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ABSTRACT

The double-dispersion representation is applied to the problem of pion-pion scattering, and it is shown that, if inelastic effects are important only at very high energies, a set of integral equations for the low-energy amplitudes can be derived. The solution of these equations appears to depend on only one arbitrary real parameter, which may be defined as the pion-pion coupling constant. The order of magnitude of the new constant is established, and a procedure for solving the integral equations by iteration is outlined

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I. INTRODUCTION

It has become evident in recent times that no further substantial progress will be made in the theory of strong-interaction phenomena involving pions and nucleons until something is understood about the pion-pion interaction. Previous theoretical work on this problem has lacked a framework in which to make plausible approximations, so the results of calculations done up to now are not considered reliable. Recently, however, one of us has proposed a generalization of dispersion relations that allows the simultaneous extension of energy and momentum transfer variables into the complex plane.[1] If the double-dispersion representation is accepted as correct, it becomes possible to formulate an approximation method for elementary-particle scattering at low energies that is extremely plausible. We propose in this paper to apply the new method to the pion-pion interaction.

The underlying motivation of the new approach is the property of an analytic function that its behavior in a limited region of the complex plane is dominated by nearby singularities. This circumstance is the basis of all "effective-range" theories for partial-wave scattering amplitudes. Effective-range theory leads to approximate formulas for partial amplitudes, valid in a small range of energies, that include nearby poles and branch points but ignore distant singularities. These formulas approximate the influence of the neglected

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1 S. Mandelstam, Phys. Rev. (1959), to be published, and University of California Physics Department preprint, Berkeley (1959). Also Phys. Rev. 112, 1344 (1958).

singularities by arbitrary constants to be fitted by experiment. The content of the double-dispersion representation is essentially to give the location and character of <u>all</u> the singularities of a scattering amplitude as well as the behavior at infinity. Armed with this information, one may extend the usual "effective-range" approach so as to reduce drastically the number of free parameters. Of course one can never include all the distant singularities, but in the pion-pion problem the first difficult branch point occurs at such a high energy that we believe the omitted effects can to a good approximation be absorbed into a single real parameter.

In the conventional Lagrangian formulation of field theory an independent constant appears in the pion-pion interaction, so one may be tempted to regard an effective-range approach with a single free parameter as the equivalent of a complete dynamical calculation. We prefer not to delve here into this very difficult question of principle but leave to the reader the theoretical interpretation of the constant λ that is to be introduced. Our definition of λ will be unambiguous from the experimental point of view.

As the price for including more of the nearby singularities than is usually attempted in effective-range theories, we shall have to solve nonlinear integral equations to find the pion-pion scattering amplitude. These equations will perhaps seem complicated, but they can be put into a nonsingular form amenable to numerical solution. The results of the numerical solutions for various values of λ are given in a subsequent paper.

II. SYMMETRIES AND KINEMATICS

Pion-pion elastic scattering may be represented by the diagram of Fig. 1, where the ingoing four-momenta and isotopic-spin indices are[2] (p_1, α) and (p_2, β) and the outgoing are $(-p_3, \gamma)$ and $(-p_4, \delta)$. It is convenient for discussions of symmetry to use a notation in which all momenta are formally directed inward, although in the physical region p_1 and p_2 are positive timelike, with p_3 and p_4 negative timelike. The convenient invariant dynamical variables for the double-dispersion representation are the squares of the total center-of-mass energies for the three reactions:

$$\begin{array}{ll} \text{I.} & (p_1, \alpha) + (p_2, \beta) \to (-p_3, \gamma) + (-p_4, \delta) \\ \text{II.} & (p_1, \alpha) + (p_4, \delta) \to (-p_2, \beta) + (-p_3, \gamma) \\ \text{III.} & (p_1, \alpha) + (p_3, \gamma) \to (-p_2, \beta) + (-p_4, \delta) . \end{array} \qquad \text{(II.1)}$$

Thus we define

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 = 4(q^2 + \mu^2) ,$$

$$\bar{t} = (p_1 + p_4)^2 = (p_2 + p_3)^2 = -2q^2(1 + \cos\theta) , \qquad \text{(II.2)}$$

and

$$t = (p_1 + p_3)^2 = (p_2 + p_4)^2 = -2q^2(1 - \cos\theta) ,$$

where q is the magnitude of the three-momentum and θ the angle of scattering in the barycentric system. Note the important supplementary condition

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[2] The isotopic indices α, β, γ, and δ can each assume the values 1, 2, or 3. The value 3 corresponds to the neutral pion, while linear combinations of 1 and 2 correspond in the usual way to charged pions.

$$s + t + \bar{t} = 4\mu^2 , \tag{II.3}$$

which means that only two of the three variables $s, t, \bar{t}$ are independent even when extensions are made into the complex plane.

Since isotopic spin is conserved and the three values $I = 0, 1, 2$ can occur, we expect to have three independent invariant functions of $s, t, \bar{t}$. These functions are conveniently introduced by writing the complete amplitude as

$$A(s, t, \bar{t})\delta_{\alpha\beta}\,\delta_{\gamma\delta} + B(s, t, \bar{t})\delta_{\alpha\gamma}\,\delta_{\beta\delta} + C(s, t, \bar{t})\delta_{\alpha\delta}\,\delta_{\beta\gamma} . \tag{II.4}$$

Crossing symmetry leads at once to the relations

$$\left.\begin{array}{l} A \to A \\ B \to C \end{array}\right\} \quad t \to \bar{t} , \quad s \to s , \tag{II.5}$$

$$\left.\begin{array}{l} A \to B \\ C \to C \end{array}\right\} \quad s \to t , \quad \bar{t} \to \bar{t} , \tag{II.6}$$

and

$$\left.\begin{array}{l} A \to C \\ B \to B \end{array}\right\} \quad s \to \bar{t} , \quad t \to t . \tag{II.7}$$

The first of these relations simply expresses the Pauli principle, but the remaining two place a powerful new condition on the combined energy and angular dependence of the amplitude. Such a condition, even though it arises from very simple considerations, is not known outside field theory.

An elementary calculation gives the connection between A, B, C and the three amplitudes A^I corresponding to well-defined I spin:

$$\begin{aligned} A^0 &= 3A + B + C , \\ A^1 &= B - C , \\ A^2 &= B + C . \end{aligned} \tag{II.8}$$

At this point one may verify that (II.5), together with (II.2), means that only even powers of $\cos\theta$ appear in the amplitudes for $I = 0, 2$ and only odd powers of $\cos\theta$ for $I = 1$. The implications of (II.6) and (II.7) are much more subtle, as we shall see later.

The unitarity condition on the pion-pion amplitude is most usefully expressed in terms of the partial-wave expansion of the amplitudes A^I when these are considered as functions of q^2 and $\cos\theta$:

$$A^I(q^2, \cos\theta) = \sum_{\substack{\ell \text{ even, } I=0,2 \\ \ell \text{ odd, } I=1}} (2\ell + 1) A_\ell^I(q^2) P_\ell(\cos\theta) \; . \qquad \text{(II.9)}$$

Unitarity allows the partial amplitudes $A_\ell^I(q^2)$ to be written in terms of phase shifts δ_ℓ^I according to[3]

$$A_\ell^I(q^2) = \frac{\sqrt{q^2 + \mu^2}}{q} \, e^{i\delta_\ell^I} \sin \delta_\ell^I \; , \qquad \text{(II.10)}$$

$s=(4\mu)^2$

where the phase shifts are real for $q^2 < 3\mu^2$, the threshold for inelastic scattering with the production of two additional pions.[4] At higher energies the phase shifts are complex, but the content of (II.10) can generally be expressed by the relation

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[3] The normalization of (II.10) is arbitrary, but the dependence on q follows from the Lorentz invariance of the S matrix.

[4] Production of any odd number of pions is forbidden by the G parity of Lee and Yang, Nuovo Cimento 3, 749 (1956). Single-pion production happens also to be forbidden by ordinary parity conservation.

$$\mathrm{Im}\, A_\ell^{\ I} = \frac{q}{\sqrt{q^2 + \mu^2}}\, R_\ell^{\ I} \left| A_\ell^{\ I}(q^2) \right|^2 ,$$

or

$$\mathrm{Im}\, (A_\ell^{\ I})^{-1} = -\frac{q}{\sqrt{q^2 + \mu^2}}\, R_\ell^{\ I} , \qquad (\mathrm{II.11})$$

where $R_\ell^{\ I}$ is the ratio of the total to the elastic partial-wave cross section.

