{-(|t|/\tau(k,0))}/m_0^2$$
$$+\sqrt{\kappa\eta}\,e^{i(\nu+\Delta E(k,1)-\Delta E(k,0))t}e^{-(|t|/2)(1/\tau(k,0)+1/\tau(k,1))}\Big],\tag{33a}$$

in the case of $n_1=n_2=n_3=0$, where

$$1/2\pi\tau(\boldsymbol{k},0)=\sum_{ul_1l_2l_3}C_u^2e^{-u^2\eta/2\nu}f_u(l_1l_2l_3)(e^\beta-1)^{-1}(\delta_u^+(1,\nu;l_1l_2l_3)$$
$$+e^\beta\delta_u^-(1,\nu;l_1l_2l_3))+\sum_{ul_1l_2l_3}D_u^2e^{-u^2\eta/2\nu}f_u(l_1l_2l_3)(e^{\omega\beta}-1)^{-1}$$
$$\times(\delta_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}\delta_u^-(\omega,\nu;l_1l_2l_3)), \tag{33b}$$

where we have used the notation $\epsilon_k=\boldsymbol{k}^2/2m_0$ and the abbreviations are

$$\delta_u^\pm(x,y;l_1l_2l_3)=\delta(\epsilon_k\pm x-\epsilon_{k\pm u}-(l_1+l_2+l_3)y). \tag{33c}$$

and

$$\Delta E(\boldsymbol{k},0)=\sum_{ul_1l_2l_3}C_u^2e^{-u^2\eta/2\nu}f_u(l_1l_2l_3)(e^\beta-1)^{-1}(p_u^+(1,\nu;l_1l_2l_3)+e^\beta p_u^-(1,\nu;l_1l_2l_3)), \tag{33d}$$

where the abbreviations are

$$p_u^\pm(xy;l_1l_2l_3)\equiv P/(\epsilon_k\pm x-\epsilon_{k\pm u}-(l_1+l_2+l_3)y), \tag{33e}$$

$$1/2\pi\tau(\boldsymbol{k},1)=\sum_{ul_1l_2l_3}C_u^2e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu)f_u(l_1l_2l_3)(e^\beta-1)^{-1}(\delta_u^+(1,\nu;l_1l_2l_3)$$
$$+e^\beta\delta_u^-(1,\nu;l_1l_2l_3))+\sum_{ul_1l_2l_3}D_u^2e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu)f_u(l_1l_2l_3)$$
$$\times(\delta_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}\delta_u^-(\omega,\nu;l_1l_2l_3))+\sum_{ul_1l_2l_3}C_u^2e^{-u^2\eta/2\nu}g_u(l_1l_2l_3)$$
$$\times(\delta_u^+(1,\nu;l_1l_2l_3)+e^\beta\delta_u^-(1,\nu;l_1l_2l_3))+\sum_{ul_1l_2l_3}D_u^2e^{-u^2\eta/2\nu}g_u(l_1l_2l_3)$$
$$\times(\delta_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}\delta_u^-(\omega,\nu;l_1l_2l_3)), \tag{33f}$$

$$\Delta E(\boldsymbol{k},1)=\sum_{ul_1l_2l_3}C_u^2e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu)f_u(l_1l_2l_3)(e^\beta-1)^{-1}(p_u^+(1,\nu;l_1l_2l_3)$$
$$+e^\beta p_u^-(1,\nu;l_1l_2l_3))+\sum_{ul_1l_2l_3}D_u^2e^{-u^2\eta/2\nu}(\eta u_x^2/2\nu)f_u(l_1l_2l_3)$$
$$\times(p_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}p_u^-(\omega,\nu;l_1l_2l_3))+\sum_{ul_1l_2l_3}C_u^2e^{-u^2\eta/2\nu}g_u(l_1l_2l_3)$$
$$\times(p_u^+(1,\nu;l_1l_2l_3)+e^\beta p_u^-(1,\nu;l_1l_2l_3))+\sum_{ul_1l_2l_3}D_u^2e^{-u^2\eta/2\nu}g_u(l_1l_2l_3)$$
$$\times(p_u^+(\omega,\nu;l_1l_2l_3)+e^{\omega\beta}p_u^-(\omega,\nu;l_1l_2l_3)). \tag{33g}$$

