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THEORY OF PREHOMOGENEOUS VECTOR SPACES (ALGEBRAIC PART)—THE ENGLISH TRANSLATION OF SATO'S LECTURE FROM SHINTANI'S NOTE

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Contents

Introduction

The purpose of this paper is to introduce a-functions and b-functions of prehomogeneous vector spaces in the original way of M. Sato and give a proof of the structure theorem of them. All the results were obtained by M. Sato when he constructed the theory of prehomogeneous vector spaces in 60's. However he did not write a paper on his outcomes at that time. His theory was distributed through his lectures and informal seminars. Only small number of people could know it. The only publication left for us is a mimeographed note of his lecture [Sa-Sh1] written by T. Shintani in Japanese. Sato and Shintani published the paper [Sa-Sh2] in 1974 on zeta functions associated with prehomogeneous vector spaces, but a very narrow class of prehomomgeneous vector space was dealt with there. In [Sa-Sh1], Sato gave the exact definitions of a-functions and b-functions for a wider class of prehomogeneous vector spaces and gave a remarkable theorem of their structures. But it seems to have been forgotten for a long time.

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This paper stems from the chapter one of [Sa-Sh1], or we may say that all of the part of this paper is a modified translation from [Sa-Sh1] not only in the contents but also in the formulation. However, the responsibility for this paper rests with the translator. The translator tried to state faithfully the original idea of M. Sato based on the lecture note which is left to us by T. Shintani. The proof of the structure theorem is given in English for the first time here. The key of the proof is the theorem in Appendix; it is useful for clarifying the structure not only of b-functions but also more wider class of functions satisfying the cocycle condition. The idea due to Sato (and partly to Shintani), that has never been written in a formal publication or widely circulated journals, appears for the first time in English. In particular, his original proof is given here for the fact that the b-functions for relative invariants are divided to a product of inhomogeneous linear forms. This follows from the theorem (Theorem or its corollary in Appendix) on a family of rational functions with a cyclic condition, which itself is a useful proposition. In order to clarify the role of this theorem, the translator extracts it from the original proof. It can be proved independently from the theory of prehomogeneous vector spaces. Recently it becomes clear that it has a fruitful application. See Aomoto [A1] and [A2].

The translator wishes to express his gratitude to Professor M. Sato for his permission to publish his results in this form. He is also grateful to Professor Aomoto for his suggestion to publish this paper.

§ 1. Fundamental idea of prehomogeneous vector spaces

Let Ω be a universal domain of characteristic 0 and let V be an n-dimensional vector space over Ω. Let $G \subset GL(V)$ be a connected linear algebraic group defined over Ω, and we denote by $g \cdot x$ the action of G on V with $g \in G$ and $x \in V$. For a point x in V, we denote by H_x the *isotropy subgroup*, i.e., $H_x := \{g \in G;\ g \cdot x = x\}$. In the theory of algebraic groups, the following lemma is well-known.

LEMMA 1. *Let x be a point in V. The G-orbit $G \cdot x$ generated by x can be written as $G \cdot x = E_x - F_x$ where E_x is a G-invariant algebraic subset in V and F_x is a G-invariant proper algebraic subset of E_x. Here* $\dim E_x = \dim G - \dim H_x$. *Here* dim *means the dimension. The totality of $x \in V$ such that the dimension of E_x are maximal is a G-invariant Zariski-open set in V.*

DEFINITION 1 (prehomogeneous vector space). If there exists a point $x \in V$ such that $\dim H_x = \dim G - \dim V$, then we say that V is *prehomogeneous* with respect to the action of G and call the pair (G, V) a *prehomogeneous vector space*. We call the set of points $x \in V$ such that $\dim H_x > \dim G - \dim V$ the *singular set* and denote it by S.

Henceforth, let (G, V) be a prehomogeneous vector space and let S be its singular set. Let $x \in V$ and put $G \cdot x = E_x - F_x$. Then, from the definition, the following four conditions are equivalent;

(1) $x \in V - S$,

(2) $\dim H_x = \dim G - \dim V$

(3) $\dim E_x = \dim V$

(4) $E_x = V$.

Note that $\dim E_x \leq \dim V$. Then, by Lemma 1, $V - S$ is a G-invariant Zariski-open subset in V, which implies that S is a G-invariant proper algebraic subset. If $x \in V - S$, then $G \cdot x \subset V - S$. Since $G \cdot x = V - F_x$, we have $F_x \supset S$. Now, suppose that $F_x - S \neq \varnothing$ and $y \in F_x - S$. Then $G \cdot y = V - F_y$ is a Zariski-open set in V for $y \notin S$. On the other hand, since $y \in F_x$ and since F_x is G-invariant, $G \cdot y$ is contained in the proper algebraic subset F_x of V. This is a contradiction. Therefore we have $F_x - S = \varnothing$, which yields that $F_x = S$. This means that $G \cdot x = V - S$ if $x \in V - S$. Consequently $V - S$ is a G-orbit. Arranging the above arguments, we have the following proposition.

PROPOSITION 1. *Let (G, V) be a prehomogeneous vector space and let S be the singular set. Then S is a G-invariant proper algebraic subset in V and $V - S$ is a G-orbit in V.*

DEFINITION 2 (rational characters and character groups). We call a rational homomorphism from G to $\Omega^{\times} := \Omega - \{0\}$ a *rational character*. We denote by $X(G)$ the set of all rational characters which forms a multiplicative group. Let χ be a rational character of G. We call a non-zero rational function $P(x)$ on V a *relative invariant* or a *relatively invariant rational function* corresponding to the character χ if $P(g \cdot x) = \chi(g) \cdot P(x)$ for any $g \in V$. In particular, if $P(x)$ is a polynomial, we also call it a *relatively invariant polynomial*. Let $\chi_1, \cdots, \chi_r$ be rational characters belonging to $X(G)$. We say that they are *multiplicatively independent* when they generate a free Abelian group of rank r in $X(G)$.

PROPOSITION 2. (1) *Any relative invariant corresponding to a character is determined up to a constant factor by the character.*

(2) *Any prime divisor of a relative invariant is a relative invariant.*

(3) *Relative invariants are homogeneous functions.*

(4) *Relative invariants corresponding to multiplicatively independent characters are algebraically independent.*

Proof. (1) From the definition, the zeros and poles of a relative invariant are G-invariant proper algebraic subsets. They are contained in S. Let $R_1(x)$ and $R_2(x)$ be two relative invariants corresponding to a rational character $\chi \in X(G)$. Let $x_0 \in V - S$ and let $Q(x) = R_2(x_0)R_1(x) - R_1(x_0)R_2(x)$. Then we have $Q(g\cdot x) = \chi(g)\cdot Q(x)$ for all $g \in G$ and all $x \in V$, and $Q(x_0) = 0$. Therefore, we have $Q(x) = 0$ for all x in $V - S$ because $V - S$ coincides with $G\cdot x_0$. Thus we have $Q = 0$. Since $R_1(x_0)$ is not 0 or ∞, we have $R_1(x) = (R_1(x_0)/R_2(x_0))\cdot R_2(x)$.

(2) Let $R(x)$ be a relative invariant corresponding to a character $\chi \in X(G)$ and let $\prod_{i=1}^{k} R_i(x)^{n_i}$ be the decomposition into prime divisors of $R(x)$. Namely, $R_1(x), \cdots, R_k(x)$ are mutually different irreducible polynomials and n_i's are non-zero integers such that $R(x) = \prod_{i=1}^{k} R_i(x)^{n_i}$. From the definition, we have $\prod_{i=1}^{k} R_i(g\cdot x)^{n_i} = \chi(g) \prod_{i=1}^{k} R_i(x)^{n_i}$. Since each $R_i(g\cdot x)$ $(i = 1, \cdots, k)$ is an irreducible polynomial on V, it must coincide with one of the polynomials $R_1(x), \cdots, R_k(x)$ up to a constant factor. However, since the group G is a connected algebraic group, we have $R_i(g\cdot x) = \chi_i(g)R_i(x)$ for all $g \in G$. Here, $\chi_i(g)$ is an $\Omega^\times$-valued rational function on G and evidently is a rational character of G. Thus $R_1(x), \cdots, R_k(x)$ are all relative invariants.

(3) Let $P(x)$ be a relative invariant corresponding to a rational character $\chi \in X(G)$. Let t be an element of $\Omega^\times$ and define a rational function $P_t(x) := P(tx)$. Then we have $P_t(g\cdot x) = P(g\cdot(tx)) = \chi(g)P(tx) = \chi(g)P_t(x)$ for all $x \in G$ and all $x \in V$. Therefore both $P(x)$ and $P_t(x)$ are relative invariants corresponding to χ, and hence they coincide with each other up to a constant factor. Consequently we have $P(tx) = c\cdot P(x)$ with a constant c depending only on t, which means $P(x)$ is a homogeneous polynomial.

(4) Let $\chi_1, \cdots, \chi_r$ be rational characters in $X(G)$ which are multiplicatively independent. Let $R_1(x), \cdots, R_r(x)$ be relative invariants corresponding to the characters $\chi_1, \cdots, \chi_r$, respectively. Suppose that $R_1(x), \cdots, R_r(x)$ are algebraically dependent. Then we may take monomials

$U_i(R_1, \cdots, R_r)$ $(i = 1, \cdots, s)$ in $R_1(x), \cdots, R_r(x)$ such that $U_1, \cdots, U_s$ are linearly dependent and any $(s-1)$ of them are linearly independent. Now, we let $W := \{(c_1, \cdots, c_s) \in \Omega^s;\ \sum_{i=1}^{s} c_i U_i = 0\}$. Then, from the assumption, W is a one-dimensional vector subspace in Ω^s. On the other hand, each U_i is a relative invariant. We let ν_i be the corresponding character of $U_i(x)$. If $(c_1, \cdots, c_s) \in W$, then we have $(c_1\nu_1(g), \cdots, c_s\nu_s(g)) \in W$ for any $g \in G$ from the definition. Since $\dim W = 1$, we have $\nu_1 = \cdots = \nu_s$. However, $U_1, \cdots, U_s$ are different from one another as monomials in $R_1, \cdots, R_r$, which means $\nu_1, \cdots, \nu_s$ are different from one another since $\chi_1, \cdots, \chi_s$ are multiplicatively independent. This is a contradiction. Thus $R_1(x), \cdots, R_r(x)$ are algebraically independent. (q.e.d.)

DEFINITION 3 (singular set). Let (G, V) be a prehomogeneous vector space and let S be its singular set. We let $S_{(0)}$ be the *union of irreducible components in S of codimension one* in V and let $S_{(1)}$ be the *union of irreducible components in S of codimension more than two.* Let $S_1, \cdots, S_m$ be irreducible components of $S_{(0)}$ and let $P_i(x)$ be an *irreducible polynomial defining the irreducible hypersurface S_i.* Namely;

$$S = S_{(0)} \cup S_{(1)} \quad \text{and} \quad S_{(0)} = S_1 \cup \cdots \cup S_m\,,$$
$$S_i := \{x \in V;\, P_i(x) = 0\} \quad (i = 1, \cdots, m)\,,$$
$$\operatorname{codim} S_{(0)} \geq 2\,.$$

PROPOSITION 3. (1) *The defining polynomials $P_1(x), \cdots, P_m(x)$ in Definition 3 are relatively invariant polynomials of (G, V) and they are algebraically independent.*

(2) *The group of relative invariants under multiplications coincides with the free Abelian group of rank m generated by $P_1(x), \cdots, P_m(x)$. Here we consider polynomials modulo constant factors.*

Proof. (1) Since G is a connected algebraic group and since S_i is an irreducible algebraic subset, the algebraic closure $(\overline{G\cdot S_i})$ of the set $G\cdot S_i := \{g\cdot x;\, g \in G,\, x \in S_i\}$ is also irreducible. On the other hand, $S_i \subset G\cdot S_i \subset S$ implies that $S_i \subset (\overline{G\cdot S_i}) \subset S$, and S_i is an irreducible component. Then we have $S_i = (\overline{G\cdot S_i})$. Thus we have $g\cdot S_i = S_i$ for any $g \in G$. Since $g\cdot S_i = \{x \in V;\ P_i(g^{-1}\cdot x) = 0\}$ and $S_i = \{x \in V;\ P_i(x) = 0\}$, and since $P_i(x)$ is an irreducible polynomial, the two polynomials $P_i(x)$ and $P_i(g^{-1}\cdot x)$ coincide with each other up to a constant factor. Therefore, there exists a character $\chi_i(g)$ such that $P_i(g\cdot x) = \chi_i(g)P_i(x)$ for all $g \in G$

and all $x \in V$. This means that $P_1(x), \cdots, P_m(x)$ are all relative invariants. Since $P_1(x), \cdots, P_m(x)$ are mutually different irreducible polynomials, $\chi_1, \cdots, \chi_m$ are multiplicatively independent. Thus $P_1(x), \cdots, P_m(x)$ are algebraically independent by Proposition 2, (4).

(2) Let $P(x)$ be an irreducible relatively invariant polynomial. Since $\{x \in V; P(x) = 0\}$ is a G-invariant proper algebraic subset, it is contained in S, and since it is an irreducible hypersurface, it coincides with one of $S_1, \cdots, S_m$. Therefore $P(x)$ coincides with one of $P_1(x), \cdots, P_m(x)$. From Proposition 2, (2), any relative invariant is written as a product of integer powers of irreducible relatively invariant polynomials. Thus any relative invariant coincides with a product of powers of $P_1(x), \cdots, P_m(x)$.

(q.e.d.)

DEFINITION 4 (basic relative invariants and their character group $X_1(G)$).

(1) The polynomials of $P_1(x), \cdots, P_m(x)$ are called *basic relative invariants* of (G, V). We denote by $\chi_1, \cdots, \chi_m$ the rational characters corresponding to the relative invariants $P_1(x), \cdots, P_m(x)$, respectively. The set $\{P_1, \cdots, P_m\}$ of all basic relatively invariants is called the *complete system of basic relative invariants.*

(2) We denote by $[G, G]$ the *commutator group* of G. Let $x_0 \in V - S$ be a fixed point. Then the subgroup of G generated by $[G, G]$ and the isotropy subgroup H_{x_0} does not depend on the choice of $x_0 \in V - S$. We denote it by G_1, i.e., $G_1 := [G, G] \cdot H_{x_0}$. The group G_1 is a normal algebraic subgroup of G, and G/G_1 is a connected abelian algebaric group. We denote by $X_1(G)$ the group of rational characters of G/G_1. Namely, $X_1(G) = \{\chi \in X(G); \chi(g) = 1 \text{ for all } g \in G_1\}$.

PROPOSITION 4. *The character group $X_1(G)$ defined in Definition* 4 *is the free Abelian group of rank m generated by $\chi_1, \cdots, \chi_m$.*

Proof. Let $P(x)$ be a relative invariant corresponding to a character $\chi(g)$. Let $x \in V - S$. Then $P(g \cdot x) = \chi(g)P(x)$ and $P(x) \neq 0$ or ∞. Thus if $g \in H_x$, then $\chi(g) = 1$. That is to say, χ is trivial on H_x. On the other hand, since $\chi(g_1 g_2 g_1^{-1} g_2^{-1}) = 1$ for all $g_1, g_2 \in G$, χ is trivial on $[G, G]$. Thus χ is trivial on $G_1 = [G, G] \cdot H_x$. We have $\chi \in X_1(G)$. Conversely let χ be an arbitrary element of $X_1(G)$. Let $x_0 \in V - S$. Then χ can be viewed as a rational regular function on $G/H_{x_0} \simeq V - S$. We denote it by P, a rational regular function on $V - S$, which is evaluated by $P(g \cdot x_0) := \chi(g)$

with $g \in G$. Here we identify $g \cdot x_0 \in V - S$ with the representative $[g] \in G/H_{x_0}$ of $g \in G$. We have $P(g \cdot x) = \chi(g)P(x)$ for any $g \in G$ by definition. The rational function $P(x)$ on $x \in V - S$ is extended to the rational function $P(x)$ on $x \in V$ keeping the relation $P(g \cdot x) = \chi(g)P(x)$. Namely, there is a relative invariant $P(x)$ corresponding to the rational character χ. By Theorem 1, the subgroup of $X(G)$ consisting of characters corresponding to relative invariants coincides with the free Abelian group generated by $\chi_1, \cdots, \chi_m$. Thus we complete the proof. (q.e.d.)

