Theory of Two-Dimensional Multirate Filter Banks

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Abstract—**New results are presented on two-dimensional finite-impulse-response filter banks for multirate applications. The theory is valid for all sampling lattices, and conditions for alias-free and perfect signal reconstruction are derived. Synthesis structures for paraunitary and nonparaunitary polynomial matrices are derived, which yield perfect reconstruction filter banks. The degrees of freedom are given for these systems. Linear phase conditions are posed on the polyphase form of filter banks, which is used to derive a design structure for the restricted, but important, case of linear phase filter banks.**

I. Introduction

In recent years, subband coding has gained attention as a powerful method for compressing still images and video (see, for example, [22], [26], [28], [9], [12], [14], [27], and [2]). The technique of subband coding is explained in Fig. 1. A signal is passed through a bank of band-pass filters, the analysis filters. Owing to the reduced bandwidth, each resulting component may be subsampled to its new Nyquist frequency, thus yielding the subband signals. Following that, each subband would be encoded, transmitted, and, at the destination, decoded. To finally reconstruct the signal, each subband is upsampled to the sampling rate of the input. All upsampled components are passed through the synthesis filter bank, where they are interpolated, and subsequently added to form the reconstructed signal.

Most of the previously reported work on subband coding of images has relied on separable processing. However, it is only proper that two-dimensional signals, such as images, should be processed with truly two-dimensional systems. The advantage with nonseparable filters is that the subband analysis may have directional properties which are not limited to the vertical and horizontal directions of separable filters, and general, nonrectangular subsampling patterns can be used. In addition, nonseparable filters can have better frequency characteristics than their separable counterpart: consider that a nonseparable impulse response with $M \times M$ coefficients has M^2 free variables while its separable counterpart has only $2M$. Multidirectional subband analysis can thereby be obtained which may give enhanced coding performance. For example, a separable paraunitary filter bank (to be explained in Section V-A) cannot have linear phase response in the case of separable subsampling by a factor 2 in each dimension, while the nonseparable counterpart can possess both of these properties. Filter bank theory for one-dimensional systems has been thoroughly researched. For an overview of recent results, which are pertinent to the theory of this paper, the reader is referred to [18], [23], [19], [7], and [24] and references therein. Discussions on two-dimensional filter banks related to the present work have appeared in [20], [1], [25], [2], [13], [14], and [3].

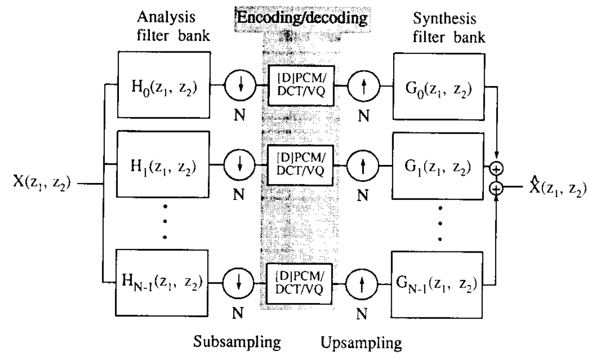

Fig. 1. A subband coding system consists of filter banks, sample rate conversions, and compression. The latter is commonly performed by PCM, DPCM, DCT, vector quantization, or any combination thereof.

Subband analysis and synthesis may be approached with two levels of quality in mind: alias-free and perfect signal reconstruction. Alias-free reconstruction means that the system is shift-invariant in the absence of coding loss. In this case it is also possible to eliminate amplitude and phase distortion, which is referred to as perfect reconstruction. The subband coding system in Fig. 1 may be viewed as consisting of three distinct parts where the first is the filter banks which are used for analysis and synthesis of a signal, the second is the sample rate conversion of the subbands (i.e., the subsampling and upsampling), and the third part is the encoding and decoding of the subbands. In this paper the focus is on the former two parts; no coding results are included. The theory pertains mainly to perfect reconstruction filter banks with finite impulse responses and critical subsampling. The sampling structures of the input and the subsampling must be representable as lattices, and structures which can only be represented as unions of shifted lattices will not be considered.

The outline of the article is as follows. Section II covers the general two-dimensional subsampling and upsampling. Given this, the related polyphase decomposition of

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a filter bank is derived in Section III. In Section IV, the input/output relationship of the entire subband analysis and synthesis system is derived in the polyphase domain, upon which we pose conditions for alias-free and perfect signal reconstruction. Section V gives structures which can be used for designing perfect reconstruction filter banks. In Section VI we derive a test condition in the polyphase domain for the linear phase response of two-dimensional systems, which is used to develop a design structure for linear phase systems.

We will adhere to the following notation. Matrices are denoted by boldface italic capital letters (e.g., $\boldsymbol{A}$), determinants by vertical bars (e.g., $|\boldsymbol{A}|$), and column vectors by an overbar (e.g., $\bar{a}$). Functions in the space domain are named by lowercase letters with their z-transform equivalent given in capital letters.[1] The exceptions to this naming convention are N, which denotes the subsampling factor, and W_N, which denotes the Nth root of unity (i.e., $W_N = e^{-j(2\pi/N)}$). Functions in the polyphase domain [6] are indicated by the subscript p. The axes of the input lattice are n_1 and n_2, and the axes on the subsampling lattice are u_1 and u_2. Matrix transpose is marked by superscript T, $\otimes$ is the Kronecker matrix product [10], and all other notations are explained in the text or given by their context.

II. Two-Dimensional Subsampling

An integral part of subband coding is the subsampling of the analyzed signal and the reciprocal upsampling before the synthesis. For one-dimensional signals, uniform, or equidistant, sampling structures are most common. In two-dimensions the counterpart is a signal sampled on a uniform grid with one sample located at each vertex point of the grid. In this article such a grid will be referred to as a lattice. For a good introduction to lattice theory the reader is referred to [5], which, however, goes much beyond the knowledge needed for this article.

Assume a two-dimensional signal which is defined on an arbitrary sampling lattice. A subsampling of this signal can be seen as a linear transformation from the input lattice to a subsampling lattice (i.e., the subsampling lattice is a coset of the input lattice) [8]. The subsampling should not be restricted to be along the axes of the input lattice, which corresponds to scaling, but also rotation of the axes should be possible. Thus, given a location (u_1, u_2) on the subsampling lattice, the corresponding location (n_1, n_2) on the sampling lattice of the input is given by

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \boldsymbol{D}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \text{where } \boldsymbol{D} = \begin{pmatrix} d_{00} & d_{01} \\ d_{10} & d_{11} \end{pmatrix}. \quad (1)$$

All elements d_{ij} are integers, and we will require that $d_{10} = 0$, $d_{00} > 0$, $d_{11} > 0$, and $0 \leq d_{01} < d_{00}$. (The requirement that $d_{10} = 0$ makes the axes n_1 and u_1 collinear.) The requirements do not lead to loss of generality since a given sublattice can be described by more than one $\boldsymbol{D}$ matrix [5], as shown in Example 1. (In n dimensions, $\boldsymbol{D}$ is upper (or lower) triangular with nonzero elements on the main diagonal [5].) Hence, we have just chosen a form that will simplify our analysis (especially the definition of the polyphase decomposition in Section III.) The different transformation matrices for a given sublattice are all related to one another by matrices with integer elements and unity determinants [5], [8]. The subsampling factor is given by the determinant of $\boldsymbol{D}$, and, owing to the aforementioned restrictions, it is equal to $N = |\boldsymbol{D}| = d_{00}d_{11}$. It is important to note that the subsampling factor does not uniquely determine the subsampling structure. This is illustrated in Example 1 for the case of $N = 4$, where $\boldsymbol{D}_1$ and $\boldsymbol{D}_2$ describe the same subsampling lattice, while $\boldsymbol{D}_3$ describes a completely different lattice (obtained by one-dimensional subsampling). Consequently, these two cases will yield entirely different conditions for the filter bank design.

