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ABSTRACT

It is shown that when P-wave pion-pion scattering is large at low energies, the integral equations previously formulated by the authors require a cutoff. Because of the cutoff and the unstable nature of the solution, the numerical integration procedure becomes much more involved. The original equations are therefore replaced by a series of conditions at the symmetry point, and the unphysical cuts of the partial-wave amplitudes are replaced by a corresponding series of poles. Within this framework one need not speak of a cutoff, but one new parameter appears. Self-consistent solutions can be found in which a P-wave resonance is sustained by a "bootstrap" mechanism; that is, a strong attractive force in the $I = 1$ state results from the exchange of a resonating pair of P-wave pions. The symmetry-point conditions used would be modified by the cutoff and quantitative accuracy is not attempted; however, this and other corrections are not expected to change the qualitative nature of our solutions. Rough estimates of the corrections are made.

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I. INTRODUCTION

In an earlier paper,[1] hereafter to be referred to as CM-I, a single-parameter set of integral equations for the low-energy pion-pion scattering amplitudes has been derived. These equations satisfy the requirements of analyticity and crossing symmetry but are based on the assumption that the imaginary part of the amplitude is adequately represented by keeping elastic S and P waves only. It has been shown by explicit calculation[2] that there exists a class of solutions of these equations consistent with this assumption. These solutions, however, have very small P phase shifts, whereas the only information available so far about the π-π interaction suggests that there is a low-energy P resonance.[3] In this paper we examine the problem further and show that if the P phase shift is large at low energies, the original assumption about the imaginary part is inconsistent; the CM equations require modification and a new parameter appears.

It is possible to make the necessary modification through a cutoff of the partial-wave imaginary parts on the left-hand (unphysical) cut. Actually, three cutoffs would be needed, for the I = 0 and I = 2 S states

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as well as for the P state, but crossing symmetry may be used to correlate the three in terms of a single real parameter. It will be made plausible that in a correct calculation, where inelastic processes and the imaginary parts of higher partial waves are included, unitarity will make any new parameters unnecessary.

At the present level of approximation it will be argued that the cutoffs of the imaginary parts on the left-hand cut probably occur for values of $\omega = -q^2$ substantially greater than 9, the mathematical limit of convergence of the polynomial expansion. It will be shown that in such a case a strong intermediate-range attractive force capable of producing a P-wave resonance occurs in the I = 1 state. The force is due to the exchange of a pair of P-wave pions resonating in transit (or in other words to the left-cut contribution obtained by crossing relations from the P-wave absorptive part on the right), so we have a "bootstrap" mechanism. The corresponding strong intermediate range force in the I = 0 state is repulsive and in the I = 2 case attractive.

We assume that the short-range contributions from D and higher waves, except for their role in producing the cutoff, may be absorbed into the parameter λ already introduced in CM-I. That is, we replace them, together with all other exchange mechanisms of high energy, by a phenomenological zero-range force. The consistency of such an approach can be investigated a posteriori by calculating the higher angular-momentum contributions to the force once the S and P phase shifts have been determined. Rough estimates of this kind are reported below.

In Section II, the necessity for the cutoff, when the P wave is "bootstrapping" itself, will be demonstrated and the relation of the cutoff

to our approximations is discussed. Section III treats certain exact and almost exact crossing conditions that put powerful restrictions on the scattering amplitudes. In Section IV further conditions at the symmetry point, corresponding to the basic approximation of CM-I, are developed and used in the replacement of left-hand discontinuities by a small number of delta functions. The straightforward iteration procedure previously employed does not converge when the P amplitude is large because the "bootstrap" character of the mechanism makes the solution very unstable. Section V tests the new method on the "known" S-dominant problem, while Section VI deals with the P-dominant situation. It is shown that a reasonable choice of the new parameter leads to a P resonance with a position and width roughly that required by the electromagnetic structure of the nucleon. Our results are not quantitatively accurate, because the crossing conditions at the symmetry point should be modified by the cutoff, and also because we neglect the S waves in the crossing relations. Corrections to the results obtained are discussed in Section VII, and future calculations are outlined.

II. PROPERTIES OF THE INTEGRAL EQUATIONS WHEN THERE IS STRONG P-WAVE SCATTERING

To derive the integral equations in CM-I, the partial-wave amplitudes were written in the form

$$\frac{1}{\nu^{\ell}} \; A^{(\ell)I}(\nu) \; = \; \frac{N_{\ell}^{I}(\nu)}{D_{\ell}^{I}(\nu)} \; , \qquad \text{(II.1)}$$

where N has a cut along the negative real axis and D along the positive real axis only. The index ℓ denotes the angular momentum and I the isotopic spin. On defining $\omega = -\nu$, $E_{\ell}^{I}(\omega) = D_{\ell}^{I}(\nu)$, we found the following integral equations [CM-I, Eqs. (V.14) and V.26)]:

$$E_0^I(\omega) = 1 + (\omega + \nu_0)K(\omega, -\nu_0)a_I + \frac{\omega + \nu_0}{\pi} \int_1^\infty d\omega' \frac{K(\omega, \omega') f_0^I(\omega') E_0^I(\omega')}{\omega' + \nu_0} \tag{II.2}$$

and

$$E_1^I(\omega) = 1 + \frac{\omega}{\pi} \int_1^\infty d\omega' \frac{K(\omega, \omega') f_1^1(\omega') E_1(\omega')}{\omega'} , \tag{II.3}$$

where

$$K(\omega, \omega') = \frac{2}{\pi(\omega - \omega')} \left\{ \sqrt{\frac{\omega}{\omega - 1}} \ln \left[\sqrt{\omega} + \sqrt{\omega - 1} \right] - \sqrt{\frac{\omega'}{\omega' - 1}} \ln \left[\sqrt{\omega'} + \sqrt{\omega' - 1} \right] \right. . \tag{II.4}$$

In these equations, a_I is the S-wave amplitude for isotopic spin I at $\nu = \nu_0$, and f_ℓ^I the discontinuity across the cut of $A^{(\ell)I}(\nu)$ for $-\omega = \nu < 0$.

The f's are not known explicitly, but must be calculated by crossing symmetry from the absorptive parts for positive energies. In CM-I, we denoted the absorptive parts of the complete amplitude by $A_s^I(\nu, \cos\theta)$ and wrote the necessary equations in the form

$$f_\ell^I(\omega) = -\frac{1}{\omega} \int_0^{\omega-1} d\nu' \, P_\ell\left(1 - 2\frac{\nu' + 1}{\omega}\right) \sum_{I'=0,1,2} \alpha_{II'} A_s^{I'}(\nu', 1 - 2\frac{\omega - 1}{\nu'}) , \tag{II.5}$$

$$(\omega > 1)$$

where the crossing matrix $\alpha_{II'}$ is

$$\alpha_{II'} = \begin{pmatrix} 2/3 & 2 & 10/3 \\ 2/3 & 1 & -5/3 \\ 2/3 & -1 & 1/3 \end{pmatrix} . \qquad \text{(II.6)}$$

If $A_s^{I'}$ in Eq. (II.5) is resolved into partial waves and the expansion cut off after the P-wave, the formula becomes

$$f_\ell^I(\omega) = -\frac{1}{\omega} \int_0^{\omega-1} d\nu' \; P_\ell\left(1 - 2\frac{\nu'+1}{\omega}\right) \left\{ \alpha_{I0} \operatorname{Im} A^{(0)0}(\nu') + \alpha_{I2} \operatorname{Im} A^{(0)2}(\nu') + 3\left(1 - 2\frac{\omega-1}{\nu'}\right) \alpha_{I1} \operatorname{Im} A^{(1)1}(\nu') \right\} . \qquad \text{(II.7)}$$

The right-hand side of Eq. (II.7) involves the imaginary parts of the partial-wave amplitudes for positive energies, which are given by the simple formula,

$$\operatorname{Im} A^{(\ell)I}(\nu) = \sqrt{\frac{\nu+1}{\nu}} \sin^2 \delta_\ell^I . \qquad \text{(II.8)}$$

The phase shifts, δ_ℓ^I, can be calculated from the functions E by the formulae CM-I (V.20) and (V.26), so we have a self-consistency problem; the functions E are determined in terms of the f's by Eqs. (II.2) and (II.3), while the f's are determined in terms of the E's from crossing by Eqs. (II.6) to (II.8).

In our previous calculations an iteration procedure was used to obtain self-consistency. The convergence was rapid, and the solutions had the property that the P-wave amplitude was extremely small. The reason

was that the terms $\mathrm{Im}\, A^{(0)0}(\nu')$ and $\mathrm{Im}\, A^{(0)2}(\nu')$ in Eq. (II.7) are bounded by unitarity and, further, have opposite signs for I = 1 so that $f_\ell{}^1(\omega)$ cannot become large. According to CM-I (V.26), the P-wave amplitude then remains small, and its smallness in turn means that the third term on the right of Eq. (II.7) is small.