Next we consider the contragredient representation of G on the dual vector space V^*. The action of $g \in G$ for $y \in V^*$ is denoted by $g^* \cdot y$. We have $\langle g \cdot x, g^* \cdot y \rangle = \langle x, y \rangle$ for all $g \in G$, $x \in V$ and $y \in V^*$. Here $\langle \, , \rangle$ stands for the canonical bilinear form on $V \times V^*$ to Ω. The group G is viewed as a connected linear algebraic subgroup of $GL(V^*)$. For a fixed point $y_0 \in V^*$, we denote by $H_{y_0 *}$ the isotropy subgroup at y_0, i.e., $H_{y_0 *} := \{g \in G; g^* \cdot y_0 = y_0\}$. *The dual pair* (G, V^*) may not be a prehomogeneous vector space even if (G, V) is prehomogeneous (see the example at the end of this section). In this paper, we are interested in the cases that at least one of (G, V) and (G, V^*) is a prehomogeneous vector space. We give the notations for (G, V^*) here when (G, V^*) is a prehomogeneous vector space.

We suppose that (G, V^*) is a prehomogeneous vector space. (However, we do not have to suppose that (G, V) is prehomogeneous.) let S^* be the singular set of (G, V^*). By Proposition 1, S^* is a G-invariant proper algebraic subset in V^* and $V^* - S^*$ is a G-orbit in V^*. We denote by $S^*_{(0)}$ the union of irreducible components of S^* whose codimension in V^* is one. The set $S^*_{(1)}$ is the union of irreducible components of S^* of codimension more than two in V^*. Let $S^*_1, \cdots, S^*_m$ be irreducible components of $S^*_{(0)}$ and let $Q_i(y)$ be an irreducible polynomial defining the hypersurface S^*_i. Namely;

$$\begin{aligned} &S^* = S^*_{(0)} \cup S^*_{(1)} \quad \text{and} \quad S^*_{(0)} = S^*_1 \cup \cdots \cup S^*_{m'} \,, \\ &S_i := \{y \in V^*;\ Q_i(y) = 0\} \quad (i = 1, \cdots, m') \,, \\ &\operatorname{codim} S^*_{(1)} \geq 2 \,. \end{aligned}$$

Applying Theorem 1 to the prehomogeneous vector space (G, V^*), $Q_1(y)$, $\cdots$, $Q_{m'}(y)$ are basic relative invariants of the prehomogeneous vector space (G, V^*), and $(Q_1, \cdots, Q_{m'})$ is the complete system of basic relative invariants of (G, V^*). We denote by $\mu_1, \cdots, \mu_{m'}$ the corresponding rational characters of the relative invariants $Q_1(y), \cdots, Q_{m'}(y)$, respectively, i.e., $Q_i(g^*\cdot y) = \mu_i(g)Q_i(y)$ for all $g \in G$ and all $y \in V$. Let $y_0 \in V^* - S^*$ be a fixed point. Then the subgroup $G_{1*} := [G, G]\cdot H_{y_0*}$ does not depend on the choice of $y_0 \in V^* - S^*$. Here H_{y_0*} means the isotropy subgroup of G at $y_0 \in V$. G_{1*} is a normal subgroup of G and G/G_{1*} is a connected Abelian algebraic group. We denote by $X_{1*}(G)$ the subgroup of $X(G)$ consisting of rational characters which is trivial on G_{1*}, i.e., $X_{1*}(G) ;= \{\mu \in X(G); \mu(g) = 1 \text{ for all } g \in G_{1*}\}$. Applying Proposition 4 to (G, V^*), $X_{1*}(G)$ is the free Abelian group of rank m' generated by $\mu_1, \cdots, \mu_{m'}$.

We close this section by giving an examples of prehomogeneous vector spaces. A systematical classification of prehomogeneous vector spaces have been done by [Sa-Ki].

EXAMPLE (a prehomogeneous vector space whose contragredient action is not prehomogeneous). Let $G := \left\{g = \begin{pmatrix} 1, & a \\ 0, & b \end{pmatrix};\ a,\ b \in \Omega,\ b \neq 0\right\}$ and let $V := \left\{x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix};\ x_1,\ x_2 \in \Omega\right\}$. Then G is an algebraic subgroup in $GL(V)$. The Zariski-dense subset $V' := \{x \in V;\ x_2 \neq 0\}$ is a G-orbit, hence (G, V) is a prehomogeneous vector space whose singular set is $\{x \in V;\ x_2 = 0\}$. Let $V^* := \left\{y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix};\ y_1, y_2 \in \Omega\right\}$ be the dual vector space. The contragredient action g^* is given by $g^*\cdot y = {}^t g^{-1}\cdot y$. Then each orbit in V^* is parametrized by the value y_1. This means there are no Zariski-dense orbits in V^*. Thus (G, V^*) is not a prehomogeneous vector space.

§ 2. Quasi regular prehomogeneous vector space

Let V be an n-dimensional vector space defined over the universal domain Ω and let V^* be its dual vector space. Let G be a linear algebraic subgroup of $GL(V)$, which naturally acts on V^* by the contragredient action. We suppose that (G, V) (resp. (G, V^*)) is a prehomogeneous vector space defined on Ω. We use the same notations as in § 1.

Let $\mathfrak{g}$ be the Lie algebra of G and let $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{1*}$) be the Lie algebra of G_1 (resp. G_{1*}). Let $\mathfrak{g}^\vee$ be the dual vector space of $\mathfrak{g}$. We denote by $A\cdot x$ (rsep. $A^*\cdot y$) the action of an element $A \in \mathfrak{g}$ for $x \in V$ ($y \in V^*$), i.e., $A\cdot x := (d/dt)(\exp(tA)\cdot x)|_{t=0}$ (resp. $A^*\cdot y := (d/dt)(\exp(tA)^*\cdot x)|_{t=0}$). We have the relation $\langle A\cdot x, y\rangle + \langle x, A^*\cdot y\rangle = 0$ for all $A \in \mathfrak{g}$ and $x \in V$, $y \in V^*$.

Let φ (resp. ψ) be a rational map from V to V^* (resp. V^* to V). In other words, φ (resp. ψ) is a V^*-valued (resp. V-valued) rational function on V (resp. V^*). We say that φ (resp. ψ) is a *G-admissible map* if φ (resp. ψ) is a regular function on $V-S$ (resp. V^*-S^*) and $\varphi(g\cdot x)=g^*\cdot\varphi(x)$ (resp. $\psi(g^*\cdot y)=g\cdot\psi(y)$) for all $g\in G$.

Let $\omega\in\mathfrak{g}^\vee$. When a G-admissible map φ (resp. ψ) satisfies the condition $\langle A\cdot x,\varphi(x)\rangle=\langle A,\omega\rangle$ (resp. $\langle A^*\cdot y,\psi(y)\rangle=-\langle A,\omega\rangle$) for all $A\in\mathfrak{g}$ and $x\in V-S$ (resp. $y\in V^*-S^*$), we denote it by φ_ω (resp. ψ_ω). We denote by $\overline{X}_1$ (resp. $\overline{X}_{1*}$) the totality of the elements in $\mathfrak{g}^\vee$ which are null on $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{1*}$). In other words, $\overline{X}_1$ (resp. $\overline{X}_{1*}$) is the dual space $(\mathfrak{g}/\mathfrak{g}_1)^\vee$ (resp. $(\mathfrak{g}/\mathfrak{g}_{1*})^\vee$). For an element $\chi\in X_1(G)$ (resp. $\mu\in X_{1*}(G)$), we define an element $\delta\chi\in\overline{X}_1$ (resp. $\delta\mu\in\overline{X}_{1*}$) by $\delta\chi(A):=(d/dt)\chi(\exp(tA))|_{t=0}$ (resp. $\delta\mu(A):=(d/dt)\mu(\exp(tA))|_{t=0}$) for $A\in\mathfrak{g}$. The map δ gives an injective homomorphism from $X_1(G)$ (resp. $X_{1*}(G)$) to $\overline{X}_1$ (resp. $\overline{X}_{1*}$). We call the *infinitesimal character* of χ.

PROPOSITION 5. *In order that there exists a G-admissible map φ_ω (resp. ψ_ω) satisfying the condition $\langle A\cdot x,\varphi_\omega(x)\rangle=\langle A,\omega\rangle$ (resp. $\langle A^*\cdot y,\psi_\omega(y)\rangle=-\langle A,\omega\rangle$) for all $A\in\mathfrak{g}$ and $x\in V-S$ (resp, $y\in V^*-S^*$), it is necessary and sufficient that $\omega\in\overline{X}_1$ (resp. $\omega\in\overline{X}_{1*}$). The map φ_ω (resp. ψ_ω) is determined uniquely if it exists.*

Proof. We shall give the proof only for the map φ_ω since almost the same proof for ψ_ω can be obtained easily.

(Necessity). Let φ_ω be a G-admissible map satisfying the condition $\langle A\cdot x,\varphi_\omega(x)\rangle=\langle A,\omega\rangle$ for all $A\in\mathfrak{g}$ and $\omega\in V-S$. Let $x\in V-S$ and we denote by $\mathfrak{h}_x$ the Lie algebra of H_x, i.e., $\mathfrak{h}_x=\{A\in\mathfrak{g};\ A\cdot x=0\}$. Then $\langle\omega,A\rangle=0$ for all $A\in\mathfrak{h}_x$. On the other hand, since $\varphi_\omega(g\cdot x)=g^*\cdot\varphi_\omega(x)$ for all $x\in V-S$ and $g\in G$, we have $\langle\omega,\mathrm{Ad}(g)\cdot A\rangle=\langle\omega,A\rangle$ for any $g\in G$ and $A\in\mathfrak{g}$. Here Ad means the adjoint representation of G on $\mathfrak{g}$. Indeed, $\langle\omega,\mathrm{Ad}(g)\cdot A\rangle=\langle\omega,gAg^{-1}\rangle=\langle gAg^{-1}\cdot x,\varphi_\omega(x)\rangle=\langle Ag^{-1}\cdot x,g^*\cdot\varphi_\omega^{-1}(x)\rangle=\langle Ag^{-1}\cdot x,\varphi_\omega(g^{-1}\cdot x)\rangle=\langle\omega,A\rangle$. Thus we have $\langle\omega,[\mathfrak{g},\mathfrak{g}]\rangle=0$. This means that ω is null both on $\mathfrak{h}_x$ and on $[\mathfrak{g},\mathfrak{g}]$. The Lie algebra $\mathfrak{g}_1$ is the Lie algebra generated by $\mathfrak{h}_x$ and $[\mathfrak{g},\mathfrak{g}]$, and hence ω is null on $\mathfrak{g}_1$.

(Sufficiency) Suppose that $\omega\in\mathfrak{g}^\vee$ is null on $\mathfrak{g}_1$. Take an element x in $V-S$. Since $\dim\mathfrak{g}-\dim\mathfrak{h}_x=\dim V$, the map $A\mapsto A\cdot x$ from $\mathfrak{g}/\mathfrak{h}_x$ to V gives a one-to-one linear map. Since ω is null on $\mathfrak{h}_x$, there exists an element $\varphi(x)\in V^*$ satisfying $\langle A\cdot x,\varphi(x)\rangle=\langle A,\omega\rangle$ for all $A\in\mathfrak{g}$ and it is determined uniquely. On the other hand, since ω is null on $\mathfrak{g}_1$, we have $\langle\omega,\mathrm{Ad}(g)\cdot A\rangle=\langle\omega,A\rangle$ for all $A\in\mathfrak{g}$ and $g\in G$. Therefore we have

$\langle\varphi(g\cdot x), Ag\cdot x\rangle = \langle\omega, A\rangle = \langle\omega, \mathrm{Ad}(g^{-1})A\rangle = \langle g^*\varphi(x), Ag\cdot x\rangle$. Thus $\varphi(g\cdot x) = g^*\cdot\varphi(x)$ for all $x\in V-S$ and $g\in G$. Since it is clear that φ is a regular rational function on $V-S$, φ is a G-admissible map from V to V^* which satisfies the condition in $\langle A\cdot x, \varphi_\omega(x)\rangle = \langle A, \omega\rangle$ for all $A\in\mathfrak{g}$ and $x\in V-S$. At the same time we have proved that φ is determined uniquely by ω. (q.e.d.)

For $x\in V-S$ (resp. $y\in V^*-S^*$), we define $d\varphi_\omega(x)$ (resp. $d\psi_\omega(y)$) to be the linear map from $V\simeq T_xV$ (resp. $V^*\simeq T_yV^*$) to $V^*\simeq T_{\varphi_\omega(x)}V^*$ (resp. $V\simeq T_{\psi_\omega(y)}V$) given by

$$u\longmapsto d\varphi_\omega(x)(u) := \left\{\frac{d}{dt}\varphi_\omega(x+tu)\right\}\Big|_{t=0} \qquad (u\in V).$$

$$\left(\text{resp.}\quad v\longmapsto d\psi_\omega(y)(v) := \left\{\frac{d}{dt}\psi_\omega(y+tv)\right\}\Big|_{t=0} \qquad (v\in V^*).\right)$$

Let $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$ be the coordinates on V and V^* respectively, such that they are dual coordinates to each other. The map φ_ω (resp. ψ_ω) is written as $((\varphi_\omega)_1,\cdots,(\varphi_\omega)_n)$ (resp. $((\psi_\omega)_1,\cdots,(\psi_\omega)_n)$) with respect to the coordinate $(y_1,\cdots,y_n)$ (resp. $(x_1,\cdots,x_n)$). Let $\varphi_\omega(x)\cdot dx := \sum_{i=1}^n(\varphi_\omega(x))_i dx_i$ (resp. $\psi_\omega(y)\cdot dy := \sum_{i=1}^n(\psi_\omega(y))_i dy_i$), a G-invariant differential form on $V-S$ (resp. V^*-S^*). This definition does not depend on the choice of the coordinate. Each component $(\varphi_\omega(x))_i$ (resp. $(\psi_\omega(y))_i$) is a homogeneous polynomial of degree one with respect to ω.

PROPOSITION 6. *Let (G, V) resp. (G, V^*)) be a prehomogeneous vector space.*

(1) *Let $\omega\in\overline{X}_1$ (resp. $\omega\in\overline{X}_{1*}$) and let $d\varphi_\omega(x)$ (resp. $d\psi_\omega(y)$) be the linear map defined above. Then $d\varphi_\omega(g\cdot x)(g^*\cdot u) = g^*\cdot\{d\varphi_\omega(x)(u)\}$ (resp. $d\psi_\omega(g^*\cdot y)\cdot(g\cdot v) = g\cdot\{d\varphi_\omega(y)(v)\}$) for all $g\in G$, $x\in V-S$ and $u\in V$ (resp. $y\in V^*-S^*$ and $v\in V^*$).*

(2) *The differential form $\varphi_\omega(x)\cdot dx$ (resp. $\psi_\omega(y)\cdot dy$) defined above is a closed form on V (resp. V^*). Hence, the map $(u,v)\in V\times V\mapsto\langle u, d\varphi_\omega(x)(v)\rangle\in\Omega$ (resp. $(u,v)\in V^*\times V^*\mapsto\langle u, d\psi_\omega(y)(v)\rangle\in\Omega$) is a symmetric bilinear form on $V\times V$ (resp. $V^*\times V^*$).*

Proof. We shall prove this theorem for φ_ω. The proof for ψ_ω is obtained in a similar way.

(1) It is clear from the definition.

(2) We take coordinates $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$ on V and V^* respectively, such that they are dual to each other. We may identify

them with the same vector space Ω^n. The linear actions of G and $\mathfrak{g}$ on $V=\Omega^n$ are represented as $n\times n$ matrices by $g=(g_{ij})$ and $A=(A_{ij})$. Then the i-th coordinates of $(g\cdot x)$ and $(A\cdot x)$ are given by

$$(g\cdot x)_i=\sum_{j=1}^n g_{ij}\cdot x_j\,,\qquad (g\in G)\,,$$
$$(A\cdot x)_i=\sum_{j=1}^n A_{ij}\cdot x_j\,,\qquad (A\in \mathfrak{g})\,,$$

When we let $\varphi_\omega(x):=(\varphi_\omega(x)_1,\cdots,\varphi_\omega(x)_n)$ for $x=(x_1,\cdots,x_n)\in V$, we have from the definition $\sum_{i,j=1}^n\varphi_\omega(x)_iA_{ij}x_j=\langle\omega,A\rangle$ for all $A\in\mathfrak{g}$ and $x=(x_1,\cdots,x_n)\in V-S$. Therefore we have

$$\sum_{ij=1}^n\frac{\partial\varphi_\omega(x)_i}{\partial x_l}\cdot A_{ij}x_j+\sum_{i=1}^n\varphi_\omega(x)_iA_{il}=0\,.$$

On the other hand, since $\varphi_\omega(g\cdot x)=g^*\varphi_\omega(x)$, we have:

$$\sum_{ij=1}^n\frac{\partial\varphi_\omega(x)_l}{\partial x_i}\cdot A_{ij}x_j=-\sum_{i=1}^n A_{il}\varphi_\omega(x)_i\,.$$

Thus we have:

$$\sum_{ij=1}^n\frac{\partial\varphi_\omega(x)_i}{\partial x_l}\cdot A_{ij}x_j=\sum_{ij=1}^n\frac{\partial\varphi_\omega(x)_l}{\partial x_i}\cdot A_{ij}x_j$$

for all $A\in\mathfrak{g}$ and $x\in V-S$. The vector $(\sum_{j=1}^n A_{1j}x_j,\cdots,\sum_{j=1}^n A_{nj}x_j)$ may take every value in $V=\Omega^n$ when A runs through $\mathfrak{g}$. We get:

$$\sum_{ij=1}^n\frac{\partial\varphi_\omega(x)_i}{\partial x_l}=\sum_{ij=1}^n\frac{\partial\varphi_\omega(x)_l}{\partial x_i}\,,$$

for all $i,l=1,\cdots,n$. Thus $\varphi_\omega(x)dx=\sum_{i=1}^n\varphi_\omega(x)_idx_i$ is a closed differential form. Lastly, note that

$$\langle y,d\varphi_\omega(x)z\rangle=\sum_{ij=1}^n y_i\cdot\frac{\partial\varphi_\omega(x)_i}{\partial x_j}\cdot z_j\,.$$

Then we have that $d\varphi_\omega(x)$ gives a symmetric bilinear form. (q.e.d.)