The terms including $D_u$ of $\tau(\boldsymbol{k},0)$ and $\tau(\boldsymbol{k},1)$ are shown to be zero, by taking into account that $D_u^2$ contains a $\delta$-function $\delta(\boldsymbol{u})$ (see Eq. (25)) and $\nu\neq\omega$. We retain these terms, however, in order to make it easy to get insight into a relation between the energy shift and the collision time.

It is noted that $\Delta E(\boldsymbol{k},0)$ and $\tau(\boldsymbol{k},0)$ correspond to the energy shift and the collision time of the state $|\boldsymbol{k},n_1=n_2=n_3=0)$, respectively, and $\Delta E(\boldsymbol{k},1)$ and $\tau(\boldsymbol{k},1)$ the energy shift and the collision time of the state $|\boldsymbol{k},n_1=1,n_2=n_3=0)$ respectively. The energy shift $\Delta E(\boldsymbol{k},0)$ at $\boldsymbol{k}=0$ near 0°K is given, after some mathematical manipulations, by

$$\Delta E(\boldsymbol{k}=0,\,0)=-\sum_{u} C_{u}^{2}\int_{0}^{\infty}\exp[-u^{2}\{\eta(1-e^{-\nu t})/2\nu+t/2m_{0}\}]e^{-t}\,dt$$

$$-\sum_{u} D_{u}^{2}\int_{0}^{\infty}\exp[-u^{2}\{\eta(1-e^{-\nu t})/2\nu+t/2m_{0}\}]e^{-\omega t}\,dt,$$

$$=-\alpha\nu/\sqrt{\pi}\,\omega\left\{\int_{0}^{\infty}d\tau\, e^{-\tau}/\sqrt{\tau(1+(\nu^{2}-\omega^{2})(1-e^{-\nu\tau})/\nu\omega^{2}\tau)}\right\}+(3C/\nu\omega). \tag{34}$$

The energy of the ground state determined by the trial action is $(3/2)(\nu-\omega)$, which has been calculated by Feynman.[3] Therefore, the self-energy of the polaron state at 0°K is $(3/2)(\nu-\omega)+\Delta E(\boldsymbol{k}=0,\,0)$. This result agrees with that obtained by Feynman, that is, the approximation in our calculation of mobility may be considered to correspond to that which gives Feynman's result for the self-energy at 0°K.

Further, $\Delta E(\boldsymbol{k}=0,\,1)$ near 0°K is given by

$$\Delta E(\boldsymbol{k}=0,\,1)=-\sum_{u} C_{u}^{2}\int_{0}^{\infty}(1+(\eta u_{x}^{2}/\nu))\exp[-u^{2}\{\eta(1-e^{-\nu t})/2\nu+t/2m_{0}\}]e^{-(\nu+1)t}\,dt$$

$$-\sum_{u} D_{u}^{2}\int_{0}^{\infty}(1+(\eta u_{x}^{2}/\nu))\exp[-u^{2}\{\eta(1-e^{-\nu t})/2\nu+t/2m_{0}\}]e^{-(\nu+\omega)t}\,dt. \tag{35}$$

Explicit calculation of Eq. (35) is not necessary, because the contribution to the mobility at very low temperature from the term including $\Delta E(\boldsymbol{k}=0,\,1)$ is neglected as we shall mention below.