For a system with critical subsampling, where the number of samples is conserved, the number of subbands is equal to the subsampling factor [23]. Only critically subsampled systems will be considered in this article. The effective subsampling factors along the directions n_1 and n_2 (i.e., the distance between two samples along those axes) can be found to be $N_{n_1} = d_{00}$ and $N_{n_2} = c\, d_{00}d_{11}/d_{01}$ for $d_{01} > 0$, where c is the smallest integer such that N_{n_2} is integer, and $N_{n_1} = d_{00}$ and $N_{n_2} = d_{11}$ for $d_{01} = 0$. Note that $N \neq N_{n_1}N_{n_2}$ except when the subsampling is separable.

Assume that a signal $x(n_1, n_2)$ is subsampled, as described by $\boldsymbol{D}$ in (1) with the aforementioned restrictions, and immediately upsampled by $\boldsymbol{D}^{-1}$. The resulting signal, $y(n_1, n_2)$, can be expressed as the input signal modulated by a function $f(n_1, n_2)$, which is defined on the input lattice, I. The function has to be unity-valued at the points of the subsampling lattice, $f(n_1, n_2) = 1$, $(n_1, n_2) \in S$, and zero at all other points of the input lattice, $f(n_1, n_2) = 0$, $(n_1, n_2) \in \bar{S}(= I \setminus S)$, where $S \subseteq I$ (the symbol $\setminus$ denotes set-minus). It is easily verified that $f(n_1, n_2)$ can be written as

$$\begin{aligned} f(n_1, n_2) &= \frac{1}{N}\sum_{k=0}^{d_{00}-1}\sum_{l=0}^{d_{11}-1} \exp\left(-j2\pi(n_1\ n_2)(\boldsymbol{D}^{-1})^T\begin{pmatrix} k \\ l \end{pmatrix}\right) \\ &= \frac{1}{N}\sum_{k=0}^{d_{00}-1}\sum_{l=0}^{d_{11}-1} W_N^{d_{11}n_1k + d_{00}n_2l - d_{01}n_2k} \end{aligned} \quad (2)$$

where W_N is the Nth root of unity [13], [14], [3]. If the z transforms of $x(n_1, n_2)$ and $y(n_1, n_2)$ are $X(z_1, z_2)$ and $Y(z_1, z_2)$, respectively, then, following the modulation theorem, the output can be expressed as

$$Y(z_1, z_2) = \frac{1}{N}\sum_{k=0}^{d_{00}-1}\sum_{l=0}^{d_{11}-1} X(W_N^{-d_{11}k}z_1, W_N^{-(d_{00}l - d_{01}k)}z_2). \quad (3)$$

[1]The two-dimensional z transform of a discrete-space function, $a(n_1, n_2)$, is defined as $A(z_1, z_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} a(n_1, n_2)z_1^{-n_1}z_2^{-n_2}$.

This expression for the modulation will be used to derive the behavior of the entire subband coding system in Section IV. The z transform expressions for a subsampled signal and upsampled signal, respectively, are given in the Appendix. There it is also shown that $Y(z_1, z_2)$ in (3) is a polynomial in powers of $z_1^{d_{00}}$ and $z_1^{d_{01}} z_2^{d_{11}}$.

III. Polyphase Representation of Two-Dimensional Filter Banks

For the defined subsampling pattern, the analysis filters can be decomposed into polyphase components. This is a generalization of the familiar one-dimensional polyphase concept [6]. The polyphase components together cover all the points on the input lattice, where each polyphase component is formed from the points of a coset of that lattice. Let the coefficients of the impulse analysis filter bank be indexed as $h_i(n_1, n_2)$, where i is the filter index ($i \in [0, N-1]$ for critical subsampling), and n_1 and n_2 are the row and column indices of the impulse response. Given (1), the polyphase components of the ith filter can be defined as

$$H_{pi,k,l}(z_1, z_2) = \sum_{u_1=-\infty}^{\infty} \sum_{u_2=-\infty}^{\infty} h_i(\underbrace{k + d_{00}u_1 + d_{01}u_2}_{n_1}, \underbrace{l + d_{11}u_2}_{n_2}) \cdot z_1^{-u_1} z_2^{-u_2} \quad (4)$$

where $k = 0, \cdots, d_{00} - 1$ and $l = 0, \cdots, d_{11} - 1$. So, k and l span a unit cell of the input lattice, as illustrated in Fig. 2. The z transform of the impulse response is calculated from the polyphase components by [13], [14]

$$\begin{aligned} H_i(z_1, z_2) &= \overline{Z}_1(z_1)^T \boldsymbol{H}_{pi}(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}}) \overline{Z}_2(z_2) \\ &= \text{vec}\left[\boldsymbol{H}_{pi}(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}})\right]^T \overline{Z}_2(z_2) \otimes \overline{Z}_1(z_1). \end{aligned} \quad (5)$$

The vec $[\cdot]$ operator creates a column vector out of a matrix by stacking the columns on top of one another (i.e., for a matrix with N rows, its element a_{ij} becomes element a_{i+jN} of the column vector), and vec $[\boldsymbol{XYZ}]^T = \text{vec}[\boldsymbol{Y}]^T \cdot \boldsymbol{Z} \otimes \boldsymbol{X}^T$ [10]. In the above expression, the polyphase components of analysis filter i are given in matrix form:

$$\boldsymbol{H}_{pi}(z_1, z_2) = \begin{pmatrix} H_{pi,0,0}(z_1, z_2) & \cdots & H_{pi,0,d_{11}-1}(z_1, z_2) \\ \vdots & \ddots & \vdots \\ H_{pi,d_{00}-1,0}(z_1, z_2) & \cdots & H_{pi,d_{00}-1,d_{11}-1}(z_1, z_2) \end{pmatrix} \quad (6)$$

and

$$\begin{aligned} \overline{Z}_1(z_1) &= (1 \quad z_1^{-1} \cdots z_1^{-d_{00}+1})^T \\ \overline{Z}_2(z_2) &= (1 \quad z_2^{-1} \cdots z_2^{-d_{11}+1})^T. \end{aligned} \quad (7)$$

The analysis filter bank, which is a column vector of the impulse responses, H_i, can be expressed in terms of its polyphase components as

