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Theory of Resonance Probe

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A theoretical analysis is made of the resonance phenomena in the radio-frequency probe experiments of Takayama et al. The Boltzmann-Vlasov equation is solved under the action of an external rf electric field. The solution gives the resonance peak of the dc component of the electron current to the probe at the plasma frequency. For a partially ionized plasma, the peak-height δj and the half-width $\varDelta\omega_{1/2}$ are given by the following formulae.

$$\delta j = j_0 \frac{1}{\sqrt{2}} \frac{\omega_p}{\nu} \frac{\lambda_d}{L} \sqrt{\frac{eV}{\kappa T}} \frac{e\delta V}{\kappa T} I_1\left(\frac{e\delta V}{\kappa T}\right),$$

$$\varDelta\omega_{1/2} = 2\nu.$$

For fully ionized plasmas, they are determined as follows,

$$\delta j = j_0 \frac{1}{\sqrt{2}} \sqrt{\frac{eV}{\kappa T}} \frac{e\delta V}{\kappa T} I_1\left(\frac{e\delta V}{\kappa T}\right),$$

$$\varDelta\omega_{1/2} = 4\left(\frac{\lambda_d}{L}\right)^2 \omega_p.$$

In the above expressions, T is the electron temperature, ω_p is the electron plasma frequency, and λ_d is the Debye length. j_0 is an electron current density to the probe when no oscillating field is superposed on it, δV is the amplitude of the superposed rf voltage, $I_1(z)$ is the modified Bessel function of the first order. L is an effective penetration depth of the external field. V is the potential difference between the plasma space and the probe, and ν is the effective collision frequency of the electron with neutral molecules.

The present theory confirms that the analysis of the resonance peak in the radio-frequency probe experiments is an effective method for the plasma diagnosis.

§ 1. Introduction

As is well known,[1] the electron current density j_0 to the Langmuir probe is given by

$$j_0 = ne\left(\frac{\kappa T}{2\pi m}\right)^{1/2} \exp\left(-\frac{eV}{\kappa T}\right) \tag{1}$$

where κ is the Boltzmann constant, e and m are the charge and the mass of the electron, respectively. With Eq. (1) the plasma electron density n and the

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electron temperature T is determined from the j_0-V characteristic curve. If the electron velocity is isotropic, the velocity distribution function is determined by

$$f\left(\sqrt{\frac{2eV}{m}}\right)=\frac{4m}{e^2}V\frac{d^2j_0}{dV^2}. \tag{2}$$

When the low voltage rf signal $\delta V \sin \omega t$ is superposed on the Langmuir probe, the second derivative of the electron current characteristic curve to the potential difference can be measured as the dc current increase $\varDelta j$,

$$\varDelta j=\frac{1}{4}(\delta V)^2\frac{d^2j_0}{dV^2}. \tag{3}$$

For the Maxwellian function, the current increase is given in another form,

$$\varDelta j=j_0\left\{I_0\left(\frac{e\delta V}{\kappa T}\right)-1\right\}, \tag{4}$$

where $I_0(z)$ is the modified Bessel function of the zeroth order. It must be noted that $\varDelta j$ is independent of the frequency ω of the rf signal.

Takayama and his collaborators[2)] have shown that Eqs. (3) and (4) are valid as far as the frequency is less than a certain critical frequency. Their typical experimental results are shown in Fig. 1.