The question now arises whether there are any other solutions of our equations. An immediate possibility is a solution dominated by the P wave, in which $f_1{}^1(\omega)$ is large and receives its main contribution from the third term of Eq. (II.7), i.e., from the imaginary part of the P wave itself. Owing to the large numerical factor multiplying the third term, the unitarity limitation no longer makes $f_1{}^1(\omega)$ small. To determine the sign of the P-wave phase shift in such a solution, we notice that $f_1{}^1(\omega)$ according to Eq. (II.7) is negative for small ω and positive for large ω, and that, as may easily be verified, the positive part always predominates in the sense that the dispersion integral

$$\frac{1}{\pi} \int_1^\infty d\omega' \, \frac{f_1(\omega')}{\omega'(\omega' + \nu)}$$

is positive. According to our equations, it then follows that the phase shift is positive. We are led to the possibility of a P-wave resonance, which Frazer and Fulco[3] require in order to bring the calculations on nucleon electromagnetic structure into agreement with experiment. Rough examination indicates that we can achieve self-consistency in our equations with such a resonance.

The type of solution suggested here would exist even without any coupling to the S waves, though of course it becomes modified by such coupling. For each value of the constant λ, there are two solutions--

one of the type discussed previously with small P waves, and one of the type under consideration here with large P waves. One might raise the objection that, if λ is interpreted as a coupling constant, it would be expected to define uniquely the solution. However, it must be borne in mind that we are dealing with a renormalized coupling constant whose definition is largely a matter of convenience. There seems to be no reason why such a quantity cannot be the same for two different solutions of our equations. To put it another way, it would be quite possible that two different values of the unrenormalized coupling constant--if such a quantity had a meaning--should give the same value of the renormalized coupling constant. The question as to which of the two solutions is actually realized in nature is on the same level as the question of the value of the coupling constant, and at present must be determined by experiment.

The qualitative nature of the solution with large P-wave phase shifts is encouraging from the point of view of the nucleon electromagnetic structure. The fact that $f_1^1(\omega)$ is not positive over its entire range but is negative if ω is sufficiently small has the effect of considerably narrowing the resonance. This property follows from Eq. (II.3), or it may be seen by expressing the problem in more conventional language. The potential corresponding to an $f_1^1(\omega)$ of our form has a repulsive outer part and an attractive inner part, and it is hardly necessary to remark that such a potential favors a narrow resonance. Now Frazer and Fulco have shown that a resonance sufficiently narrow to explain the electromagnetic structure cannot be obtained with a purely positive $f_1(\omega)$, corresponding to a purely attractive force, without making the predominant values of ω unreasonably high--at least 150 and, for a good fit, nearer 600.[3] If f_1 changes sign in the manner described, a narrow resonance can be obtained without going

to such high values of ω. This class of solutions to the pion-pion problem seems therefore to be qualitatively just what is required to fit the electromagnetic structure data.

There are unfortunately two difficulties that must be overcome before we can obtain a solution of the type described in the foregoing paragraphs. The first is purely practical in nature; the iteration procedure used previously does not converge now, as mentioned in the introduction. We shall have to use some trial-and-error procedure to obtain a consistent solution, and the numerical work is therefore considerably increased. The second difficulty is one of principle. We have pointed out in CM-I that the integral Eqs. (II.2) and (II.3) above become singular if f_0^I of f_1^1 approach a constant value with infinite ω. It follows from Eq. (II.7) that the contribution to f_ℓ^I from the third term does approach a constant, even if only a finite range of values of ν' is taken. The behavior of f_ℓ^I at infinity is therefore just bad enough to make our integral equations singular, and in such a case the integral equation usually has a unique solution if and only if the coefficient of the singular term is sufficiently small.

By replacing our integral equation by one with the same asymptotic behavior but which is exactly soluble, we can show that a unique solution exists provided that the limit of $f_1(\omega)$ as ω becomes infinite is less than unity. (A negative limit never gives trouble.) If the function $\mathrm{Im}\, A^{(1)1}(\nu')$ on the right of Eq. (II.7) is obtained from a solution that has approximately the characteristics required by Frazer and Fulco,[3] the limit of $f_1(\omega)$ is found to be considerably greater than unity--of the order of magnitude of six. We are therefore well within the range where the equation does not have a unique solution.

The situation here is precisely analogous to that occurring in the relativistic scattering by a potential with a $1/r$ behavior at the origin. Again we find a unique solution in the attractive case only if the coefficient of the singularity is sufficiently small (less than $\hbar c$). In the repulsive case, which corresponds to a negative $f_1^1(\omega)$, there is of course no trouble.

One may easily see the reason for the limit 1 on $f_1^1(\omega)$ at infinity by considering the dispersion relation for $A^{(1)1}(\nu)$:

$$\frac{1}{\nu} A^{(1)1}(\nu) = \frac{1}{\pi} \int_1^\infty d\omega' \frac{f_1^1(\omega')}{\omega'(\omega' + \nu)} + \frac{1}{\pi} \int_0^\infty d\nu' \frac{\operatorname{Im} A_1^{(1)1}(\nu')}{\nu'(\nu' - \nu)} . \tag{II.9}$$

If $f_1(\omega')$ approaches the constant c without oscillation as ω' tends to infinity, the first term will behave like $\frac{1}{\pi} \frac{c}{\nu} \log \nu$ as ν tends to infinity. Since $| A^{(1)1}(\nu) |$ is bounded by unitarity and cannot be greater than $\sqrt{\left(\frac{\nu + 1}{\nu} \right)}$ in the physical region, this logarithmic behavior must be cancelled by an opposite logarithmic behavior of the second term. The function $\operatorname{Im} A^{(1)1}(\nu)$ must therefore approach c as ν tends to infinity. However, $| \operatorname{Im} A^{(1)1}(\nu) |$ is of course also less than $\sqrt{\left(\frac{\nu + 1}{\nu} \right)}$, so that c cannot be greater than unity. This argument seems to apply to the repulsive as well as the attractive case, but the presence of "ghosts" in the former complicates the situation and the singularity at infinity does not have any further adverse effect.

The source of the singularity in the integral equation appears to be the use of the Legendre expansion for $A_s^{I'}$ in Eq. (II.5) at all values of ω, whereas we know it to be justified only if ω is less than 9. If we could use the full expression (II.5) and calculate $f_1(\omega)$ correctly at high values of ω, a uniquely soluble equation would result. Our procedure

for calculating $f_1(\omega)$ is, however, not accurate for high ω. At this stage, therefore, there appears to be no alternative to cutting off $f_1(\omega)$ at some point. This cutoff is meant to replace the excluded contributions, which should remove the difficulty in the integral equation. The cutoff is a second parameter (in addition to λ), which seems unavoidable at present. If the calculation could be taken to higher approximations, it should be possible to see the natural cutoff (or high-energy oscillations) appearing, so that the extra parameter would be unnecessary. It represents our lack of knowledge at present of processes at high energies.

An important physical consideration is whether the cutoff may occur at such a low value of ω as to remove the attractive part of the interaction in the $I = 1$ state. To investigate this point, let us calculate $f_1^1(\omega)$ from Eq. (II.7), keeping only the third term on the right and assuming a sharp resonance at $\nu = \nu_R$. The functional form of $f_1^1(\omega)$ is then roughly

$$f_1^1(\omega) \sim \frac{1}{\omega}\left(1 - 2\frac{\nu_R + 1}{\omega}\right)\left(2\frac{\omega - 1}{\nu_R} - 1\right), \qquad \omega > \nu_R + 1$$
$$\sim 0, \qquad \omega < \nu_R + 1, \tag{II.10}$$

which changes sign at $\omega = 2(\nu_R + 1)$, being attractive for larger values of ω and repulsive for smaller values. According to Frazer and Fulco,[3] the position of the resonance should correspond to $\nu_R \lesssim 2$, so the attractive region on the left cut begins at $\omega \lesssim 6$. Therefore, if the cutoff occurs at $\omega \gtrsim 12$ there will be a substantial region of attraction.

Now, the polynomial expansion of $A_s^{I'}$ in Eq. (II.5) formally breaks down at $\omega = 9$, but if there is in fact a P resonance and the higher

partial waves are not of abnormal size, one expects the S and P waves to give a reasonable approximation to the full absorptive part up to somewhat higher values of ω. One may easily, in fact, estimate the D phase shifts produced by the exchange of a resonating P-wave pair and investigate how large ω must be before the D-wave contributions to $A_s^{I'}$ become important. The result suggests that the cutoff will not occur until $\omega \gtrsim 20$.