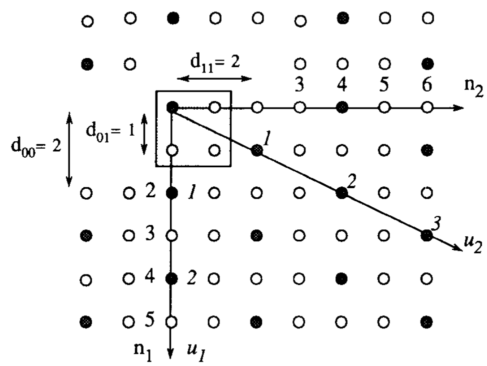


Fig. 2. Hexagonal subsampling on a rectangular lattice. The circles denote samples of the input of which the shaded ones are retained in the subsampling. The shaded rectangle is a unit cell—a polyphase component will be defined for each location in this cell (see Section III).

$$\overline{H}(z_1, z_2) = \boldsymbol{H}_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}}) \overline{Z}_2(z_2) \otimes \overline{Z}_1(z_1) \quad (8)$$

where all polyphase components of the filter bank form a polynomial matrix of size $N \times N$:

$$\boldsymbol{H}_p(z_1, z_2) = \begin{pmatrix} \text{vec}\left[\boldsymbol{H}_{p0}(z_1, z_2)\right]^T \\ \text{vec}\left[\boldsymbol{H}_{p1}(z_1, z_2)\right]^T \\ \vdots \\ \text{vec}\left[\boldsymbol{H}_{pN-1}(z_1, z_2)\right]^T \end{pmatrix}. \quad (9)$$

Analogously, the bank of synthesis filters, $\overline{G}(z_1, z_2)$, is defined as

$$\overline{G}(z_1, z_2) = z_1^{-d_{00}+1} z_2^{-d_{11}+1} \boldsymbol{G}_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}}) \overline{Z}_2(z_2^{-1}) \otimes \overline{Z}_1(z_1^{-1}). \quad (10)$$

This representation will be used next when considering the input/output relationship of the entire subband coding scheme; the analysis, subsampling, upsampling, and synthesis. The polyphase decomposition of a four-channel filter bank for hexagonal subsampling is given in Example 2.

IV. Subband Analysis and Synthesis

An input signal, $X(z_1, z_2)$, is split into subbands by a bank of analysis filters. The resulting subbands are then subsampled and upsampled, synthesis filtered, and recombined to form the output signal $\hat{X}(z_1, z_2)$, as shown in Fig. 1. By (3), (8), and (10), this whole system, from input to output, can be described by

$$\hat{X}(z_1, z_2) = \frac{1}{N} \overline{G}(z_1, z_2)^T \sum_{i=0}^{d_{00}-1} \sum_{j=0}^{d_{11}-1} \overline{H}(W_N^{-d_{11}i} z_1, W_N^{-(d_{00}j - d_{01}i)} z_2) X(W_N^{-d_{11}i} z_1, W_N^{-(d_{00}j - d_{01}i)} z_2)$$

$$= \frac{1}{N} z_1^{-d_{00}+1} z_2^{-d_{11}+1} \bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T$$
$$T_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}})$$
$$\sum_{i=0}^{d_{00}-1} \sum_{j=0}^{d_{11}-1} \bar{Z}_2(W_N^{-(d_{00}j - d_{01}i)} z_2)$$
$$\otimes \bar{Z}_1(W_N^{-d_{11}i} z_1)\, X(W_N^{-d_{11}i} z_1, W_N^{-(d_{00}j - d_{01}i)} z_2). \tag{11}$$

In the second equality we used the transfer matrix T_p, which is defined as

$$T_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}}) = G_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}})^T H_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}}). \tag{12}$$

Note that the polyphase matrix H_p is unaffected by the modulation since its elements are polynomials in powers of $z_1^{d_{00}}$ and $z_1^{d_{01}} z_2^{d_{11}}$ [13], [14]. The polyphase equivalent of the system in Fig. 1 is illustrated in Fig. 3, which corresponds to the second equality of (11). The system becomes shift invariant if the aliasing terms are eliminated (i.e., the terms for $i \neq 0$ or $j \neq 0$). The following condition is necessary and sufficient for aliasing cancellation.

Lemma: The transfer matrix in (12) represents an alias-free system *if and only if*

$$\bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T T_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}})$$
$$= T(z_1, z_2)\, \bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T \tag{13}$$

where $T(z_1, z_2)$ is a scalar-valued polynomial. This condition implies that $\bar{Z}_2(z_2^{-1}) \otimes \bar{Z}_1(z_1^{-1})$ is a left eigenvector of $T_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}})$, with the corresponding eigenvalue $T(z_1, z_2)$.

Proof: Sufficiency is shown by moving $\bar{Z}_1(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T$ inside the summations in (11). For fixed i and j within a unit cell (i.e., $0 \leq i < d_{00}$ and $0 \leq j < d_{11}$) we get

$$\left(\bar{Z}_2(z_2^{-1})^T \otimes \bar{Z}_1(z_1^{-1})^T\right)\left(\bar{Z}_2(W_N^{-(d_{00}j - d_{01}i)} z_2) \otimes \bar{Z}_1(W_N^{-d_{11}i} z_1)\right)$$
$$= \underbrace{\sum_{m=0}^{d_{11}-1} W_N^{-(d_{00}j - d_{01}i)m}}_{=0,\ j \neq 0} \underbrace{\sum_{n=0}^{d_{00}-1} W_N^{-d_{11}in}}_{=0,\ i \neq 0}$$
$$= \begin{cases} N, & i = j = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

(The product $(A \otimes B)(C \otimes D) = AC \otimes BD$ if all matrix products are defined [10].)

The necessity of (13) is shown as follows. Let the right-hand side of the equation be replaced by a general row vector, $\bar{V}(z_1, z_2)^T$. If this vector is moved inside the summation of (11), then for fixed i and j the condition for aliasing cancellation yields that

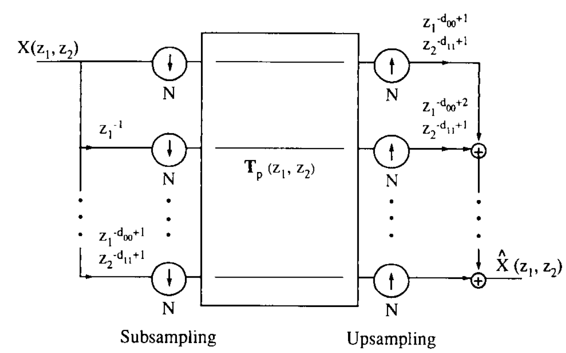


Fig. 3. The system described by (11) with T_p defined in (12). If the box were replaced by short circuits of each branch, it would correspond to $T_p = I$. Note that the product of delays is the same along all horizontal branches.