III. THE DOUBLE-DISPERSION REPRESENTATION

A prescription for extending the scattering amplitude to complex values of s, t and $\bar{t}$, subject to (II.3), has been given by one of us.[1] This rule is embodied by the representation[5]

$$A(s, t, \bar{t}) = \frac{1}{\pi^2} \iint ds'\, dt' \frac{A_{13}(s', t')}{(s' - s)(t' - t)} + \frac{1}{\pi^2} \iint ds'\, d\bar{t}' \frac{A_{12}(\bar{t}', s')}{(s' - s)(\bar{t}' - \bar{t})}$$

$$+ \frac{1}{\pi^2} \iint dt'\, d\bar{t}' \frac{A_{23}(t', \bar{t}')}{(\bar{t}' - \bar{t})(t' - t)} , \qquad (\mathrm{III.1})$$

where the integrations in the primed variables extend in each case over regions of the positive real axis extending to infinity, and the weight functions A_{ij} are real. The functions B and C have similar representations, but the

[5] As shown in reference 1, the correct π-π representation probably requires also single dispersion integrals and an over-all subtraction term. See the remarks below, following Eq. (III.5), in this connection, as well as those following (IV.7).

crossing conditions tell us that only two out of the total of nine weight functions are independent, with one of these a symmetric function of its two arguments. In particular, in order to satisfy (II.5), (II.6) and (II.7), we require

$$\rho(x, y) = A_{13}(x, Y) = A_{12}(y, x) = B_{13}(y, x) = B_{23}(x, y) = C_{12}(x, y) = C_{23}(y, x) ,$$

$$\rho_s(x, y) = \rho_s(y, x) = A_{23}(x, y) = B_{12}(x, y) = C_{13}(x, y) . \tag{III.2}$$

The region of the (x, y) plane in which the weight functions fail to vanish is bounded by $x = 4\mu^2$ and $y = 4\mu^2$, but the region is not rectangular. According to the rules developed by one of us on the basis of perturbation theory,[1] the boundary is given by the curves,

$$x = \frac{16\mu^2 y}{y - 4\mu^2} , \quad \text{for} \quad x > y ,$$

and

$$y = \frac{16\mu^2 x}{x - 4\mu^2} , \quad \text{for} \quad y > x , \tag{III.3}$$

as shown in Fig. 2. The large distance to the boundary from the corner, $y = x = 4\mu^2$, is associated with the absence of a three-pion vertex and considerably simplifies our problem. The absence of a three-particle vertex also is responsible for the absence of poles in (II.1).[6]

[6] We assume that there exist no strongly interacting particles with the same quantum numbers as a pair of pions. If such a particle should be found, corresponding poles must be added to (III.1), whether or not the new particle is interpreted as a two-pion bound state.

A point of maximum symmetry in the s, t, $\bar{t}$ variables is the nonphysical point, $s = t = \bar{t} = 4\mu^2/3$, where A, B, and C are all real and equal to each other. It is appropriate then to introduce the pion-pion coupling constant λ through the definition[7]

$$\lambda = -A\left(\frac{4}{3}\mu^2, \frac{4}{3}\mu^2, \frac{4}{3}\mu^2\right) = -B\left(\frac{4}{3}\mu^2, \frac{4}{3}\mu^2, \frac{4}{3}\mu^2\right)$$
$$= -C\left(\frac{4}{3}\mu^2, \frac{4}{3}\mu^2, \frac{4}{3}\mu^2\right) . \tag{III.4}$$

It follows from (II.8) that at this symmetry point we have

$$A^0 = -5\lambda , \quad A^1 = 0 , \quad A^2 = -2\lambda . \tag{III.5}$$

Normally a coupling constant is defined through the residue of a pole, but here there are no poles. The new constant λ may be explicitly introduced into (III.1), if desired, by making a subtraction at the symmetry point. Subtractions are probably necessary to give a meaning to the double-dispersion representation (III.1), but we only need this expression in order to locate the singularities of the scattering amplitude. Thus we proceed at once to consider the analyticity properties of the partial-wave amplitudes $A_\ell^I(q^2)$, which can be correctly obtained by inspection of (III.1).

[7] This λ is, in conventional terminology, a renormalized unrationalized coupling constant. It corresponds to a term in the Lagrangian of the form $4\pi\lambda(\phi_\mu \phi_\mu)^2$.

IV. ANALYTICITY PROPERTIES OF THE PARTIAL-WAVE AMPLITUDES

In this paper we shall concentrate most of our attention on the low angular-momentum states. In principle, the approximation scheme based on the double-dispersion representation does not consist of taking more and more angular-momentum states into account--such a procedure would be inadequate owing to the failure (to be discussed below) of the Legendre expansion to converge in the unphysical region. It will therefore ultimately be necessary to calculate the spectral functions in (III.1), and so to include effects of all the angular-momentum waves. An approximation scheme for calculating the spectral functions can be worked out and was outlined in reference 1. Even if the spectral functions were known, however, it would still be necessary to treat separately the low angular-momentum states. The reason is that, when the single-dispersion integrals are included in (III.1), the absorptive parts of the low angular-momentum states will no longer be determined by the spectral functions, as has been explained in reference 1. We shall see below that, because of special properties of the π-π system, the calculation in the lowest approximation can be based entirely on the low angular-momentum states.

From (II.9) it follows that

$$A_\ell^I(q^2) = \frac{1}{2}\int_{-1}^{+1} d\cos\theta\, A^I(q^2, \cos\theta)\, P_\ell(\cos\theta) , \qquad \text{(IV.1)}$$

so that, in view of (II.2), the projection of a given partial wave amounts to an integration at fixed s over either dt or $d\bar{t}$. The two variables, t and $\bar{t}$ each cover the range between 0 and $-4q^2$, moving in opposite directions. It is straightforward, then, by inspection of [illegible] to establish the nature and location of the singularities of $A_\ell^I(q^2)$ [illegible] q^2 [illegible]. 8

8 It is possible, of course, to carry out the integration over [illegible] in (III.1) explicitly.

* lowest branch pt. in $\bar{t} = 4\mu^2$. $s = 4\mu^2 - \bar{t} - t = -t$.
and the lowest value of $t = 0$ at $\cos\theta = +1$. So, $s = 0 \Rightarrow q^2 = -\mu^2$.

It is obvious, first of all, that all the singularities lie on the real axis.[9] Next it will be recognized that there are three sets of branch points. The first set is associated with the vanishing of denominators containing s, with the lowest branch point occurring at $q^2 = 0$, the threshold of the physical region. The next branch point of this set will be at $q^2 = 3\mu^2$, the threshold for producing two additional pions, and so on. It is evidently appropriate to choose a cut running along the positive real axis from 0 to ∞. We shall refer to this as the "right-hand" or "physical" cut.

The other two sets of branch points are associated with the vanishing of denominators containing t or $\bar{t}$ and are coincident, lying on the negative real axis. The first pair of branch points is at $q^2 = -\mu^2$,* the second at $q^2 = -4\mu^2$, etc., the spacing being the same as on the positive axis. A second cut may then be chosen to run from $-\mu^2$ to $-\infty$; this will be called the "left-hand" or "unphysical" cut.

Finally it should be recognized that our partial-wave amplitude is a <u>real</u> analytic function of q^2, whose boundary value as the physical cut is approached from above is the complex physical amplitude, but which is real in the gap between $-\mu^2$ and 0 on the real axis. The discontinuity in going across either cut is twice the imaginary part of the limit as the cut is approached. The required imaginary part is given for the right-hand cut by (II.11).

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9 With unequal masses, as in pion-nucleon scattering, the singularities in the partial-wave amplitudes do not all lie on the real axis, but they can be located without difficulty. See, for example, S. W. McDowell, University of Birmingham Physics Department preprint (1959).

The calculation of the imaginary part on the left-hand cut is much more involved, as the unitarity condition cannot be used for negative values of q^2. We shall have to use crossing symmetry to obtain the imaginary part on the left-hand cut in terms of that on the right-hand cut, and the most convenient way to handle this problem is in terms of the absorptive parts for the three reactions I, II, and III, defined in reference 1.