Another consideration is the influence of inelastic processes, which have been neglected but which certainly will be important at high energies. In CM-I it was estimated that on the right cut the elastic approximation should be adequate for $\nu \lesssim 10$. The crossing relation (II.5) tells us that a value ω on the left corresponds to an "average" value of ν on the right equal to $\frac{1}{2}(\omega - 1)$. Thus a breakdown of the elastic approximation at $\nu \sim 10$ corresponds to a failure of our formulas on the unphysical cut at $\omega \sim 20$. On this score as well, therefore, there is reason for confidence in the intermediate-range attractive force, which is the crucial element in the problem.

III. EXACT AND ALMOST-EXACT CROSSING CONDITIONS AT THE SYMMETRY POINT

With no cutoff, the equations of CM-I satisfy crossing symmetry exactly. We shall lose this feature if cutoffs are introduced in an arbitrary fashion into the different partial-wave amplitudes, so it is desirable to establish in advance certain important consequences of crossing symmetry that can be used as a guide.

The general crossing conditions are given by Eq. (II.5) to (II.7) of CM-I. It was also pointed out there that a singularity free point of maximum symmetry in the $\pi\pi$ problem occurs at $s = t = u = 4/3$ or at

$$\cos\theta = 0 , \quad \nu = \nu_0 = -2/3 .$$

Advantage was taken of the first crossing condition at this point, namely that $A = B = C$, in order to define the $\pi\pi$ coupling constant through CM-I (III.4). An infinite number of further conditions on the derivatives of the amplitudes are also derivable, as we now show.

Consider the condition

$$A(s, t, u) = B(t, s, u) . \tag{III.1}$$

This may also be written in terms of the variables ν and $\cos\theta$ which are connected to s, t, and u through CM-I (II.2). The result is

$$A(\nu, \cos\theta) = B(\nu', \cos\theta') , \tag{III.2}$$

where

$$\nu' = \frac{\nu}{2}(1 + \cos\theta) - (\nu + 1) \tag{III.3}$$

and

$$\cos\theta' = \frac{\frac{\nu}{2}(1 + \cos\theta) + (\nu + 1)}{\frac{\nu}{2}(1 + \cos\theta) - (\nu + 1)} . \tag{III.4}$$

Evidently, at the symmetry point $\nu = \nu' = -2/3$ and $\cos\theta = \cos\theta' = 0$, so we have immediately

$$A(-\tfrac{2}{3}, 0) = B(-\tfrac{2}{3}, 0) , \tag{III.5}$$

or, in view of CM-I (II.8), remembering that

$$C(\nu, \cos\theta) = B(\nu, -\cos\theta) , \tag{III.6}$$

we have

$$\tfrac{1}{5}A^0(-\tfrac{2}{3}, 0) = \tfrac{1}{2}A^2(-\tfrac{2}{3}, 0) = \lambda , \tag{III.7}$$

the result already stated in CM-I (III.5).

Next let us differentiate Eq. (III.2) above with respect to ν and evaluate at the symmetry point. We find

$$\left[\frac{\partial A}{\partial \nu} = -\frac{1}{2} \frac{\partial B}{\partial \nu} - \frac{9}{4} \frac{\partial B}{\partial \cos\theta} \right]_{\substack{\nu = -2/3 \\ \cos\theta = 0}} . \qquad \text{(III.8)}$$

Similarly, by differentiating with respect to $\cos\theta$ we find

$$\left[-\frac{1}{3} \frac{\partial B}{\partial \nu} + \frac{1}{2} \frac{\partial B}{\partial \cos\theta} \right]_{\substack{\nu = -2/3 \\ \cos\theta = 0}} . \qquad \text{(III.9)}$$

Replacing A and B by A_1, A_2, and A_3 through CM-I (II.8), one may then deduce the two symmetry-point conditions,

$$\frac{\partial A^0}{\partial \nu} = 2 \frac{\partial}{\partial \cos\theta} \left(\frac{A^1}{\nu} \right) , \qquad \text{(III.10)}$$

and

$$\frac{\partial A^2}{\partial \nu} = -\frac{\partial}{\partial \cos\theta} \left(\frac{A^1}{\nu} \right) . \qquad \text{(III.11)}$$

There are three second-derivative conditions, corresponding to the operations, $\partial^2/\partial\nu^2$, $\partial^2/\partial \cos^2\theta$, and $\partial^2/\partial \cos\theta \, \partial\nu$ on Eq. (III.1). Remembering that all odd derivatives of A_0 and A_2 with respect to $\cos\theta$ vanish at $\cos\theta = 0$, as do all even derivatives of A_1, we can write these three new symmetry-point conditions as

$$\frac{\partial^2 A^0}{\partial \nu^2} - \frac{5}{2} \frac{\partial^2 A^2}{\partial \nu^2} = -\frac{9}{2} \frac{\partial^2}{\partial \cos\theta \, \partial \nu} \left(\frac{A^1}{\nu} \right) , \qquad \text{(III.12)}$$

$$\frac{\partial^2}{\partial \cos^2\theta} \left(\frac{A^0}{\nu^2} \right) - \frac{5}{2} \frac{\partial^2}{\partial \cos^2\theta} \left(\frac{A^2}{\nu^2} \right) = \frac{3}{2} \frac{\partial^2}{\partial \cos\theta \, \partial \nu} \left(\frac{A^1}{\nu} \right) , \qquad \text{(III.13)}$$

and

$$\frac{\partial^2}{\partial \cos^2\theta}\left(\frac{A^0}{\nu^2}\right) - 7\frac{\partial^2}{\partial \cos^2\theta}\left(\frac{A^2}{\nu^2}\right) = -\frac{\partial^2 A^0}{\partial \nu^2} + \frac{\partial^2 A^2}{\partial \nu^2} \quad . \tag{III.14}$$

Evidently such a procedure can be extended indefinitely, giving an infinite number of conditions on the derivatives of the scattering amplitudes. The conditions written above, however, seem the most interesting for the time being because we shall be concerned principally with S and P waves; higher derivatives give conditions that mainly involve higher ℓ values.

It was pointed out in CM-I that Eq. (III.7) above implies at $\nu = -\frac{2}{3}$ a simple relation between the two S amplitudes that holds to a high degree of accuracy even though it is not exact. So long as D and higher waves are not of abnormal size, the relation is

$$\frac{1}{5}\, a_0 \approx \frac{1}{2}\, a_2 \approx \lambda \quad . \tag{III.15}$$

Similarly, if we define

$$\lambda_1 = \left[\frac{\partial}{\partial \cos\theta}\left(\frac{A^1}{\nu}\right)\right]_{\substack{\nu = -2/3 \\ \cos\theta = 0}} \quad , \tag{III.16}$$

then the conditions (II.10) and (II.11), to a good approximation, become

$$\frac{1}{2}\, a_0' \approx -a_2' \approx \lambda_1 \quad , \tag{III.17}$$

where a_0' and a_2' are the derivatives of the S amplitudes at $\nu = -\frac{2}{3}$.

The very simple conditions (III.15) and (III.17), while not exact, have a higher order of reliability than the other approximations to be made. Because of them, the low-energy S phase shifts are fairly well

determined once the two constants λ and λ_1 are known.

The new constant, λ_1, is closely related to the P amplitude at the symmetry point. Neglecting F and higher waves, we have

$$a_1 = \left(\frac{A^{(1)1}}{\nu}\right)_{\nu=-2/3} \approx \frac{1}{3}\lambda_1 \quad . \tag{III.18}$$

The task of the following sections might be described as that of developing a procedure for calculating a_1', the derivative of the P amplitude at the symmetry point, in terms of λ and λ_1. We shall strive, in other words, for a two-parameter theory, but the symmetry conditions of this section are already sufficient to allow the construction of reasonable S and P effective-range formulas with a <u>total</u> of no more than <u>three</u> arbitrary parameters.

The second-derivative condition (III.12) is more sensitive to the D wave than our first three conditions, but a correction can be made using condition (III.13). We then find

$$a_0'' - \frac{5}{2}a_2'' \approx -12\, a_1' \quad . \tag{III.19}$$

Estimates of the D amplitudes, themselves, are given by conditions (III.13) and (III.14) in terms of a_1', a_0'', and a_2''. It does not seem possible, however, to get conditions on a_0'' and a_2'' separately in terms of a_1'. This circumstance illustrates again the incompleteness of the exact crossing relations if we confine ourself to the symmetry point. Some of the physics certainly lies elsewhere.

IV. APPROXIMATE CONDITIONS AT THE SYMMETRY POINT

A straightforward approach to the large P-wave problem is to set the left-hand partial-wave imaginary parts equal to zero beyond a certain value of ω and to use Eq. (II.7) for smaller values. The integral equations of CM-I are then nonsingular and can be solved without difficulty. Furthermore, as explained in Section II above, we know that the consequence of the higher partial waves is to produce such a cutoff. A complication arises, however, in the necessity for correlating the cutoffs in the three states (I = 0, 1, and 2) so as to satisfy the exact crossing conditions (III.10) and (III.11). These conditions mean that only one new arbitrary parameter occurs, not three. (Higher derivative conditions are relatively insensitive to the cutoffs.) A further difficulty is the instability of the large P-wave problem because of its "bootstrap" aspect. It seems impossible to construct a convergent iteration procedure by the straightforward approach used in the S-dominant problem.