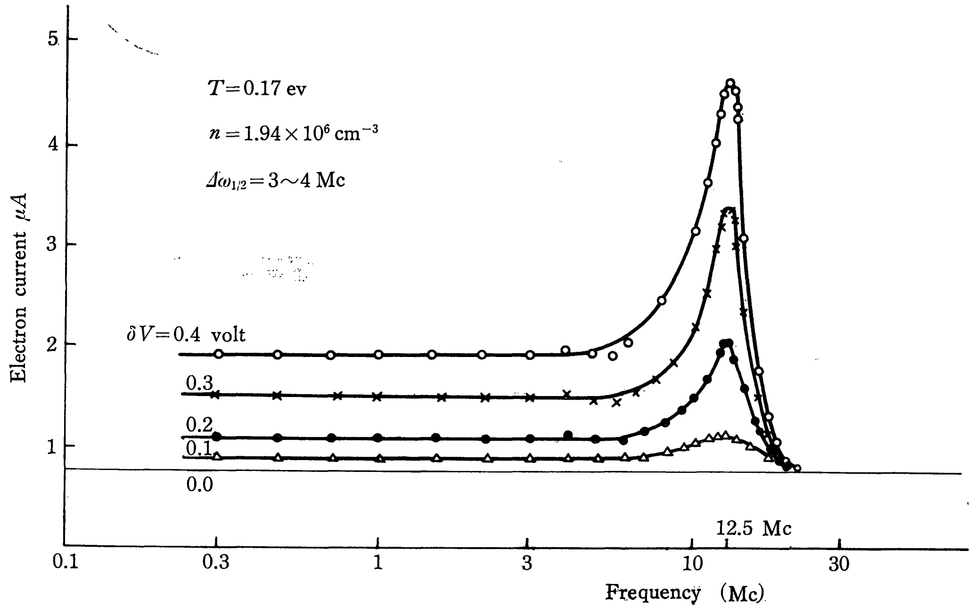

Fig. 1. Experimental curves for a plasma in the mercury discharge tube with a diameter of 30 cm. The parameters are $T=0.17$ ev and $n=1.94\times10^6$ cm^{-3}. The half-width is determined and the electron collision frequency ν is given by Eq. (26). The value $\nu=1.5\sim2$ Mc is consistent with that estimated by atomic theory.

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The characteristic curves in Fig. 1 consist of the following three frequency ranges: The first range where the electron current keeps a constant value independent of the frequency, the second range where a resonance peak appears, and the third range where the superposed rf voltage has no effect on the dc current to the probe.

The first range is in accord with Eq. (4) and the temperature of the electron can be determined. The resonance peak in the second range appears at the electron plasma frequency. The electron density determined by the resonance frequency coincides with that obtained by the Langmuir probe.

Now, our paper aims to give a theory of the resonance in the second frequency range. Our theory is confirmed by a series of recent experiments,[3] in which it is shown that the resonance increase of the dc current to the probe occurs also when the rf voltage is applied not to the probe, but to the pair of electrodes in a plasma. The mechanism of a beam-plasma interaction discussed by Takayama et al.[2] may be unable to explain the fact.

§ 2. Model of the resonance probe

Since what we are concerned with is the resonance phenomena at the electron plasma frequency, it is apparent that the collective interaction between electrons has an essential effect. As for the effects of the rf voltage superposed on the probe, one may set up several models. When the probe potential is negative, positive ions are attracted and form a sheath around it. The oscillation of the probe potential may induce charge fluctuations, as the electrons flow in and out of the sheath. Takayama and his collaborators have considered that such electrons may be accelerated by the oscillating electric field and they will be injected into the plasma as a beam. The electron beam will excite the plasma oscillation. They have assumed that the resonance excitation of the oscillation may occur when the frequency of the rf voltage is just equal to the plasma frequency. Bohm[4] has discussed a possible operation of the acceleration mechanism in a transition region between the sheath and the plasma. If such a mechanism of the beam-plasma interaction as considered by Takayama and his collaborators were essential in the resonance phenomena, the characteristics of the resonance peak would depend on the details of the acceleration mechanism.

Our new model will be figured as follows: When the rf voltage is superposed on the probe, the external electric field will be shielded out within a certain distance from the probe surface. In the equilibrium state, the distance is the Debye length, while in a steady state, it will be the Langmuir's sheath depth. For the case of the oscillating electric field, it may be called a penetration depth. The rf voltage is shared across the distance in the following way.

When the frequency is lower than the electron plasma frequency, the electric field will be shielded out within the sheath. This is the reason why the first range in Fig. 1 exists, or why Eq. (4) can be successfully applied to the ex-

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periments. The shielding is caused by the accumulation of electrons induced at the surface of the probe by the external electric field and they share the dc current increase. In higher frequency ranges, only a small portion of the electric field is shielded in the sheath, as the sheath depth is not large enough to shield it out, and the external electric field will stretch into the plasma region, inducing the electric charge fluctuations. Near the plasma frequency the fluctuation becomes maximum and produces greatest electron flow, which is called the resonance. For a much higher frequency range, a very small portion of the electric field is shielded in the sheath where the dielectric constant is nearly equal to unity, and the electrons which jump into the probe experience so many periods of the rf field during it across the effective penetration depth of the field. This means that on the average the electrons behave as if they were in a static electric field and the flow of electrons results in no dc current and the third frequency range will be realized. The same situation occurs[3] when the probe is at the space potential, where no sheath is formed around the probe.