The commutative connected algebraic group (G/G_1) (resp. (G/G_{1*})) is the direct sum of its torus subgroup $(G/G_1)_t$ (resp. $(G/G_{1*})_t$) and its unipotent subgroup $(G/G_1)_u$ (resp. $(G/G_{1*})_u$). Then we have the corresponding decomposition of the Lie algebra $(\mathfrak{g}/\mathfrak{g}_1)=(\mathfrak{g}/\mathfrak{g}_1)_t\oplus(\mathfrak{g}/\mathfrak{g}_1)_u$ (resp. $(\mathfrak{g}/\mathfrak{g}_1)=(\mathfrak{g}/\mathfrak{g}_1)_t\oplus(\mathfrak{g}/\mathfrak{g}_1)_u$). Then the dual vector space $\overline{X}_1=(\mathfrak{g}/\mathfrak{g}_1)^\vee$ (resp. $\overline{X}_{1*}=(\mathfrak{g}/\mathfrak{g}_{1*})^\vee$) decomposes into two parts; $\overline{X}_1=(\overline{X}_1)_t\oplus(\overline{X}_1)_u$ (resp. $\overline{X}_{1*}=(\overline{X}_{1*})_t\oplus(\overline{X}_{1*})_u$) where $(\overline{X}_1)_t:=\{\omega\in\overline{X}_1;\,\omega|_{(\mathfrak{g}/\mathfrak{g}_1)_u}\equiv 0\}$ (resp. $(\overline{X}_{1*})_t:=\{\omega\in\overline{X}_{1*};\,\omega|_{(\mathfrak{g}/\mathfrak{g}_1)_u}\equiv 0\}$) and $(\overline{X}_1)_u:=\{\omega\in\overline{X}_1;\,\omega|_{(\mathfrak{g}/\mathfrak{g}_1)_t}\equiv 0\}$ (resp. $(\overline{X}_{1*})_u:=\{\omega\in\overline{X}_{1*};\,\omega|_{(\mathfrak{g}/\mathfrak{g}_1)_t}\equiv 0\}$).

PROPOSITION 7. *Let* $\{P_1(x), \cdots, P_m(x)\}$ *(resp.* $\{Q_1(y), \cdots, Q_{m'}(y)\}$*) be the complete system of basic relative invariants and let* χ_i $(i = 1, \cdots, m)$ *(resp.* μ_j $(j = 1, \cdots, m')$*) be the corresponding character of* $P_i(x)$ *(resp.* $Q_j(y)$*).*

(1) *The vector space* $(\overline{X}_1)_t$ *(resp.* $(\overline{X}_{1*})_t$*) is the subspace of* $\overline{X}_1$ *(resp.* $\overline{X}_{1*}$*) spanned by* $\delta\chi_1, \cdots, \delta\chi_m$ *(resp.* $\delta\mu_1, \cdots, \delta\mu_{m'}$*) and hence it is an m-dimensional (resp. m'-dimensional) vector space.*

(2) *For* $\omega = \sum_{i=1}^m s_i \cdot \delta\chi_i \in (\overline{X}_1)_t$ *(resp.* $\omega = \sum_{j=1}^{m'} s_j^* \cdot \delta\mu_j \in (\overline{X}_{1*})_t$*) we have* $\varphi_\omega(x) \cdot dx = \sum_{i=1}^m (s_i/P_i(x))dP_i(x))$ *(resp.* $\psi_\omega(y) \cdot dy = -\sum_{j=1}^{m'} (s_j^*/Q_j(y))dQ_j(y))$*. In other words,* $(\varphi_\omega(x))_k = \sum_{i=1}^m (s_i/P_i(x))(\partial P_i/\partial x_k)$ *(resp.* $(\psi_\omega(y))_l = -\sum_{j=1}^{m'} (s_j^*/Q_j(y))(\partial Q_j/\partial y_l)$*).*

Proof. (1) Any rational character of (G/G_1) is trivial on $(G/G_1)_u$. Thus any element of $(\overline{X}_1)_t$ is given by a linear combination of $\delta\chi_1, \cdots, \delta\chi_m$.

(2) We identify V and V^* by their dual bases. From that $P_i(g\cdot x) = \chi_1(g)P_i(x)$ $(i = 1, \cdots, m)$, we have the following equation: $\sum_{l,j=1}^n (\partial P_i/\partial x_l) \cdot A_{lj}x_j = \delta\chi_i(A)P_i(x)$ for all $A \in \mathfrak{g}$. Thus if $x \in V - S$, then $\omega(A) = \sum_{i=1}^m s_i \cdot \delta\chi_i(A) = \sum_{j,l=1}^n \sum_{i=1}^m \{s_i(1/P_i(x))(\partial P_i/\partial x_l)\}A_{lj}x_j$ for all $A \in \mathfrak{g}$. Thus from the definition we have $(\varphi_\omega(x))_l = \sum_{i=1}^m s_i(1/P_i(x))(\partial P_i/\partial x_l)$. Therefore we have: $(\varphi_\omega(x))dx = \sum_{i=1}^m (s_i P_i(x))dP_i$.

The similar proof is possible for the dual space V^*. (q.e.d.)

Let $x \in V - S$ (resp. $y \in V^* - S^*$) and let $\omega \in \overline{X}_1$ (resp. $\omega \in \overline{X}_{1*}$). If the linear map $d\varphi_\omega(x)$ (resp. $d\psi_\omega(y)$) from V to V^* (resp. from V^* to V) is invertible, we say that $d\varphi_\omega(x)$ (resp. $d\psi_\omega(y)$) is *non-degenerate*. By Proposition 6, the set $\{x \in V - S;\ d\varphi_\omega(x)$ is non-degenerate$\}$ (resp. $\{y \in V^* - S^*;\ d\psi_\omega(y)$ is non-degenerate) is a G-invariant subset in $V - S$ (resp. $V^* - S^*$). Then it coincides with $V - S$ (resp. $V^* - S^*$) itself or the empty set $\emptyset$. When it coincides with $V - S$ (resp. $V^* - S^*$), we say that φ_ω (resp. ψ_ω) is *non-degenerate*.

The set $\{\omega \in \overline{X}_1;\ \varphi_\omega$ is non-degenerate$\}$ (resp. $\{\omega \in \overline{X}_{1*};\ \psi_\omega$ is non-degenerate$\}$) is a Zariski-open set in $\overline{X}_1$ if it is not an empty set. Indeed, let $\omega \in \overline{X}_1$ be an element such that φ_ω is non-degenerate. Let $x \in V - S$. When we write the linear map $d\varphi_\omega(x);\ V \to V^*$ by an $n \times n$ matrix with respect to suitable bases of V and V^*, each component is a polynomial of degree one with respect to ω. Then the determinant of $d\varphi_\omega(x)$ is a polynomial in ω. Thus $\{\omega \in \overline{X}_1;\ \varphi_\omega$ is non-degenerate$\} = \{\omega \in \overline{X}_1;\ \det(d\varphi_\omega(x)) \neq 0\}$, and hence it is a Zariski-open set in $\overline{X}_1$ if it is not empty. We can prove the parallel fact for the map ψ_ω in the same way. In addition, we can prove that tbe set $\{\omega \in (\overline{X}_1)_t;\ \varphi_\omega$ is non-degenerate$\}$ (resp. $\{\omega \in (\overline{X}_{1*})_t;\ \psi_\omega$ it non-degenerate$\}$) is Zariski-dense in $(\overline{X}_1)_t$ (resp. $(\overline{X}_{1*})_t$) if it is non-empty similarly.

DEFINITION 5 (quasi-regular prehomogeneous vector space). We say that the prehomogeneous vector space (G, V) (resp. (G, V^*)) is *quasi-regular* if there exists $\omega \in \overline{X}_1$ (resp. $\omega \in \overline{X}_{1*}$) such that φ_ω (resp. ψ_ω) is non-degenerate.

PROPOSITION 8. *If (G, V) (resp. (G, V^*)) is a quasi-regular prehomogeneous vector space, then (G, V^*) (resp. (G, V)) is also a quasi-regular prehomogeneous vector space. Moreover we have $\overline{X}_1 = \overline{X}_{1*}$ and $G_1 = G_{1*}$. Let $\omega \in \overline{X}_1$ be an element such that φ_ω (resp. ψ_ω) is a non-degenerate map from V (resp, V^*) to V^* (resp. V). Then φ_ω (resp. ψ_ω) gives a biholomorphic rational map from $V - S$ to $V^* - S^*$. Moreover $\psi_\omega = \varphi_\omega^{-1}$.*

Proof. Since $(V - S)$ is a G-orbit in V and since φ_ω is G-admissible, $\varphi_\omega(V - S)$ is a G-orbit in V^*. Then there exists an algebraic subset E^* and a proper algebraic subset F^* in E^* satisfying $\varphi_\omega(V - S) = E^* - F^*$. On the other hand the linear map $d\varphi_\omega(x)$ from V to V^* is invertible at any point $x \in V - S$, which means the dimension of the image of φ_ω is the same as the dimension of $V - S$. Hence we have $\dim(E^*) = \dim(V - S) = n$, which yields that $E^* = V^*$. This means that (G, V^*) is a prehomogeneous vector space with respect to the contragredient action of G and F^* coincides with the singular set S^*. Thus we have $\varphi_\omega(V - S) = V^* - S^*$. For a point $x \in V - S$, we let $y := \varphi_\omega(x) \in V^* - S^*$ and $H_{y*} := \{g \in G;\ g^* \cdot y = y\}$. Then since $H_{y*} \supset H_x$ and $\dim H_{y*} = \dim G - \dim V^* = \dim H_x$, the Lie algebra of H_x coincides with the Lie algebra of H_{y*}. Thus when we let $G_{1*} := [G, G] \cdot H_{y*}$, the Lie algebra $\mathfrak{g}_{1*}$ of G_{1*} coincides with $\mathfrak{g}_1$. Then we have $\overline{X}_1 = (\mathfrak{g}/\mathfrak{g}_1)^\vee = (\mathfrak{g}/\mathfrak{g}_{1*})^\vee = \overline{X}_{1*}$. Since $\omega \in \overline{X}_{1*}$, there exists a G-admissible map ψ_ω from V^* to V satisfying $\langle \psi_\omega(y), A^* \cdot y \rangle = -\langle \omega, A \rangle$ for all $y \in V^* - S^*$ and $A \in \mathfrak{g}$ by Proposition 5. Then the map ψ_ω is determined uniquely. On the other hand, from $\langle \varphi_\omega(x), A \cdot x \rangle = \langle \omega, A \rangle$, we have $\langle x, A^* \cdot \varphi_\omega(x) \rangle = -\langle \omega, A \rangle$ for all $x \in V - S$ nad $A \in \mathfrak{g}$ and hence $y = \varphi_\omega(x)$ implies $x = \psi_\omega(y)$ and the converse is true. That is to say, φ_ω and ψ_ω are the inverse maps of each other. Thus φ_ω gives a biholomorphic rational map from $V - S$ onto $V^* - S^*$. At the same time, we have proved that $H_{y*} = H_x$ if $y = \varphi_\omega(x)$ and hence $G_{1*} = G_1$. (q.e.d.)

COROLLARY. *Let (G, V) be a quasi-regular prehomogeneous vector space and let (G, V^*) be its dual prehomogeneous vector space. Then the number m' of irreducible components in S^* of codimension one in V^* coincides with the number m of irreducible components in S of codimension one in V.*

Proof. By Theorem 1, m (resp. m') coincides with the rank of the character group of the abelian algebraic group (G/G_1) (resp. (G/G_{1*})). We have proved that $G_{1*} = G_1$ in the proof of Proposition 7. Thus we have $m = m'$. (q.e.d.)

DEFINITION 6 (regular prehomogeneous vector spaces). Let (G, V) (resp. (G, V^*)) be a quasi-regular prehomogeneous vector space. If there exists $\omega \in (\overline{X}_1)_t$ such that φ_ω (resp. ψ_ω) is non-degenerate, we say that (G, V) (resp. (G, V^*)) is a *regular prehomogeneous vector space.*

The following proposition is easily proved.

PROPOSITION 9. (1) *In order that (G, V) is a regular prehomogeneous vector space, it is necessary and sufficient that there exists a relative invariant $P(x)$ whose Hessian $\det(\partial P/\partial x_i \partial x_j)$ does not vanish on $V - S$.*

(2) *If (G, V) is a regular prehomogeneous vector space, then (G, V^*) is a regular prehomogeneous vector space.*

§ 3. The *a*-functions $a_\chi(\omega)$ and the *b*-functions $b_\chi(\omega)$

In this section we suppose that (G, V) is a *quasi-regular prehomogeneous vector space* and we denote by (G, V^*) the dual prehomogeneous vector space. Let $\{P_1(x), \cdots, P_m(x)\}$ (resp. $\{Q_1(y), \cdots, Q_m(y)\}$) be the complete system of basic relative invariants of (G, V) (resp. (G, V^*)) and let χ_i (resp. μ_i) be the corresponding character of $P_i(x)$ (resp. $Q_i(y)$) for $i = 1, \cdots, m$.

Let $x \in X_1(G)$. Since $X_1(G) = X_{1*}(G)$, we can write $\chi = \prod_{i=1}^m \chi_i^{n_i} = \prod_{j=1}^m \mu_j^{n_j^*}$ where n_i and n_j are integers. We let $P_\chi(x) := \prod_{i=1}^m P_i(x)^{n_i}$ and $Q_\chi(y) := \prod_{j=1}^m Q_j(y)^{n_j^*}$. Then $P_\chi(x)$ (resp. $Q_\chi(y)$) is a relative invariant of (G, V) (resp. (G, V^*)) corresponding to the character χ. When $P_\chi(x)$ (resp. $Q_\chi(y)$) is a polynomial, we say that (χ) (resp. $(\chi)^*$) is *non-negative* and write polynomial, we say that (χ) (resp. $(\chi)^*$) is *non-negative* and write $(\chi) \geq 0$ (resp. $(\chi)^* \geq 0$). Namely, $(\chi) \geq 0$ (resp. $(\chi)^* \geq 0$) if and only if $n_1, \cdots, n_m \geq 0$ (resp. $n_1^*, \cdots, n_m^* \geq 0$).

PROPOSITION 10. *Let* $\chi \in X_1(G)$ *and let* $\omega \in \overline{X}_1$. *Then* $P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x))$ *(resp.* $Q_\chi(y)P_{\chi^{-1}}(\psi_\omega(y))$*) is a homogeneous rational function with respect to* ω*, which does not depend on* $x \in V$ *(resp.* $y \in V^*$*). If* $(\chi^{-1})^* \geq 0$ *(resp.* $(\chi^{-1})^* \geq 0$*), then* $P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x))$ *(resp.* $Q_\chi(y)P_{\chi^{-1}}(\psi_\omega(y))$*) is a homogeneous polynomial whose degree coincides with the degree of* $Q_{\chi^{-1}}(y)$ *(resp.* $P_{\chi^{-1}}(x)$*).*

Proof. Let $x \in V - S$ and let $g \in G$. Then we have:

$$\begin{aligned} P_\chi(g\cdot x)Q_{\chi^{-1}}(\varphi_\omega(g\cdot x)) &= \chi(g)P_\chi(x)Q_{\chi^{-1}}(g^*\cdot\varphi_\omega(x)) \\ &= \chi(g)\cdot\chi^{-1}(g)P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x)) = P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x)). \end{aligned}$$

Since $V - S$ is a G-orbit, $P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x))$ $(x \in V - S)$ coincides with a constant which depends only on χ and ω. Since $Q_{\chi^{-1}}(y)$ is a homogeneous rational function with respect to $y \in V^*$ and since each component of $\varphi_\omega(y)$ is a homogeneous function of degree one with respect to ω, $P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x))$ is a homogeneous rational function on ω. The degree coincides with the degree of $Q_{\chi^{-1}}(y)$. The similar proof is possible for $a_\chi^*(\omega)$. (q.e.d.)