The absorptive parts A_I, B_I, and C_I may be identified with the imaginary parts of the corresponding amplitudes in the physical region of reaction I, $q^2 > 0$. Similarly, the absorptive parts with subscripts II and III are equal to imaginary parts in the physical regions of reactions II and III respectively. These will be regions of negative q^2. It is possible to deduce from (III.2) the following crossing rules, which correspond to the relations (II.5) to (II.7):

$$\left.\begin{array}{l} A_I \rightarrow A_I \\ B_I \rightarrow C_I \\ C_I \rightarrow B_I \end{array}\right\} \quad \text{for} \quad t \rightarrow \bar{t}, \quad s \rightarrow s,$$

$$\left.\begin{array}{l} A_{III} \rightarrow B_I \\ B_{III} \rightarrow A_I \\ C_{III} \rightarrow C_I \end{array}\right\} \quad \text{for} \quad s \rightarrow t, \quad \bar{t} \rightarrow \bar{t}, \qquad \text{[illegible]}$$

and

$$\left.\begin{array}{l} A_{II} \rightarrow C_I \\ B_{II} \rightarrow B_I \\ C_{II} \rightarrow A_I \end{array}\right\} \quad \text{for} \quad s \rightarrow \bar{t}, \quad t \rightarrow t.$$

The other relation needed is that connecting the imaginary part of the amplitudes for $q^2 < 0$ with the absorptive parts for reactions II and III. By examination of (III.1) we find

$$\text{Im}\, A(q^2, \cos\theta) = -A_{II}(q^2, \cos\theta) - A_{III}(q^2, \cos\theta) , \qquad \text{(IV.3)}$$

$$\text{for} \quad q^2 < 0 ,$$

with similar relations for Im B and Im C.

If we now define

$$q'^2 = \frac{t}{4} - \mu^2 ,$$

$$\bar{q}'^2 = \frac{\bar{t}}{4} - \mu^2 ,$$

$$\cos\theta' = 1 + \frac{s}{2q'^2} = 1 + 2\,\frac{q^2 + \mu^2}{q'^2} ,$$

and

$$\cos\bar{\theta}' = -1 - \frac{s}{2\bar{q}'^2} = -1 - 2\,\frac{q^2 + \mu^2}{\bar{q}'^2} ,$$

and recall from (II.2) that

$$q^2 = \frac{s}{4} - \mu^2 ,$$

and

$$\cos\theta = 1 + 2\,\frac{q'^2 + \mu^2}{q^2} = -1 - 2\,\frac{\bar{q}'^2 + \mu^2}{q^2} ,$$

then the crossing rules (IV.2) allow us to write in place of (IV.3),

$$\text{Im}\, A(q^2, \cos\theta) = -C_I(\bar{q}'^2, \cos\bar{\theta}') - B_I(q'^2, \cos\theta') , \qquad \text{(IV.3')}$$

where q'^2 ranges from $-q^2 - \mu^2$ to $-\mu^2$ as $\cos\theta$ goes from -1 to $+1$, while $\bar{q}'^2$ covers the same range but in the opposite direction. It can be seen by inspection of (III.1) that B_I and C_I vanish in the range between 0 and $-\mu^2$, and so we have achieved our goal of expressing the imaginary part of the amplitude for negative q^2 in terms of quantities at positive q^2.

It remains now to project out the partial waves. From (IV.1) we have for $q^2 < -\mu^2$,

$$\begin{aligned} \operatorname{Im} A_\ell(q^2) &= \frac{1}{2} \int_{-1}^{+1} d\cos\theta \, \operatorname{Im} A(q^2, \cos\theta) P_\ell(\cos\theta) \\ &= \int_0^{-q^2-\mu^2} \frac{d\bar{q}'^2}{q^2} \, C_I\left(\bar{q}'^2, -1 - 2\,\frac{q^2+\mu^2}{\bar{q}'^2}\right) P_\ell\left(-1 - 2\,\frac{\bar{q}'^2+\mu^2}{q^2}\right) \\ &+ \int_0^{-q^2-\mu^2} \frac{dq'^2}{q^2} \, B_I\left(q'^2, 1 + 2\,\frac{q^2+\mu^2}{q'^2}\right) P_\ell\left(1 + 2\,\frac{q'^2+\mu^2}{q^2}\right) . \end{aligned} \tag{IV.4}$$

The formulas for $\operatorname{Im} B_\ell(q^2)$ and $\operatorname{Im} C_\ell(q^2)$ are similar, and the corresponding expressions for amplitudes with well-defined isotopic spin are

$$\begin{aligned} \operatorname{Im} A_\ell^I(q^2) &= \int_0^{-q^2-\mu^2} \frac{dq'^2}{q^2} \, P_\ell\left(1 + 2\,\frac{q'^2+\mu^2}{q^2}\right) \\ &\times \sum_{I'=0,1,2} \alpha_{II'} \, A_I^{I'}\left(q'^2, 1 + 2\,\frac{q^2+\mu^2}{q'^2}\right) \end{aligned} \tag{IV.5}$$

for $q^2 < -\mu^2$,

where

$$\alpha_{II'} \begin{pmatrix} 2/3 & 2 & 10/3 \\ 2/3 & 1 & -5/3 \\ 2/3 & -1 & 1/3 \end{pmatrix} . \qquad \text{(IV.6)}$$

Under the integrals in (IV.5) appear the absorptive parts of scattering amplitudes at values of $\cos\theta$ less than -1. From the boundary curve of Fig. 2 and Formula (III.3) it is possible to conclude that the Legendre polynomial expansion of $A_I^{\,I}(q'^2, \cos\theta)$ converges for the values of $\cos\theta$ required in (IV.5) so long as $q^2 > -9\mu^2$.[10] For the "effective-range" approach of this paper, such a limit might as well be $-\infty$. The surprisingly large magnitude of this limit is associated, as mentioned above, with the absence of a three-pion vertex. Crudely speaking, absence of a single-pion exchange mechanism reduces the range of the force to $\sim 1/2\mu$ and greatly improves the convergence of the partial wave expansion. Also it should be remembered, as emphasized by Lehmann,[10] that the expansion of the absorptive part of the amplitude always converges better than that of the real part.

It is possible to view in a slightly different way the approximation made in keeping only the first few terms of the polynomial expansion of the absorptive parts on the right of (IV.5). As shown in reference 1, the absorptive part can be written as a dispersion integral

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[10] For a discussion of the convergence of the Legendre polynomial expansion of a scattering amplitude, see H. Lehmann, Nuovo Cimento 10, 579 (1958).

$$A_I \{ q^2(s), \cos\theta(s, t) \} = a_0(s) + a_1(s)t + \ldots$$

$$+ (t - t_0)^n \frac{1}{\pi} \int dt' \frac{A_{13}(s, t')}{(t' - t_0)^n (t' - t)}$$

$$+ (\bar{t} - \bar{t}_0)^n \frac{1}{\pi} \int dt' \frac{A_{12}(s, \bar{t}')}{(\bar{t}' - \bar{t}_0)^n (\bar{t}' - \bar{t})} . \tag{IV.7}$$

The subtraction terms are here written explicitly, and the value of the exponent n is equal to the number of such terms. Perturbation theory prescribes that only one subtraction is necessary. However, further subtractions may be made either because one distrusts perturbation theory in this connection or to increase the accuracy of the calculation.[11] In this paper we make two subtractions, as shown below explicitly in Formulas (IV.9) and (IV.10).

Let us examine the form of the region in which one of the spectral functions, A_{13} for instance, is nonzero. As explained in reference 1, this spectral function consists of a number of parts corresponding to different Feynman diagrams. The two parts extending to the lowest values of s and t are bounded by the curves AB and CD of Fig. 2. Now, the part bounded by AB begins at or above the value $s = (4\mu)^2$, the threshold for the production of two additional pions. In the following section we shall approximate the absorptive part in the physical region by neglecting inelastic processes in the unitarity condition. This is in line with the "effective-range" principle, which assumes

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11 Subtractions of this kind in one variable do not correspond to the introduction of new parameters. See reference 1.

that the behavior of the scattering amplitude at low momenta is dominated by the nearest singularities. The part of A_{13} bounded by AB is therefore zero in this approximation. Similar considerations will apply to the other spectral functions. If, further, we require the crossing relations (III.2) to be satisfied, we shall also have to assume that the part of A_{13} bounded by CD is zero in the lowest approximation, so that the spectral functions are to be neglected entirely. That is to say, all contributions to the spectral functions begin at values of s and t which are so far from the region of interest that they should be ignored in a consistent "effective-range" approach.