We can calculate conductivity using Eqs. (1), (2) and Eq. (33) in the same way as the derivation of Eq. (14). Near 0°K, we are interested in the collision time for the state $k=0$. If we put $\tau(\boldsymbol{k}=0,\,0)=\tau_0$ and $\tau(\boldsymbol{k}=0,\,1)=\tau_1$, we get

$$1/\tau_0=2\sqrt{m_0}\,\alpha/(e^{\beta}-1),$$
$$1/\tau_1=(2\sqrt{m_0}\,\alpha/(e^{\beta}-1))(1+2(\eta m_0)^2/5\nu^2). \tag{36}$$

Using Eqs. (1), (33a) and (36), we get the mobility $\mu$ near 0°K given by

$$\mu=(\mu_0\exp(m_0\eta/\nu)/m_0^{3/2})+(\beta\sqrt{\kappa\eta}\,e/2)$$
$$\times(1/\tau_1+1/\tau_0)/2\{(\nu+\Delta E(1,\,0)-\Delta E(0,\,0))^2+(1/\tau_1+1/\tau_0)^2/4\}, \tag{37}$$

where $\mu_0$ is the mobility given by Eq. (14). The second term in the right-hand side of Eq. (37) can be neglected compared with the first term at very low temperatures. Therefore, we get

$$\lim_{\beta\to\infty}(\mu/\mu_0)=\exp(m_0\,\eta/\nu)/(m_0)^{3/2}. \tag{38}$$

## § 4. Discussion

We compare the polaron mobility at 0°K obtained in the proceding section with those obtained by Shultz[4] and by Morita[5] which are shown in Fig. 1. Their results are given over a wide range of the magnitude of coupling constant. The present result is numerically given in Table I.

The treatment used by Shultz is as follows. He considers the "extended Hamiltonian" which gives the same energy-shift as that obtained by Feynman at $\boldsymbol{k}=0$. He says that the use of this Hamiltonian in a time-dependent formalism with the appropriate boundary conditions gives the same results as the evaluation of the path-integration in all orders of the difference between true action and trial action. He calculates the self-energy for all $\boldsymbol{k}$ to the first approximation of the damping theory, using this "extended" Hamiltonian. Then, he calculates the polaron mobility, using this self-energy as the unperturbed energy of the polaron and treating the interaction between electron and phonons as the perturbation determining polaron mobility. It seems rather difficult, however, to give the concrete justification of Shultz' treatment using the damping theory. The equivalence of the path-integration and his "extended" Hamiltonian is not verified in all the orders of the difference between true action and trial action. Therefore, the approximation contained in his result of the polaron mobility is not evident.

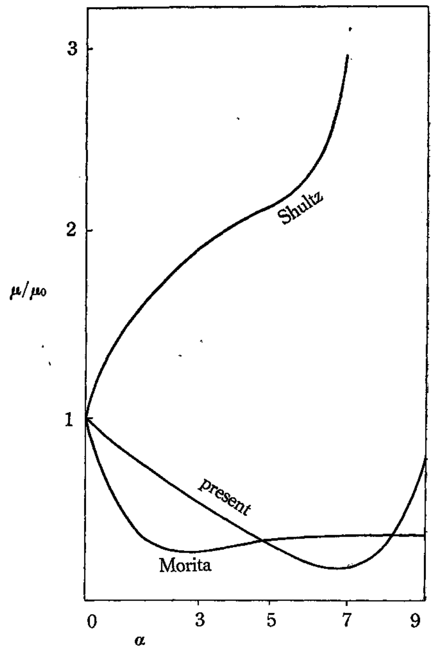

Fig. 1. Theoretical Value of Polaron mobility.

As was mentioned before, the approximation used in the present calculation for the polaron mobility at 0°K gives the same self-energy as that obtained

Table I

| $\alpha$ | $\nu$ | $\omega$ | $m_0$ | $\mu/\mu_0$ |
|---|---|---|---|---|
| 3 | 3.4 | 2.55 | 1.78 | 0.530 |
| 5 | 4.02 | 2.13 | 3.56 | 0.282 |
| 7 | 5.81 | 1.60 | 13.2 | 0.170 |
| 9 | 9.85 | 1.28 | 59.2 | 0.808 |
| 11 | 15.5 | 1.15 | 181. | 45.5 |

