19. *Theory of Tempered Ultrahyperfunctions. I*

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In this paper we consider the tempered ultrahyperfunction which was introduced by Sebastião e Silva [3] and Hasumi [1] in the name of tempered ultradistribution. We will give some precisions on the work of M. Hasumi. The same idea was developped in [2] for the Fourier ultrahyperfunction.

§ 1. **The basic spaces $H(\boldsymbol{R}^n; O')$ and $H(\boldsymbol{R}^n; K')$.** Let $K' \subset \boldsymbol{R}^n$ be a convex compact set. Put $h_{K'}(x) = \sup\{\langle x, \xi\rangle ; \xi \in K'\}$, $\langle x, \xi\rangle$ being the canonical inner product of $\boldsymbol{R}^n \times \boldsymbol{R}^n$. Remark $h_{K'}(x) = k'|x|, |x| = |x_1| + |x_2| + \cdots + |x_n|$ for $K' = [-k', k']^n$. Let $H_b(\boldsymbol{R}^n; K')$ be the space of all C^∞ functions f on $\boldsymbol{R}^n$ such that $\exp(h_{K'}(x))D^p f(x)$ is bounded in $\boldsymbol{R}^n$ for any multi-index p. D^p denotes the partial differential operator

$$\frac{\partial^{|p|}}{\partial x_1^{p_1}\partial x_2^{p_2}\cdots\partial x_n^{p_n}}, \quad p = (p_1, p_2, \cdots, p_n), \quad |p| = p_1 + p_2 + \cdots + p_n.$$

We define in $H_b(\boldsymbol{R}^n; K')$ seminorms

$$(1) \qquad \|f\|_{K',k} = \sup\{\exp(h_{K'}(x))|D^p f(x)| ; |p| \leqslant k, x \in \boldsymbol{R}^n\}$$

for $k = 0, 1, 2, \cdots$. With these seminorms, the space $H_b(\boldsymbol{R}^n; K')$ is a Fréchet space. If K_1' and K_2' are two convex compact sets in $\boldsymbol{R}^n$ such that $K_1' \subset K_2'$, then the canonical injection

$$(2) \qquad H_b(\boldsymbol{R}^n; K_2') \hookrightarrow H_b(\boldsymbol{R}^n; K_1')$$

is continuous.

Let O' be a convex open set of $\boldsymbol{R}^n$. We define

$$(3) \qquad H(\boldsymbol{R}^n; O') = \lim_{K' \subset O'} \operatorname{proj} H_b(\boldsymbol{R}^n; K'),$$

where K' runs through the convex compact sets contained in O' and the projective limit is taken following the canonical injections (2). If O_1' and O_2' are two convex open sets in $\boldsymbol{R}^n$ such that $O_1' \subset O_2'$, we have the canonical injection: $H(\boldsymbol{R}^n; O_2') \hookrightarrow H(\boldsymbol{R}^n; O_1')$.

For a convex compact set K' of $\boldsymbol{R}^n$, we put

$$(4) \qquad H(\boldsymbol{R}^n; K') = \lim_{K'' \supset K'} \operatorname{ind} H_b(\boldsymbol{R}^n; K''),$$

where K'' runs through the convex compact sets such that K' is contained in the interior of K'' and the inductive limit is taken following the canonical mappings (2). If K_1' and K_2' are convex compact sets in $\boldsymbol{R}^n$ such that $K_1' \subset K_2'$, then we have the canonical injection: $H(\boldsymbol{R}^n; K_2') \hookrightarrow H(\boldsymbol{R}^n; K_1')$.

Theorem 1. *Let O' be a convex open set in $\boldsymbol{R}^n$ and K' be a convex* *compact set in* $\boldsymbol{R}^n$. *Then the space* $H(\boldsymbol{R}^n; O')$ *is an* FS *space and the space* $H(\boldsymbol{R}^n; K')$ *is the inductive limit of* FS *spaces.*