From Eq. (IV.7), the absorptive part A_I can then be approximated by an expression of the form

$$A_I \{ q^2(s), \cos \theta(s, t) \} = a_0(s) + a_1(s)t \ldots$$

which is terminated at an early stage. The absorptive parts are thus represented by taking a small number of angular-momentum states only. This conclusion bears out the statement made at the beginning of this section that, in the lowest approximation, the calculation can be based entirely on the low angular-momentum states.

The approach just outlined enables us to understand why the absence of a three-pion vertex is critical in allowing one to terminate the Legendre expansion of the absorptive part. Had there been such a vertex, the curve bounding the shaded area in Fig. 2 would have consisted of a single part which approached asymptotically the lines $x = 4\mu^2$, $y = 4\mu^2$. The neglect of the spectral functions would then not have been justified. It would have been necessary to insert them in some approximation into Eq. (IV.7), with the resulting expressions then substituted into the integrals of Eq. (IV.5).

[Actually the fourth-order perturbation approximation could be used for the spectral functions, as all other contributions begin at values of either s or t greater than $(4\mu)^2$.] Even in the actual problem with no 3-pion vertex, if we were to go beyond the lowest approximation it would be necessary to calculate the spectral functions to an appropriate accuracy and then insert them into Eq. (IV.7).

It is worth emphasizing that we only assume the absorptive part of the scattering amplitude to be represented by its lowest angular-momentum waves. No such assumption regarding the real part is made. At the end of the calculation the real part of the first angular-momentum state omitted can be computed; if its square turns out to be small at the energies under consideration, we were presumably justified in leaving out the absorptive part. There is thus a check on the number of angular states which it is necessary to include.

To illustrate the above considerations and for future reference, we now derive formulas that clearly show the difference in our treatment of low and high partial waves. With no subtractions, one could write the following momentum-transfer dispersion relation on the basis of (III.1).

$$A(q^2, \cos\theta) = \frac{1}{\pi}\int_{4\mu^2}^{\infty} dt' \, \frac{A_{III}\left(q^2, 1 + \frac{t'}{2q^2}\right)}{t' - t} + \int_{4\mu^2}^{\infty} d\bar{t}' \, \frac{A_{II}\left(q^2, -1 - \frac{\bar{t}'}{2q^2}\right)}{\bar{t}' - \bar{t}}$$

$$= \frac{1}{\pi}\int_0^{\infty} dq'^2 \, \frac{B_I\left(q'^2, 1 + 2\,\frac{q^2 + \mu^2}{q'^2}\right)}{q'^2 + \mu^2 + \frac{q^2}{2}(1 - \cos\theta)} + \frac{1}{\pi}\int_0^{\infty} d\bar{q}'^2 \, \frac{C_I\left(\bar{q}'^2, -1 - 2\,\frac{q^2 + \mu^2}{\bar{q}'^2}\right)}{\bar{q}'^2 + \mu^2 + \frac{q^2}{2}(1 - \cos\theta)}$$

(Equation (IV.8) cont'd)

$$= \frac{1}{\pi} \int_0^\infty dq'^2 \, B_I\left(q'^2, 1 + 2\,\frac{q^2 + \mu^2}{q'^2}\right) \left[\frac{1}{q'^2 + \mu^2 + \frac{q^2}{2}(1 - \cos\theta)} \right.$$

$$\left. + \frac{1}{q'^2 + \mu^2 + \frac{q^2}{2}(1 + \cos\theta)} \right] , \qquad \text{(IV.8)}$$

with similar expressions for B and C. Now, the absorptive part B_I is, in general, complex, but the imaginary part of B_I vanishes in the lower range of the integral (IV.8) because from the equivalent of Eq. (IV.7), for $q^2 > 0$ and $q'^2 > 0$, we have

$$\text{Im}\, B_I\left(q'^2, 1 + 2\,\frac{q^2 + \mu^2}{q'^2}\right) = B_{13}(4(q'^2 + \mu^2), 4(q^2 + \mu^2)) ,$$

which is zero outside the shaded region of Fig. 2. Thus, if we make a subtraction in the dispersion relation (IV.8) to suppress the high-energy part, the remainder will be almost entirely real for small q^2. Figure 2 shows, of course, that as q^2 becomes large, the imaginary part cannot be suppressed. These considerations are identical to those following Eq. (IV.7).

Let us make the subtraction by removing the S-wave part of Eq. (IV.8).

$$A(q^2, \cos\theta) = A_0(q^2) + \frac{1}{\pi}\int_0^\infty dq'^2\, B_I\left(q'^2, 1 + 2\,\frac{q^2+\mu^2}{q'^2}\right)\left[\frac{1}{q'^2+\mu^2+\frac{q^2}{2}(1-\cos\theta)}\right.$$

$$\left. + \frac{1}{q'^2+\mu^2+\frac{q^2}{2}(1+\cos\theta)} - \frac{2}{q^2}\,\ln\left(1 + \frac{q^2}{q'^2+\mu^2}\right)\right] \tag{IV.8}$$

In the next section we shall determine $A_0(q^2)$, allowing it to be complex, but the residual amplitude (which starts here with the D wave) is real to a good approximation, for q^2 not too large. Furthermore, so long as the imaginary part of B_I is to be neglected, we are ignoring the singularities of this function in the variable q^2, and according to the arguments following Eq. (IV. [illegible]) we may consistently approximate it by a low-order polynomial. Since B_I when continued to the physical region of reaction I is the imaginary part of the amplitude, the appropriate procedure is to represent B_I in terms of precisely those partial waves that have been subtracted out, i.e., those that are allowed to be complex.

We now give the formulas for $I = 0, 2$ that correspond to Eq. (IV.8):

$$A^I(q^2, \cos\theta) = A_0^{\,I}(q^2) + \frac{1}{\pi}\int_0^\infty dq'^2 \sum_{I'} \alpha_{II'}\, A_I^{\,I'}\left(q'^2, 1 + 2\,\frac{q^2+\mu^2}{q'^2}\right)$$

$$\left\{\frac{1}{2}\left[\frac{1}{q'^2+\mu^2+\frac{q^2}{2}(1-\cos\theta)} + \frac{1}{q'^2+\mu^2+\frac{q^2}{2}(1+\cos\theta)}\right] - \frac{1}{q^2}\,\ln\left(1 + \frac{q^2}{q'^2+\mu^2}\right)\right\} . \tag{IV.9}$$

For $I = 1$, we subtract the P wave:

$$A^1(q^2, \cos\theta) = 3\cos\theta\, A_1{}^1(q^2) + \frac{1}{\pi}\int_0^\infty dq'^2 \sum_{I'} \alpha_{1I'}\, A_I{}^{I'}\left(q'^2, 1 + 2\,\frac{q^2+\mu^2}{q'^2}\right)$$

$$\left\{\frac{1}{2}\left[\frac{1}{q'^2+\mu^2+\frac{q^2}{2}(1-\cos\theta)} - \frac{1}{q'^2+\mu^2+\frac{q^2}{2}(1+\cos\theta)}\right]\right.$$

$$\left. - \frac{3\cos\theta}{q^2}\left[\left(1 + 2\,\frac{q'^2+\mu^2}{q^2}\right)\ell n\left(1+\frac{q^2}{q'^2+\mu^2}\right) - 2\right]\right\} . \tag{IV.10}$$

Under the integrals in these formulas $A_I{}^{I'}(q'^2, \cos\theta')$ will be approximated by

$$A_I{}^{0,2}(q'^2, \cos\theta') \underset{q'^2>0}{\approx} \operatorname{Im} A_0{}^{0,2}(q'^2) ,$$

$$A_I{}^{1}(q'^2, \cos\theta') \underset{q'^2>0}{\approx} 3\cos\theta' \operatorname{Im} A_1{}^{1}(q'^2) . \tag{IV.11}$$