Eventually we hope to solve the cutoff equations by a modified numerical iteration scheme, and progress in this direction is described below. However, an understanding of the essential elements of the problem may be achieved by an analytical approach making maximum use of crossing symmetry at the expense of an accurate handling of certain cutoff effects.

If the functions $f_\ell^I(\omega)$ are approximated by a finite number of delta functions--a procedure that corresponds to replacing the left-hand branch cuts by a series of poles--then the integral Eqs. (II.2) and (II.3) become algebraic and trivially soluble in terms of the residues of the poles. It will be verified below that the S-dominant solutions obtained by numerical integration of the original CM-I equations can be well approximated in this way,[2] and there seems no reason why such an approach

should be less accurate when the P wave is large. The essential question, then, is how to determine the residues and positions of the poles, or in other words the strength and range of the various contributing interactions.

Since the poles are inserted as an approximation to the left-hand branch cuts, the equations used to determine them will be relations between the left and right cuts. From these relations it is possible to obtain sufficient equations to determine the positions and residues of the poles in terms of the parameters λ and λ_1. These equations contain more information than the exact crossing relations at the symmetry point, which we showed in the last section to be insufficient.

To achieve our object, let us first consider not quite the S and P amplitudes but $A^{0,2}$ and $\frac{\partial}{\partial \cos\theta} A^1$ evaluated at $\cos\theta = 0$. From Eqs. (IV.10) and (IV.11) of CM-I, keeping only S and P imaginary parts, we may derive the following formulas, which show the relative contributions to these functions from the right- and left-hand cuts:[4]

$$\mathcal{F}_{0,2}(\nu) = A^{0,2}(\nu, 0) = \begin{pmatrix} -5 \\ -2 \end{pmatrix} \lambda + \frac{\nu - \nu_0}{\pi} \int_0^\infty \frac{d\nu'}{\nu' - \nu} \frac{\mathrm{Im}\, A^{(0)I}(\nu')}{\nu' - \nu_0}$$

$$- \frac{1}{2} \frac{\nu - \nu_0}{\pi} \int_0^\infty \frac{d\nu'}{\nu' - \nu_0 + \frac{1}{2}(\nu - \nu_0)} \left\{ \begin{pmatrix} 2/3 \\ 2/3 \end{pmatrix} \frac{\mathrm{Im}\, A^{(0)0}(\nu')}{\nu' - \nu_0} \right.$$

$$\left. + \begin{pmatrix} 10/3 \\ 1/3 \end{pmatrix} \frac{\mathrm{Im}\, A^{(0)2}(\nu')}{\nu' - \nu_0} + \begin{pmatrix} -2 \\ 1 \end{pmatrix} 9 \frac{\mathrm{Im}\, A^{(1)1}(\nu')}{\nu'} \right\} ; \qquad \text{(IV.1)}$$

$$\mathcal{F}_1(\nu) = \frac{1}{3}\left[\frac{\partial}{\partial \cos\theta}\,\frac{A^1(\nu, \cos\theta)}{\nu}\right]_{\cos\theta = 0} = \frac{1}{\pi}\int_0^\infty d\nu'\,\frac{\operatorname{Im} A^{(1)1}(\nu')}{\nu'(\nu' - \nu)}$$

$$+ \frac{1}{2\pi}\int_0^\infty \frac{d\nu'}{[\nu' - \nu_0 + \frac{1}{2}(\nu - \nu_0)]^2}\left\{\frac{2}{9}\operatorname{Im} A^{(0)0}(\nu')\right.$$

$$\left. - \frac{5}{9}\operatorname{Im} A^{(0)2}(\nu') + \frac{\nu' - \nu_0 + 2(\nu - \nu_0)}{\nu'}\operatorname{Im} A^{(1)1}(\nu')\right\}. \qquad \text{(IV.2)}$$

Evidently, the first integral in each case is the contribution from the right cut and the second integral that from the left.

The importance of these formulas lies in the fact that at the symmetry point the contributions from the left are simply related to those from the right. Defining

$$A_R^{(0)I}(\nu) = \frac{\nu - \nu_0}{\pi}\int_0^\infty \frac{d\nu'}{\nu' - \nu}\,\frac{\operatorname{Im} A^{(0)I}}{\nu' - \nu_0} \qquad \text{(IV.3)}$$

and

$$\frac{A_R^{(1)1}(\nu)}{\nu} = \frac{1}{\pi}\int_0^\infty \frac{d\nu'}{\nu' - \nu}\,\frac{\operatorname{Im} A^{(1)1}(\nu')}{\nu'}, \qquad \text{(IV.4)}$$

where the meaning of the notation is obvious, we see by inspection of Eqs. (IV.1) and (IV.2) that at $\nu = \nu_0$, we have

$$\frac{d}{d\nu}\mathcal{F}_{0,2} = \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix}\frac{d}{d\nu}A_R^{(0)0} + \begin{pmatrix} -5/3 \\ 5/6 \end{pmatrix}\frac{d}{d\nu}A_R^{(0)2} + 9\begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\left(\frac{A_R^{(1)1}}{\nu}\right), \qquad \text{(IV.5)}$$

$$\frac{d^2 \mathcal{F}_{0,2}}{d\nu^2} = \begin{pmatrix} 7/6 \\ \\ 1/6 \end{pmatrix} \frac{d^2}{d\nu^2} A_R^{(0)0} + \begin{pmatrix} 5/6 \\ \\ 13/12 \end{pmatrix} \frac{d^2}{d\nu^2} A_R^{(0)2} - 9 \begin{pmatrix} 1 \\ \\ -1/2 \end{pmatrix} \frac{d}{d\nu} \left(\frac{A_R^{(1)1}}{\nu} \right) , \qquad \text{(IV.6)}$$

$$\mathcal{F}_1 = \frac{1}{9} \frac{d}{d\nu} A_R^{(0)0} - \frac{5}{18} \frac{d}{d\nu} A_R^{(0)2} + \frac{3}{2} \frac{A_R^{(1)1}}{\nu} , \qquad \text{(IV.7)}$$

and

$$\frac{d}{d\nu} \mathcal{F}_1 = -\frac{1}{18} \frac{d^2}{d\nu^2} A_R^{(0)0} + \frac{5}{36} \frac{d^2}{d\nu^2} A_R^{(0)2} + \frac{3}{2} \frac{d}{d\nu} \left(\frac{A_R^{(1)1}}{\nu} \right) . \qquad \text{(IV.8)}$$

There are, in fact, relations of this kind for all derivatives at the symmetry point, but these four will suffice for our purposes. It may easily be verified that they satisfy the exact conditions (III.10) to (III.12) of the preceding section. Of course the above conditions have more content and correspondingly are not exact; the imaginary parts of amplitudes for $\ell > 1$ have been dropped in their derivation. The chief error is associated with the cut-off effect. That is to say, in (IV.1) and (IV.2) we should reduce the contribution from the left branch cuts by an amount that varies inversely with the cutoff. An estimate made below shows that this error is nonnegligible for the expected position of the cutoff; however, the qualitative features of the problem are not changed by disregarding the effect of a cutoff in these formulas.

Accepting the derivative relations (IV.5) to (IV.8), we can calculate the positions and residues of the poles which are to replace the unphysical

cuts of the partial-wave amplitudes. It is easy to establish that at the symmetry point the values of $\mathcal{F}_{0,1,2}(\nu)$, as well as the first two derivatives, are well approximated by the S- and P-wave parts of these functions. One may make the correction in a self-consistent calculation through formulas of the type (V.18) of CM-I, but here we ignore such refinements. If we remember that

$$A^{(0)I}(\nu) = a_I + A_R^{(0)I}(\nu) + A_L^{(0)I}(\nu) \qquad \text{(IV.9)}$$

and

$$\frac{A^{(1)1}(\nu)}{\nu} = \frac{A_R^{(1)1}}{\nu} + \frac{A_L^{(1)1}}{\nu} , \qquad \text{(IV.10)}$$

the symmetry-point conditions (IV.5) to (IV.8) become approximately

$$\frac{d}{d\nu} A_L^{(0)I} = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix} \frac{d}{d\nu} A_R^{(0)0} + \begin{pmatrix} -5/3 \\ -1/6 \end{pmatrix} \frac{d}{d\nu} A_R^{(0)2} + 9 \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} \left(\frac{A_R^{(1)1}}{\nu} \right) , \qquad \text{(IV.11)}$$

$$\frac{d^2}{d\nu^2} A_L^{(0)I} = \begin{pmatrix} 1/6 \\ 1/6 \end{pmatrix} \frac{d^2}{d\nu^2} A_R^{(0)0} + \begin{pmatrix} 5/6 \\ 1/12 \end{pmatrix} \frac{d^2}{d\nu^2} A_R^{(0)2} + 9 \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} \frac{d}{d\nu} \left(\frac{A_R^{(1)1}}{\nu} \right) , \qquad \text{(IV.12)}$$

$$\frac{A_L^{(1)1}}{\nu} = \frac{1}{9}\frac{d}{d\nu}A_R^{(0)0} - \frac{5}{18}\frac{d}{d\nu}A_R^{(0)2} + \frac{1}{2}\frac{A_R^{(1)1}}{\nu}, \tag{IV.13}$$

and

$$\frac{d}{d\nu}\left(\frac{A_L^{(1)1}}{\nu}\right) = -\frac{1}{18}\frac{d^2}{d\nu^2}A_R^{(0)0} + \frac{5}{36}\frac{d^2}{d\nu^2}A_R^{(0)2} + \frac{1}{2}\frac{d}{d\nu}\left(\frac{A_R^{(1)1}}{\nu}\right). \tag{IV.14}$$