DEFINITION 7 (a-function $a_\chi(\omega)$). For a rational character $\chi \in X_1(G)$, we define a homogeneous rational function in $\omega \in \overline{X}_1$: $a_\chi(\omega) := P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x))$ and $a_\chi^*(\omega) := Q_\chi(y)P_{\chi^{-1}}(\psi_\omega(y))$ and call them *a-function*. It is determined up to a constant factor, but if we fix the complete systems of basic relative invariants $\{P_1(x), \cdots, P_m(x)\}$ and $\{Q_1(y), \cdots, Q_m(y)\}$, then a-function is determined uniquely.

In particular, $a_\chi(\omega) = a_{\chi^{-1}}^*(\omega)$. Indeed, if φ_ω is non-degenerate and $x \in V - S$, then by substituting $y = \varphi_\omega(x)$ to the definition of $a_\chi(\omega)$, we get the definition of $a_{\chi^{-1}}^*(\omega)$. Thus $a_\chi(\omega) = a_{\chi^{-1}}^*(\omega)$ if φ_ω is non-degenerate. Since the set $\omega \in \overline{X}_1$ such that φ_ω is non-degenerate is Zariski-dense in $\overline{X}_1$, $a_\chi(\omega) = a_{\chi^{-1}}^*(\omega)$ for all $\omega \in \overline{X}_1$.

PROPOSITION 11. *Let* $\chi_0(g) := \deg(g)$. *Then* $\chi_0^2 \in X_1(G)$ *and there exist homogeneous polynomials* $C(\omega)$ *and* $C^*(\omega)$ *of degree n satisfying*

$$\begin{aligned} J_\omega(x) &:= \det d\varphi_\omega(x)) = C(\omega)(P_{\chi_0^2}(x))^{-1}, \\ J_\omega^*(y) &:= \det(d\psi_\omega(x)) = C^*(\omega)(Q_{\chi_0^2}(y))^{-1} \end{aligned}$$

for any $\omega \in \overline{X}_1$ *such that* φ_ω *is non-degenerate* $(\psi_\omega = (\varphi_\omega)^{-1})$. $C(\omega)$ *and* $C^*(\omega)$ *are determined uniquely up to a constant factor.*

Proof. Suppose that φ_ω ($\omega \in \overline{X}_1$) is non-degenerate. We identify V and V^* with Ω^n by their dual bases and regard $d\varphi_\omega(x)$ ($x \in V - S$) as a linear transformation from Ω^n to Ω^n. When we let $J_\omega(x) := \det\{d\varphi_\omega(x)\}$, $J_\omega(x)$ is a rational function on $x \in V$, which is regular and non-zero on $V - S$. Moreover, $J_\omega(x)$ is of homogeneous degree n with respect to ω as we have proved in § 2. From Proposition 6, (1), we have $d\varphi_\omega(g \cdot x) = {}^t(g)^{-1} \cdot d\varphi_\omega(x) \cdot (g)^{-1}$ for all $g \in G$ and $x \in V - S$, hence we have $J_\omega(g \cdot x) = \deg(g)^{-2} J_\omega(x)$. Namely when we let $\chi_0(g) := \det(g) \in X(G)$, $J_\omega(x)$ is a relatively invariant rational function corresponding to the character χ_0^{-2}. Thus we have $\chi_0^2 \in X_1(G)$. The rational functions $J_\omega(x)$ and $(P_{\chi_0}(x))^{-1}$ coincide with each other up to a constant factor. Therefore, there exists a homogeneous polynomial $C(\omega)$ of degree n such that $P_{\chi_0^2}(x) J_\omega(x) = C(\omega)$. (q.e.d.)

For $\omega \in \overline{X}_1$, we define a linear differential operator $D_\omega(x) := \operatorname{grad}_x + \varphi_\omega(x)$ (resp. $D_\omega^*(y) := \operatorname{grad}_y + \psi(y)$) from the linear space of rational functions on V (resp. V^*) to that of V^*-valued (resp. V-valued) rational functions on V (resp. V^*) in the following way. Let $(x_1, \cdots, x_n)$ be a coordinate of V and let $(y_1, \cdots, y_n)$ be its dual coordinate of V^*. We denote by $(\varphi_\omega(x))_i$ (resp. $(\psi_\omega(y))_i$) the i-th coordinate of the value $\varphi_\omega(x) \in V^*$ (resp. $\psi_\omega(y) \in V$). Let f be a rational function on V (resp. V^*). We let $(D_\omega(x))_i f(x) = (D_\omega(x) f(x))_i := (\partial/\partial x_i) f(x) + \varphi_\omega(x)_i f(x)$ (resp. $(D_\omega^*(y))_i f(y) = (D_\omega^*(y) f(y))_i := (\partial/\partial y) f(y_i) + \psi_\omega(y)_i f(y)$) for $i = 1, \cdots, n$. Then this definition does not depend on the choice of the coordinates.

LEMMA 2. *Let f be a function on V (resp. V^*).*

(1) $D_\omega(g \cdot x) f_g(x) = g^* \cdot D_\omega(x) f(x)$
(resp. $D_\omega^*(g^* \cdot y) f(y) = g \cdot D_\omega^*(y) \cdot f(y)$*), for all* $g \in G$.

(2) *Let* $x \in X_1(G)$. *Then we have*

$$D_\omega(x)(P_\chi(x) f(x)) = P_\chi(x)(D_{\omega+\delta\chi}(x) f(x))$$

$$(resp.\ D_\omega^*(y)(Q_\chi(y) f(y)) = P_\chi(y)(D_{\omega+\delta\chi}^*(y) f(y))).$$

Proof. (1) We denote by $\{f(x)\}_g$ the function $f(g \cdot x)$. Let $x' = g \cdot x$. Then

$$\begin{aligned} D_\omega(g \cdot x) f(x) &= D_\omega(x') \cdot f(g^{-1} \cdot x') \\ &= (\operatorname{grad}_{x'} + \varphi_x(x')) f(g^{-1} \cdot x') \\ &= g^* \cdot \{(\operatorname{grad}_{x'} + \varphi_\omega(x')) f(x')\}_g \\ &= g^* \cdot D_\omega(x) f(x) \end{aligned}$$

We can prove $D^*_\omega(g^*\cdot y)f(y) = g\cdot(D^*_\omega(y)f(y))$ in the same way.

(2) $$\begin{aligned} D_\omega(x)(P_\chi(x)f(x)) &= \mathrm{grad}_x(P_\chi(x))f(x) + P_\chi(x)(\mathrm{grad}_x + \varphi_\omega(x))f(x) \\ &= P_\chi(x)\cdot(\varphi_{\delta\chi}(x)f(x)) + P_\chi(x)(\mathrm{grad}_x + \varphi_\omega(x))f(x) \\ &= P_\chi(x)(\mathrm{grad}_\chi + \varphi_{\delta\chi+\omega}(x))f(x) \\ &= P_\chi(x)(D_{\omega+\delta\chi}(x)f(x)). \end{aligned}$$

We can prove $D^*_\omega(y)(Q_\chi(y)f(y)) = Q_\chi(y)(D^*_{\omega+\delta\chi}(y)f(y))$ in the same way. (q.e.d.)

Recall that the i-th component of $D_\omega(x)$ (resp. $D^*_\omega(y)$) with respect to the coordinate $(y_1, \cdots, y_n)$ (resp. $(x_1, \cdots, x_n)$) is denoted by $(D_\omega(x))_i$ (resp. $(D^*_\omega(y))_i$) $(i = 1, \cdots, n)$. Then the differential operators $(D_\omega(x))_1, \cdots, (D_\omega(x))_n$ (resp. $(D^*_\omega(y))_1, \cdots, (D^*_\omega(y))_n$) commutes with one another. Then, for a polynomial $R(y)$ (resp. $R(x)$) on V^* (resp. V), $R(D_\omega(x))$ (resp. $R(D^*_\omega(y))$) is well defined as a differential operator on V (resp. V^*).

PROPOSITION 12.

(1) *Let $x \in X_1(G)$. If $(\chi^{-1})^* \geq 0$ (resp. $(\chi^{-1}) \geq 0$), then $P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))$ (resp. $Q_\chi(y)P_{\chi^{-1}}(D^*_\omega(y))$) is a differential operator on V (resp. V^*). The application of this operator to the constant function 1: $P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1$ (resp. $Q_\chi(y)P_{\chi^{-1}}(D^*_\omega(y))\cdot 1$), is a polynomial of the same degree as $Q_{\chi^{-1}}(x)$ (resp. $P_{\chi^{-1}}(y)$). We denote it by $b_\chi(\omega)$ (resp. $b^*_\chi(\omega)$).*

(2) *The highest degree part of $b_\chi(\omega)$ (resp. $b^*_\chi(\omega)$) coincides with $a_\chi(\omega)$ (resp. $a^*_\chi(\omega)$).*

(3) *Let χ and χ' be two elements in $X_1(G)$. If $(\chi^{-1})^* \geq 0$ and $(\chi'^{-1})^* \geq 0$ (resp. $(\chi^{-1}) \geq 0$ and $(\chi'^{-1}) \geq 0$), then we have*

(♯) $$b_{\chi\cdot\chi'}(\omega) = b_\chi(\omega)b_{\chi'}(\omega - \delta\chi)$$
$$(\textit{resp. } b^*_{\chi\cdot\chi'}(\omega) = b^*_\chi(\omega)b^*_{\chi'}(\omega - \delta\chi)).$$

Here $\delta\chi$ is a corresponding infinitesimal element in $\overline{X}_1$ of χ.

Proof. First we shall prove (1) and (2). Let $F(x) := P_\chi(x)Q_{\chi^{-1}}(D(x))\cdot 1$. Then, by Lemma 2, (1), we have

$$\begin{aligned} F(g\cdot x) &= P_\chi(g\cdot x)Q_{\chi^{-1}}(D_\omega(g\cdot\chi))\cdot 1 \\ &= P_\chi(g\cdot x)Q_{\chi^{-1}}(g^*\cdot D_\omega(x))\cdot 1 \\ &= \chi(g)\chi^{-1}(g)P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1 = F(x). \end{aligned}$$

Then $F(x)$ is a G-invariant rational function on V. Hence it is a constant function which depends only on $\chi \in X_1(G)$ and $\omega \in \overline{X}_1$. We denote it by $b_\chi(\omega)$.

If $(\chi^{-1})^* \geq 0$, then $b_\chi(\omega)$ is a polynomial in ω since $Q_{\chi^{-1}}(y)$ is a polynomial and each component of $\mathrm{grad}_x + \varphi_\omega$ is a polynomial of degree one in ω. By a direct computation of $P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1$, the highest degree term of it coincides with $a_\chi(\omega) = P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x))$. Since $a_\chi(\omega)$ is the homogeneous function of the same degree as $Q_{\chi^{-1}}(y)$, $b_\chi(\omega)$'s degree is that of $Q_{\chi^{-1}}(y)$. Thus we get (1) and (2) for $b_\chi(\omega)$. We can prove those for $b_\chi^*(\omega)$ in the same way.

Next we shall prove (3). By Lemma 2, (2), we have $P_\chi(x)\cdot D_\omega \cdot P_\chi(x)^{-1} = D_{\omega-\delta\chi}$. Then, for χ, $\chi' \in X_1(G)$ satisfying $(\chi^{-1})^* \geq 0$, $(\chi'^{-1})^* \geq 0$, we have

$$\begin{aligned}
b_{\chi\chi'}(\omega) &= P_{\chi\chi'}(x)Q_{(\chi\chi')^{-1}}(D_\omega(x))\cdot 1 \\
&= P_\chi(x)P_{\chi'}(x)Q_{\chi^{-1}}(D_\omega(x))Q_{\chi'^{-1}}(D_\omega(x))\cdot 1 \\
&= P_{\chi'}(x)P_\chi(x)Q_{\chi'^{-1}}(D_\omega(x))P_\chi(x)^{-1}P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1 \\
&= P_{\chi'}(x)Q_{\chi'^{-1}}(P_\chi(x)D_\omega(x)P_\chi(x)^{-1})P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1 \\
&= P_{\chi'}(x)Q_{\chi'^{-1}}(D_{\omega-\delta\chi}(x))P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1 \\
&= P_{\chi'}(x)Q_{\chi'^{-1}}(D_{\omega-\delta\chi}(x))\cdot b_\chi(\omega) \\
&= b_{\chi'}(\omega-\delta\chi)b_\chi(\omega) = b_\chi(\omega)b_{\chi'}(\omega-\delta\chi). \qquad \text{(q.e.d.)}
\end{aligned}$$

DEFINITION 8 (b-function $b_\chi(\omega)$).

(1) For $\chi \in X_1(G)$ with $(\chi^{-1}) \geq 0$ (resp. $(\chi^{-1})^* \geq 0$), we define

$$b_\chi(\omega) := P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1$$
$$(\text{resp. } b_\chi^*(\omega) := Q_\chi(y)P_{\chi^{-1}}(D_\omega^*(y))\cdot 1)$$

(2) For $\chi \in X_1(G)$, we can write $\chi = \lambda\nu^{-1}$ by using λ, $\nu \in X_1(G)$ satisfying $(\lambda^{-1}) \geq 0$ and $(\nu^{-1}) \geq 0$ (resp. $(\lambda^{-1})^* \geq 0$ and $(\nu^{-1})^* \geq 0$). Then we define

$$b_\chi(\omega) := b_\lambda(\omega)/b_\nu(\omega - \delta\lambda + \delta\nu)$$
$$(\text{resp. } b_\chi^*(\omega) := b_\lambda^*(\omega)/b_\nu^*(\omega - \delta\lambda + \delta\nu))$$

PROPOSITION 13.

(1) *For any* χ, $\chi' \in X_1(G)$, *we have*

$$(\sharp)' \quad b_{\chi\chi'}(\omega) = b_\chi(\omega)b_{\chi'}(\omega - \delta\chi)$$
$$(\textit{resp. } b_{\chi\chi'}^*(\omega) = b_\chi^*(\omega)b_{\chi'}^*(\omega - \delta\chi))$$

(2) *Let* $\{b_\chi(\omega)\}_{\chi \in X_1(G)}$ (*resp.* $\{b_\chi^*(\omega)\}_{\chi \in X_1(G)}$) *be a family of rational functions such that* $b_\chi(\omega)$ *is the one defined by Definition* 8, (1) *if* $(\chi^{-1}) \geq 0$ (*resp.* $(\chi^{-1})^* \geq 0$). *If we suppose the relation* $(\sharp)'$, *then* $b_\chi(\omega)$ (*resp.* $b_\chi^*(\omega)$) *is determined uniquely for all* $\chi \in X_1(G)$.

Proof. We shall prove these propositions for $\{b_\chi(\omega)\}_{\chi\in X_1(G)}$. The same propositions for $\{b_\chi^*(\omega)\}_{\chi\in X_1(G)}$ can be proved in the parallel way.

Let $\chi=\lambda\nu^{-1}$ and $\chi'=\lambda'\nu'^{-1}$ where λ, ν, λ', $\nu'\in X_1(G)$ with (λ^{-1}), (ν^{-1}), (λ'^{-1}), $(\nu'^{-1})\geq 0$. Then

$$\begin{aligned}
b_{\chi\chi'}(\omega) &= b_{\lambda\lambda'}(\omega)/b_{\nu\nu'}(\omega-\delta\lambda-\delta\lambda'+\delta\nu+\delta\nu')\\
&= b_\lambda(\omega)b_{\lambda'}(\omega-\delta\lambda)/b_{\nu'}(\omega-\delta\lambda-\delta\lambda'+\delta\nu+\delta\nu')b_\nu(\omega-\delta\lambda-\delta\lambda'+\delta\nu)\\
&= \{b_\lambda(\omega)/b_\nu(\omega-\delta\lambda+\delta\nu)\}\{b_{\lambda'}(\omega-\delta\lambda+\delta\nu)/b_{\nu'}(\omega-\delta\lambda+\delta\nu-\delta\lambda'+\delta\nu')\}\\
&\quad\times\{b_\nu(\omega-\delta\lambda+\delta\nu)b_{\lambda'}(\omega-\delta\lambda)\}\{b_{\lambda'}(\omega-\delta\lambda+\delta\nu)b_\nu(\omega-\delta\lambda-\delta\lambda'+\delta\nu)\}^{-1}\\
&= b_{\lambda\nu^{-1}}(\omega)b_{\lambda'\nu'^{-1}}(\omega-\delta\chi)b_{\nu\lambda'}(\omega-\delta\lambda+\delta\nu)b_{\lambda'\nu}(\omega-\delta\lambda+\delta\nu)^{-1}\\
&= b_\chi(\omega)b_{\chi'}(\omega-\delta\chi)
\end{aligned}$$

(2) is clear since we obtain $b_\chi(\omega)=b_{\lambda\nu^{-1}}(\omega)=b_\lambda(\omega)/b_\nu(\omega-\delta\lambda+\delta\nu)$ if $\chi=\lambda\nu^{-1}$ by substituting $\chi=\lambda$ and $\chi'=\nu^{-1}$ in $(\sharp)'$. (q.e.d.)