Proof. Choose a sequence of convex compact sets K'_m such that $K'_m \subset K'_{m+1}$ and $O' = \bigcup_{m=1}^{\infty} K'_m$. For any integer $m>0$ and $f \in C^m(\boldsymbol{R}^n)$, put $\|f\|_m = \|f\|_{K'_m, m}$. Then clearly $\|f\|_{m+1} \geqslant \|f\|_m$. Put

$$X_m = \{f \in C^m(\boldsymbol{R}^n); \|f\|_m < \infty\}.$$

Then we can show easily that X_m is a Banach space and that the mapping $X_{m+1} \to X_m$ is compact. As we have $H(\boldsymbol{R}^n; O') = \text{lim proj}\, X_m$, the space $H(\boldsymbol{R}^n; O')$ is an FS space. The second statement results from the equality:

$$H(\boldsymbol{R}^n; K') = \underset{O' \supset K'}{\text{lim ind}}\, H(\boldsymbol{R}^n; O'),$$

where O' runs through the convex open sets containing K' and the inductive limit is taken following the canonical injections. q.e.d.

Theorem 2. *Let* $\mathcal{D}(\boldsymbol{R}^n)$ *be the space of all* C^∞ *functions on* $\boldsymbol{R}^n$ *with compact support. Then* $\mathcal{D}(\boldsymbol{R}^n)$ *is dense in* $H(\boldsymbol{R}^n; O')$ *and in* $H(\boldsymbol{R}^n; K')$.

Proof. Fix a function $\alpha(x) \in \mathcal{D}(\boldsymbol{R}^n)$ such that $\alpha(x) \equiv 1$ for $|x|<1$, $\alpha(x) \equiv 0$ for $|x|>2$ and $0 \leqslant \alpha(x) \leqslant 1$. If $f \in H(\boldsymbol{R}^n; O')$ (resp. $H(\boldsymbol{R}^n; K')$), then $f_m(x) = \alpha(x/m) f(x) \in \mathcal{D}(\boldsymbol{R}^n)$ converges to f in the topology of $H(\boldsymbol{R}^n; O')$ (resp. of $H(\boldsymbol{R}^n; K')$) as m tends to ∞. q.e.d.

Corollary. *The space* $H = H(\boldsymbol{R}^n; \boldsymbol{R}^n)$ *is dense in* $H(\boldsymbol{R}^n; O')$ *and in* $H(\boldsymbol{R}^n; K')$.

§ 2. Spaces of distributions of exponential growth. For a linear topological space (over the field $\boldsymbol{C}$) X, we denote by X' the dual space of X. Thanks to Theorem 2 and Corollary, we have two inclusion relations:

$$H'(\boldsymbol{R}^n; O') \subset H' \subset \mathcal{D}'(\boldsymbol{R}^n),$$
$$H'(\boldsymbol{R}^n; K') \subset H' \subset \mathcal{D}'(\boldsymbol{R}^n).$$

It means that a continuous linear functional on the space $H(\boldsymbol{R}^n; O')$ or on $H(\boldsymbol{R}^n; K')$ is considered as a Schwartz distribution on $\boldsymbol{R}^n$. Hasumi [1] studied the space $H = H(\boldsymbol{R}^n; \boldsymbol{R}^n)$ and its dual space H'. He denotes H' by Λ_∞ and determined it as the space of distributions of exponential growth on $\boldsymbol{R}^n$. (See also Zieleźny [4].) The next theorem generalizes Hasumi's results:

Theorem 3. *A distribution* $T \in \mathcal{D}'(\boldsymbol{R}^n)$ *belongs to* $H'(\boldsymbol{R}^n; O')$ *if and only if there exist a multi-index* p, *a convex compact set* $K' \subset O'$ *and a bounded continuous function* F *such that*

$$(5) \qquad T = D^p[\exp(h_{K'}(x)) F(x)].$$

A distribution T *belongs to* $H'(\boldsymbol{R}^n; K')$ *if and only if for any* $\varepsilon>0$ *there exist a multi-index* p_ε *and a bounded continuous function* F_ε *such that*

$$(5') \qquad T = D^{p_\varepsilon}[\exp(h_{K'}(x) + \varepsilon |x|) F_\varepsilon(x)].$$

§3. The Fourier image of $H(\boldsymbol{R}^n; O')$ and $H(\boldsymbol{R}^n; K')$. For $f \in H(\boldsymbol{R}^n; O')$ we put

$$(6)\qquad \mathcal{F}f(\zeta)=(2\pi)^{-n/2}\int\cdots\int_{\boldsymbol{R}^n} f(x)\exp(-i\langle x,\zeta\rangle)dx_1dx_2\cdots dx_n,$$

where $\langle x,\zeta\rangle = x_1\zeta_1+x_2\zeta_2+\cdots+x_n\zeta_n$, $\zeta\in\boldsymbol{C}^n$. The Fourier transformation $\mathcal{F}f$ of f is a function defined for ζ for which the right hand side of (6) has a meaning. To describe the Fourier images, we introduce some spaces of holomorphic functions.