The relations (IV.11) to (IV.14) could also have been derived from the fundamental crossing formula (V.8) of CM-I (or (II.7) above). Had we done so, the effect of a cutoff clearly would have been to reduce the coefficient of $A_R^{(1)1}/\nu$ on the right-hand side of relations (IV.11) and (IV.13). (At the same time the change in the higher derivative relations is much less important.) If the sharp resonance form (II.10) for $f_1^1(\omega)$ is cutoff at $\omega = \omega_c$, a simple calculation shows that coefficient of $A_R^{(1)1}/\nu$ in (IV.13) is reduced roughly by a factor $1 - 12\frac{\nu_R}{\omega_c}$. We shall have to consider such a reduction factor when evaluating the validity of results based on the above formulas.

V. THE S-WAVE PROBLEM BY THE POLE APPROXIMATION

These formulas, (IV.11) to (IV.14), for the contribution from the left cuts of the partial-wave amplitudes tell us what we need to know about the equivalent poles. Consider first the S waves, where we attempt to represent the left cuts in each case by a single pole:

$$A_L^{(0)I}(\nu) = +b_I(\nu - \nu_0)\frac{\omega_{SI} + \nu_0}{\omega_{SI} + \nu}. \tag{V.1}$$

By the use of Eqs. (IV.11) and (IV.12), the values of b_I and ω_{SI} may be determined, since at the symmetry point we have

$$b_I = \frac{d}{d\nu} A_L^{(0)I} , \tag{V.2}$$

and

$$\frac{b_I}{\omega_{SI} + \nu_0} = -\frac{1}{2} \frac{d^2}{d\nu^2} A_L^{(0)I} . \tag{V.3}$$

It may be seen from Eq. (IV.11)[5] that b_2 is always negative, corresponding to attractive forces in the $I = 2$ state, but the sign of b_0 depends on the relative magnitudes of S- and P-wave scattering. The force in the $I = 0$ state due to S-pair exchange is attractive, but that due to P-pairs is repulsive.

Using the N/D technique of CM-I, we may immediately write down the S-wave amplitudes corresponding to Eq. (V.1). Equations (V.11) and (V.14) of CM-I then become

$$N_0^I(\nu) = a_I + (\nu - \nu_0) \frac{\omega_{SI} + \nu_0}{\omega_{SI} + \nu} B_I , \tag{V.4}$$

and

$$E_0^I(\omega) = 1 + (\omega + \nu_0) \left\{ K(\omega, -\nu_0) a_I + (\omega_{SI} + \nu_0) K(\omega_{SI}, \omega) B_I \right\} , \tag{V.5}$$

where

$$B_I = b_I E_0^I(\omega_{SI}) . \tag{V.6}$$

To implement the conditions (V.2) and (V.3), we first remember Eq. (IV.9) which allows us to write

$$\frac{d^n A^{(0)I}}{d\nu^n} = \frac{d^n}{d\nu^n} A_R^{(0)I} + \frac{d^n}{d\nu^n} A_L^{(0)I} \quad . \tag{V.7}$$

In particular, at the symmetry point we have

$$\frac{d A^{(0)I}}{d\nu} = \frac{d A_R^{(0)I}}{d\nu} + b_I \quad , \tag{V.8}$$

so that from Eq. (IV.11),

$$\frac{d}{d\nu} A^{(0)0} = -2 \frac{d}{d\nu} A^{(0)2} = \frac{4}{3} b_0 - \frac{10}{3} b_2 - 18 \left(\frac{A_R^{(1)1}}{\nu} \right) \quad . \tag{V.9}$$

From Eqs. (V.4) and (V.5) we can calculate the derivative of the S amplitudes at the symmetry point:

$$\frac{d}{d\nu} A^{(0)I} = B_I + a_I \left\{ K(-\nu_0, -\nu_0) a_I + (\omega_{SI} + \nu_0) K(\omega_{SI}, -\nu_0) B_I \right. \quad . \tag{V.10}$$

Equations (V.9) and (V.10), together with (V.6), determine b_0 and b_2 once ω_{SI} and $A_R^{(1)1}/\nu$ are given.

To establish the positions of the poles, we must consider the second derivative of the amplitudes. Again at the symmetry point, from Eq. (V.7), we have

$$\frac{d^2 A^{(0)I}}{d\nu^2} = \frac{d^2 A_R^{(0)I}}{d\nu^2} - \frac{2 b_I}{\omega_{SI} + \nu_0} \quad , \tag{V.11}$$

so that from Eq. (V.3), we can write

$$\frac{d^2 A^{(0)I}}{d\nu^2} = - \begin{pmatrix} 2/3 \\ 8/3 \end{pmatrix} \frac{b_0}{\omega_{S0} + \nu_0} + \begin{pmatrix} -40/3 \\ 2/3 \end{pmatrix} \frac{b_2}{\omega_{S2} + \nu_0} - \begin{pmatrix} 1 \\ -1/2 \end{pmatrix} 54 \frac{d}{d\nu} \left(\frac{A_R^{(1)1}}{\nu} \right) . \tag{V.12}$$

The second derivative of the S amplitudes may be calculated from Eqs. (V.4) and (V.5), so we have enough conditions to determine ω_{S0} and ω_{S2} if $A_R^{(1)1}/\nu$ and its derivative are given.

In general the problem is one of self-consistency, involving the P wave. Let us consider first, however, the situation when the low-energy P phase-shift is so small that $A_R^{(1)1}$ and its derivatives may be set equal to zero. Such is the case for the solutions determined in reference 2 by numerical integration and iteration of the CM-I equations. One possible procedure for determining b_0, b_2, ω_{S0} and ω_{S2} in this simple case will now be described.

The first leg of Eq. (V.9), together with Eq. (V.10), gives a linear relation between B_0 and B_2,

$$\rho_0 B_0 + \xi_0 + 2(\rho_2 B_2 + \xi_2) = 0 , \tag{V.13}$$

where

$$\xi_I = a_I^2 K(2/3, \quad 2/3) , \tag{V.14}$$

and

$$\rho_I = 1 + a_I(\omega_{SI} - 2/3)\, K(\omega_{SI},\ 2/3) \quad . \tag{V.15}$$

The second leg of Eq. (V.9), together with Eq. (V.6), yields a quadratic relation,

$$\rho_0 B_0 + \xi_0 = \frac{4}{3} \frac{B_0}{\rho_0 + \ell_0 B_0} - \frac{10}{3} \frac{B_2}{\rho_2 + \ell_2 B_2} \quad , \tag{V.16}$$

where

$$\ell_I = (\omega_{SI} - 2/3)^2 K(\omega_{SI},\ \omega_{SI}) \quad . \tag{V.17}$$

Taken together, Eqs. (V.16) and (V.13) correspond to a cubic equation for either B_0 or B_2 separately, which can easily be solved once a_I and ω_{SI} are given. It turns out that there is only one real root of the equation.

The relations (V.12) are transcendental, so our procedure is to guess ω_{S0} and ω_{S2}, solve Eq. (V.13) and (V.16) for B_0 and B_2, and then check to see how badly Eq. (V.12) is violated. The two pole positions are then adjusted until Eq. (V.12) is finally satisfied. For small values of λ, it can easily be shown that

$$\omega_{S0} - 2/3 = \omega_{S2} - 2/3 = 2 \frac{\left[- \frac{d}{d\omega} K\left(\frac{2}{3}, \omega \right) \right]_{\omega = 2/3}}{K(2/3,\ 2/3)} = 3.43 \quad ,$$

or

$$\omega_{S0} = \omega_{S2} = 4.1 \quad .$$

As λ increases in the negative (attractive) direction, both ω_{S0} and ω_{S2} decrease, the latter slightly faster than the former. However, at the time the $I = 0$ bound state is reached ($\lambda = -0.46$), the two poles are still quite close together. For example, at $\lambda = -0.433$ we find $\omega_{S0} = 2.5$

and $\omega_{S2} = 2.7$. At the same time, we have $b_0 = 0.42$ and $b_2 = 0.27$. In Fig. 1 is shown, for this value of λ, a comparison between the pole approximation and the numerical solution obtained in reference 2. One observes that there are no important differences.