§ 3. Formulation of the problem

Let us consider the plane probe. When the rf voltage of $\delta V \sin \omega_0 t$ is superposed on the probe, the charge fluctuation induced by variation of the probe potential can be regarded as an oscillating electric double layer with a thickness of effective penetration depth L. Hence, it acts as an external electric field of the following form,

$$E_{ext}(x,t) = -\frac{\delta V}{L} D(x) \sin \omega_0 t, \tag{5·a}$$

$$D(x) = \frac{1}{\pi} \int_{+\infty}^{+\infty} \frac{\sin kL}{kL} e^{ikx} dk. \tag{5·b}$$

Here, we consider the one-dimensional problem for convenience. Our problem is to find the induced charge fluctuation in a plasma by the external electric field defined by Eqs. (5·a) and (5·b).

The Boltzmann-Vlasov equation with the external field in a linearized form is given by

$$\frac{\partial}{\partial t} f + v \frac{\partial}{\partial x} f + \frac{e}{m} E \frac{\partial}{\partial v} f_0 = -\frac{e}{m} E_{ext} \frac{\partial}{\partial v} f_0 + \frac{\partial}{\partial t} f|_{coll}, \tag{6}$$

$$\frac{\partial}{\partial x} E = 4\pi e \int f dv, \tag{7}$$

where $f(x,v,t)$ is a fluctuation of the electron distribution function from the unperturbed function $f_0(v)$, and $E(x,t)$ in Eq. (6) is the self-consistent electric

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field determined by Eq. (7).

We shall look for the solution of Eqs. (6) and (7) in the following form,

$$f(x, v, t) = \int f(k, v, \omega) e^{i(kx-\omega t)} dkd\omega, \tag{8}$$

$$E(x, t) = \int E(k, \omega) e^{i(kx-\omega t)} dkd\omega. \tag{9}$$

The collision term in the right hand side of Eq. (6) may be replaced by $-\nu f$, where ν is an effective collision frequency. Then, Eqs. (6) and (7) are rewritten as

$$i(\omega - kv + i\nu) f(k, v, \omega) = \frac{e}{m} \{E(k, \omega) + E_{ext}(k, \omega)\} \frac{\partial f_0}{\partial v}, \tag{10}$$

$$ikE(k, \omega) = 4\pi e \overline{f(k, \omega)}, \tag{11}$$

where

$$\overline{f(k, \omega)} = \int f(k, v, \omega) dv. \tag{12}$$

Substituting Eq. (11) in Eq. (10) and integrating the resultant expression over the velocity v, we get

$$\overline{f(k, \omega)} = -\frac{i}{\mathcal{E}(k, w)} \int \frac{\partial f_0/\partial v}{\omega - kv + i\nu} dv \frac{e}{m} E_{ext}(k, \omega), \tag{13}$$

where

$$\mathcal{E}(k, \omega) \doteq 1 + \frac{4\pi e^2}{m} \frac{1}{k} \int \frac{\partial f_0/\partial v}{\omega - kv + i\nu} dv. \tag{14}$$

Substitution of Eq. (10) back to Eq. (11) gives the expression for the Fourier transform of the electric field of the plasma waves induced by the external oscillating field,

$$E(k, \omega) = \left\{ \frac{1}{\mathcal{E}(k, \omega)} - 1 \right\} E_{ext}(k, \omega). \tag{15}$$

With Eqs. (10) and (15) the induced fluctuation of the electron distribution function is given by

$$f(k, v, \omega) = -i \frac{\partial f_0/\partial v}{\omega - kv + i\nu} \frac{1}{\mathcal{E}(k, \omega)} \frac{e}{m} E_{ext}(k, \omega). \tag{16}$$