Parameters at 0°K

by Feynman. Our result of the collision time (33b) includes the absorption and emission of phonons followed by the excitation of the internal state of the polaron at a finite temperature. Our result is similar to Morita's result up to $\alpha=7$, if the effective mass is put to unity in his formulation. According to the present result, the mobility of the polaron increases steeply at a large coupling constant even though the increase is slower than the result obtained by Shultz where the increase begins already at a smaller coupling constant. This increase is expected because when the coupling becomes strong the wave length of phonons, which contribute effectively to the polaron scattering, decreases more steeply than the localization length of the polaron, so that the polaron becomes insensitive to the scattering by phonons. This effective wave length is proportional to $1/\alpha^2$ and the localization length of the polaron to $1/\alpha$, as Shultz already suggested. This fact is based on the assumption that the polaron mobility is determined by the scattering of the polaron which is accompanied with phonon cloud. It has to be noted, however, that the polaron mobility related to the electronic current is determined by the scattering of the electronic part of the polaron. There will remain some problem in the behaviour of the mobility for a large coupling constant.

At finite temperatures, the deviation from $(e^{\beta}-1)$ of the temperature dependence of the mobility comes from the temperature dependence of the variational parameters $\nu$ and $\omega$, that is, the temperature change of the polaron state. Moreover, other mechanisms for the scattering, for example, the internal excitation and the spontaneous emission of phonons that occur only at finite temperatures, change the temperature dependence of the mobility. Also the second term in the right-hand side of Eq. (37), which is neglected at very low temperature, has a contribution to the mobility at finite temperature.

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## Appendix I

In this appendix, we shall prove the following two equations:

$$\int e^{[S_0(X')+i\omega\cdot X'(\tau)-X'(\sigma))]}\mathfrak{D}\left(X'(t)\,;\,\begin{matrix}X_2(u)\\X_1(0)\end{matrix}\right)$$

$$=\sum_m\int dq\,e^{-uE_m^0}\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)\,(m|e^{i\omega\cdot X(\tau)}e^{-i\omega\cdot X(\sigma)}|m), \qquad (\text{A}\cdot 1\cdot 1)$$

$$\int e^{[S_0(X')+i\omega\cdot X(\tau)]}\,\mathfrak{D}\left(X'(t)\,:\,\begin{matrix}X_2(u)\\X_1(0)\end{matrix}\right)$$

$$=\lim_{\sigma\to 0}\sum_m\int dq\,e^{-uE_m^0}\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)\,(m|e^{i\omega\cdot X(\tau)}e^{-i\omega\cdot X(\sigma)}|m),$$

$$=\lim_{\sigma\to 0}\int e^{[S_0(X')+i\omega(X'(\tau)-X'(\sigma))]}\mathfrak{D}\left(X'(t)\,:\,\begin{matrix}X_2(u)\\X_1(0)\end{matrix}\right), \qquad (\text{A}\cdot 1\cdot 2)$$

where the action $S_0$ corresponds to Hamiltonian $H_0$ and the operator $e^{i\omega\cdot X(\tau)}$ is defined by $e^{\tau H_0}e^{i\omega\cdot X}e^{-\tau H_0}$. At first, we shall derive Eq. (A·1·1).

From the meaning of path-integration, we can see that

$$\int e^{S_0(X')}(\partial^n X'(t)/\partial t^n)_{t=0}\mathfrak{D}\left(X'(t):\frac{X_2(u)}{X_1(0)}\right)=\sum_m\int dq\, e^{-uE_m^0}$$
$$\times\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)(m|\underbrace{[H_0[H_0\cdots[H_0}_{n},X]\cdots]]|m),$$

where $\psi_m{}^0(X, q)$ is the eigenfunction of Hamiltonian $H_0$. Using this equation, we can derive Eq. (A·1·1) as follows:

$$\int e^{[S_0(X')+i\omega\cdot(X'(\tau)-X'(\sigma))]}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right)=\int e^{[S_0(X')+i\omega\cdot(X'(\tau-\sigma)-X'(0))}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right),$$
$$=\sum_{n=0}^{\infty}\int e^{S_0(X')}\left[\frac{\partial^n}{\partial(\tau-\sigma)^n}e^{i\omega\cdot X'(\tau-\sigma)}\right]_{\tau-\sigma=0}e^{-i\omega\cdot X'(0)}\frac{(\tau-\sigma)^n}{n!}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right),$$
$$=\sum_{n=0}^{\infty}\sum_m\int dq\, e^{-uE_m^0}\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)$$
$$\times\left[\frac{\partial^n}{\partial(\tau-\sigma)^n}(m|e^{(\tau-\sigma)H_0}e^{i\omega X}e^{-(\tau-\sigma)H_0}e^{-i\omega X}|m)\right]_{\tau-\sigma=0}\frac{(\tau-\sigma)^n}{n!}$$
$$=\sum_m\int dq\, e^{-uE_m^0}\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)(m|e^{(\tau-\sigma)H_0}e^{i\omega\cdot X}e^{-(\tau-\sigma)H_0}e^{-i\omega\cdot X}|m),$$
$$=\sum_m\int dq\, e^{-uE_m^0}\psi_m^{0*}(X_1q)\psi_m{}^0(X_2q)(m|e^{\tau H_0}e^{i\omega\cdot X}e^{-\tau H_0}e^{\sigma H_0}e^{-i\omega\cdot X}e^{-\sigma H_0}|m).$$

Noting that $e^{(i\omega\cdot X(0))}=\exp\left[i\omega\int_{-\infty}^{u}\delta(t)X(t)dt\right]$ and the boundary condition $X'(0)=X_1$, we can see that

$$\int e^{[S_0(X')+i\omega\cdot X(\tau)]}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right)=\lim_{\sigma\to0}\int e^{[S_0(X')+i\omega\cdot(X(\tau)-X(\sigma))]}\mathfrak{D}\left(X':\frac{X_2(u)}{X_1(0)}\right).\tag{A·1·3}$$

This equation is also simply derived by using the superposition of action. We can derive Eq. (A·1·2) in the same way as the derivation of Eq. (A·1·1) by using Eq. (A·1·3).

Using Eqs. (A·1·1) and (A·1·2), we can easily derive the average values of Eq. (9). For example, we get

$$\langle e^{i\omega\cdot(X''(\tau)-X''(\sigma))}\rangle_{X_2X_1;\lambda}=\{\sum_k e^{-k^2\lambda/2+ik\cdot(X_2-X_1)}(k|e^{i\omega\cdot X(\tau)}e^{-i\omega\cdot X(\sigma)}|k)\}$$
$$\div\{\sum_k e^{-k^2\lambda/2+ik\cdot(X_2-X_1)}\}=\exp(i\omega\cdot(X_2-X_1)(\tau-\sigma)/\lambda)$$
$$\times\exp\left(\frac{-\omega^2(\tau-\sigma)}{2}\left(1-\frac{(\tau-\sigma)}{\lambda}\right)\right).$$

In the same way as the derivation of Eqs. (A·1·1) and (A·1·2), we obtain Eq. (26). Applying Eq. (26) to the case of weak interaction, we simply get Eq. (11).

## Appendix II

At first we shall derive the formulae (A·2·1) :

$$\sum_{\omega} F(\omega)\left[\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s}+\int_0^{-it} ds(-it-s)e^{-f(\omega)s}\right]\simeq 2\pi t\sum_{\omega} F(\omega)\delta(f(\omega)). \tag{A·2·1}$$

We obtain, after some elementary integrations,

$$\sum_{\omega} F(\omega)\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s}=\sum_{\omega} F(\omega)\left[\frac{(\beta+it)}{f(\omega)}+\frac{\text{[illegible]}}{f^2(\omega)}\right].$$