The value of λ we have chosen for our example, -0.433, is barely small enough not to give a bound state, and the error in the pole approximation is consequently a maximum. This circumstance occurs not only because the relative importance of the left-hand cut is greater for greater magnitudes of λ, but also because the dominant part of the right-hand cut is now at the lowest possible energy. It follows by crossing symmetry that the "average position" of the left-hand cut will be close to the low-energy physical region and the error in replacing it by a pole will be relatively large.

VI. THE P-DOMINANT PROBLEM IN THE POLE APPROXIMATION

Our task now is to repeat the approach of the preceding section when the low-energy P-wave phase shift is allowed to become large. The coupled S-P problem is quite complicated, so we begin by assuming that the contribution of S-pair exchange to the left cut of the P wave is small and that the main force in the P state comes from the exchange of a P-wave pair. In other words, we neglect the contribution to Eq. (IV.13) from the S-wave terms on the right. Such an approximation is not quantitatively reliable, but it serves to show certain essential features of the large P-wave situation. In any case, an accurate calculation must include a cutoff correction as well as the S wave.

The symmetry-point conditions (IV.13) and (IV.14) become, in this approximation,

$$\frac{A_L^{(1)1}}{\nu} = \frac{1}{2}\,\frac{A_R^{(1)1}}{\nu} , \qquad \text{(VI.1)}$$

and

$$\frac{d}{d\nu}\left(\frac{A_L^{(1)1}}{\nu}\right) = \frac{1}{2}\,\frac{d}{d\nu}\left(\frac{A_R^{(1)1}}{\nu}\right) , \qquad \text{(VI.2)}$$

and we may add the corresponding second-derivative condition,

$$\frac{d^2}{d\nu^2}\left(\frac{A_L^{(1)1}}{\nu}\right) = -\frac{5}{8}\,\frac{d^2}{d\nu^2}\left(\frac{A_R^{(1)1}}{\nu}\right) \qquad \text{(VI.3)}$$

which is easily obtained from Eq. (IV.2). Now, the fact that both $A_L^{(1)1}/\nu$ and its first derivative at the symmetry point are positive means that a single pole cannot represent the left cut,[5] since a function of the type

$$\frac{c}{\omega_p + \nu}$$

has the opposite sign to its first derivative so long as $\omega_p + \nu$ is positive. We shall therefore need two poles, the outer one attractive and the inner one repulsive in order to satisfy conditions (VI.1) and (VI.2). This circumstance could have been anticipated from formula (II.10).

We write, then,

$$\frac{A_L^{(1)1}}{\nu} = c_A\,\frac{\omega_A + \nu_0}{\omega_A + \nu} - c_R\,\frac{\omega_R + \nu_0}{\omega_R + \nu} , \qquad \text{(VI.4)}$$

where the subscripts A and R stand for "attractive" and "repulsive," respectively. Here c_A and c_R are defined so as to be both positive, and the normalization is such that at $\nu = \nu_0$, we have

$$\frac{A_L^{(1)1}}{\nu} = c_A - c_R \ , \tag{VI.5}$$

$$\frac{d}{d\nu}\left(\frac{A_L^{(1)1}}{\nu}\right) = \frac{c_R}{\omega_R + \nu_0} - \frac{c_A}{\omega_A + \nu_0} \ , \tag{VI.6}$$

and

$$-\frac{1}{2}\frac{d^2}{d\nu^2}\left(\frac{A_L^{(1)1}}{\nu}\right) = \frac{c_R}{(\omega_R + \nu_0)^2} - \frac{c_A}{(\omega_A + \nu_0)^2} \ . \tag{VI.7}$$

Observe now that the repulsive pole will dominate both in Eqs. (VI.6) and (VI.7) if ω_R is substantially smaller than ω_A, so there is effectively only the one condition (VI.5) restricting the attractive-pole position and residue, and one parameter remains arbitrary.[6]

This free parameter corresponds to the cutoff that would have to be introduced in (II.7) if we were to attempt to solve the equations of CM-I. In our pole approach here we do not speak of a cutoff, but the position of the outer (attractive) pole is a closely related concept. Actually it is more convenient to introduce the free parameter as the value of λ_1, defined by Eq. (III.16), a procedure that through Eq. (III.18) amounts to specifying $c_A - c_R$. It should be noted that λ_1 will not have an unlimited range of possible values. For example, in the approximation (VI.1), we have

$$\frac{1}{3}\lambda_1 = \frac{3}{2}\frac{A_R^{(1)1}}{\nu} \ ,$$

so λ_1 is necessarily positive. (When S waves and the cut-off correction are included, small negative values for λ_1 may become possible.)

Corresponding to Eq. (VI.4) we find, from Eq. (V.24) and (V.26) of CM-I,

$$N_1(\nu) = C_A \frac{\omega_A + \nu_0}{\omega_A + \nu} - C_R \frac{\omega_R + \nu_0}{\omega_R + \nu} , \tag{VI.8}$$

and

$$E_1(\omega) = 1 + (\omega + \nu_0) \left\{ C_A [\omega_A K(\omega_A, \omega) + \nu_0 K(\omega_A, -\nu_0)] - C_R[\omega_R K(\omega_R, \omega) + \nu_0 K(\omega_R, -\nu_0)] \right\} , \tag{VI.9}$$

where

$$C_A = c_A E_1(\omega_A) ,$$

and $$C_R = c_R E_1(\omega_R) . \tag{VI.10}$$

By using Eq. (IV.10) at the symmetry point, it is then a straightforward if tedious calculation to find values of C_A, C_R, ω_A, and ω_R that satisfy conditions (IV.1) to (IV.3) for various choices of λ_1.

Carrying out this program, we found that as λ_1 varies from zero to unity, the position ω_R of the repulsive pole moves only slightly --from 5 to about 3.5--while the attractive pole moves from infinity down to about 8. It was decided then, for simplicity, to fix the position of the repulsive pole at 4.0 and to ignore the second-derivative condition (VI.3). The maximum violation of the second derivative condition is never worse than about 20% under these circumstances, and in Fig. 2 the insensitivity of the solution to the position of the repulsive pole is demonstrated. The two functions shown correspond to the same value of λ_1 (0.84),

and both satisfy the conditions (VI.1) and (VI.2). In one case, however, the pole positions are $\omega_A = 10$, $\omega_R = 4$, while in the other they are $\omega_A = 7$, $\omega_R = 6$. In the first instance the violation of the second-derivative condition (VI.3) is 18%, while in the second it is 40%; nevertheless, the difference in the physical region is negligible. This insensitivity shows that the detailed form of the left-hand discontinuity is unimportant so long as the symmetry-point conditions on the function and its first derivative are satisfied! Thus the replacement of the left-cut by poles seems justified. Unfortunately, the main error in our approach stems from a cut-off modification of the symmetry-point condition (VI.1).

In Fig. 3 are shown P-wave solutions with $\omega_R = 4$, satisfying (VI.1) and (VI.2) for three different values of λ_1. One sees that with ω_A in the anticipated range, these are of a resonance character, although the phase shift never actually passes through 90 deg. The resonance position is satisfactory for $\lambda_1 \sim 1$, but the width is about twice that implied by nucleon electromagnetic structure, according to the calculations of Frazer and Fulco.[3] However, the presence of the repulsive pole has been tremendously effective in narrowing the resonance. Without it, we would not even approach the required width, and when the repulsion is augmented by S-wave contributions and the cut-off correction included, it may be possible to achieve the desired additional narrowing through reduction of λ_1.

The cut-off correction to Eq. (VI.1) produces a substantial narrowing of the resonance. For example, the uncoupled P-wave equations of CM-I with a cutoff have been solved numerically for $\lambda_1 = 0.53$, with the result shown in Fig. 4. Here the phase shift actually passes through 90 deg. and the resonance is about 40% narrower than in the family of

solutions shown in Fig. 3. The method of solution was to use as a trial function a two-pole P amplitude of the form (VI.8) and (VI.9), but with C_A, C_R, ω_A, and ω_R completely arbitrary. These four parameters were then varied until one complete cycle of the CM-I equations (with a cutoff and no S wave) approximately reproduced the trial function in the physical region. No attempt was made to get the best possible reproduction. Figure 4 shows the pole function for $C_A = 0.38$, $C_R = 0.20$, $\omega_A = 25$, and $\omega_R = 4$ compared to the first iteration of the uncoupled P equations, with the cutoff at $\omega_c = 44$. Also shown is a solution satisfying conditions (IV.1) and (VI.2) for a comparable value of λ_1.