$$\sum_{m=0}^{d_{00}-1} \sum_{n=0}^{d_{11}-1} V_{m+nd_{00}}(z_1, z_2)(W_N^{d_{11}i} z_1^{-1})^m (W_N^{(d_{00}j - d_{01}i)} z_2^{-1})^n$$
$$= \begin{cases} \mathcal{V}(z_1, z_2), & i = j = 0 \\ 0, & \text{otherwise} \end{cases} \tag{15}$$

where $V_{m+nd_{00}}(z_1, z_2)$ are the polynomial elements of $\bar{V}$, and $\mathcal{V}(z_1, z_2)$ is an arbitrary polynomial. In order to get cancellations, the elements of $\bar{V}$ have to be of the form $V_{m+nd_{00}}(z_1, z_2) = \Upsilon_{m+nd_{00}}(z_1, z_2) z_1^m z_2^n$. The condition in (15) is then met only if all the Υ's are equal, in which case $\Upsilon(z_1, z_2) = \mathcal{V}(z_1, z_2)$ and the cancellation of terms is given by (14). This, in turn, means that $\bar{V}(z_1, z_2)^T$ can be written as the right-hand side of (13) where $T(z_1, z_2) = \mathcal{V}(z_1, z_2)$. □

Note that the condition given in (13) does not stipulate a certain matrix structure. In the case of separable subsampling (i.e., $d_{01} = 0$), the structure has been shown to be block pseudocirculant in z_2 where each block is pseudocirculant in z_1 [15]. This form is generally not valid for nonseparable subsampling, as shown in Example 3 (except in its most restricted form: $T(z_1, z_2) I_N$, i.e., a diagonal matrix with one unique polynomial element).

After cancellation of the aliasing terms in (11), the system is linear and shift-invariant and can be represented by a (scalar) transfer function. In order to get perfect reconstruction, where the output from the system is a perfect replica of the input signal, it is necessary and sufficient that $T(z_1, z_2) = z_1^m z_2^n$, i.e., a monic monomial. Hence, the transfer function only causes a shift of the input, and (11) is reduced to

$$\hat{X}(z_1, z_2) = z_1^{-d_{00}+1} z_2^{-d_{11}+1} T(z_1, z_2)\, X(z_1, z_2)$$
$$= z_1^{m-d_{00}+1} z_2^{n-d_{11}+1} X(z_1, z_2). \tag{16}$$

The system does not produce any spectral, phase, or amplitude distortion! A similar result, given in the Fourier domain, appears in [25]. In the remainder of this paper, we will concentrate on perfect reconstruction systems obtained by transfer matrices of the form $T_p(z_1, z_2) = T(z_1, z_2) I_N$.

V. Two-Dimensional Perfect Reconstruction FIR Filter Banks

In the following sections, we are going to look at design structures which meet the condition for perfect reconstruction derived in the previous section. Application of the proposed structures to actual filter bank design is presently under investigation and therefore not included here.

A desirable property for the analysis and the synthesis banks is that they should both consist of finite-impulse-response filters. Then we do not have to be concerned with stability issues, which are most intricate in the two-dimensional case [4]. Given that the analysis filter bank is FIR, then the synthesis (i.e., inverse) bank of a perfect reconstruction system is FIR *if and only if* the determinant of the polyphase matrix of the analysis filter bank is a monominal in z_1 and z_2.

We shall obtain perfect reconstruction by two means: paraunitary systems and invertible nonparaunitary systems. Design of filter banks based on both types of systems will be studied in the following subsections. Before that, we shall compare how constrained these systems types are. Generally speaking, the more free variables available for the design, the closer, or more easily, a design goal can be met. In Table I we show the number of free variables for a polyphase matrix of size $N \times N$ whose elements have highest power m_1 in z_1 and m_2 in z_2. The system with FIR filters in both banks is constrained only through its determinant, which should be a monomial. The determinant of the polyphase matrix is a polynomial with $N(Nm_1 + 1)(Nm_2 + 1)$ terms, of which all but one term are constrained to zero. The expressions in the table for the paraunitary and invertible nonparaunitary systems are derived in the subsections that follow.

TABLE I
The Number of Free Variables for Polyphase Matrices of Various Systems Types

System type	Free variables	$N = 4$, $m_1 = m_2 = 1$
Unconstrained system	$N^2(m_1 + 1)(m_2 + 1)$	64
Analysis and synthesis FIR	$N^2 + N(N - 1)(m_1 + m_2)$	40
Invertible FIR system ($\det D \neq 0$)	$N^2 + (2N - 1)(m_1 + m_2) - 1$	29
Paraunitary FIR system	$(N-1)(N + 2(m_1 + m_2))/2$	12

The unconstrained type gives the maximally available number. Note that D refer to the inverse system in (26).

A. Paraunitary Systems for Perfect Reconstruction Filter Banks

If we wish to have impulse responses which are equal in both filter banks (within a reversal in the directions n_1 and n_2), then we should impose a paraunitary condition on the polyphase matrix [20]. However, this also means that the system is highly constrained, as shown in Table I. For the real-valued two-dimensional case, the paraunitary condition is given by

$$H_p(z_1^{-1}, z_2^{-1})^T H_p(z_1, z_2) = I. \tag{17}$$

Consequently, the synthesis filters can be chosen as $G_p(z_1, z_2) = F(z_1, z_2) H_p(z_1^{-1}, z_2^{-1})$, where $F(z_1, z_2)$ is a monic monomial of powers so that G_p is causal. Note that (17) holds for $H_p(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}})$ as well, as needed for (11). (A paraunitary matrix is, by definition, unitary on the unit bicircles, $z_1 = e^{-j\omega_1}$ and $z_2 = e^{-j\omega_2}$. Thus, the property will not be affected by the powers of z_1 and z_2.)

1) Filter Bank Design by a Cascade Structure: For the purpose of constructing paraunitary systems of any degree, we will first propose a design structure of the matrix $H_p(z_1, z_2)$ based on a factorization method. Design of one-dimensional systems can rely on the fact that all polynomials, which appear as elements in a polyphase matrix, can be decomposed into first-order factors over the complex field. The same is generally not true for two-dimensional polynomials. However, when both the analysis and the synthesis filter banks are FIR, the polyphase matrices can in fact be factorized: A polynomial matrix, $R(z_1, z_2)$, which has a determinant that can be factorized

$$|R(z_1, z_2)| = \prod_{i=1}^{k} \mathcal{R}_i(z_1, z_2) \tag{18a}$$

can itself be factorized so that

$$R(z_1, z_2) = \prod_{i=1}^{k} R_i(z_1, z_2) \quad \text{and}$$
$$|R_i(z_1, z_2)| = \mathcal{R}_i(z_1, z_2) \tag{18b}$$

where $\mathcal{R}_i(z_1, z_2)$ are arbitrary polynomials [16, theorem 4.2]. (The coefficients of the polynomial matrix elements can belong to any field, but the factorization is not unique, and it does not extend to systems with three or more dimensions [4].)