We will use the notation $T(A)=\boldsymbol{R}^n\times iA$ for a subset A of $\boldsymbol{R}^n$. For a convex compact set K' of $\boldsymbol{R}^n$, $\mathfrak{H}_b(T(K'))$ is, by definition, the space of all continuous functions φ on $T(K')$ such that φ is holomorphic in the interior $T(\mathring{K}')$ of $T(K')$ and that $\zeta^p\varphi(\zeta)$ is bounded in $T(K')$ for any multi-index p, where $\zeta^p=\zeta_1^{p_1}\zeta_2^{p_2}\cdots\zeta_n^{p_n}$ for $\zeta=(\zeta_1,\zeta_2,\cdots,\zeta_n)$ and $p=(p_1,p_2,\cdots,p_n)$. We put

$$(7)\qquad \|\varphi\|^{K';k}=\sup\{|\zeta^p\varphi(\zeta)|;\ \zeta\in T(K'), |p|\leqslant k\}\qquad \text{for } k=0,1,2,\cdots.$$

Endowed with these seminorms, the space $\mathfrak{H}_b(T(K'))$ is a Fréchet space. If $K_1'\subset K_2'$ are two convex compact sets, we have the canonical injection

$$(8)\qquad \mathfrak{H}_b(T(K_2'))\hookrightarrow\mathfrak{H}_b(T(K_1')).$$

Let O' be a convex open set in $\boldsymbol{R}^n$. Put

$$(9)\qquad \mathfrak{H}(T(O'))=\lim_{K'\subset O'}\operatorname{proj}\ \mathfrak{H}_b(T(K')),$$

where K' runs through the convex compact sets contained in O' and the projective limit is taken following the restriction mappings (8).

Hasumi [1] considered the space $\mathfrak{H}=\mathfrak{H}(T(\boldsymbol{R}^n))$ and proved the Fourier transformation establishes a topological isomorphism of H onto $\mathfrak{H}$. As the first step of generalization of Hasumi's results, we can show easily the following proposition.

Proposition 1. *If $f\in H(\boldsymbol{R}^n; O')$, the Fourier transformation $\mathcal{F}f$ of f belongs to the space $\mathfrak{H}(T(O'))$.*

Let $\varphi\in\mathfrak{H}(T(O'))$. Then, by the Cauchy integral theorem, the integrals

$$\int\cdots\int_{\boldsymbol{R}^n}\varphi(\xi+i\eta)\exp(-i\langle x,\xi+i\eta\rangle)d\xi_1d\xi_2\cdots d\xi_n$$
$$=\int\cdots\int_{\boldsymbol{R}^n+i\eta}\varphi(\zeta)\exp(-i\langle x,\zeta\rangle)d\zeta_1d\zeta_2\cdots d\zeta_n$$

are independent of $\eta\in O'$. Hence we may put for $\eta\in O'$

$$(10)\qquad \mathcal{F}\varphi(x)=(2\pi)^{-n/2}\int\cdots\int_{\boldsymbol{R}^n}\varphi(\xi+i\eta)\exp(-i\langle x,\xi+i\eta\rangle)d\xi_1d\xi_2\cdots d\xi_n$$

and call it the Fourier transformation of $\varphi\in\mathfrak{H}(T(O'))$.