Thus the final result for the induced fluctuation of the electron distribution function is

$$f(x, v, t) = -\frac{e}{m} \frac{\partial f_0}{\partial v} \iint \frac{1}{\omega - kv + i\nu} \frac{E_{ext}(k, \omega)}{\mathcal{E}(k, \omega)} e^{i(kx-\omega t)} dkd\omega. \tag{17}$$

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The electron current density to the probe is obtained by the following equation:

$$j=\left\langle\int_{\Phi(t)}^{\infty} ev\{f_0(v)+f(0, v; t)\}dv\right\rangle_{\text{time average}}, \tag{18}$$

where

$$\Phi(t)=\left\{\frac{2e}{m}(V+\delta V\sin\omega_0 t)\right\}^{1/2}. \tag{19}$$

The second term in the brackets of the integration in Eq. (18) gives the resonance increase of the dc current to the probe.

§ 4. Structure of the resonance peak

Let us now investigate in greater detail the structure of the resonance increase of the electron current to the probe.

The dielectric constant of the plasma given by Eq. (14) can be expressed approximately in the following form, in the limit of long wavelength,

$$\varepsilon(k, \omega)=1-\left(\frac{\omega(k)}{\omega}\right)^2+i\left(\frac{\omega(k)}{\omega}\right)^2\frac{2}{\omega}(\nu+\gamma_L), \tag{20}$$

where γ_L is the Landau damping coefficient,

$$\gamma_L=\sqrt{\frac{\pi}{8}}\left(\frac{k_d}{k}\right)^3\exp\left\{-\frac{1}{2}\left(\frac{k_d}{k}\right)^2\right\}\omega_p, \tag{21}$$

and

$$\omega(k)^2=\omega_p^2+v_T^2 k^2, \tag{22}$$

where $v_T^2=(3\kappa T_e/m)$ and $k_d=2\pi/\lambda_d$.

For a partially ionized plasma, as $\nu\neq 0$, we have

$$\mathrm{Im}\frac{1}{\varepsilon(k,\omega)}=\frac{2\left(\dfrac{\omega(k)}{\omega_p}\right)^2\dfrac{\nu}{\omega_p}}{\left\{1-\left(\dfrac{\omega(k)}{\omega}\right)^2\right\}^2+4\left\{\left(\dfrac{\omega(k)}{\omega}\right)^2\dfrac{\nu}{\omega_p}\right\}^2}. \tag{23}$$

In Eq. (17), we may disregard kv in comparison with ω_p and neglect ν^2 in comparison with ω_p^2. Then, neglecting the dispersion effect due to the thermal motion of electrons, we get

$$f(0, v, t)=-\frac{e\delta V}{\kappa T}\frac{1}{\omega_0 L}\{\varepsilon_r^2+\varepsilon_i^2\}^{-1/2}vf_0(v)\sin(\omega_0 t-\delta), \tag{24}$$

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where ω_0 is the frequency of the external electric field and δ is given by

$$\tan\delta=\frac{\varepsilon_r}{\varepsilon_i},$$

where ε_r and ε_i are the real and imaginary part of the dielectric constant of plasma respectively. If we substitute Eq. (24) in Eq. (18) and neglect ν^2 in comparison with ω_0^2, the current density to the probe is given as follows,

$$j=j_0\, I_0\left(\frac{e\delta V}{\kappa T}\right)$$

$$+j_0\frac{1}{\sqrt{2}}\frac{2\dfrac{\nu}{\omega_0}}{\left\{1-\left(\dfrac{\omega_p}{\omega_0}\right)^2\right\}^2+4\left(\dfrac{\nu}{\omega_0}\right)^2}\frac{\lambda_D}{L}\frac{\omega_p}{\omega_0}\sqrt{\frac{eV}{\kappa T}}\frac{e\delta V}{\kappa T}I_1\left(\frac{e\delta V}{\kappa T}\right). \tag{25}$$

The second term in the right-hand side of Eq. (25) gives the resonance increase of the current density at $\omega_0=\omega_p$. The half-width of the peak are determined by the collision frequency as follows,

$$\Delta\omega_{1/2}=2\nu, \tag{26}$$

while the peak height is given by

$$\delta j=j_0\frac{1}{\sqrt{2}}\frac{\omega_p}{2\nu}\frac{\lambda_D}{L}\sqrt{\frac{eV}{\kappa T}}\left(\frac{e\delta V}{\kappa T}\right)I_1\left(\frac{e\delta V}{\kappa T}\right). \tag{27}$$

The above result that the half-width is determined solely by the collision frequency suggests that the resonance probe is useful not only for the measurement of the plasma density, but also for the measurement of the collision frequency ν.