V. FORMULATION OF INTEGRAL EQUATIONS

We now have the task of translating our knowledge about partial-wave amplitudes into integral equations. After introducing the variable $\nu = q^2/\mu^2$, the preceding statements about the location of singularities are equivalent to the dispersion relations,

$$A_\ell^I(\nu) = \frac{1}{\pi}\int_{-\infty}^{-1} d\nu' \frac{\text{Im}\, A_\ell^I(\nu')}{\nu' - \nu} + \frac{1}{\pi}\int_{0}^{\infty} d\nu' \frac{\text{Im}\, A_\ell^I(\nu')}{\nu' - \nu} , \tag{V.1}$$

provided the functions in question behave properly at infinity. The unitarity condition (II.10) guarantees that the partial-wave amplitudes behave asymptotically no worse than like constants. In order to estimate the error in our approximation, we shall assume that on the right-hand (physical) cut,

$$\text{Im}\, A_\ell^I(\nu) \to \frac{1}{2}$$

and (V.2)

$$\text{Re}\, A_\ell^I(\nu) \to 0 ,$$

in other words, the limit of pure diffraction scattering.[12] Such behavior, i.e. the ratio of the real to the imaginary part going asymptotically to zero, can be consistent with Eq. (V.1) only if the limits on the left-hand cut are the same.[13]

A partial-wave amplitude of order ℓ vanishes at the origin like ν^ℓ, so we may consider new quantities

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[12] Such behavior is expected because of the overwhelming competition from inelastic channels that sets in at very high energies.

[13] Considerations of this kind were first emphasized by I. Pomeranchuk, J. Exptl. Theoret. Phys. (USSR) 34, 725 (1958), in connection with forward-dispersion relations.

$$A'^{I}_{\ell}(\nu) = \frac{1}{\nu^{\ell}} A^{I}_{\ell}(\nu) \quad , \tag{V.3}$$

which also satisfy relations of the type (V.1) but whose imaginary parts, except for $\ell = 0$, now vanish at infinity like $\nu^{-\ell}$. It is clear that the higher the angular momentum, the smaller is the relative contribution from high values of ν' in the dispersion integrals (when ν is small). It is only for the S wave that distant contributions are expected to be important, so for the S wave we make a subtraction at the symmetry point

$$\nu_0 = -2/3 \quad , \tag{V.4}$$

to obtain[14]

$$A_0^{I}(\nu) = a_I + \frac{\nu - \nu_0}{\pi} \int_{-\infty}^{-1} d\nu' \frac{\operatorname{Im} A_0^{I}(\nu')}{(\nu' - \nu)(\nu' - \nu_0)}$$

$$+ \frac{\nu - \nu_0}{\pi} \int_{0}^{\infty} d\nu' \frac{\operatorname{Im} A_0^{I}(\nu')}{(\nu' - \nu)(\nu' - \nu_0)} \quad . \tag{V.5}$$

It is possible that even an S-wave subtraction is unnecessary in a treatment which includes in a serious way very-high-energy inelastic processes such as nucleon-antinucleon pair production. We do not believe, however, that such a treatment will be practical for a long time to come. Certainly nothing so ambitious will be attempted here.

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[14] The two subtraction constants a_0 and a_2 are not independent but are related to λ through Eq. (III.5). The relation is given below in Formula (V.18).

Thus, either by dividing by ν^ℓ or by subtracting we hope to suppress very high energies under the dispersion integrals. Specifically, we hope that taking finite limits for the integrals will not cause a large error, and so we consider, instead of (V.1), expressions of the form

$$A_\ell'^I(\nu) = \frac{\nu^\ell}{\pi} \int_{-L}^{-1} d\nu' \, \frac{\operatorname{Im} A_\ell'^I(\nu')}{\nu'^\ell(\nu' - \nu)} + \frac{\nu^\ell}{\pi} \int_0^L d\nu' \, \frac{\operatorname{Im} A_\ell'^I(\nu')}{\nu'^\ell(\nu' - \nu)} ,$$

or the corresponding subtracted expressions for S waves. These are supposed adequately to represent the physical scattering amplitudes, so long as we have $\nu << L$. The exact choice of L, of course, should not be important, or a new parameter would have been introduced into the problem. Using Eq. (V.2) one can easily estimate the order of magnitude of the neglected contributions to be

$$\delta A_0 \sim \frac{1}{\pi} \frac{\nu - \nu_0}{L} ,$$

and (V.6)

$$\delta A_\ell \sim \frac{1}{\pi} \frac{1}{\ell} \left(\frac{\nu}{L} \right)^\ell ,$$

which are small provided L can be made sufficiently large.

In this first attempt at solving the pion-pion problem, we shall choose L in the range where inelastic scattering first becomes important. The inelastic threshold is at $\nu = 3$, but experience with pion-nucleon scattering suggests that double-pion production won't represent a substantial fraction of the cross section until $\nu \sim 10$. Thus, with L in this range, we may use the unitarity condition (II.11) with R_ℓ set equal to unity:

$$\operatorname{Im} \left[A_\ell^I(\nu) \right]^{-1} = - \sqrt{\frac{\nu}{\nu + 1}} , \quad \text{for } 0 < \nu < L . \tag{V.7}$$

Furthermore, as discussed in the preceding section, the imaginary parts on the left-hand cut as given by Eq. (IV.5) may be evaluated by the use of Legendre polynomial expansions under the integrals. In particular we shall keep only $\ell = 0$ and $\ell = 1$ terms in these integrals; the legitimacy of this approximation may be checked a posteriori by calculating the D waves that emerge from our system of equations.

In terms of the variable ν the formulas (IV.5) for the first few partial waves become, in this approximation:

$$\operatorname{Im} A_\ell^I(\nu)_{\nu<-1} = \frac{1}{\nu}\int_0^{-\nu-1} d\nu' \, P_\ell\left(1 + 2\,\frac{\nu'+1}{\nu}\right) \left\{ \alpha_{I0} \operatorname{Im} A_0^0(\nu') + \alpha_{I2} \operatorname{Im} A_0^2(\nu') + 3\left(1 + 2\,\frac{\nu+1}{\nu'}\right) \alpha_{I1} \operatorname{Im} A_1^1(\nu') \right\} \qquad (V.8)$$

Now we put all the above information together in order to obtain a procedure for calculating phase shifts in terms of the empirical constant λ.

Consider first the two S-wave amplitudes. We attempt to represent each of these by a quotient

$$A_0^I(\nu) = \frac{N_0^I(\nu)}{D_0^I(\nu)} , \qquad (V.9)$$

where $N_0^I(\nu)$ and $D_0^I(\nu)$ are both real analytic functions, the numerator contains the branch point at $\nu = -1$ with the left-hand cut, and the denominator contains the branch point at $\nu = 0$ with the right-hand cut. It is also necessary, of course, that $D_0^I(\nu)$ have no zeros. By assumption, then, we have

$$\left.\begin{aligned} \operatorname{Im} N_0^I(\nu) &= D_0^I(\nu) \operatorname{Im} A_0^I(\nu) , \\ \operatorname{Im} D_0^I(\nu) &= 0 , \end{aligned}\right\} \quad \text{for} \quad \nu < -1$$

$$\operatorname{Im} N_0^I(\nu) = \operatorname{Im} D_0^I(\nu) = 0 , \qquad \text{for} \quad -1 < \nu < 0 \tag{V.10}$$

$$\left.\begin{aligned} \operatorname{Im} N_0^I(\nu) &= 0 , \\ \operatorname{Im} D_0^I(\nu) &= N_0^I(\nu) \operatorname{Im} \frac{1}{A_0^I(\nu)} \end{aligned}\right\} \quad \text{for} \quad \nu > 0 ,$$

and, according to our approximation of neglecting high-energy contributions, we set both imaginary parts equal to zero for $\nu > L$ and $\nu < -L$.