Proposition 2. *The Fourier transformation $\mathcal{F}$ defined by* (10) *maps $\mathfrak{H}(T(O'))$ into $H(\boldsymbol{R}^n; -O')$. Moreover we have*

$$(11)\qquad \mathcal{F}(\mathcal{F}f)(x)=f(-x)\qquad \textit{for } f\in H(\boldsymbol{R}^n; O'),$$

$$(11')\qquad \mathcal{F}(\mathcal{F}\varphi)(\zeta)=\varphi(-\zeta)\qquad \textit{for } \varphi\in\mathfrak{H}(T(O')).$$

From Propositions 1 and 2 we can conclude the following theorem:

Theorem 4. *The Fourier transformation* (6) *establishes a topological isomorphism of* $H(\boldsymbol{R}^n; O')$ *onto* $\mathfrak{H}(T(O'))$ *and the inverse mapping is given by*

(10′) $$\overline{\mathcal{F}}\varphi(x)=(2\pi)^{-n/2}\int\cdots\int_{\boldsymbol{R}^n}\varphi(\xi+i\eta)\exp(i\langle x,\xi+i\eta\rangle)d\xi_1 d\xi_2\cdots d\xi_n,$$

where $\eta\in O'$.

Let K' be a convex compact set in $\boldsymbol{R}^n$. We define

(12) $$\mathfrak{H}(T(K'))=\lim_{K''\supset K'}\operatorname{ind}\mathfrak{H}_b(T(K'')),$$

where K'' runs through the convex compact sets such that K' is contained in the interior of K'' and the inductive limit is taken following the restriction mappings (8).

For $\varepsilon>0$ we will denote by K'_ε the ε-neighborhood of K', namely, $K'_\varepsilon=K'+\{\xi\in\boldsymbol{R}^n; |\xi|<\varepsilon\}$. K'_ε is a convex open set such that $K'_\varepsilon\supset K'$. As we have

(13) $$H(\boldsymbol{R}^n; K')=\lim_{\varepsilon>0}\operatorname{ind}H(\boldsymbol{R}^n; K'_\varepsilon)$$

and

(14) $$\mathfrak{H}(T(K'))=\lim_{\varepsilon>0}\operatorname{ind}\mathfrak{H}(T(K'_\varepsilon)),$$

we can conclude the following theorem from Theorem 4.

Theorem 4′. *The Fourier transformation* $\mathcal{F}$ *defined by* (6) *establishes a topological isomorphism of* $H(\boldsymbol{R}^n; K')$ *onto* $\mathfrak{H}(T(K'))$. *The inverse mapping* $\overline{\mathcal{F}}$ *is given by* (10′).

Corollary. *The space* $\mathfrak{H}(T(O'))$ *is an* FS *space and* $\mathfrak{H}(T(K'))$ *is the inductive limit of* FS *spaces.*

Via the Fourier transformation, Corollary to Theorem 2 gives the following Runge type theorem.

Theorem 5. *The space* $\mathfrak{H}=\mathfrak{H}(T(\boldsymbol{R}^n))$ *is dense in* $\mathfrak{H}(T(O'))$ *and in* $\mathfrak{H}(T(K'))$.

§ 4. The dual Fourier transformation. For $T\in H'(\boldsymbol{R}^n; O')$ (resp. $H'(\boldsymbol{R}^n; K')$), we define the dual Fourier transformation $\mathcal{F}_d T$ as a continuous linear functional on $\mathfrak{H}(T(-O'))$ (resp. $\mathfrak{H}(T(-K'))$) by the formula

(15) $$(\mathcal{F}_d T,\varphi)=(T,\mathcal{F}\varphi)\qquad\text{for }\varphi\in\mathfrak{H}(T(-O'))\text{ (resp. }\mathfrak{H}(T(-K'))).$$

As a consequence of Theorems 4 and 4′, we have the following theorem.

Theorem 6. *The dual Fourier transformation* (15) *gives topological isomorphisms*

(16) $$\mathcal{F}_d: H'(\boldsymbol{R}^n; O')\to\mathfrak{H}'(T(-O'))$$

and

(16′) $$\mathcal{F}_d: H'(\boldsymbol{R}^n; K')\to\mathfrak{H}'(T(-K')).$$

We will define, in the forthcoming papers, the Fourier transformation of $T\in H'(\boldsymbol{R}^n; O')$ and $T\in H'(\boldsymbol{R}^n; K')$ without use of the duality. It will turn out to be identical with the dual Fourier transformation.

References

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