For a fully ionized collision-free plasma, we may take $\nu\to 0$. In the limit of the long wavelength, γ_L becomes zero, and we use the following expression in place of Eq. (23),

$$\begin{aligned}\mathrm{Im}\frac{1}{\varepsilon(k,\omega)}&=\pi\delta(\varepsilon_r(k,\omega))\\&=\frac{\pi}{2}\omega(k)\{\delta(\omega-\omega(k))+\delta(\omega+\omega(k))\}.\end{aligned} \tag{28}$$

If the thermal motion of electrons are disregarded completely as before, the shape of the resonance peak becomes $\delta(\omega-\omega_p)$. Hence, for the fully ionized collision-free plasma, it is important to take into account the dispersion effect of the thermal motion. Then, we get

$$f(0,v,t)=-\frac{e\delta V}{\kappa T}\frac{\sin\dfrac{\sqrt{|\omega_0^2-\omega_p^2|}\cdot L}{v_T}}{\dfrac{\sqrt{|\omega_0^2-\omega_p^2|}\cdot L}{\ }}\frac{v}{v_T}f_0(v)\sin\omega_0 t, \tag{29}$$

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and the electron current density to the probe

$$j=j_0 I_0\left(\frac{e\delta V}{\kappa T}\right)+j_0\frac{1}{\sqrt{2}}\frac{\sin\dfrac{\sqrt{|\omega_0^2-\omega_p^2|}\cdot L}{v_T}}{\dfrac{\sqrt{|\omega_0^2-\omega_p^2|}\cdot L}{v_T}}\sqrt{\frac{eV}{\kappa T}}\frac{e\delta V}{\kappa T}I_1\left(\frac{e\delta V}{\kappa T}\right). \tag{30}$$

In this case also, the dc current has its resonance increase at $\omega_0=\omega_p$. The half-width of the resonance peak $\Delta\omega_{1/2}$ is, however, given by

$$\Delta\omega_{1/2}=4\left(\frac{\lambda_D}{L}\right)^2\omega_p, \tag{31}$$

while the peak height becomes independent of L and is given by

$$\delta j=j_0\frac{1}{\sqrt{2}}\sqrt{\frac{eV}{\kappa T}}\frac{e\delta V}{\kappa T}I_1\left(\frac{e\delta V}{\kappa T}\right). \tag{32}$$

For the fully ionized plasma, however, we should take into account the electron-ion and electron-electron correlation effects. They determine the effective collision frequency of the order of $10\omega_p\sqrt{(e^2/\kappa T)^3 n}$. Therefore, even in the fully ionized plasma, the resonance increase of the dc electron current to the probe may be determined by Eq. (25) with $\nu\doteqdot 10\omega_p\sqrt{(e^2/\kappa T)^3 n}$, not by Eq. (30).

§ 5. Discussion

It has been discussed by Takayama and his collaborators[2] that the resonance probe method is a powerful technique of measurements of the plasma density, no matter how high concentration the plasma has. The present theory has shown explicitly that the half-width of resonance peak is determined solely by the collision frequency. Therefore, now it has become clear that the resonance probe method is also useful for direct measurements of the collision frequency.

The present theory can successfully explain the experiments[2),3)] of the resonance probe. It will be most interesting to study experimentally the behavior of the ac electron current to the probe. Can it be alive when the dc current increase is decreased? Our theory answers, "Yes".

As to the dependence of the effective penetration depth L on the frequency, quantitative discussion will need more experimental knowledge.

In conclusion the authors consider that it is their pleasant duty to thank Prof. S. Kojima and Prof. K. Takayama for suggesting the problem and the discussion of the results. Thanks are also due to Prof. S. Nakajima and Dr. M. Sumi for their critical discussions in the course of the present investigation.

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