The subtracted dispersion relation (V.5) normalizes the S-wave amplitudes to a_I at the point $\nu = \nu_0$. We accomplish this normalization in our quotient by setting $N_0^I(\nu_0) = a_I$ and $D_0^I(\nu_0) = 1$. Furthermore, with a cutoff at L the amplitudes $A_0^I(\nu)$ approach real constants at infinity,[15] so we may assign constant asymptotic behavior to both numerator and denominator. Then, introducing,

$$f_\ell^I(\nu) = \operatorname{Im} A_\ell^I(-\nu) , \qquad (\nu > 0) \tag{V.11}$$

[15] Such behavior is inconsistent with Eq. (V.2) and incorrect physically, but our modified functions are only supposed to be accurate at low energies.

we are led first to write

$$N_0^I(\nu) = a_I + \frac{\nu - \nu_0}{\pi} \int_{-L}^{-1} d\nu' \frac{f_0^I(-\nu')\, D_0^I(\nu')}{(\nu' - \nu)(\nu' - \nu_0)} ,$$

$$= a_I + \frac{\nu - \nu_0}{\pi} \int_{1}^{L} d\nu' \frac{f_0^I(\nu')\, D_0^I(-\nu')}{(\nu' + \nu)(\nu' + \nu_0)} . \tag{V.12}$$

Second, remembering (V.7), we have

$$D_0^I(\nu) = 1 - \frac{\nu - \nu_0}{\pi} \int_0^L d\nu' \sqrt{\frac{\nu'}{\nu' + 1}} \frac{N_0^I(\nu')}{(\nu' - \nu)(\nu' - \nu_0)} . \tag{V.13}$$

On defining $E_0^I(\nu) = D_0^I(-\nu)$ and substituting Eq. (V.12) into Eq. (V.13), the following integral equation is obtained:

$$E_0^I(\nu) = 1 + (\nu + \nu_0)K(\nu, -\nu_0)a_I + \frac{\nu + \nu_0}{\pi} \int_1^L d\nu' \frac{K(\nu, \nu')f_0^I(\nu')E_0^I(\nu')}{\nu' + \nu_0} , \tag{V.14}$$

with

$$K(\nu, \nu') = \frac{1}{\pi} \int_0^L d\nu'' \frac{\sqrt{\frac{\nu''}{1 + \nu''}}}{(\nu'' + \nu)(\nu'' + \nu')} . \tag{V.15}$$

If the function $f_0^I(\nu')$ were known, Eq. (V.14) would be a nonsingular Fredholm equation, soluble by any number of standard methods. It will be shown in the following paper that, even in the limit $L \to \infty$, the equation can be cast into a nonsingular form.

We cannot lessen the reliability of our result at this stage by taking L infinite rather than in the neighborhood of 10, since the associated change in the amplitude will be smaller than the error (V.6) which we have agreed to tolerate.[16] It is possible, on the other hand, that the result may be improved by taking $L = \infty$ if elastic scattering is dominant up to higher energies than might conservatively be guessed. For these reasons, plus the esthetic consideration that one does not like a calculation to depend formally on an unnecessary parameter, we shall henceforth set $L = \infty$, even though the error in our approach should be estimated from Eq. (V.6) with some finite L.

It is unfortunately true that $f_0^I(\nu')$ is not known in advance but is given only through Eqs. (V.11) and (V.8) in terms of the amplitudes we are looking for. Thus our system of equations is actually nonlinear. In a subsequent paper, however, it will be shown that the problem can be solved by an iteration procedure in which at every stage the linear equations (V.14) are solved with the f_0^I corresponding to the previous stage. We must, of course, also formulate an equation for the P amplitude since this is required in Eq. (V.8).

Before considering the P-amplitude, however, a few general remarks about the S-wave problem are in order. First, an inspection of (V.14) with f_0^I set equal to zero and $L = \infty$ shows that E_0^I will develop a zero for $\nu < \nu_0$ if a_I is negative. According to Eq. (V.8), both f_0^0 and f_0^2 will be negative if, as is likely, the S contributions under the integrals are dominant.[17]

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16 This conclusion is not quite air tight but seems very plausible since the high energy elastic partial-wave cross sections given by our equations are smaller than the total cross sections used in the estimates (V.6).

17 Recall that the imaginary part of a partial-wave amplitude in the physical region is positive definite.

The zero will therefore not be removed when f_0^I is included, but if the zero appears sufficiently far out along the negative real axis--beyond the limit L at which our calculation of Im A_0 ceases to be accurate--the associated pole in A_0 is of no physical significance and cannot be excluded. A crude estimate, based on Eq. (V.14) and neglecting f_0^I, indicates that for $-0.7 < a_I < 0$, the zero in $E_0^I(\nu)$ will occur for $\nu > 10$.

If a_I is positive, the requirement that there be no zero of $E_0^I(\nu)$ in the region $\nu_0 < \nu < 0$ (i.e. no bound state of the π-π system) puts an upper limit on a_I. As a_I increases, the zero will appear first at $\nu = 0$, so we examine the condition that $E_0^I(0)$ be positive. Here the neglect of f_0^I is a good approximation, so one may deduce from Eq. (V.14) the requirement

$$1 - \frac{2}{3} K(0, \frac{2}{3}) a_I > 0 \quad ,$$

or, since we have

$$K(0, \frac{2}{3}) = \frac{3\sqrt{2}}{\pi} \tan^{-1} \frac{1}{\sqrt{2}} \quad ,$$

we can write

$$a_I < \frac{\pi}{2\sqrt{2}} \Big/ \tan^{-1} \frac{1}{\sqrt{2}} = 1.8 \quad .$$

One may inquire also about the possibility of zeros in $D_0^I(\nu)$ that are not on the real axis. Inspection of Eq. (V.13) shows that such zeros are impossible so long as $N_0^I(\nu)$ has no zeros on the positive real axis.[18] Should we find a solution that does have zeros in the physical region, this point would have to be investigated further.

Let us now determine the relation between a_I and λ and the consequent restrictions on λ that follow from the above limitations on a_I. According to (III.5), we have

$$A^0(\nu_0, 0) = -5\lambda ,$$

$$A^1(\nu_0, 0) = 0 ,$$

and (V.16)

$$A^2(\nu_0, 0) = -2\lambda .$$

The second of these relations is identically satisfied, since A^1 contains only odd powers of $\cos\theta$. The first and the third, however, give us the required information about a_0 and a_2 which are defined by Eq. (V.5) to be

$$a_0 = A_0^0(\nu_0)$$

and (V.17)

$$a_2 = A_0^2(\nu_0) .$$

Thus to a good approximation $a_0 \approx -5\lambda$ and $a_2 \approx -2\lambda$, since we expect D and higher partial-wave amplitudes to be small.

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18 For $\nu = \nu_R + i\nu_I$, the imaginary part of $D_0^I(\nu)$ is given by

$$-\frac{\nu_I}{\pi}\int_0^\infty d\nu' \sqrt{\frac{\nu'}{\nu'+1}} \frac{N_0^I(\nu')}{(\nu'-\nu_R)^2+\nu_I^2} ,$$

and therefore vanishes only for $\nu_I = 0$ if $N_0^I(\nu')$ has a single sign.

It is possible to correct for the higher waves within the approximation outlined at the end of Section IV. Formula (IV.9), when evaluated at $\cos\theta = 0$ and $\nu = \nu_0$, leads to the following result:

$$a_0 = -5\lambda + \frac{1}{\pi}\int_0^\infty d\nu' \left\{ \frac{1}{\nu_0}\, \ell n\left(1 + \frac{\nu_0}{\nu' + 1}\right) - \frac{1}{\nu' + 1 + \nu_0/2} \right\}$$

$$\times \left\{ \frac{2}{3}\,\mathrm{Im}\, A_0^{\,0}(\nu') + \frac{10}{3}\,\mathrm{Im}\, A_0^{\,2}(\nu') + 6\left(1 + 2\,\frac{\nu_0 + 1}{\nu'}\right)\mathrm{Im}\, A_1^{\,1}(\nu') \right\} \tag{V.18}$$

$$a_2 = -2\lambda + \frac{1}{\pi}\int_0^\infty d\nu' \left\{ \frac{1}{\nu_0}\, \ell n\left(1 + \frac{\nu_0}{\nu' + 1}\right) - \frac{1}{\nu' + 1 + \nu_0/2} \right\}$$

$$\left\{ \frac{2}{3}\,\mathrm{Im}\, A_0^{\,0}(\nu') + \frac{1}{3}\,\mathrm{Im}\, A_0^{\,2}(\nu') - 3\left(1 + 2\,\frac{\nu_0 + 1}{\nu'}\right)\mathrm{Im}\, A_1^{\,1}(\nu') \right\}$$

The integral correction given by Eq. (V.18) to the simple relation between the a_I and λ is very small[19] and may be ignored except for highly refined considerations. The most restrictive conditions on λ are obtained by considering the $I = 0$ state, for which $a_0 \approx -5\lambda$. The absence of zeros on the negative real axis for $|\nu| < L$, as discussed above, then leads to the limits

$$-\frac{1}{5}(1.8) \lesssim \lambda \lesssim -\frac{1}{5}(-0.7)$$

or (V.19)

$$-0.36 \lesssim \lambda \lesssim 0.14 .$$

---

[19] The smallness is due to the expression in the first curly bracket in the integrand of Eq. (V.18), which has a maximum value of 0.15 at $\nu' = 0$ and falls rapidly to zero as ν' increases.