As mentioned above, FIR perfect reconstruction systems have $|H_p(z_1, z_2)| = cz_1^{-m_1} z_2^{-m_2}$, where $c = 1$ for paraunitary systems. We therefore know that the above factorization exists. Based on this fact, we can construct a two-dimensional design structure which lets us build a paraunitary polyphase matrix from simple factors. The method is a straightforward generalization of the one-dimensional factorization given in [21]. The design structure is given by

$$H_p(z_1, z_2) = H_0 \prod_{i=0}^{w-1} \{I - (1 - z_1^{-1})\bar{u}_i\bar{u}_i^T\}$$
$$\{I - (1 - z_2^{-1})\bar{v}_i\bar{v}_i^T\} \tag{19}$$

where H_0 is an orthogonal matrix of size $N \times N$, $\bar{u}_i^T\bar{u}_i = \{1, \text{or } 0\}$ and $\bar{v}_i^T\bar{v}_i = \{1, \text{or } 0\}$, $\forall i \in [0, w - 1]$. All column vectors $\bar{u}_i$ and $\bar{v}_i$ have length N. Note that we have to allow $\bar{u}_i^T\bar{u}_i = 0$ and $\bar{v}_i^T\bar{v}_i = 0$ so that we can get neighboring factors in the same variable. The system may have different orders in z_1 and z_2, say w_1 and w_2, respectively, which is accommodated by having $w - w_1$ factors in (19) where $\bar{u}_i = \bar{0}$ and $w - w_2$ factors where $\bar{v}_i = \bar{0}$. Note that the entire system is nonseparable since the factors do not commute. The orthogonal matrix, H_0, may be arbitrarily applied by premultiplication or postmultiplication (all the systems yielded by postmultiplication may equally well be found by premultiplication) [14].

The cascade structure resembles the factorization that we would expect from (18b). However, we do not know, at this point, how complete this design structure is; nor do we know the importance of any excluded system. Completeness aside, the cascade structure also has other shortcomings. A certain length of the cascade, say w_1 factors in z_1 and w_2 in z_2, will give systems of highest orders anywhere in the range $(1 \cdots w_1)$ in z_1 and $(1 \cdots w_2)$ in z_2. Every pair of orthogonal vectors $\bar{u}_i$ will lower the degree in z_1 and, analogously, orthogonal vectors $\bar{v}_i$ lower the degree in z_2. Thus, to design a paraunitary matrix of a given order, one has to allow cascades of much higher order which can be reduced to the desired order by appropriate orthogonality constraints. The advantage of the structure is that the filter bank design can be performed iteratively, with one factor at a time. Although the structure has the outlined shortcomings, it may nevertheless yield useful nonseparable filters.

2) Design Based on Paraunitary State-Space Description: Since the previous structure may not generate all possible FIR paraunitary systems of a given degree, we will seek to design polyphase matrices from the state-space description as an alternative. The derivation is a generalization of [7] where it is performed for one-dimensional systems.

The $N \times N$ matrix transfer function of a two-dimensional system can be written as [17]

$$H_p(z_1, z_2) = D + C\left\{\begin{pmatrix} z_1 I_{m_1} & 0 \\ 0 & z_2 I_{m_2} \end{pmatrix} - A\right\}^{-1} B \quad (20)$$

where A is an $m \times m$ matrix, D has size $N \times N$, C has size $N \times m$, and B has size $m \times N$, and $m = m_1 + m_2$. The system consists of FIR filters if A is lower (or upper) triangular with all elements on the main diagonal equal to zero. That is,

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ * & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ * & \cdots & * & 0 \end{pmatrix} = L_\Delta. \quad (21)$$

(The asterisk marks elements which are not required to be zero.) The transfer function in (20) is paraunitary if the $p \times p$ matrix

$$R = \begin{matrix} & N & m \\ m & B & A \\ N & D & C \end{matrix} \;\; R = \begin{pmatrix} B & A \\ D & C \end{pmatrix} \quad (22)$$

is orthogonal, which is verified in (46) to (48) of the Appendix. (In 1-D, this is *necessary and sufficient* and known as the lossless bounded-real lemma [18].)

Doğanata *et al.* give a design structure for orthogonal matrices, R, which have the submatrix A of the form given by (21) [7]:

$$R = \prod_{i=p-2}^{0} \prod_{j=p-1}^{i+1} \Theta_{i,j} \quad \text{and} \quad \Theta_{i,j} = I$$
$$\text{for } 0 \le i < m,\ N + i \le j < p. \quad (23)$$

In the above, $p = N + m$ and $\Theta_{i,j}$ is a rotation matrix, which is an identity matrix except for rows i and j, which are given by

$$\begin{matrix} & 0 & 1 & & i & & j & & p-1 \\ i & 0 & 0 & \cdots & \cos\theta_{i,j} & \cdots & -\sin\theta_{i,j} & \cdots & 0 \\ j & 0 & 0 & \cdots & \sin\theta_{i,j} & \cdots & \cos\theta_{i,j} & \cdots & 0 \end{matrix}. \quad (24)$$

For R in (23), the number of free variable (angles in this case) is

$$F_p = \binom{m+N}{2} - \binom{m+1}{2}$$
$$= \tfrac{1}{2}(N + 2(m_1 + m_2))(N - 1). \quad (25)$$

The outlined design method is less tractable for filter bank design than the cascade structure. The issue is that, in this method, one first designs an orthogonal matrix. This matrix has then to be partitioned into the submatrices, $[A, B, C, D]$, which can be subsequently used in the state-space description to find the polyphase matrix. In short, the indirect effect on the filter properties of a particular angle in R may be hard to foresee. However, the structure may give filter banks not obtainable from the cascade structure.

B. Nonparaunitary Systems for Perfect Reconstruction Filter Banks

Next we will search for analysis and synthesis FIR structures which are not paraunitary. Paraunitary systems are appealing mainly because the analysis and synthesis filter banks are the same. However, this property is obtained at a cost: the system is highly constrained. Since none of these constraints pertain to the frequency responses of the filters, we may end up with poorly performing filters only because we impose constraints which simplify the design procedure. By going over to nonparaunitary systems, less structural conditions are imposed, so one has to accept the occurrence of undesirable effects. The first potential problem is that the inverse system may not correspond to a filter bank with clearly defined frequency characteristics. Secondly, the synthesis filters, it is feared, may be of vastly larger size than the analysis filters. The first problem can be avoided by specifying design goals for both filter banks. However, the design will be more complex than in the paraunitary case since, in effect, two filter banks are to be designed simultaneously. Then we shall see that for the proposed design structure, which is mildly constrained, the size of the largest synthesis filter will not exceed that of the largest analysis filter.

1) Design Based on State-Space Description of Inverse System: Given the state-space description of an analysis filter bank, as in (20), the synthesis filter bank for perfect reconstruction could be taken simply as the transpose of the inverse system, i.e., $G_p(z_1, z_2) = H_p(z_1, z_2)^{-T}$. The inverse of the "forward" system in (20) is given by

$$H_p(z_1, z_2)^{-1} = D^{-1} - D^{-1}C\left\{\begin{pmatrix} z_1 I_{m_1} & 0 \\ 0 & z_2 I_{m_2}\end{pmatrix} - A + BD^{-1}C\right\}^{-1} BD^{-1} \qquad (26)$$

provided that D is nonsingular [11]. If the "forward" system is FIR, the inverse system will be FIR if $BD^{-1}C = L_\Delta$ (cf. (21)). Let $C = DE$, where E has dimensions $N \times m$, just as C. Hence, the condition for FIR becomes $BE = L_\Delta$, which can be written more explicitly as

$$(\bar{b}_0\, \bar{b}_1 \cdots \bar{b}_{m-1})^T(\bar{e}_0\, \bar{e}_1 \cdots \bar{e}_{m-1}) = L_\Delta \Rightarrow \bar{b}_i^T\bar{e}_j = 0, \quad j \geq i, \forall i \in (0, m]. \qquad (27)$$

At this point, we can note that the highest powers of the inverse system in (26) are no higher than those of the "forward" system, namely $z_1^{-m_1}$ and $z_2^{-m_2}$ (the lower power is still 0 in both z_1 and z_2). These powers are given by the inverse of the determinant of the matrix $\{diag\,(z_1 I \;\; z_2 I) - A + BE\}$. Furthermore, all its cofactors have positive powers of z_1 and z_2 which are strictly lower than the determinant. Therefore, H_p^{-1} does not contain positive powers of z_1 or z_2. In short, the synthesis filters will not be larger than the analysis filters.