We are interested in the asymptotic form of the above equation for a large time $t$. Since $\exp(-if(\omega)t)$ oscillates rapidly in time, the term including this factor remains finite near the region in which $f(\omega)$ nearly equals zero. Therefore we can put $f(\omega)\beta\to 0$ in the numerical factor of the term including $\exp(-if(\omega)t)$ of this equation. Therefore, it holds

$$\sum_{\omega} F(\omega)\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s}\simeq\sum_{\omega}\left[\frac{(\beta+it)}{f(\omega)}+\frac{1-e^{-if(\omega)t}}{f^2(\omega)}\right]F(\omega)$$

$$=\sum_{\omega}\left[\frac{i}{f(\omega)}\left(t+\frac{\sin f(\omega)t}{f(\omega)}\right)+\frac{(1-\cos f(\omega)t)}{f^2(\omega)}\right]F(\omega)+\sum_{\omega}\frac{\beta}{f(\omega)}F(\omega). \tag{A·2·2}$$

Further, we get

$$\sum_{\omega} F(\omega)\int_0^{-it}(-it-s)e^{-f(\omega)s}ds$$

$$=\sum_{\omega}\left[-\frac{i}{f(\omega)}\left\{t+\frac{\sin f(\omega)t}{f(\omega)}\right\}+\frac{(1-\cos f(\omega)t)}{f^2(\omega)}\right]F(\omega). \tag{A·2·3}$$

From the above equations, we obtain

$$\sum_{\omega} F(\omega)\left[\int_0^{\beta+it} ds(\beta+it-s)e^{-f(\omega)s}+\int_0^{-it} ds(-it-s)e^{-f(\omega)s}\right]$$

$$\simeq\sum_{\omega}\frac{\beta}{f(\omega)}F(\omega)+2\sum_{\omega}\frac{F(\omega)[1-\cos f(\omega)t]}{[f(\omega)]^2}.$$

If we employ the procedure that is the same as that used in the theory of ordinary time dependent perturbation, to the term including $1-\cos f(\omega)t$, this equation is transformed into

$$\sum_{\omega}\frac{\beta F(\omega)}{f(\omega)}+2\pi t\sum_{\omega}F(\omega)\,\delta(f(\omega)).$$

For a large time $t$, we obtain

$$\sum_{\omega}F(\omega)\left[\int_0^{\beta+it}ds(\beta+it-s)e^{-f(\omega)s}+\int_0^{-it}ds(-it-s)e^{-f(\omega)s}\right]\simeq 2\pi t\sum_{\omega}F(\omega)\,\delta(f(\omega)). \tag{A·2·1}$$

The last two terms in Eq. (11) appearing in § 2 have the following form:

$$\sum_{\omega}F(\omega)\int_0^{\beta+it}e^{-f_1(\omega)s_1}ds_1\int_0^{-it}e^{-f_2(\omega)s_2}ds_2.$$

After elementary integrations, we get

$$\sum_{\omega}F(\omega)\int_0^{\beta+it}e^{-f_1(\omega)s_1}ds_1\int_0^{-it}e^{-f_2(\omega)s_2}ds_2=\sum_{\omega}iF(\omega)\left(\frac{1-f_1(\omega)\beta}{f_1(\omega)}\right)$$

$$\times\int_0^{t}e^{if_2(\omega)z}dz+\sum_{\omega}e^{-\beta f_1(\omega)}F(\omega)\int_0^{t}e^{-if_1(\omega)z_1}dz_1\int_0^{t}e^{if_2(\omega)z_2}dz_2.$$

For a large time $t$, the right-hand side of this equation tends to

$$\sum_{\omega}iF(\omega)\left(\frac{1-f_1(\omega)\beta}{f_1(\omega)}\right)[\pi\delta(f_2(\omega))+iP/f_2(\omega)]$$

$$+\sum_{\omega}F(\omega)e^{-\beta f_1(\omega)}[\pi\delta(f_1(\omega))+iP/f_1(\omega)][\pi\delta(f_2(\omega))+iP/f_2(\omega)].$$

These terms can be neglected with comparison to the terms which are proportional to $t$, for a large time $t$.

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