One may ask how badly our solution of the cut-off equations violates the condition (VI.1). The answer is in rough agreement with the estimate of Section IV that a cut-off correction factor of $\sim (1 - 12 \frac{\nu_R}{\omega_c})$ should be applied to the right-hand side of condition (VI.1). In the example of Fig. 4, we have $\omega_c = 44$ and $\nu_R \sim 3$. It is not surprising then to find that for our solution we have $A_L^{(1)1} \approx 0$; in other words, the contributions from the attractive and repulsive interactions now just about cancel each other at the symmetry point. One can show from conditions (VI.1), (VI.2), and (VI.3) that, before the cut-off correction, the contribution from the attractive pole is generally about twice that of the repulsive. Thus, the cutoff reduced the attraction by about a factor of two (and of course left the repulsion alone).

If one takes literally the results of Frazer and Fulco,[3] some further narrowing of the P resonance is desirable. This might be accomplished through an increase of our long-range repulsion as a result of S-pair exchange. Our experience with S-dominant solutions showed that the force

due to S-pair exchange cannot by itself produce a P-resonance,[2] but if exchange in the $I = 2$ S state is more important than that for $I = 0$, there will be a net long-range repulsion in the $I = 1$ state than can narrow the P resonance, provided the basic intermediate-range attraction is supplied by some other source. We have shown above that P-pair exchange can provide the necessary intermediate-range attraction as well as some long-range repulsion. It will require further calculation to establish whether values for λ and λ_1 actually can be found that make the $I = 2$ S-pair exchange sufficiently important to give the desired additional repulsion. Such calculations are being undertaken and will be described in a later paper. Qualitatively it seems likely that if P-resonance narrowing requires a preponderance of $I = 2$ over $I = 0$ S pairs, the value of λ will be positive (repulsive). One sees from formula (IV.11) that in this case a strong P-exchange force alters the 2:5 symmetry-point ratio of the $I = 2$ and $I = 0$ amplitudes so as to favor the former in the physical region.

VII. CONCLUSION

The reader may well find confusing the question as to how many independent parameters there are in the $\pi\pi$ problem. On a completely fundamental level, when all other particles and interactions are considered, there may be none; even the pion mass may someday be related to other masses. However, one does not see at present even the outline of a procedure for calculating m_π, and the same statement may be made for the constant λ. This constant is not supposed to be calculable in conventional field theory, and in the S-matrix dispersion-theory approach one correspondingly finds that the combined requirements of Lorentz invariance, analyticity, and unitarity permit one independent real parameter in the π-π interaction.

Further independent parameters are not allowed, but as explained in Section II, our definition of λ is somewhat arbitrary, so it is possible that more than one solution exists for a given λ.

In this paper we have introduced a second parameter, λ_1, which has been treated as independent of λ for the large P-wave type of solution. However, we believe that λ_1 represents parts of the interaction that are at present difficult to calculate, and that eventually it will be possible to determine λ_1 once λ is given. For the S-dominant type of solution, the procedure for calculating λ_1 has already been established.

The question still remains, assuming that λ_1 is a function of λ, as to whether the large P-wave solutions can be reached by a continuous variation of λ, starting with S-dominant solutions. We do not know the answer to this question, and it appears that the answer will not be known until enough of the high-energy contributions can be included to produce a natural cutoff.

On the practical level, a further possible source of confusion arises from the circumstance that for the large P-wave solution, we have here discussed a crude approximation in which the P phase shift is controlled entirely by λ_1, without any specification of λ. Needless to say, such is not the case in general. It is also worth emphasizing that in this second type of solution, the S phase shifts will depend as much on λ_1 as on λ. In a later paper this dependence will be investigated, together with the P phase-shift dependence on λ.

In conclusion we remind the reader that if an as yet undiscovered "elementary" particle exists, with the quantum numbers of a two-pion system and a mass greater than $2m_\pi$ (so that it is unstable), then the $\pi\pi$ scattering amplitude must include further independent parameters associated

with this particle. Our approach could be generalized to accommodate such a situation, but for the moment we may hope that there are no "hidden" elementary particles.

Note added in proof:

Calculations of the coupled S-P problem in the pole approximation have turned out to be less difficult than anticipated and will be described in a supplement to this report.

FOOTNOTES

1. G. F. Chew and S. Mandelstam, Theory of the Low-Energy Pion-Pion Interaction, UCRL-8728, April 1959; Phys. Rev., to be published (1960).

2. G. F. Chew, S. Mandelstam, and H. P. Noyes, S-Wave Dominant Solutions of the Pion-Pion Integral Equations, UCRL-9001, November 1959; Phys. Rev., to be published (1960).

3. W. R. Frazer and J. R. Fulco, Effect of a Pion-Pion Scattering Resonance on Nucleon Structure, UCRL-8880, August 1959; Phys. Rev., to be published (1960).

4. In deriving these formulas, we use the fact that the contribution to the first term of Eqs. (IV.10) and (IV.11) of CM-I from the left-hand cut just cancels the integral over the logarithm in these equations.

5. Note that from Eq. (IV.4), at the symmetry point, we have

$$\frac{d^n}{d\nu^n} A_R^{(0)I} = \frac{n!}{\pi} \int_0^\infty d\nu' \sqrt{\frac{\nu'+1}{\nu'}} \frac{\sin^2 \delta_0^I}{(\nu' + \frac{2}{3})^{n+1}}$$

for $n \geq 1$ and

$$\frac{d^n}{d\nu^n} (A_R^{(1)1}/\nu) = \frac{n!}{\pi} \int_0^\infty d\nu' \sqrt{\frac{\nu'+1}{\nu'^3}} \frac{\sin^2 \delta_1'}{(\nu' + \frac{2}{3})^{n+1}}$$

for $n \geq 0$,

so all "right-hand" functions and derivatives are positive.

6. If higher-derivative conditions were invoked, these would serve only to determine the parameters of new poles that are closer to the physical region than ω_R.

FIGURE LEGENDS

1. The single-pole approximation to the S-dominant solution for $\lambda = 0.433$, compared to the numerical solution of reference 2.
2. Two double-pole, P-wave solutions for $\lambda_1 = 0.84$. Both solutions satisfy conditions (VI.1) and (V.2) but violate (VI.3) by widely varying amounts.
3. Double-pole solutions based on conditions (VI.1) and (VI.2) for three different values of λ_1, with the repulsive-pole position fixed at $\omega_R = 4$.
4. Numerical solution of the uncoupled and cut-off P-wave equations for $\lambda_1 = 0.54$. A double-pole solution based on conditions (VI.1) and (VI.2) for $\lambda_1 = 0.50$ is shown for comparison.

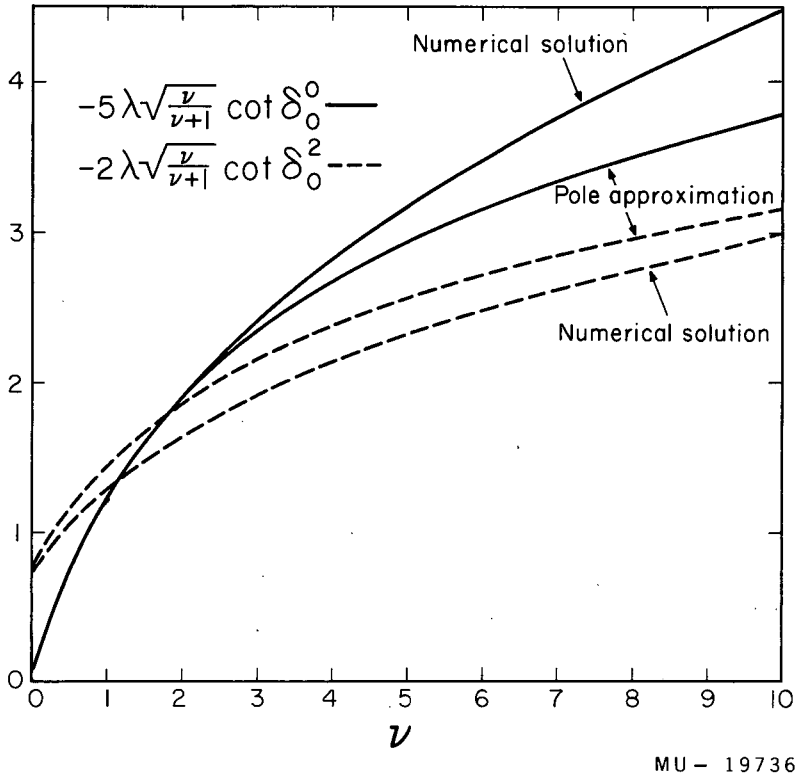
Numerical solution
$-5\lambda\sqrt{\frac{\nu}{\nu+1}}\cot\delta_0^0$
$-2\lambda\sqrt{\frac{\nu}{\nu+1}}\cot\delta_0^2$
Pole approximation
Numerical solution
0
1
2
3
4
5
6
7
8
9
10
ν
MU – 19736