The b-function on a quasi-regular prehomogeneous vector space given in Definition 8 is a rational function in $\omega\in\overline{X}_1$. This is a more general definition than the one we usually use, for example, in [Sa-Sh2]. When we suppose that Ω is the complex number field $\mathbb{C}$ and restrict $b_\chi(\omega)$ to $\omega\in(\overline{X}_1)_t$, we obtain the usual definition. Namely we have the following theorem.

PROPOSITION 14. *We suppose that Ω is the complex number field $\mathbb{C}$. Let ω be an element of $(\overline{X}_1)_t$. Then ω is written as $\sum_{i=1}^m s_i\delta\chi_i$ (resp. $\sum_{j=1}^m s_j^*\delta\mu_j$) where each $s_i\in\mathbb{C}$ (resp. $s_j^*\in\mathbb{C}$). We let $P_\omega(x):=\prod_{i=1}^m P_i(x)^{s_i}$ (resp. $Q_\omega(y):=\prod_{j=1}^m Q_j(\omega)^{s_j^*}$), which is well-defined as a function on the universal covering space of $V-S$ (resp. V^*-S^*). If $\chi\in X_1(G)$ such that $(\chi^{-1})\geq 0$ (resp. $(\chi^{-1})^*\geq 0$), then we have $Q_{\chi^{-1}}(\mathrm{grad}_x)P_\omega(x)=b_\chi(\omega)P_{\omega-\delta\chi}(x)$ (resp. $P_{\chi^{-1}}(\mathrm{grad}_y)Q_\omega(y)=b_\chi^*(\omega)Q_{\omega-\delta\chi}(y)$).*

Proof. Let $(x_1,\cdots,x_n)$ and $(y_1,\cdots,y_n)$ be the coordinates of V and V^*, respectively, which are supposed to be dual to each other. Then, by Proposition 8,

$$\begin{aligned}
(\partial/\partial x_j)P_\omega(x) &= \textstyle\sum_{k=1}^m s_k(\partial P_k/\partial x_j)P_k(x)^{s_k-1}\prod_{i\neq k}P_i(x)^{s_i}+P_\omega(x)(\partial/\partial x_j)\\
&= P_\omega(x)(\textstyle\sum_{k=1}^m s_k(1/P_k(x))(\partial P_k/\partial x_j)+(\partial/\partial x_j))\\
&= P_\omega(x)((\partial/\partial x_j)+(\varphi_\omega(x))_j)\\
&= P_\omega(x)(D_\omega)_j\,.
\end{aligned}$$

Suppose that $(\chi^{-1})\geq 0$. Then,

$$\begin{aligned} Q_{\chi^{-1}}(\mathrm{grad}_x)P_\omega(x) &= P_\omega(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1 \\ &= P_{\omega-\delta\chi}(x)\{P_\chi(x)Q_{\chi^{-1}}(D_\omega(x))\cdot 1\} \\ &= b_\chi(\omega)P_{\omega-\delta\chi}(x)\,. \end{aligned}$$

In the same way, we have

$$P_{\chi^{-1}}(\mathrm{grad}_y)Q_\omega(y) = b_\chi^*(\omega)Q_{\omega-\delta\chi}(y)\,. \qquad \text{(q.e.d.)}$$

Remark. The a-function $a_\chi(\omega)$ and the b-function $b_\chi(\omega)$ are well defined without the assumption "quasi-regular". Suppose that both (G, V) and (G, V^*) are prehomogeneous vector spaces. For $\chi \in X_1(G) \cap X_{1*}(G)$ and for $\omega \in \overline{X}_1$, we can define $a_\chi(\omega) := P_\chi(x)Q_{\chi^{-1}}(\varphi_\omega(x))$ and $b_\chi(\omega) := P_\chi(x)Q_{\omega^{-1}}(D_\omega 1)\cdot 1$. Then they do not depend on $x \in V$ and satisfy the following properties.

(1) If $a_\chi(\omega) \not\equiv 0$, then $a_\chi(\omega)$ is a homogeneous function whose degree coincides with that of $Q_{\chi^{-1}}(y)$. The highest degree term of $b_\chi(\omega)$ is $a_\chi(\omega)$.

(2) For $\chi, \lambda \in X_1(G) \cap X_{1*}(G)$ with $(\chi^{-1}), (\lambda^{-1}) \geq 0$, we have $b_{\chi\lambda}(\omega) = b_\chi(\omega)b_\lambda(\omega - \delta\chi)$.

§ 4. The structure theorem of a-functions and b-functions

In this section we suppose that (G, V) *is a regular prehomogeneous vector space.* In the preceding section, we have defined the a-function $a_\chi(\omega)$ and the b-function $b_\chi(\omega)$, which are rational functions on the vector space $\overline{X}_1$. The purpose of this section is to give the structure theorem of $a_\chi(\omega)$ and $b_\chi(\omega)$; we shall prove that the restrictions of $a_\chi(\omega)$ and $b_\chi(\omega)$ to $(\overline{X}_1)_t$ decomposes into a product of polynomials of degree one. We do not say nothing about $a_\chi(\omega)$ and $b_\chi(\omega)$ on the whole $\overline{X}_1$, but if $(\overline{X}_1)_t = \overline{X}_1$, then we get a complete structure theorem. In fact, a lot of important examples satisfy this condition.

Recall that $X_1(G)$ is the group consisting of rational characters of G which is null on G_1. Let $X_1^\vee(G)$ the set of homomorphisms from $X_1(G)$ to the additive group $\mathbb{Z}$. Then $X_1^\vee(G)$ is a $\mathbb{Z}$-module. Let $(\overline{X}_1)_t^\vee$ be the dual vector space of $(\overline{X}_1)_t$, which is isomorphic to Ω^m. For an element $e \in X_1^\vee(G)$ we define an element $\bar{e}$ in $(\overline{X}_1)_t^\vee$ by $\bar{e}(\sum_{i=1}^m s_i\cdot\delta\chi_i) := \sum_{i=1}^m s_i\cdot e(\chi_i)$ where $\sum_{i=1}^m s_i\cdot\delta\chi_i$ is an expression of an element of $(\overline{X}_1)_t$ (see Proposition 7 (1)). Then we may identify $X_1^\vee(G)$ with a lattice in $(\overline{X}_1)_t^\vee$ by the correspondence $e \mapsto \bar{e}$. Hereafter we regard $X_1^\vee(G)$ as a discrete subset in $(\overline{X}_1)_t^\vee$.

THEOREM 1. *Let (G, V) be a regular prehomogeneous vector space and let $\chi \in X_1(G)$.*

(1) *The restriction of the a-function to $(\overline{X}_1)_t$, $a_\chi(\omega)|_{\omega\in(\overline{X}_1)_t}$, is written by:*

$$a_\chi(\omega)|_{\omega\in(\overline{X}_1)_t} := C(\chi) \prod_{i=1}^{p} (\bar{e}_i(\omega))^{m_i e_i(\chi)},$$

Here $e_1, \cdots, e_p \in X_1^\vee(G)$ such that $\bar{e}_1, \cdots, \bar{e}_p$ are different linear forms; $C(\chi)$ is a homomorphism from $X_1(G)$ to $\Omega^\times$; p and m_i $(i = 1, \cdots, p)$ are natural numbers.

(2) *Each $e_i \in X_1^\vee(G)$ $(i = 1, \cdots, p)$ satisfies the following two conditions: if $(\chi) \geq 0$ or $(\chi^{-1})^* \geq 0$, then we have $e_i(\chi) \geq 0$ $(i = 1, \cdots, p)$ and $\sum_{i=1}^{p} m_i \cdot e_i(\chi) = \sum_{i=1}^{m} n_i \cdot \deg(P_i) = \sum_{i=1}^{m} n_i^* \cdot \deg(Q_i)$ for $\chi = \prod_{i=1}^{m} \chi_i^{n_i} = \prod_{i=1}^{m} \mu_i^{n_i^*}$.*

(3) *The restrictions of $a_\chi^*(\omega)|_{\omega\in(\overline{X}_1)_t}$ is given by*

$$a_\chi^*(\omega)|_{\omega\in(\overline{X}_1)_t} = C^*(\chi) \prod_{i=1}^{p} (\bar{f}_i(\omega))^{m_i f_i(\chi)}$$

where $f_i(\chi) = -e_i(\chi)$ and $C^*(\chi) = C(\chi^{-1})(-1)^{-\sum_{i=1}^{p} m_i e_i(\chi)}$.

Proof.

(1) Let $\chi \in X_1(G)$. From the definition of a-function,

$$P_\chi(x) Q_{\chi^{-1}}(\varphi_\omega(x)) = Q_{\chi^{-1}}(y) P_\chi(\psi_\omega(y)) = a_\chi(\omega), \tag{4.1}$$

for all $\omega \in (\overline{X}_1)_t$, since the set $\{\omega \in (\overline{X}_1)_t;\ \varphi_\omega$ is non-degenerate$\}$ is a Zariski-open subset of $(\overline{X}_1)_t$ (see § 2). Since $a_{\chi\cdot\chi'}(\omega) = a_\chi(\omega)\cdot a_{\chi'}(\omega)$ for every $\chi, \chi' \in X_1(G)$, $\chi = \prod_{i=1}^{m} \chi_i^{n_i}$ $(n_i \in \mathbb{Z})$ implies that $a_\chi(\omega) = \prod_{i=1}^{m} a_{\chi_i}(\omega)^{n_i}$. Let $f_1(\omega), \cdots, f_p(\omega)$ be mutually different prime divisors appearing in one of the rational functions $a_{\chi_1}(\omega), \cdots, a_{\chi_m}(\omega)$. Then there exists a suitable element $e_i' \in X_1^\vee(G)$ satisfying $a_\chi(\omega) = \prod_{i=1}^{p} f_i(\omega)^{e_i'(\chi)}$. Thus we have

$$P_\chi(x) Q_{\chi^{-1}}(\varphi_\omega(x)) = \prod_{i=1}^{p} f_i(\omega)^{e_i'(\chi)}$$

if (x, ω) belongs to the Zariski-open set of $U := (V - S) \times \{\omega \in (\overline{X}_1)_t;\ \varphi_\omega$ is non-degenerate$\}$ in $V \times (\overline{X}_1)_t$.

By taking the "logarithmic differential" of the above equation, the left hand side is

$$\begin{aligned} d\log(P_\chi(x) Q_{\chi^{-1}}(y)) &= d\log(P_\chi(x)) + d\log(Q_{\chi^{-1}}(y)) \\ &= (1/P_\chi(x)) \textstyle\sum_{j=1}^{n} (\partial P_\chi(x)/\partial x_j) dx_j \\ &\quad + (1/Q_{\chi^{-1}}(y)) \textstyle\sum_{j=1}^{n} (\partial Q_{\chi^{-1}}(y)/\partial y_j) dy_j \\ &= \varphi_{\delta\chi}(x)\cdot dx - \psi_{-\delta\chi}(y)\cdot dy = \varphi_{\delta\chi}(x)\cdot dx + \psi_{\delta\chi}(y)\cdot dy \end{aligned}$$

by Proposition 7, (2), and the right hand side is

$$\begin{aligned} d\log\left(\prod_{i=1}^{k} f_i(\omega)e^{i'(\chi)}\right) &\doteq \sum_{i=1}^{p} e_i'(\chi)\cdot(df_i(\omega)/f_i(\omega)) \\ &= \sum_{i=1}^{p} \bar{e}_i'(\delta\chi)\cdot(df_i(\omega)/f_i(\omega)). \end{aligned}$$

Let $\chi = \prod_{j=1}^{m} \chi_j^{n_j}$ where each $n_j \in \mathbb{Z}$. Then $\delta\chi = \sum_{j=1}^{m} n_j\cdot\delta\chi_j$ and

$$\sum_{j=1}^{m} n_j\left(\sum_{i=1}^{p} \bar{e}_i'(\delta\chi_j)\cdot df_i(\omega)/f_i(\omega)\right) = \sum_{j=1}^{m} n_j\cdot(\varphi_{\delta\chi_j}(x)dx + \psi_{\delta\chi_j}(y)dy).$$

Thus

$$\sum_{i=1}^{p} \bar{e}_i'(\delta\chi_j)\cdot df_i(\omega)/f_i(\omega) = \varphi_{\delta\chi_j}(x)\cdot dx + \psi_{\delta\chi_j}(y)\cdot dy,$$

for each $j = 1, \cdots, m$. Since $\bar{e}'(\cdot)$, $\varphi_{(\cdot)}(x)$ and $\psi_{(\cdot)}(y)$ are all linear forms, we have, for all $\omega' = \sum_{j=1}^{m} s_j\cdot\delta\chi_j \in (\overline{X}_1)_t$ with $(s_1, \cdots, s_m) \in \Omega^m$,

$$\sum_{i=1}^{p} \bar{e}_i'(\omega')(df_i(\omega)/f_i(\omega)) = \varphi_{\omega'}(x)\cdot dx + \psi_{\omega'}(y)\cdot dy.$$

In particular, we may take $\omega' = \omega$. Then we have

$$\begin{aligned} d\langle x, y\rangle &= \langle x, dy\rangle + \langle y, dx\rangle \\ &= \varphi_\omega(x)\cdot dx + \psi_\omega(y)\cdot dy \\ &= \sum_{i=1}^{p} \bar{e}_i'(\omega)\cdot(df_i(\omega)/f_i(\omega)). \end{aligned}$$

On the other hand, substituting $\omega = \sum_{j=1}^{m} s_j\cdot\delta\chi_j$ to $\langle x, \varphi_\omega(x)\rangle$, we have

$$\begin{aligned} \langle x, y\rangle &= \langle x, \varphi_\omega(x)\rangle \\ &= \sum_{i=1}^{n}\sum_{j=1}^{m} s_j\cdot(1/P_j(x))(\partial P_j(x)/\partial x_j)\cdot x_j \\ &= \sum_{j=1}^{m} s_j\cdot\deg(P_j). \end{aligned}$$

Then

$$\begin{aligned} d\langle x, y\rangle &= \sum_{j=1}^{m} \deg(P_j)\cdot ds_j \\ &= \sum_{i=1}^{p} (\bar{e}_i'(\omega)/f_i(\omega))\cdot df_i(\omega) \\ &= \sum_{j=1}^{m}\left(\sum_{i=1}^{p} (\bar{e}_i'(\omega)/f_i(\omega))\cdot \partial f_i(\omega)/\partial s_i)ds_j\right). \end{aligned}$$

This means that $f_i(\omega)$ is a divisor of $\bar{e}_i'(\omega)\cdot(\partial f_i(\omega)/\partial s_j)$ since $f_1(\omega), \cdots, f_p(\omega)$ are prime divisors which have no common divisors. Hence $f_i(\omega)$ is a divisor of $\bar{e}_i'(\omega)$. Then $f_i(\omega)$ coincides with $\bar{e}_i'(\omega)$ up to a constant factor. Therefore $\bar{e}_i'(\omega)$ $(i = 1, \cdots, k)$ are mutually different linear forms on $(\overline{X}_1)_t$ and we have

$$a_\chi(\omega) = C(\chi) \times \prod_{i=1}^{p} (\bar{e}_i'(\omega))^{e_i'(\chi)}.$$

Here, $C(\chi)$ is a constant which depends only on χ. We can take a natural number m_i such that $e_i'(X_1) = m_i\mathbb{Z}$. Let $e_i := 1/m_i \cdot e_i'$ for each $i = 1, \cdots, p$. Then e_i gives a homomorphism from $X_1(G)$ to $\mathbb{Z}$ and we have the equation

$$a_\chi(\omega) = C(\chi) \cdot \prod_{i=1}^{p} \bar{e}_i(\omega)^{m_i e_i(\chi)} .$$

Here the map $\chi \mapsto C(\chi)$ gives a homomorphism from $X_1(G)$ to $\Omega^\times$.