Given (27), we can determine the number of free variables we have for a FIR system whose inverse is also FIR. We get that B has $Nm - m(m+1)/2$ free variables ($\bar{b}_0$ has $N - m$ free variables, $\bar{b}_1$ has $N - m + 1$, and so on). E, which is unconstrained, has Nm free variables, D has $N^2 - 1$ (one constraint to force nonsingularity), and, finally, A has $m(m-1)/2$ free variables (its first column has $m - 1$, the second $m - 2$, and so forth). The total number is therefore

$$F_{np} = N^2 + (2N - 1)m - 1 = N^2 + (2N - 1)(m_1 + m_2) - 1. \qquad (28)$$

Clearly, the nonparaunitary system has considerably more free variables than the paraunitary system (cf. (25) and Table I), which is an indication of how well a specific design goal can be met.

The only limitation with the described design method in (26) and (27) is that it presumes that the matrix D is nonsingular. It is not clear what this condition implies in terms of restricted filter properties. For example, the matrix D is singular for paraunitary systems (the only requirement is that $D^TD + B^TB = I$). It is worth noting that the constant terms of the matrix transfer function, H_p, come from D. These terms correspond to the filter coefficients within the first unit cell (i.e., $h_i(n_1, n_2)$ for $0 \leq n_1 < d_{00}$ and $0 \leq n_2 < d_{11}$, $\forall i \in [0, N-1]$).

C. Discussion of Design Based on the Smith Form

The polyphase matrix of a system can be written in its Smith form [16]:

$$H_p(z_1, z_2) = R(z_1, z_2)\, \Lambda(z_1, z_2)\, S(z_1, z_2). \qquad (29)$$

$R(z_1, z_2)$ and $S(z_1, z_2)$ are matrices with determinants as functions of z_2 only, namely $\mathcal{R}(z_2)$ and $\mathcal{S}(z_2)$, respectively. The Smith form is $\Lambda(z_1, z_2) = \text{diag}\,(\lambda_0 \;\; \lambda_1 \cdots \lambda_{N-1})$, where the polynomials $\lambda_i(z_1, z_2)$ are monic and such that λ_i divides λ_{i+1}. Recall that the determinant of $H_p(z_1, z_2)$ is a monomial in z_1 and z_2. If the determinant of the Smith form is $\mathcal{L}(z_1, z_2)$, then

$$\left|H_p(z_1, z_2)\right| = \mathcal{R}(z_2)\, \mathcal{L}(z_1, z_2)\, \mathcal{S}(z_2) = c z_1^{-m_1} z_2^{-m_2}. \qquad (30)$$

This implies that $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{L}$ are all monomials. In particular, it implies that all the λ_i's are monomials; thus, the Smith form is paraunitary. Consequently, for every nonparaunitary perfect reconstruction FIR system, there exists a Smith-form equivalent paraunitary system, and when both $R(z_1, z_2)$ and $S(z_1, z_2)$ are paraunitary, so is $H_p(z_1, z_2)$.

By analogy with the one-dimensional case, we propose that $R(z_1, z_2)$ and $S(z_1, z_2)$ could be designed as products of finite numbers of elementary matrices, each corresponding to an elementary row or column operation (see [11]). There are three types of elementary row and column operations:

1) Multiply a row by a polynomial in z_2 (i.e., $R = \text{diag}\,(1 \cdots 1 \;\; P(z_2) \;\; 1 \cdots 1)$).

2) Permute two rows (R is constructed from an identity matrix by exchanging rows k and l).

3) Add one row, multiplied by an arbitrary polynomial, to any other row (i.e., $R(z_1, z_2) = I + Q(z_1, z_2)_{i,j}$, where i, j is the row and column location of the polynomial, and $i \neq j$).

Elementary matrices of type 2 and 3 have determinants equal to unity, and only type 1 has a determinant as a function of z_2. Since the determinant is limited to be a monomial, we get that $P(z_2) = a z_2^{-1}$ (a is an arbitrary constant). The inverse of the unimodular matrices would then be given by a cascade of the inverse elementary matrices. Note that these also belong to the three categories above. Lastly, we wish to point out that we do not know if the three elementary matrix types are sufficient to construct all two-dimensional matrices which have one-dimensional determinants.

VI. Linear Phase Systems

When images are subband coded it is often desirable to have filters with linear phase response. With nonlinear phase response, coding loss can result in phase error in addition to amplitude error. However, phase error is usually regarded as more visible in images than amplitude error [8]. It should therefore be avoided by using linear phase filters. In what follows, a condition will be derived to test filter banks for linear phase in the polyphase domain, as given by the previously discussed design procedures, and a structure is developed for direct design of linear phase filter banks.

A. Linear Phase Conditions for FIR Filter Banks

Assume that $h_i(n_1, n_2)$ is a coefficient in the two-dimensional impulse response of filter i and that the filter has finite size $M_1 \times M_2$, specified on the input lattice. If the filter has linear phase response, the coefficients have the following symmetry:

$$h_i(n_1, n_2) = \pm h_i(M_1 - n_1 - 1, M_2 - n_2 - 1). \quad (31)$$

The possible sign change appears if the impulse response has odd symmetry. The sign is thus either + or − for the entire filter. This condition is easily translated into the z-transform domain [14]:

$$H_i(z_1, z_2) = \pm z_1^{-M_1+1} z_2^{-M_2+1} H_i(z_1^{-1}, z_2^{-1}). \quad (32)$$

The condition of (31) is in fact valid for any shape of the impulse response. In the general case, M_1 and M_2 are the dimensions of the parallelogram which bounds the impulse response on the input lattice, and the points outside the impulse response are taken to be zero.