Fig. 1

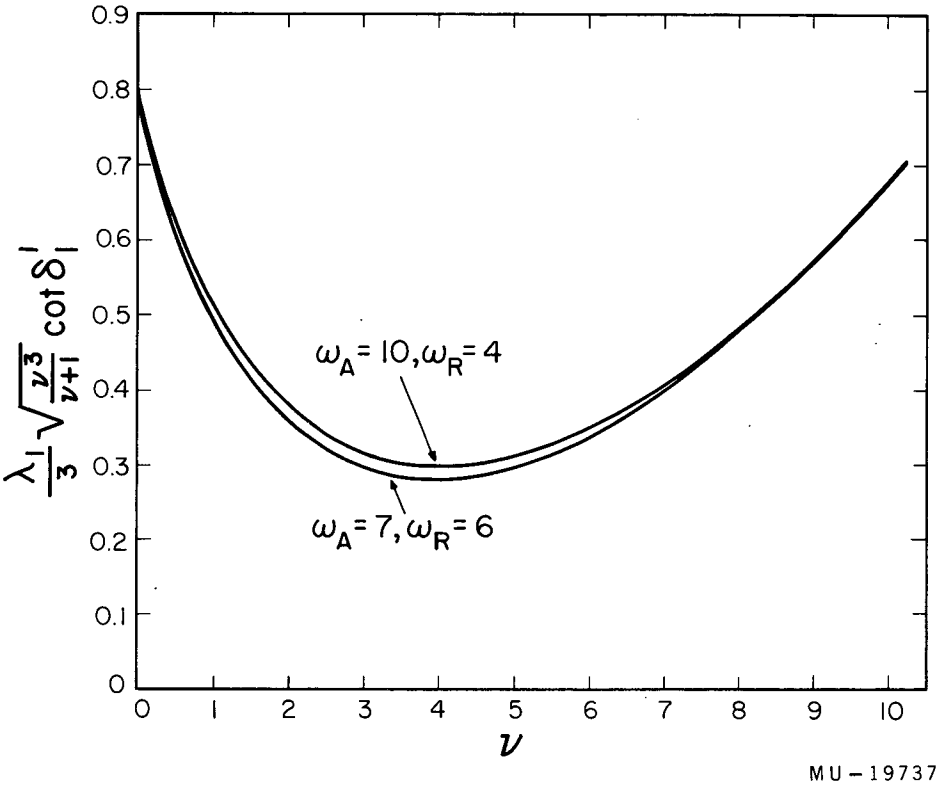
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0 1 2 3 4 5 6 7 8 9 10
ν
$\frac{\lambda_1}{3}\sqrt{\frac{\nu 3}{\nu+1}}\cot\delta_1'$
$\omega_A = 10, \omega_R = 4$
$\omega_A = 7, \omega_R = 6$
MU-19737

Fig. 2

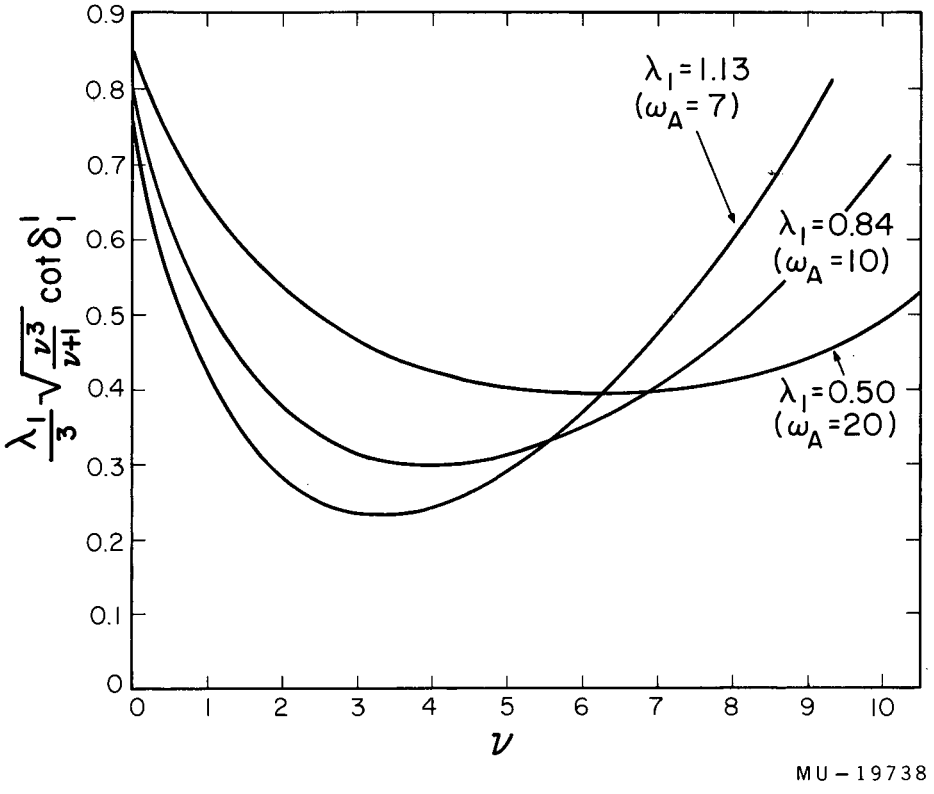
$\frac{\lambda_1}{3}\sqrt{\frac{\nu 3}{\nu+1}}\cot\delta_1'$
ν
$\lambda_1 = 1.13$
$(\omega_A = 7)$
$\lambda_1 = 0.84$
$(\omega_A = 10)$
$\lambda_1 = 0.50$
$(\omega_A = 20)$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
0
1
2
3
4
5
6
7
8
9
10
MU-19738

Fig. 3

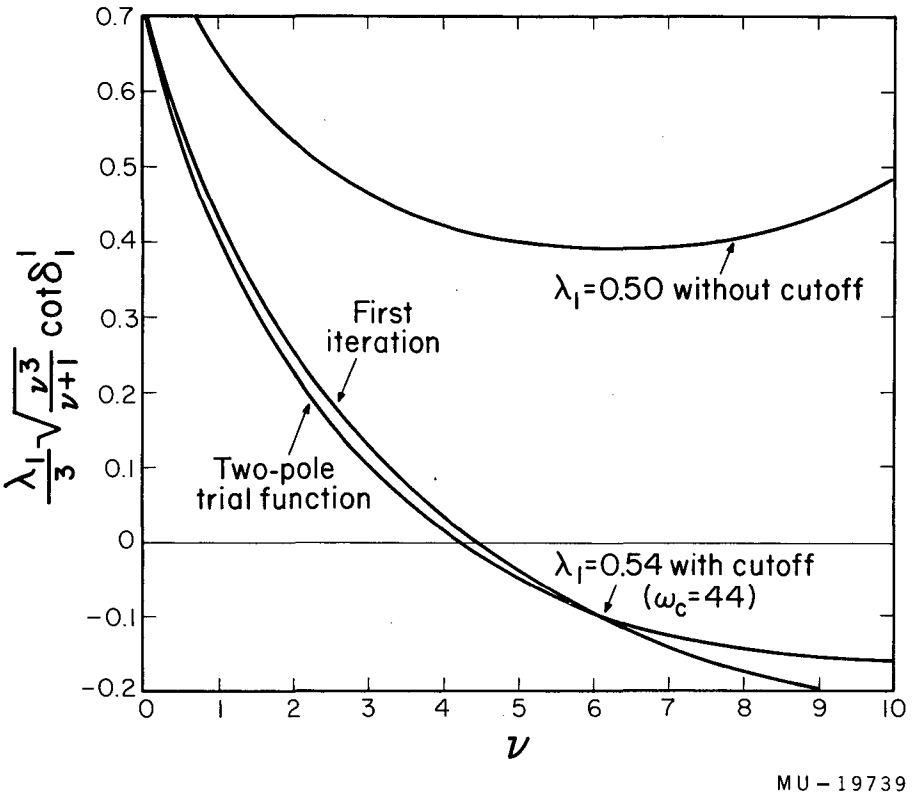
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
0
1
2
3
4
5
6
7
8
9
10
ν
$\frac{\lambda_l}{3}\sqrt{\frac{\nu^3}{\nu+1}}\cot\delta_l^l$
λ_l=0.50 without cutoff
First iteration
Two-pole trial function
λ_l=0.54 with cutoff
(ω_c=44)
MU-19739

Fig. 4