(2) By (4.1), if $P_\chi(x)$ or $Q_{\chi^{-1}}(y)$ is a polynomial, then $a_\chi(\omega)$ is a polynomial in ω. That is to say, if $(\chi) \geq 0$ or $(\chi^{-1})^* \geq 0$, then $e_i(\chi) \geq 0$ $(i = 1, \cdots, m)$. By comparing the degrees with respect to ω of the both sides of (4.1), when $\chi = \prod_{i=1}^{m} \chi_i^{n_i} = \prod_{i=1}^{m} (\mu_i)^{n_i^*}$, we have $\sum_{i=1}^{m} n \cdot \deg(P_i) = \sum_{i=1}^{m} n_i^* \cdot \deg(Q_i)$. It is the degree of the polynomial $P_\chi(x)$ (resp. $Q_{\chi^{-1}}(y)$) if $(\chi) \geq 0$ (resp. $(\chi^{-1})^* \geq 0$) and coincides with the degree of $a_\chi(\omega)$. Thus we have

$$\sum_{i=1}^{m} n_i \cdot \deg(P_i) = \sum_{i=1}^{m} - n_i \cdot \deg(Q_i) = \sum_{i=1}^{p} m_i \cdot e_i(\chi) .$$

(3) Since $a_\chi^*(\omega) = a_{\chi^{-1}}(\omega)$, we have

$$\begin{aligned} a_\chi^*(\omega)|_{\omega \in (\bar{X}_1)_t} &= C(\chi^{-1}) \prod_{i=1}^{p} (\bar{e}_i(\omega))^{m_i e_i(\chi^{-1})} \\ &= C(\chi^{-1}) \prod_{i=1}^{p} (-\bar{f}_i(\omega))^{-m_i e_i(\chi)} \\ &= C(\chi^{-1})(-1)^{-\Sigma_{i=1}^{p} m_i e_i(\chi)} \prod_{i=1}^{p} (\bar{f}_i(\omega))^{m_i f_i(\chi)} \\ &= C^*(\chi) \prod_{i=1}^{p} (\bar{f}_i(\omega))^{m_i f_i(\chi)} . \end{aligned} \qquad \text{(q.e.d.)}$$

COROLLARY. *Let $C(\omega)$ and $C^*(\omega)$ be polynomials in $\omega \in \bar{X}_1$ introduced in Proposition* 11. *If (G, V) is a regular prehomogeneous vector space, then for $\omega \in (\bar{X}_1)_t$ we have:*

$$C(\omega)|_{\omega \in (\bar{X}_1)_t} = C \cdot \prod_{i=1}^{k} e_i(\omega)^{\varepsilon_i} \quad \textit{and} \quad C^*(\omega)|_{\omega \in (\bar{X}_1)_t} = C^* \cdot \prod_{i=1}^{p} e_i(\omega)^{\varepsilon_i^*}$$

where $C, C^* \in \Omega^\times$ and ε_i, ε_i^* $(i = 1, \cdots, k)$ are positive integers satisfying $\varepsilon_i + \varepsilon_i^* = m_i e_i(\chi_0^2)$ with $\chi_0(g) := \det(g)$.

Proof. From the definition, if $\omega \in (\bar{X}_1)_t$, then we have $C(\omega)C^*(\omega) = a_{\chi_0^2}(\omega) = C(\chi_0^2) \prod_{i=1}^{k} \bar{e}_i(\omega)^{m_i e_i(\chi_0^2)}$. Note that $C(\omega)$ and $C^*(\omega)$ are polynomials. Then, we have:

$$C(\omega) = C \cdot \prod_{i=1}^{k} e_i(\omega)^{\varepsilon_i} \quad \text{and} \quad C^*(\omega) = C^* \cdot \prod_{i=1}^{k} e_i(\omega)^{\varepsilon_i^*},$$

where C, C^* are constants in $\Omega^\times$ and ε_i, ε_i^* are non-negative integers satisfying $\varepsilon_i + \varepsilon_i^* = m_i e_i(\chi_0^2)$. We suppose that $\varepsilon_i = 0$ for some i. Note that $C(\omega) \neq 0$ is a necessary and sufficient condition in order that φ_ω is non-degenerate. Take an element $\omega \in (\bar{X}_1)_t$ such that $\bar{e}_i(\omega) = 0$ and $\bar{e}_j(\omega) \neq 0$ for all j which are different from i. Such element ω exists because

$\bar{e}_1(\omega), \cdots, \bar{e}_k(\omega)$ are linear forms which have no common divisors. Since $\varepsilon_i = 0$, we have φ_ω is non-degenerate. On the other hand, taking an element $\chi \in X_1$ satisfying $e_i(\chi) > 0$, (such character χ always exists.) we have $a_\chi(\omega) = C(\chi) \prod_{i=1}^k \bar{e}_i(\omega)^{m_i e_i(\chi)} = 0$. Thus we have $P_\chi(x) Q_{\chi^{-1}}(\varphi_\omega(x)) = a_\chi(\omega) = 0$ for $x \in V - S$. When φ_ω is non-degenerate, $\varphi_\omega(x) \in V^* - S^*$ for $x \in V - S$. Thus $P(x) \neq 0$ and $Q_{\chi^{-1}}(\varphi_\omega(x)) \neq 0$, which yields a contradiction. Then each ε_i is a positive integer. We can prove that ε_i^* is positive. (q.e.d.)

THEOREM 2. *Let (G, V) be a regular prehomogeneous vector space and let $\chi \in X_1(G)$. The restriction of $b_\chi(\omega)$ (resp. $b_\chi^*(\omega)$) to $(\bar{X}_1)_t$ is written in the following form:*

$$b_\chi(\omega)|_{\omega \in (\bar{X}_1)_t} = C(\chi) \cdot \prod_{i=1}^p \left(\prod_{\nu=1}^{\bar{e}_i(\delta\chi)-1} \varphi_i(\bar{e}_i(\omega) - \nu) \right). \tag{4.2}$$

$$(\textit{resp. } b_\chi^*(\omega)|_{\omega \in (\bar{X}_1)_t} = C^*(\chi) \cdot \prod_{i=1}^p \left(\prod_{\nu=0}^{\bar{f}_i(\delta\chi)-1} \varphi_i^*(\bar{f}_i(\omega) - \nu) \right).)$$

Here, $C(\chi)$ (resp. $C^*(\chi)$) and $e_i(\chi)$ (resp. $f_i(\chi)$) is a map from $X_1(G)$ to $\Omega^\times$ given in Theorem 1; each φ_i (resp. φ_i^*) is a rational function of one variable on Ω of degree m_i, where m_i is the natural number defined in Theorem 1; for non-positive $l-1$, the product $\prod_{\nu=0}^{l-1} f(x-\nu)$ means 1 if $l = 1$ and it stands for $\prod_{\nu=l}^{-1} f(x+\nu)$ if $l < 1$.

Proof. In the Corollary to Theorem in Appendix, we can take $X_1(G) = \Xi$, $(\bar{X}_1)_t = \Omega^m$ and define the map δ: $X_1(G) \to (\bar{X}_1)_t$ to be the map given in § 2. Proposition 7, (1) says that the map δ satisfies the condition required by the Theorem in Appendix. Since $b_\chi(\omega)|_{\omega \in (\bar{X}_1)_t}$ (resp. $b_\chi^*(\omega)|_{\omega \in (\bar{X}_1)_t}$) satisfies the relation $b_{\chi \cdot \chi'}(\omega) = b_\chi(\omega) b_{\chi'}(\omega - \delta\chi)$ (resp. $b_{\chi \cdot \chi'}^*(\omega) = b_\chi^*(\omega) b_{\chi'}(\omega - \delta\chi)$), we can apply the Theorem in Appendix to our case. In addition, $b_\chi(\omega)$ (resp. $b_\chi^*(\omega)$) is a polynomial if $(\chi^{-1})^* \geq 0$ (resp. $(\chi^{-1}) \geq 0$), which means that $b_{\mu_i^{-1}}(\omega)$ (resp. $b_{\chi_i^{-1}}^*(\omega)$) is a polynomial for each $i = 1, \cdots, m$ where $\{\mu_1, \cdots, \mu_m\}$ (resp. $\{\chi_1, \cdots, \chi_m\}$) are the set of generators of $X_1(G)$ defined in § 1. Then by the Corollary to the Theorem in Appendix, we have the expression like (A.15). Namely we have,

$$b_\chi(\omega) = C(\chi) \cdot \prod_{i=1}^p \prod_{\nu=0}^{\bar{e}_i(\delta\chi)-1} \varphi_i(\bar{e}_i(\omega) - \nu)$$

$$(\text{resp. } b_\chi^*(\omega) = C^*(\chi) \cdot \prod_{i=1}^p \prod_{\nu=0}^{\bar{f}_i(\delta\chi)-1} \varphi_i^*(\bar{f}_i(\omega) - \nu))$$

by replacing $C(\xi)$ with $C(\chi)$ (resp. $C^*(\chi)$), ψ_i with φ_i (resp. φ_i^*) and $\bar{e}_i^\vee$ with $\bar{e}_i$ (resp. $\bar{f}_i$) in the formula (A.15). By comparing the leading term of $b_\chi(\omega)$ with $a_\chi(\omega)$ (resp. $b_\chi^*(\omega)$ with $a_\chi^*(\omega)$), we see that $\varphi_i(z)$ (resp. $\varphi_i^*(z)$) is a rational functional function of degree m_i. (q.e.d.)

COROLLARY. (1) *Let* $\{f_{ij}\}_{j=1,\cdots,r(i)}$ *be the set of all the locations of poles and zeros of* φ_i *which are not congruent to one another modulo* 1. *Then we can write* $\varphi_i(x) = \prod_{j=1}^{r(i)} \prod_{\mu\in\mathbb{Z}}(x - f_{ij} + \mu)^{n_{ij}(\mu)}$ *where* $\mu \to n_{ij}(\mu)$ *is a map from* $\mathbb{Z}$ *to* $\mathbb{Z}$ *such that each* $n_{ij}(\mu)$ *is zero except for a finite number of* μ; $r(i)$ *is a positive integer. Let* $\chi \in X_1(G)$. *If* $(\chi^{-1})^* \geq 0$ *and if* $e_i(\chi) = l > 0$, *then* $\sum_{\nu=0}^{l-1} n_{ij}(\mu + \nu) \geq 0$ *for all* $\mu \in \mathbb{Z}$. *In particular, if there exists* $\chi \in X_1(G)$ *such that* $(\chi^{-1})^* \geq 0$ *and* $e_i(\chi) = 1$, *then* $\varphi_i(z)$ *is a polynomial. Moreover* $\sum_{j=1}^{r(i)} \sum_{\mu\in\mathbb{Z}} n_{ij}(\mu) = m_i$ *for* $i = 1, \cdots, k$.

(2) *In particular, we assume that* $\Omega = \mathbb{C}$. *For the rational function* $\varphi_i(z) = (\prod_{j=1}^{\alpha_i}(z - c_{ij})/\prod_{j=1}^{\beta_i}(z - d_{ij}))$ *with* $\alpha_i - \beta_i = m_i$, *we define the corresponding gamma-factor*

$$\gamma(\omega) = \prod_{i=1}^{p}(\prod_{j=1}^{\alpha_i}\Gamma(\bar{e}_i(\omega) - c_{ij} + 1)/\prod_{i=1}^{\beta_i}\Gamma(\bar{e}_i(\omega) - d_{ij} + 1)\,,$$

where $\Gamma(z)$ *is the gamma function. Then* $b_\chi(\omega)|_{\omega\in(X_1)_t} = C(\chi)\cdot(\gamma(\omega)/\gamma(\omega - \delta\chi))$.

Proof. (1) We suppose that $(\chi^{-1})^* \geq 0$ and $e_i(\chi) = l > 0$. Then $b_\chi(\omega)$ is a polynomial in ω and it is written as $b_\chi(\omega) = C(\chi)\cdot\prod_{i=1}^{p}\prod_{\nu=0}^{e_i(\chi)-1}\varphi_i(\bar{e}_i(\omega) - \nu)$. If $i \neq j$, then $\prod_{\nu=0}^{e_i(\chi)-1}\varphi_i(\bar{e}_i(\omega) - \nu)$ and $\prod_{\nu=0}^{e_i(\chi)-1}\varphi_j(\bar{e}_j(\omega) - \nu)$ have no common divisors. Then each $\prod_{\nu=0}^{e_i(\chi)-1}\varphi_i(\bar{e}_i(\omega) - \nu)$ for $i = 1, \cdots, m$ is a polynomial in ω. That is to say,

$$\begin{aligned}\prod_{\nu=0}^{e_i(\chi)-1}\varphi_i(\bar{e}_i(\omega) - \nu) &= \prod_{\nu=0}^{l-1}\prod_{j=1}^{r(i)}\prod_{\mu\in\mathbb{Z}}(\bar{e}_i(\omega) - f_{ij} - \nu + \mu)^{n_{ij}(\mu)} \\ &= \prod_{j=1}^{r(i)}\prod_{\lambda\in\mathbb{Z}}(\bar{e}_i(\omega) - f_{ij} + \lambda)^{\sum_{\nu=0}^{l-1} n_{ij}(\lambda+\nu)}\end{aligned}$$

is a polynomial in ω. Since f_{ij} $(j = 1, \cdots, r(i))$ are not congruent to one another modulo 1, we have $\sum_{\nu=0}^{l-1} n_{ij}(\mu + \nu) \geq 0$ for all $\mu \in \mathbb{Z}$. In particular, if there exists $\chi \in X_1(G)$ such that $(\chi^{-1})^* > 0$ such that $e_i(\chi) = 1$, then $\varphi_i(z) := \prod_{j=1}^{r(i)}\prod_{\mu\in\mathbb{Z}}(z - f_{ij} + \mu)^{n_{ij}(\mu)}$ is a polynomial since $n_{ij}(\mu) \geq 0$ for all $\mu \in \mathbb{Z}$. Lastly, since the highest degree part of $b_\chi(\omega)$ coincides with $a_\chi(\omega) = C(\chi)\cdot\prod_{i=1}^{k}\bar{e}_i(\omega)^{m_i e_i(\chi)}$, we have $\sum_{j=1}^{r(i)}\sum_{\mu\in\mathbb{Z}} n_{ij}(\mu) = m_i$.

(2) It is clear. (q.e.d.)

Remark. For the function $b_\chi^*(\omega)$, we have the same result as the corollary to Theorem 2.

Appendix. Sato's theorem on a family of rational functions satisfying the cocycle conditions

In this appendix, we shall prove a theorem concerning a family of rational functions satisfying some condition. This theorem can be understood without any knowledge on prehomogeneous vector spaces.

Let Ξ be an abelian group generated by $\xi_1, \cdots, \xi_m$; an element $\xi \in \Xi$ is written by $\xi_1^{n_1} \cdot \xi_2^{n_2} \cdots \xi_m^{n_m}$ with $(n_1, n_2, \cdots, n_m) \in \mathbb{Z}^m$. Let $\delta\xi_1, \delta\xi_2, \cdots, \delta\xi_m$ be linearly independent elements in an m-dimensional vector space Ω^m; the map

$$\text{(A.1)} \qquad \delta\colon\ \theta := \xi_1^{n_1} \cdot \xi_2^{n_2} \cdots \xi_m^{n_m} \longmapsto \delta\theta := \textstyle\sum_{i=1}^m n_i \cdot \delta\xi_i\,,$$

gives an injective homomorphism from Ξ to Ω^m. The elements $\delta\xi_1, \cdots, \delta\xi_m$ form a basis of the vector space Ω^m. We denote by $R^\times(\Omega^m)$ the abelian group consisting of rational functions on Ω^m under the multiplication law. For $f(\omega) \in R^\times(\Omega^m)$, we define the action of $\xi \in \Xi$ by $f^\xi(\omega) := f(\omega - \delta\xi)$ for $f(\omega) \in R^\times(\Omega^m)$. Let $\{f_\xi(\omega)\}_{\xi \in \Xi}$ be a family of elements in $R^\times(\Omega^m)$ with a parameter $\xi \in \Xi$. If $\{f_\xi(\omega)\}_{\xi \in \Xi}$ satisfies the condition

$$\text{(A.2)} \qquad f_{\xi \cdot \xi'}(\omega) = f_\xi(\omega) \cdot f_{\xi'}^\xi(\omega)$$

for all $\xi, \xi' \in \Xi$, we say that it satisfies the *cocycle condition.* The purpose of this section is to prove the following theorem; the proof given here is essentially due to M. Sato.