The aim is to see how the condition in (31) translates for the polyphase representation of the filter bank. For a coefficient $h_i(n_1, n_2)$, contained in polyphase component (k, l), the corresponding coefficient $h_i(M_1 - n_1 - 1, M_2 - n_2 - 1)$ is contained in polyphase component (k', l') and the two are related by

$$\begin{pmatrix} k' \\ l' \end{pmatrix} = \begin{pmatrix} M_1 - k - 1 \\ M_2 - l - 1 \end{pmatrix} - D \begin{pmatrix} \left\lfloor \dfrac{M_1 - k - 1}{d_{00}} \right\rfloor \\ \left\lfloor \dfrac{M_2 - l - 1}{d_{11}} \right\rfloor \end{pmatrix}. \quad (33)$$

D is the subsampling matrix of (1), and $\lfloor a \rfloor$ gives the integer part of a. The polyphase index k' should be evaluated modulo d_{00} and the index l' modulo d_{11}. By analogy with (32), the linear phase condition for polyphase component (k, l) becomes

$$H_{pi,k,l}(z_1, z_2) = \pm z_1^{-(u_{1\max} + u_{1\min})} z_2^{-(u_{2\max} + u_{2\min})} H_{pi,k',l'}(z_1^{-1}, z_2^{-1}). \quad (34)$$

The highest and lowest powers of z_1 in the polyphase component are denoted by $u_{1\max}$ and $u_{1\min}$, respectively, and analogously for z_2.

The size of a finite impulse response can be defined on the input lattice, or the filter size can be specified in terms of unit cells on the subsampling lattice. The former case can become cumbersome for nonseparable subsampling, and the reader is referred to [14], where it is treated. Thus, we shall only be concerned with the latter case, which is in fact more likely with the design procedures previously described. So, assume filter size given in terms of unit cells, say w_1 along u_1 and w_2 along u_2. In this case, polyphase component (k, l) is defined by (4) for $0 \le u_1 < w_1$ and $0 \le u_2 < w_2$. The region of support for the impulse response is bounded on the input lattice by $M_1 = w_1 d_{00} + (w_2 - 1) d_{01}$ and $M_2 = w_2 d_{11}$. The relation between polyphase components (k', l') and (k, l), as given by (33), is reduced to $k' = d_{00} - k - 1$ and $l' = d_{11} - l - 1$. The linear phase condition for an entire filter bank is then given by

$$H_p(z_1, z_2) = z_1^{-(w_1-1)} z_2^{-(w_2-1)} \operatorname{diag}(\pm 1 \cdots \pm 1) H_p(z_1^{-1}, z_2^{-1}) J. \quad (35)$$

The diagonal matrix has value −1 in row i if the ith filter has odd symmetry (as given by (31)). J is a matrix with unit elements along the antidiagonal and it corresponds to the reordering of the polyphase components. Example 4 illustrates the linear phase condition for a filter bank when hexagonal subsampling is being used.

B. Design of Linear Phase Systems

Aided by the linear phase condition of the previous section, it is possible to design linear phase systems. Assume that we already have a linear phase perfect reconstruction system, $H_p(z_1, z_2)$, which meets the condition of (35). In order to create a filter bank with larger impulse responses the matrix $H_p(z_1, z_2)$ is multiplied by the "extension" $E_p(z_1, z_2)$. The new system should remain a perfect reconstruction one with linear phase. The filters of the new system are bounded by $M_1' \times M_2'$, which will be set to $\mu_1 d_{00} + (\mu_2 - 1) d_{01} \times \mu_2 d_{11}$, where $\mu_1 \ge w_1$ and $\mu_2 \ge w_2$. When the linear phase condition is imposed, we get

$$H_p(z_1, z_2) E_p(z_1, z_2) = z_1^{-(\mu_1-1)} z_2^{-(\mu_2-1)} \operatorname{diag}(\pm 1 \cdots \pm 1) \times H_p(z_1^{-1}, z_2^{-1}) E_p(z_1^{-1}, z_2^{-1}) J. \quad (36)$$

It is assumed that the symmetries of the filters are not affected by the extension (i.e., the diagonal matrix of ±1's is unaffected). Then, by comparing (36) with (35), the following condition on $E_p(z_1, z_2)$ falls out:

$$E_p(z_1, z_2) = z_1^{-(\mu_1 - w_1)} z_2^{-(\mu_2 - w_2)} J E_p(z_1^{-1}, z_2^{-1}) J. \quad (37)$$

We assume that $E_p(z_1, z_2)$ can be designed as

$$E_p(z_1, z_2) = \Delta(z_1, z_2) E \quad (38)$$

where Δ is an $N \times N$ diagonal matrix with monic monomials as elements and where E is an $N \times N$ scalar matrix with elements e_{ij}. The condition in (37) then leads to

$$E = JEJ \Rightarrow e_{ij} = e_{(N-i-1)(N-j-1)}. \quad (39)$$

Furthermore, we get

$$\Delta(z_1, z_2) = z_1^{-(\mu_1 - w_1)} z_2^{-(\mu_2 - w_2)} J \Delta(z_1^{-1}, z_2^{-1}) J \quad (40a)$$

where each element δ_i must meet

$$\delta_i(z_1, z_2) = z_1^{-(\mu_1 - w_1)} z_2^{-(\mu_2 - w_2)} \delta_{N-i-1}(z_1^{-1}, z_2^{-1}). \quad (40b)$$

It is therefore possible to iteratively design linear phase filter banks which give perfect reconstruction by the structure

$$H_p'(z_1, z_2) = H_p(z_1, z_2) \Delta(z_1, z_2) E \quad (41)$$

where E meets (39), and $\Delta(z_1, z_2)$ is a diagonal matrix of delays which meet (40b). When H_p is paraunitary and E is orthogonal, the system is paraunitary; otherwise it is nonparaunitary. In the paraunitary case, the inverse system is given by

$$G_p(z_1, z_2) = F(z_1, z_2)\, H_p(z_1^{-1}, z_2^{-1})\, \Delta(z_1^{-1}, z_2^{-1}) E \tag{42a}$$

where the monic monomial $F(z_1, z_2)$ should be chosen so that the synthesis system remains causal. For an invertible system, the synthesis is given by

$$G_p(z_1, z_2) = F(z_1, z_2)\, H_p(z_1, z_2)^{-T}\, \Delta(z_1^{-1}, z_2^{-1}) E^{-T} \tag{42b}$$

provided that E is nonsingular (the superscript $-T$ denotes the transposed inverse).

The design structure in (41) is a generalization of the one-dimensional iterative method for the design of linear phase nonparaunitary filter banks, given by Vetterli and Le Gall in [24]. The above method is not complete in that certain solutions cannot be reached. Nevertheless, the structure seems promising and may yield useful linear phase paraunitary filter banks, as illustrated in Example 5.

VII. Conclusions

This paper has presented new results on the theory of general two-dimensional multirate filter banks. The multirate theory is general, with no restrictions on the geometries of the input and subsampling lattices. By decomposition of a filter bank into its polyphase components, the input/output relation of a subband analysis and synthesis system is determined leading to necessary and sufficient conditions for alias-free and perfect signal reconstruction. Then, working on the polyphase decomposition, design structures were given which gave rise to the design of two-dimensional perfect reconstruction filter banks. The four design structures treated in this paper are for paraunitary as well as nonparaunitary systems. The advantage with paraunitary systems is that the synthesis filter bank has the same properties as the analysis bank. Nonparaunitary systems are interesting mainly for their higher number of free variables compared to their paraunitary counterpart. For both types of systems, a nonseparable system has more free variables than does a comparable separable system.