A study of the formula for the cotangent of the S phase shifts reveals another interesting circumstance. We have for $\nu > 0$

$$\sqrt{\frac{\nu}{\nu + 1}} \cot \delta_0^I = \frac{\mathrm{Re}\, D_0^I(\nu)}{N_0^I(\nu)}$$

$$= \frac{1 - (\nu - \nu_0)\, I(\nu, -\nu_0)\, a_I - \dfrac{\nu - \nu_0}{\pi} \displaystyle\int_1^\infty d\nu' \frac{I(\nu, \nu') f_0^I(\nu') E_0^I(\nu')}{\nu' + \nu_0}}{a_I + \dfrac{\nu - \nu_0}{\pi} \displaystyle\int_1^\infty d\nu' \frac{f_0^I(\nu') E_0^I(\nu')}{(\nu' + \nu)(\nu' + \nu_0)}} , \tag{V.20}$$

where $I(\nu, \nu') = \lim_{L \to \infty} K(-\nu, \nu')$

$$I(\nu, \nu') = \frac{P}{\pi} \int_0^\infty d\nu'' \frac{\sqrt{\dfrac{\nu''}{\nu'' + 1}}}{(\nu'' - \nu)(\nu'' + \nu')} . \tag{V.21}$$

Again in the approximation where f_0^I is neglected we may study the possibility of a resonance developing, that is, $\cos \delta_0^I$ vanishing. We have

$$\sqrt{\frac{\nu}{\nu + 1}} \cot \delta_0^I \approx \frac{1}{a_I} - \frac{2}{\pi} \left\{ \sqrt{2} \tan^{-1} \frac{1}{\sqrt{2}} - \sqrt{\frac{\nu}{\nu + 1}}\, \ell n(\sqrt{\nu} + \sqrt{\nu + 1}) \right\} , \tag{V.22}$$

an expression that does not vanish for $\nu > 0$ if it is positive at $\nu = 0$. The condition of being positive at $\nu = 0$ for a_I positive is, however, exactly the condition that there shall be no bound state. Thus it seems unlikely that a resonance will develop in either S state for negative λ unless the effects of the f_0^I are very strong.

For positive λ and negative a_I, formula (V.22) has a zero but only for $\nu > L$ if the condition (V.19) is obeyed. Thus we tentatively conclude that there are no low-energy S-wave resonances in pion-pion scattering.[20]

We turn now to the P wave and again attempt to represent the amplitude by a ratio

$$\frac{1}{\nu} A_1(\nu) = \frac{N_1(\nu)}{D_1(\nu)} , \tag{V.23}$$

with the same division of singularities between the numerator and denominator as for the S wave. By arguments analogous to those used above, we may derive the equations

$$\begin{aligned} N_1(\nu) &= \frac{1}{\pi} \int_{-L}^{-1} d\nu' \frac{f_1^{\,1}(-\nu')D(\nu')}{\nu'(\nu' - \nu)} , \\ &= \frac{1}{\pi} \int_{1}^{L} d\nu' \frac{f_1^{\,1}(\nu')D(-\nu')}{\nu'(\nu' + \nu)} , \end{aligned} \tag{V.24}$$

and

$$D_1(\nu) = 1 - \frac{\nu}{\pi} \int_0^L d\nu' \sqrt{\frac{\nu'}{\nu' + 1}} \frac{N_1(\nu')}{\nu' - \nu} , \tag{V.25}$$

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[20] The absence of S-state resonances in simple two-body systems is a very general circumstance and may be traced to the lack of a centrifugal barrier that can "confine" a positive energy state. The only way to get an S-wave resonance is to have the force sufficiently complicated so that a strong inner attraction is surrounded by an outer repulsion. P-wave resonances, in contrast, arise naturally whenever there is a sufficiently strong attraction.

where N_1 has been assigned a $1/\nu$ behavior at infinity and D_1 a constant behavior. Introducing $E_1(\nu) = D_1(-\nu)$, the following integral equation is obtained by substituting Eq. (V.24) into Eq. (V.25):

$$E_1(\nu) = 1 + \frac{\nu}{\pi} \int_1^L d\nu' \frac{K(\nu, \nu') f_1^{\,1}(\nu') E_1(\nu')}{\nu'} . \tag{V.26}$$

The P phase shift in the physical region for $\nu > 0$ is given by the formula

$$\sqrt{\frac{\nu^3}{\nu + 1}} \cot \delta_1 = \frac{\operatorname{Re} D_1(\nu)}{N_1(\nu)} ,$$

$$= \frac{1 - \dfrac{\nu}{\pi} \displaystyle\int_1^\infty d\nu' \frac{I(\nu, \nu') f_1^{\,1}(\nu') E_1(\nu')}{\nu'}}{\dfrac{1}{\pi} \displaystyle\int_1^\infty d\nu' \frac{f_1^{\,1}(\nu') E_1(\nu')}{\nu'(\nu' + \nu)}} . \tag{V.27}$$

If $f_1^{\,1}$ were positive and sufficiently large it would be possible to show that a resonance develops in the P wave. Examination of Eq. (V.8) shows that $f_1^{\,1}(\nu)$ may change sign as ν increases but is definitely positive for large values of ν. Its magnitude is uncertain. We cannot say with confidence, therefore, that a P-wave resonance <u>will</u> develop until the equations have been integrated, but the possibility appears strong.

The sum of the higher partial-wave amplitudes is to be calculated from Eqs. (IV.9) and (IV.10). If individual phase shifts are desired, the appropriate projection from these formulas is straightforward. In a subsequent paper, D phase shifts are calculated in this manner.

VI. CONCLUSION

A set of coupled integral equations for the S- and P-wave pion-pion amplitudes has been formulated and in a subsequent paper the numerical solution of these equations for various values of λ will be described. The D and higher phase shifts can consistently be calculated by integration over the left-hand cut only, where the discontinuity across this cut is expressed in terms of the S and P amplitudes.

The physical meaning of our approximation in conventional language is that we consider explicitly only the exchange of pairs of virtual pions between the two physical pions being scattered, lumping 4-pion and higher multiplicity exchanges into the constant λ. Furthermore we only attempt to calculate accurately the exchanged pairs of lower energy--those which are mainly in S and P states. The higher energy pairs are included in λ along with all sorts of other high-energy exchanges. In terms of the range of various contributing mechanisms to the pion-pion force, what we are trying to do, of course, is to calculate the longest-range effects in detail and to represent the short-range effects by an empirical constant. If there is an intrinsically incalculable zero-range force, as suggested by Lagrangian field theory, this also is included in λ.

Beside the solution discussed in the foregoing paragraphs, there are also an infinite number of other possible solutions, corresponding to the Castillejo, Dalitz, and Dyson (CDD) ambiguity.[21] We can add to the right-hand side of Eq. (V.13) any number of terms of the form $a_r/(\nu - \nu_r)$, since the only effect of such terms is to introduce zeros into the scattering amplitude.

[21] Castillejo, Dalitz, and Dyson, Phys. Rev. 101, 453 (1956).

While a rigorous treatment of the CDD ambiguity has not been given for relativistic field theory, the problem has been solved for several models,[22] and there seems to be little doubt as to the meaning of the extra solutions. They correspond to theories in which, before the coupling is turned on, there are one or more particles with the same quantum numbers as two pions. Once the coupling is turned on, these particles become unstable, and appear experimentally as resonances. These "kinematical" resonances differ from "dynamical" resonances, such as that which we have suggested might appear in the P state of this problem, in that they occur for arbitrarily small values of the coupling constant. The absence of such unstable particles must be regarded as an additional postulate to be inserted into the theory.