THEOREM (M. Sato). *Let $\{f_\xi(\omega)\}_{\xi \in \Xi}$ be a family of non-trivial rational functions on Ω^m with the parameter ξ. We assume that $\{f_\xi(\omega)\}_{\xi \in \Xi}$ satisfies the cocycle condition* (A.2). *Then $f_\xi(\omega)$ is written in the following form;*

$$\text{(A.3)} \qquad \begin{aligned} f_\xi(\omega) = C(\xi) \cdot \textstyle\prod_{i=1}^p \prod_{k=0}^{\bar{e}_i^\vee(\delta\xi)-1} \psi_i(\bar{e}_i^\vee(\omega) - k) \\ \times \textstyle\prod_{i=1}^q \prod_{j=1}^{t_i} \{h_i(\omega - \delta\lambda_i(j))/h_i(\omega - \delta\lambda_i(j) - \delta\xi)\}^{n_i(j)}. \end{aligned}$$

Here, $C(\xi)$ is an Ω-valued function depending only on $\xi \in \Xi$ which satisfies $C(\xi \cdot \xi') = C(\xi) \cdot C(\xi')$; $\bar{e}_1^\vee(\omega), \cdots, \bar{e}_p^\vee(\omega)$ are linear forms on $\omega \in \Omega^m$ which are integer-valued on the $\mathbb{Z}$-lattice $\delta(\Xi)$ in Ω^m; $\psi_1(x), \cdots, \psi_p(x)$ are rational functions of one variable $x \in \Omega$; $h_1(\omega), \cdots, h_q(\omega)$ are irreducible polynomials in $\omega \in \Omega^m$ which are not converted to one another by the action of Ξ; p, q and $t_1, \cdots, t_q$ are positive integers; $\lambda_i(j)$ $(1 \le i \le q,\ 1 \le j \le t_i)$ are elements in Ξ; $n_i(j)$ $(1 \le i \le q,\ 1 \le j \le t_i)$ are integers. The product $\prod_{k=0}^{l-1} \psi_i(x - k)$ means 1 if $l = 0$ and it means $\prod_{l \le k \le -1} \psi_i(x + k)^{-1}$ if l is a negative integer.

Proof. First, we prepare some notations before starting the proof. Let Θ be a group and let A be an abelian group. We assume that Θ operates on A as an automorphism group of A, that is to say, for an element $\theta \in \Theta$, a map $a \mapsto a^\theta$ from A to A is defined and satisfies $(a \cdot b)^\theta = a^\theta \cdot b^\theta$ and $a^{(\theta \cdot \tau)} = (a^\theta)^\tau$ with $a, b \in A$ and $\theta, \tau \in \Theta$. Let $a_{(\cdot)}\colon \theta \mapsto a_{(\theta)}$ be a map from Θ to A. When the map $a_{(\cdot)}$ satisfies $a_{(\theta\cdot\tau)} = a_{(\theta)}\cdot a_{(\tau)}^{\theta}$ for any θ, $\tau \in \Theta$, we say that $a_{(\cdot)}$ is a *cocycle*. Cocycles form a module under the multiplication law in A. We denote by $Z(\Theta, A)$ the module of cocycles. In particular, a map $a_{(\cdot)}$ is a cocycle if it is written as $a_{(\theta)} = b^{\theta}\cdot b^{-1}$ by using an element $b \in A$. We call such an $a_{(\cdot)}$ a *coboundary*. Coboundaries form a subgroup of $Z(\Theta, A)$. We denote it by $B(\Theta, A)$. The first cohomology group $H^1(\Theta, A)$ is defined by $Z(\Theta, A)/B(\Theta, A)$. The action of Θ on A is extended to the action of the group algebra $\mathbb{Z}[\Theta]$, in the following way: for $\sigma := \sum_{i=1}^{l} m_i\cdot\theta_i \in \mathbb{Z}[\theta]$, we let $a^{\sigma} = \prod_{i=1}^{l}(a^{\theta_i})^{m_i}$; here $m_i \in \mathbb{Z}$ and $\theta_i \in \Theta$ for each i in $1 \leq i \leq m$. That is all what we have to prepare before the proof.

In the above situation, we take Ξ as the group Θ and $R^{\times}(\Omega^m)$ as the abelian group A. Then, for an element $\xi \in \Xi$, we associate an automorphism of $R^{\times}(\Omega^m)$: $f(\omega) \mapsto f^{\xi}(\omega) := f(\omega - \delta\xi)$. The group Ξ acts on $R^{\times}(\Omega^m)$ as an automorphism group of $R^{\times}(\Omega^m)$ and it is extended to the action of $\mathbb{Z}[\Xi]$ on $R^{\times}(\Omega^m)$ in the above way. The assumption (A.2) indicates that this action satisfies the cocycle condition: $f_{\xi\cdot\xi'}(\omega) = f_{\xi'}(\omega)\cdot f_{\xi'}(\omega - \delta\xi)$ for each $\xi, \xi' \in \Xi$. This means f_{ξ} belongs to the module of cocycles $Z(\Xi, R^{\times}(\Omega^m))$.

Any element ξ in Ξ is written as an integer power of $\xi_1, \cdots, \xi_m$. Let $h_1(\omega)$ and $h_2(\omega)$ be two polynomials. We say that $h_1(\omega)$ is converted to $h_2(\omega)$ by Ξ if there exists $\xi \in \Xi$ such that $h_1^{\xi} = h_2$. Together with the cocycle condition (A.2), f_{ξ} is written by an integer power of $f_{\xi_1}, \cdots, f_{\xi_m}$ and their conversions by the actions of Ξ. However there may be duplication among the divisors appearing in $f_{\xi_1}, \cdots, f_{\xi_m}$. In order to reduce the duplication, we choose non-constant irreducible polynomials $h_1(\omega), \cdots, h_l(\omega)$ on Ω^m such that

1) each h_j $(j = 1, \cdots, l)$ is a prime divisor of one of the rational functions $f_{\xi_1}, \cdots, f_{\xi_m}$ and they are not converted into one another by the action of Ξ up to a constant factor.

2) any prime divisor appearing in $f_{\xi_1}, \cdots, f_{\xi_m}$ is obatined by converting one of the h_j's by the action of an element of Ξ up to a constant factor.

Then we have,

$$f_{\xi}(\omega) = (\text{const.}) \times \prod_{i=1}^{l} h_i^{\alpha_i(\xi)}(\omega)\,, \tag{A.4}$$

by taking suitable maps $\alpha_1(\cdot), \cdots, \alpha_l(\cdot)$ from Ξ to $\mathbb{Z}[\Xi]$.

We shall calculate each terms $h_i^{\alpha_i(\xi)}(\omega)$ in (A.4). Before the calculation, we need to show some properties of $\alpha_i(\xi)$ and related lemmas. We devote the following until Lemma A-4 to them. First, we shall show that each $\alpha_i(\cdot)$ satisfies a cocycle condition in a different form. We let $\mathcal{T}_i := \{\sigma \in \mathbb{Z}[\Xi];\ h_i^\sigma$ is a constant function$\}$. Then $\mathcal{T}_i$ is an ideal in $\mathbb{Z}[\Xi]$. Indeed, once h_i^σ becomes a constant function by an element $\sigma \in \mathbb{Z}[\Xi]$, it remains to be a constant function after the action of any element $\sigma \in \mathbb{Z}[\Xi]$. We denote by $(\bar{\sigma})_i$ the image of $\sigma \in \mathbb{Z}[\Xi]$ under the natural projection map $\mathbb{Z}[\Xi] \to \mathbb{Z}[\Xi]/\mathcal{T}_i$. Then we may write

$$f_\xi(\omega) = (\text{const.}) \times \prod_{i=1}^{l} h_i^{(\overline{\alpha_i(\xi)})_i}(\omega)\,, \tag{A.5}$$

and, conversely, we can determine the map $(\overline{\alpha_i(\cdot)})_i$ from Ξ to $\mathbb{Z}[\Xi]/\mathcal{T}_i$ uniquely so that the equation (A.5) is established.

We say that a map $\alpha(\cdot)$ from Ξ to $\mathbb{Z}[\Xi]/\mathcal{T}_i$ is a *cocycle* if $\alpha(\xi\cdot\xi') = \alpha(\xi) + (\bar{\xi})_i\cdot\alpha(\xi')$ is satisfied. We denote by $Z(\Xi, \mathbb{Z}[\Xi]/\mathcal{T}_i)$ the subalgebra of cocycles in the algebra of the maps from Ξ to $\mathbb{Z}[\Xi]/\mathcal{T}_i$. In this case, a *coboundary* in $Z(\Xi, \mathbb{Z}[\Xi]/\mathcal{T}_i)$ means a map $\alpha(\cdot)$ given by $\alpha(\xi) := (\bar{\xi})_i\cdot(-\beta) - (-\beta) = (1 - (\bar{\xi})_i)\cdot\beta$ with an element $\beta \in \mathbb{Z}[\Xi]/\mathcal{T}_i$. The cocycle condition (A.2) is transformed to the cocycle condition for $(\overline{\alpha_i(\cdot)})_i$:

$$(\overline{\alpha_i(\xi\cdot\xi')})_i = (\overline{\alpha_i(\xi)})_i + (\bar{\xi})_i\cdot(\overline{\alpha_i(\xi')})_i \tag{A.6}$$

for $\xi, \xi' \in \Xi$, which means that $(\overline{\alpha_i(\cdot)})_i \in Z(\Xi, \mathbb{Z}[\Xi]/\mathcal{T}_i)$.

Next we let $\Xi_i := \{\xi \in \Xi;\ h_i^\xi = h_i\}$. Then Ξ_i is a subgroup of Ξ. We have the following lemmas.

LEMMA A-1. *Ξ_i is a proper subgroup of Ξ.*

Proof. Suppose that $\Xi = \Xi_i$. Then h_i is a Ξ-invariant polynomial. From the assumption $\delta(\Xi)$ is a $\mathbb{Z}$-lattice of rank m in Ω^m. Therefore $h_i(\omega)$ is a rational function on the m-dimensional torus $\Omega^m/\delta(\Xi)$, which means h_i is a constant function. This is a contradiction.

(q.e.d. of Lemma A-1)

LEMMA A-2. *The following sequence is exact,*

$$0 \longrightarrow \mathcal{T}_i \longrightarrow \mathbb{Z}[\Xi] \xrightarrow{\pi_i} \mathbb{Z}[\Xi/\Xi_i] \longrightarrow 0\,. \tag{A.7}$$

Hence we have $\mathbb{Z}[\Xi/\Xi_i]$ is isomorphic to $\mathbb{Z}[\Xi]/\mathcal{T}_i$.

Proof. Let $\mathcal{X}$ be an element of $\mathbb{Z}[\Xi]$ which is given by $\mathcal{X} := \sum_{k=1}^{s} n_k\cdot\lambda_k$

with $n_k \cdot \mathbb{Z}$ and $\lambda_k \in \Xi$. Let $\{\tilde{\lambda}_1, \cdots, \tilde{\lambda}_t\}$ be the set of all the non zero elements appearing in $\pi_i(\lambda_1), \cdots, \pi_i(\lambda_m)$. Then we have $\pi(\mathscr{X}) = \sum_{k=1}^{t} N_k \tilde{\lambda}_k$ with $N_k := \sum n_j$ where j runs through the set $\{j;\ \pi(\lambda_j) = \tilde{\lambda}_k\}$ and $h_i^{\mathscr{X}} = \prod_{k=1}^{t} (h_i^{\tilde{\lambda}_k})^{N_k}$. If $\pi_i(\mathscr{X}) = 0$, then $N_k = 0$ for all k in $1 \leq k \leq t$. Then $h_i^{\mathscr{X}}$ is a constant function and hence $\mathscr{X} \in \mathscr{T}_i$. Conversely, if $\mathscr{X} \in \mathscr{T}_i$, then $h_i^{\mathscr{X}} = \prod_{k=1}^{t} (h_i^{\tilde{\lambda}_k})^{N_k}$ must be a constant function. Note that $\pi_i(\xi) = \pi_i(\mu)$ if and only if $h_i^{\xi} = h_i^{\mu}$ for $\xi, \mu \in \Xi$. Since h_i is an irreducible polynomial, h_i^{ξ} and h_i^{μ} coincide with each other if they have a non-trivial common divisors. Thus any two of $h_i^{\tilde{\lambda}_1}, \cdots, h_i^{\tilde{\lambda}_t}$ have no non-trivial common divisors. This implies that $N_k = 0$ for each k in $0 \leq k \leq t$ and hence $\pi(\mathscr{X}) = 0$.

(q.e.d. of Lemma A-2)

Then we can identify $Z(\Xi, \mathbb{Z}[\Xi]/\mathscr{T}_i)$ and $Z(\Xi, \mathbb{Z}[\Xi/\Xi_i])$. The natural projection map $(\bar{\cdot})_i$ from $\mathbb{Z}[\Xi]$ to $\mathbb{Z}[\Xi]/\mathscr{T}_i$ is identified with the above map $\pi_i(\cdot)$. In particular, $(\bar{\cdot})_i$ is the natural porjection map from Ξ to Ξ/Ξ_i when we restrict it to Ξ from $\mathbb{Z}[\Xi]$.

LEMMA A-3. *Ξ/Ξ_i is a free abelian group for $i = 1, \cdots, l$.*

Proof. Let $\xi \in \Xi$ and suppose that $\xi^k \in \Xi_i$ for a non zero integer k. Then $h_i(\omega - k \cdot \delta\xi) = h_i(\omega)$ for all $\omega \in \Omega^m$. For any integer n, we have $h_i(\omega - (nk) \cdot \delta\xi) = h_i(\omega)$. Since h_i is a polynomial, we have $h_i(\omega - c\delta\xi) = h_i(\omega)$ for all $c \in \Omega$. Thus $h_i^{\xi}(\omega) = h_i(\omega - \delta\xi) = h_i(\omega)$, and hence $\xi \in \Xi_i$. Therefore Ξ/Ξ_i is torsion free. (q.e.d. of Lemma A-3)

LEMMA A-4. *If $\alpha(\cdot) \in \mathbb{Z}(\Xi, \mathbb{Z}[\Xi/\Xi_i])$ and $\xi \in \Xi_i$, then $\alpha(\xi) = 0$.*

Proof. Take an element such that μ is not contained in Ξ_i. This is possible by Lemma A-1. Then for an element $\xi \in \Xi$, since $(\bar{\xi})_i = 1$ if and only if $\xi \in \Xi_i$, we have $\alpha(\xi \cdot \mu) = \alpha(\xi) + (\bar{\xi})_i \cdot \alpha(\mu) = \alpha(\xi) + \alpha(\mu)$ and $\alpha(\xi \cdot \mu) = \alpha(\mu \cdot \xi) = \alpha(\mu) + (\bar{\mu})_i \cdot \alpha(\xi)$. Then we have $(1 - (\bar{\mu})_i)\alpha(\xi) = 0$. Note that $\mathbb{Z}[\Xi/\Xi_i]$ has no zero divisor for Ξ/Ξ_i is torsion free. Since $(\bar{\mu})_i \neq 1$ for $\mu \notin \Xi_i$, we have $\alpha(\xi) = 0$. (q.e.d. of Lemma A-4)

Now we begin the calculation on $h_i^{(\overline{\alpha_i(\xi)})_i}(\omega)$ appearing in (A.4). We shall compute $h_i^{(\overline{\alpha_i(\xi)})_i}(\omega)$ in the following two cases.

$$\text{(A.8)} \qquad 1)\ \operatorname{rank}(\Xi/\Xi_i) = 1\,, \qquad 2)\ \operatorname{rank}(\Xi/\Xi_i) \geq 2\,.$$

First we consider the case that $\operatorname{rank}(\Xi/\Xi_i) = 1$. We denote by $\Xi^\vee$ the group of homomorphisms from Ξ to $\mathbb{Z}$. Take the element $e_i^\vee \in \Xi^\vee$

such that $e_i^\vee|_{\Xi_i} = 0$ and which gives an isomorphism from Ξ/Ξ_i to $\mathbb{Z}$; it is uniquely determined. Let $\bar{e}_i^\vee$ be the linear form on Ω^m satisfying $\bar{e}_i^\vee(\sum_{j=1}^m a_j\delta\xi_j) = \sum_{j=1}^m a_j e_i^\vee(\xi_j)$ for $(a_1, \cdots, a_m) \in \Omega^m$. It is also determined uniquely. It is easy to see that $\bar{e}_i^\vee$ is invariant when we regard it as an element in $R^\times(\Omega^m)$. From the definition of Ξ_i, $h_i(\omega)$ is an irreducible polynomial which is invariant under the action of Ξ_i.

LEMMA A-5. *If* $\mathrm{rank}\,(\Xi/\Xi_i) = 1$, *then* $h_i(\omega)$ *is written by an irreducible polynomial of one variable, that is to say,* $h_i(\omega) = (\text{const.}) \times (\bar{e}_i^\vee(\omega) - c_i)$ *with a constant* c_i.