A knowledge of the pion-pion scattering amplitude will allow a systematic calculation of many important properties of nucleons. The application to the nucleon electromagnetic structure has been emphasized already by Frazer and Fulco.[23] This application, however, actually requires a prior knowledge of the full amplitude for the graph shown in Fig. 3, which describes not only pion-nucleon scattering but also nucleon-antinucleon annihilation to form two pions. One of us has outlined a procedure for attacking this problem which is identical in spirit to that described here for the π-π problem.[24] The procedure requires a knowledge of π-π scattering and may now be implemented. It is hoped that a reasonably accurate description of the low-energy π-N phase shifts in terms of a single additional parameter, the pion-nucleon coupling constant, will result

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[22] N. G. Van Kampen, Physica 23, 157 (1957).

[23] W. R. Frazer and J. R. Fulco, The Effect of a Pion-Pion Scattering Resonance on Nuclear Structure, UCRL-8688, March 1959, and Phys. Rev. Letters, [illegible] (1959).

[24] S. Mandelstam, Phys. Rev. 112, 1344 (1958).

With an understanding of the graph of Fig. 3 one can proceed to a systematic calculation not only of nucleon electromagnetic structure but also of the two-pion exchange terms in the nuclear force. One can also, of course, make a solid theory of photopion production. All these problems are under investigation.

There is no reason why the generalized effective-range approach based on the double dispersion representation cannot be used in more complicated problems, such as those involving strange particles. As the structure of the nearby singularities becomes more complicated, of course, it becomes more and more difficult to include enough of them to constitute a good approximation. It is doubtful that any other problem can be found that is as favorable in this respect as π-π scattering.

FIGURE LEGENDS

Fig. 1. The pion-pion interaction, $\pi + \pi \leftrightarrow \pi + \pi$.

Fig. 2. The domain in which the spectral functions of the two-dimensional π-π representation are nonvanishing.

Fig. 3. Diagram for the reactions, $\pi + N \leftrightarrow \pi + N$ and $\pi + \pi \leftrightarrow N + \bar{N}$.

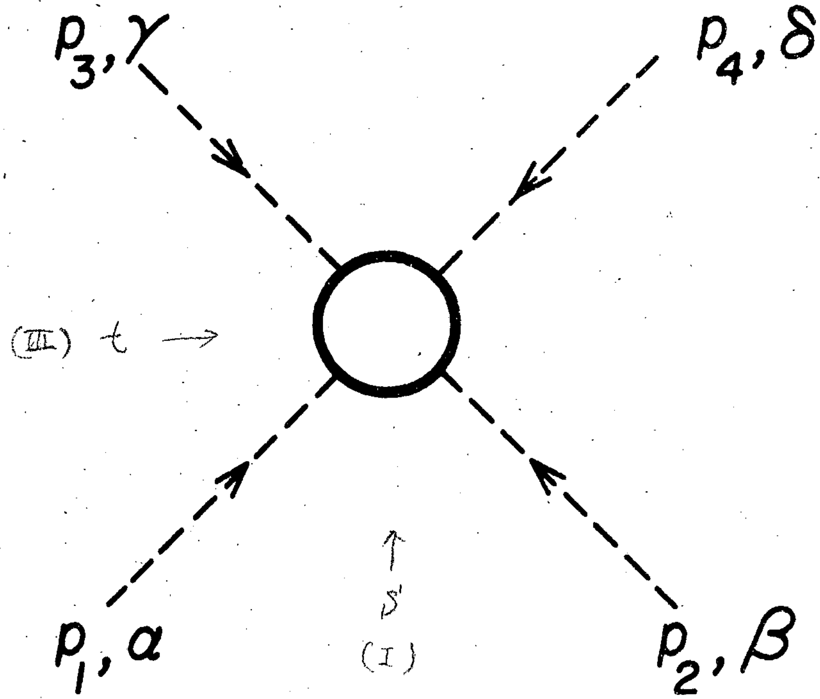

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Fig. 1

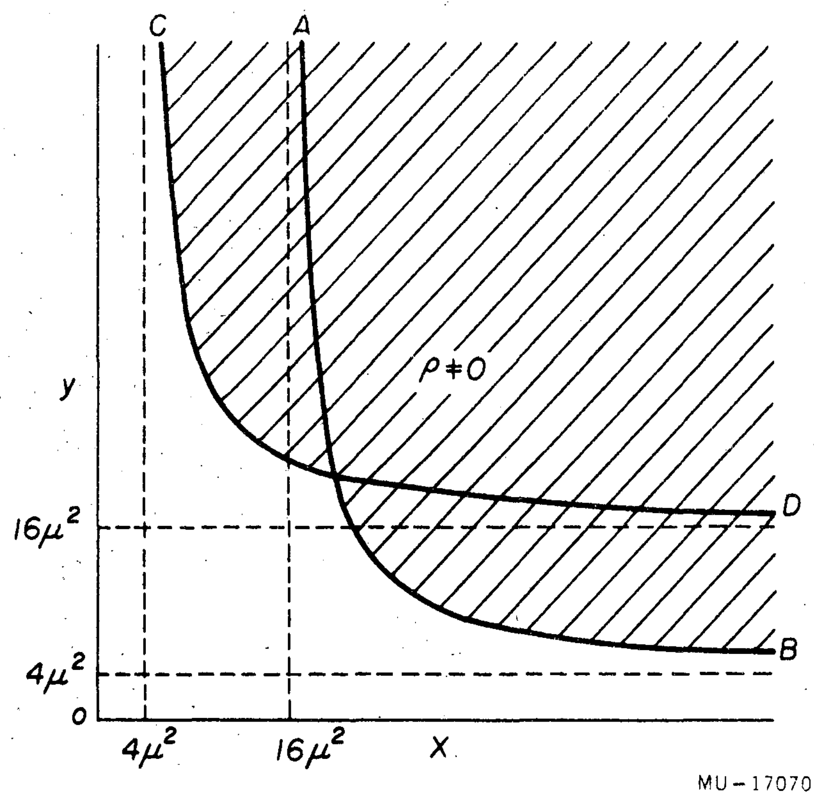

Fig. 2

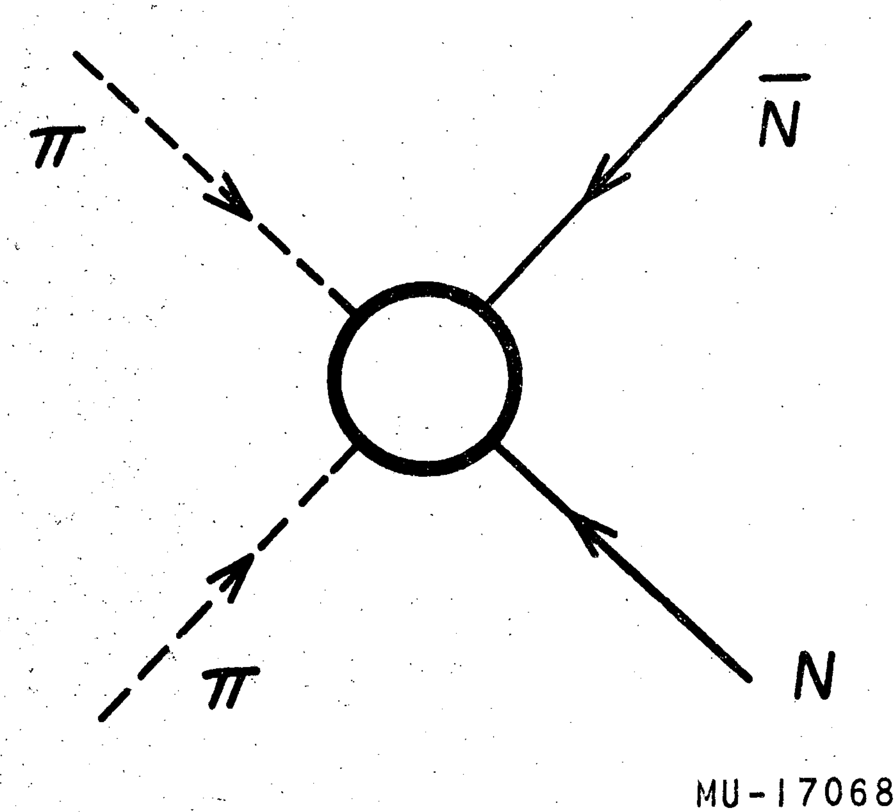

Fig. 3