Proof. Let $r_1(\omega) = \bar{e}_i^\vee(\omega), r_2(\omega), \cdots, r_m(\omega)$ be linearly independent linear forms on Ω^m. Then they form another linear coordinate system on Ω^m. The polynomial $h_i(\omega)$ is a polynomial of m-variables $r_1(\omega), \cdots, r_m(\omega)$. Since $\mathrm{rank}\,(\Xi/\Xi_i) = 1$, the functions $r_2, \cdots, r_m$ are not invariant by the action of Ξ_i. If $h_i(\omega)$ depends on r_k $(k \geq 2)$, then h_i is a periodic function with respect to the variable r_k. Therefore h_i does not depend on $r_2, \cdots, r_m$. This means that h_i is an irreducible polynomial of the one variable $r_1 = \bar{e}_i^\vee(\omega)$. (q.e.d. of Lemma A-5)

Let λ_i be an element in Ξ such that $\bar{e}_i^\vee(\lambda_i) = 1$.

LEMMA A-6. *For any* $\xi \in \Xi$, $\xi \cdot \lambda_i^{-\bar{e}_i^\vee(\delta\lambda_i)}$ *belongs to* Ξ_i.

Proof. We may take elements $\nu_2, \cdots, \nu_m \in \Xi_i$ such that $\lambda_i, \nu_2, \cdots, \nu_m$ form generators of the group Ξ. An element $\xi \in \Xi$ is written by $\xi = (\lambda_i)^{l_1}(\nu_2)^{l_2} \cdots (\nu_m)^{l_m}$ where l_i's are integers. Since

$$\bar{e}_i^\vee(\delta\xi) = \bar{e}_i^\vee(l_1\delta\lambda_i + l_2\delta\nu_2 + \cdots + l_m\delta\nu_m) = l_1\bar{e}_i^\vee(\delta\lambda_i) = l_1 ,$$

we have $\xi \cdot (\lambda_i)^{-\bar{e}_i^\vee(\delta\xi)} = (\nu_2)^{l_2} \cdots (\nu_m)^{l_m}$ belongs to Ξ_i. (q.e.d. of Lemma A-6)

Thus we have, for any $\xi \in \Xi$, we have

$$\begin{aligned}\overline{(\alpha_i(\xi))}_i &= \overline{(\alpha_i((\lambda_i^{\bar{e}_i^\vee(\delta\xi)})\cdot(\xi\cdot\lambda_i^{-\bar{e}_i^\vee(\delta\xi)}))}_i \\ &= \overline{(\alpha_i((\lambda_i^{\bar{e}_i^\vee(\delta\xi)}))}_i + \overline{(\lambda_i^{\bar{e}_i^\vee(\delta\xi)})\cdot(\alpha_i(\xi\cdot\lambda_i^{-\bar{e}_i^\vee(\delta\xi)}))}_i\end{aligned}$$

from the cocycle condition. By Lemma A-4 and Lemma A-6, this implies $\overline{(\alpha_i(\xi))}_i = \overline{(\alpha_i((\lambda_i^{\bar{e}_i^\vee(\delta\xi)}))}_i$. Applying the cocycle condition (A.6) repeatedly, we have

$$\text{(A.10)}\quad \begin{aligned}\overline{(\alpha_i(\xi))}_i &= (1 + (\bar{\lambda}_i)_i + \cdots + (\bar{\lambda}_i)_i^{\bar{e}_i^\vee(\delta\xi)-1})\cdot\overline{(\alpha_i(\lambda_i))}_i && \text{if } \bar{e}_i^\vee(\delta\xi) > 0 , \\ &= 1 && \text{if } \bar{e}_i^\vee(\delta\xi) = 0 , \\ &= -((\bar{\lambda}_i)_i + \cdots + (\bar{\lambda}_i)_i^{-\bar{e}_i^\vee(\delta\xi)})\cdot\overline{(\alpha_i(\lambda_i))}_i && \text{if } \bar{e}_i^\vee(\delta\xi) < 0 .\end{aligned}$$

We shall calculate $\overline{(\alpha_i(\xi))}_i$. From the definition, $\overline{(\alpha_i(\xi))}_i$ is an element in $\mathbb{Z}[\Xi/\Xi_i]$. Since Ξ/Ξ_i is generated by $(\bar{\lambda}_i)_i$, $\overline{(\alpha_i(\lambda_i))}_i$ is regarded as a polynomial in $(\bar{\lambda}_i)_i$ and $(\bar{\lambda}_i)_i^{-1}$ of integer coefficients. Therefore we can write

$$\overline{(\alpha_i(\lambda_i))}_i = \sum_{j\in\mathbb{Z}} n_i(j)\cdot(\bar{\lambda}_i)_i^j, \tag{A.11}$$

where $n_i(j)$'s $(j\in\mathbb{Z})$ are integers and all of $n_i(j)$'s $(j\in\mathbb{Z})$ except for finite ones are zero. Together with (A.10) and (A.11) and Lemma A-5, we have

$$\begin{aligned} h_i^{\overline{(\alpha_i(\xi))}_i}(\omega) &= (\text{const.})\times(\bar{e}_i^\vee - c_i)^{\overline{(\alpha_i(\xi))}_i}(\omega) \\ &= (\text{const.})\times\prod_{k=0}^{\bar{e}_i^\vee(\delta\xi)-1}(\bar{e}_i^\vee - c_i - k)^{\overline{(\alpha_i(\lambda_i))}_i}(\omega) \end{aligned}$$

and

$$\begin{aligned} (\bar{e}_i^\vee - c_i - k)^{\overline{(\alpha_i(\lambda_i))}_i}(\omega) &= \prod_{j\in\mathbb{Z}}(\bar{e}_i^\vee(\omega - j\delta\lambda_i) - c_i - k)^{n_i(j)} \\ &= \prod_{j\in\mathbb{Z}}((\bar{e}_i^\vee(\omega) - k) - j - c_i)^{n_i(j)}. \end{aligned}$$

After all, if $\operatorname{rank}(\Xi/\Xi_i) = 1$, then by putting $\psi_i(x) := \prod_{j\in\mathbb{Z}}(x - j - c_i)^{n_i(j)}$, we have,

$$h_i^{\overline{(\alpha_i(\xi))}_i}(\omega) = (\text{const.})\times\prod_{k=0}^{\bar{e}_i^\vee(\delta\xi)-1}\psi_i(\bar{e}_i^\vee(\omega) - k). \tag{A.12}$$

Here, the product $\prod_{k=0}^{l-1}\psi_i(x-k)$ means 1 if $l=0$ and it stands for $\prod_{l\le k\le -1}\psi_i(x+k)^{-1}$ if l is a negative integer.

Next we consider the case of $\operatorname{rank}(\Xi/\Xi_i)\ge 2$.

LEMMA A-7. *If* $\operatorname{rank}(\Xi/\Xi_i)\ge 2$, *then* $H^1(\Xi, \mathbb{Z}[\Xi/\Xi_i]) = 0$. *In other words, for all* $\alpha(\cdot)\in Z(\Xi, \mathbb{Z}[\Xi/\Xi_i])$, *there exists an element* $\beta\in\mathbb{Z}[\Xi/\Xi_i]$ *such that* $\alpha(\cdot) = (1 - (\bar{\cdot})_i)\cdot\beta$.

Proof. Let $\alpha(\cdot)\in Z(\Xi, \mathbb{Z}[\Xi/\Xi_i])$ and put $r := \operatorname{rank}(\Xi/\Xi_i)$. We take $\xi_1, \cdots, \xi_r\in\Xi$ such that $(\bar{\xi}_1)_i, \cdots, (\bar{\xi}_r)_i$ generate the abelian group Ξ/Ξ_i. For positive integers s and t satisfying $1\le s\le t\le r$, we have

$$\alpha(\xi_s\cdot\xi_t) = \alpha(\xi_s) + (\bar{\xi}_s)_i\cdot\alpha(\xi_t) = \alpha(\xi_t) + (\bar{\xi}_t)_i\cdot\alpha(\xi_s),$$

from the cocycle condition. Then we have

$$(1 - (\bar{\xi}_t)_i)\cdot\alpha(\xi_s) = (1 - (\bar{\xi}_s)_i)\cdot\alpha(\xi_t). \tag{A.13}$$

Note that $\mathbb{Z}[\Xi/\Xi_i]$ is isomorphic to the ring of polynomials generated by $(\bar{\xi}_1)_i, \cdots, (\bar{\xi}_r)_i$ and $(\bar{\xi}_1)_i^{-1}, \cdots, (\bar{\xi}_r)_i^{-1}$ with $\mathbb{Z}$-coefficients. Then, as an example, in the equation (A.13), $\alpha(\xi_s)$ and $\alpha(\xi_t)$ are such polynomials. Let β be the greatest common divisor of $\alpha(\xi_s)$ and $\alpha(\xi_t)$. Then we have $\alpha(\xi_s)=(1-(\bar{\xi}_s)_i)\cdot\beta$ and $\alpha(\xi_t)=(1-(\bar{\xi}_t)_i)\cdot\beta$. We may take any s and t as they are different. Then we have

$$\alpha(\xi_k)=(1-(\bar{\xi}_k)_i)\cdot\beta \qquad \text{for all } k \text{ in } 1\leq k\leq r.$$

SUBLEMMA 1. *Let* $\alpha(\cdot)\in Z(\Xi, \mathbb{Z}[\Xi/\Xi_i])$ *and let* $\xi, \mu\in\Xi$. *If* $\alpha(\xi)=(1-(\bar{\xi})_i)\cdot\beta$ *and* $\alpha(\mu)=(1-(\bar{\mu})_i)\cdot\beta$ *with* $\beta\in\mathbb{Z}[\Xi/\Xi_i]$. *Then* $\alpha(\xi\cdot\mu^{-1})=(1-(\overline{\xi\cdot\mu^{-1}})_i)\cdot\beta$.

Proof. From the cocycle condition we have

$\alpha(1)=\alpha(\mu\cdot\mu^{-1})=\alpha(\mu)+(\bar{\mu})_i\cdot\alpha(\mu^{-1})=\alpha(\mu^{-1})+(\overline{\mu^{-1}})_i\cdot\alpha(\mu)$. Then we have $(1-(\overline{\mu^{-1}})_i)\cdot\alpha(\mu)=(1-(\bar{\mu})_i)\cdot\alpha(\mu^{-1})$ and hence $(1-(\bar{\mu})_i)\cdot\alpha(\mu^{-1})=(1-(\overline{\mu^{-1}})_i)\cdot(1-(\bar{\mu})_i)\cdot\beta$. Since $\mathbb{Z}[\Xi/\Xi_i]$ has no zero divisor, we have $\alpha(\mu^{-1})=(1-(\overline{\mu^{-1}})_i)\cdot\beta$. Then we have

$$\begin{aligned}\alpha(\xi\cdot\mu^{-1})&=\alpha(\xi)+(\bar{\xi})_i\cdot\alpha(\mu^{-1})\\&=(1-(\bar{\mu})_i)\cdot\beta+(\bar{\xi})_i\cdot(1-(\overline{\mu^{-1}})_i)\cdot\beta\\&=(1-(\bar{\xi})_i\cdot(\overline{\mu^{-1}})_i)\cdot\beta\\&=(1-(\overline{\xi\cdot\mu^{-1}})_i)\cdot\beta\end{aligned}$$ (q.e.d. of Sublemma 1)

We proceed the proof of Lemma A-7. Let Ξ^i be the subgroup of Ξ generated by $\xi_1,\cdots,\xi_r$. From Sublemma 1, we have $\alpha(\zeta)=(1-(\bar{\zeta})_i)\cdot\beta$ for any $\zeta\in\Xi^i$. From Lemma A-4, if $\zeta\in\Xi^i$ and $\lambda\in\Xi_i$, then we have $\alpha(\lambda\cdot\zeta)=\alpha(\lambda)+(\bar{\lambda})_i\cdot\alpha(\zeta)=\alpha(\zeta)=(1-(\bar{\zeta})_i)\cdot\beta=(1-(\overline{\lambda\cdot\zeta})_i)\cdot\beta$. Since Ξ is a direct product of Ξ^i and Ξ_i, we have $\alpha(\xi)=(1-(\bar{\xi})_i)\cdot\beta$ for all $\xi\in\Xi$. Thus we have $\alpha(\cdot)\in B(\Xi,\mathbb{Z}[\Xi/\Xi_i])$ if $\alpha(\cdot)\in Z(\Xi,\mathbb{Z}[\Xi/\Xi_i])$, which implies that $H^1(\Xi,\mathbb{Z}[\Xi/\Xi_i])=0$. (q.e.d. of Lemma A-7)

Now we go back to the calculation of $h_i^{(\overline{\alpha_i(\xi)})_i}(\omega)$. By Lemma A-7, if rank $(\Xi/\Xi_i)\geq 2$, then we can write as $(\overline{\alpha_i(\xi)})_i=(1-(\bar{\xi})_i)\cdot\beta_i$ by taking a suitable element $\beta_i\in\mathbb{Z}[\Xi/\Xi_i]$. Then

$$\begin{aligned}h_i^{(\overline{\alpha_i(\xi)})_i}(\omega)&=h_i^{(1-(\bar{\xi})_i)\cdot\beta_i}(\omega)\\&=h_i^{\beta_i}(\omega)/h_i^{\beta_i}(\omega-\delta\xi).\end{aligned}$$

We can write $\beta_i=\sum_{j=1}^{t_i}n_i(j)\cdot\lambda_i(j)$ by using suitable $\lambda_i(j)\ni\Xi$ and $n_i(j)\in\mathbb{Z}$. After all, if rank$(\Xi/\Xi_i)\geq 2$, we have

$$h_i^{\overline{(\alpha_i(\xi))}t}(\omega) = \prod_{i=1}^{t_i} \{h_i(\omega - \delta\lambda_i(j))/h_i(\omega - \delta\lambda_i(j) - \delta\xi)\}^{n_i(j)}, \tag{A.14}$$

From (A.12) and (A.14), we have the result. (q.e.d. of Theorem)

Corollary to Theorem. *We use the same notations as Theorem. Suppose that* $\{f_\xi\}_{\xi \in \Xi}$ *satisfies the condition: there exists a set of generators* $\{\mu_1, \cdots, \mu_m\}$ *of* Ξ *such that each* $f_{\mu_i}(\omega)$ *is a polynomial in* $\omega \in \Omega$. *Then* f_ξ *is written in the following form*:

$$f_\xi(\omega) = C(\xi) \cdot \prod_{i=1}^{p} \prod_{\nu=0}^{\bar{e}_i^\vee(\delta\xi)-1} \psi_i(\bar{e}_i^\vee(\omega) - \nu). \tag{A.15}$$

Proof. In the expression (A.3) of $f_\xi(\omega)$, all the terms appearing in the left hand side are polynomials if $f_\xi(\omega)$ is a polynomial in ω since they have no common divisors from the definition. From the assumption, if $\xi = \mu_1^{n_1} \cdots \mu_m^{n_m}$ where $n_1, \cdots, n_m$ are non-negative integers, then $f_\xi(\omega)$ is a polynomial, and hence

$$\prod_{j=1}^{t_i} \{h_i(\omega - \delta\lambda_i(j))/h_i(\omega - \delta\lambda_i(j) - \delta\xi)\}^{n_i(j)} \tag{A.16}$$

is a polynomial for each $i = 1, \cdots, q$. However it is necessary that $n_i(j) = 0$ for $j = 1, \cdots, t_i$ since there exists $\xi \in \Xi$ such that $h_i(\omega - \delta\lambda_i(j)) \neq h_i(\omega - \delta_i(j) - \delta\xi)$ for all $j = 1, \cdots, t_i$. This means that the term of the form (A.16) does not appear in the expression (A.3). Then we obtain (A.15).

References

[A1] Aomoto, K., Finiteness of a cohomology associated with certain Jackson integrals, preprint 1989.

[A2] ——, Connection coefficients for Jackson integrals of extended Selberg type, preprint 1989.

[Sa-Ki] Sato, M. and Kimura, T., A classification of irreducible prehomogeneous vector spaces and their relative invariants, Nagoya Math. J., **65** (1977), 1–155.

[Sa-Sh1] Sato, M. and Shintani, T., Gaikinshitsu bekutoru kuukan no riron (Theory of prehomogeneous vector spaces) (in Japanese), Sugaku no Ayumi, **15–1** (1970), 85–157.

[Sa-Sh2] ——, On zeta functions associated with prehomogeneous vector spaces, Ann. of Math., **100** (1974), 131–170.

[Sa-Ka-ki-Os] Sato, M., Kashiwara, M., Kimura, T. and Oshima, T., Micro-local analysis of prehomogeneous vector spaces, Invent. Math., **62** (1980), 117–179.

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