A condition was derived to test these filter banks in the polyphase domain for linear phase, which also permitted us to develop a restricted design structure for linear phase systems. This structure was illustrated in an example which pointed out one case where nonseparable systems have properties unobtainable by separable systems. In this case, a paraunitary linear phase system was given which in two dimensions only exists in the nonseparable case.

It is hoped that the described work will help and inspire more research in nonseparable filter banks. Such work is currently under way and preliminary results are presented in Example 5.

Appendix

In the z-transform domain, subsampling by N, as given by the subsampling matrix D, can be expressed as

$$Y(z_1, z_2) = \frac{1}{N} \sum_{k=0}^{d_{00}-1} \sum_{l=0}^{d_{11}-1} X\left(W_N^{-d_{11}k} z_1^{1/d_{00}},\; W_N^{-(d_{00}l - d_{01}k)} z_1^{-d_{01}/N} z_2^{1/d_{11}}\right). \tag{43}$$

The reciprocal upsampling gives

$$Y(z_1, z_2) = X(z_1^{d_{00}}, z_1^{d_{01}} z_2^{d_{11}}). \tag{44}$$

One way to verify the above is to substitute the expression in (44) into (43). The resulting expression is reduced to $X(z_1, z_2)$, since upsampling followed by subsampling by the same pattern is an identity operation.

Next we show that the polynomial in (3) can only contain powers of $z_1^{d_{00}}$ and $z_1^{d_{01}} z_2^{d_{11}}$. We show it for a monomial $X(z_1, z_2) = z_1^a z_2^b$ by determining the powers a and b for which $Y(z_1, z_2) \neq 0$:

$$\begin{aligned} Y(z_1, z_2) &= \frac{1}{N} \sum_{k=0}^{d_{00}-1} \sum_{l=0}^{d_{11}-1} (W_N^{-d_{11}k} z_1)^a \cdot (W_N^{-(d_{00}l - d_{01}k)} z_2)^b \\ &= \frac{z_1^a z_2^b}{N} \sum_{k=0}^{d_{00}-1} W_N^{-(d_{11}a - d_{01}b)\,k} \underbrace{\sum_{l=0}^{d_{11}-1} W_{d_{11}}^{-bl}}_{=0 \text{ for } b \neq jd_{11}} \\ &= \frac{z_1^a z_2^{jd_{11}}}{d_{00}} \underbrace{\sum_{k=0}^{d_{00}-1} W_{d_{00}}^{-(a - jd_{01})k}}_{= \text{ for } a - jd01 \neq id_{00}} \\ &= (z_1^{d_{00}})^i (z_1^{d_{01}} z_2^{d_{11}})^j. \end{aligned} \tag{45}$$

We have thereby proven the statement for a monomial and by superposition it is clear that it holds for any two-dimensional polynomial. □

Finally we show that orthogonality of the matrix R in (22) is indeed sufficient to ensure that the system in (21) is paraunitary.

Let the $m \times N$ polynomial matrix $Q(z_1, z_2)$ be defined as

$$Q(z_1, z_2) = \left\{ \begin{pmatrix} z_1 I_{m_1} & 0 \\ 0 & z_2 I_{m_2} \end{pmatrix} - A \right\}^{-1} B. \tag{46}$$

Then we can write

$$\underbrace{\begin{pmatrix} \operatorname{diag}(z_1 I, z_2 I) Q(z_1, z_2) \\ H_p(z_1, z_2) \end{pmatrix}}_{W(z_1, z_2)} = \underbrace{\begin{pmatrix} B & A \\ D & C \end{pmatrix}}_{R} \begin{pmatrix} I \\ Q(z_1, z_2) \end{pmatrix} \tag{47}$$

where $H_p(z_1, z_2)$ is the transfer function in (21). Since R is assumed to be orthogonal, we have that $W(z_1^{-1}, z_2^{-1})^T$

$\boldsymbol{W}(z_1, z_2)$ is reduced to

$$\boldsymbol{Q}(z_1^{-1}, z_2^{-1})^T \boldsymbol{Q}(z_1, z_2) + \boldsymbol{H}_p(z_1^{-1}, z_2^{-1})^T \boldsymbol{H}_p(z_1, z_2) = \boldsymbol{Q}(z_1^{-1}, z_2^{-1})^T \boldsymbol{Q}(z_1, z_2) + \boldsymbol{I} \quad (48)$$

from which the statement immediately follows. □

Examples

Example 1

Hexagonal subsampling can be described by either $\boldsymbol{D}_1$ or $\boldsymbol{D}_2$ below, of which we opt for the form $\boldsymbol{D}_2$. This subsampling is also illustrated in Fig. 2. The two matrices are related to one another by a matrix with integer elements and determinant equal to 1:

$$\boldsymbol{D}_1 = \begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix} \qquad \boldsymbol{D}_2 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \quad \text{and}$$

$$\boldsymbol{D}_2 = \boldsymbol{D}_1 \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

$\boldsymbol{D}_2$ gives subsampling by factors 2 and 4 in the directions of n_1 and n_2, respectively. Note that a one-dimensional subsampling by $N = 4$, as given by $\boldsymbol{D}_3 = \text{diag}\,(4 \quad 1)$, cannot be related to the truly two-dimensional structures given above; thus, it describes a completely different sublattice.

Example 2

For hexagonal subsampling the analysis filter bank is described by its polyphase components as

$$\bar{H}(z_1, z_2) = \begin{pmatrix} H_{p0,0,0}(z_1^2, z_1 z_2^2) & \cdots & H_{p0,1,1}(z_1^2, z_1 z_2^2) \\ \vdots & \cdots & \vdots \\ H_{p3,0,0}(z_1^2, z_1 z_2^2) & \cdots & H_{p3,1,1}(z_1^2, z_1 z_2^2) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ z_1^{-1} \\ z_2^{-1} \\ z_1^{-1} z_2^{-1} \end{pmatrix}.$$

Example 3

For hexagonal subsampling, the following form of $\boldsymbol{T}_p$ is sufficient to meet the condition for aliasing cancellation in (13):

$$\boldsymbol{T}_p(z_1, z_2) = \begin{pmatrix} T_0(z_1, z_2) & T_1(z_1, z_2) & T_2(z_1, z_2) & T_3(z_1, z_2) \\ T_1(z_1, z_2) z_1^{-1} & T_0(z_1, z_2) & T_3(z_1, z_2) z_1^{-1} & T_2(z_1, z_2) \\ T_3(z_1, z_2) z_2^{-1} & T_2(z_1, z_2) z_1 z_2^{-1} & T_0(z_1, z_2) & T_1(z_1, z_2) \\ T_2(z_1, z_2) z_2^{-1} & T_3(z_1, z_2) z_2^{-1} & T_1(z_1, z_2) z_1^{-1} & T_0(z_1, z_2) \end{pmatrix}$$

which yields

$$\begin{aligned} &(1 \quad z_1 \quad z_2 \quad z_1 z_2) \boldsymbol{T}_p(z_1^2, z_1 z_2^2) \\ &= [T_0(z_1^2, z_1 z_2^2) + z_1^{-1} T_1(z_1^2, z_1 z_2^2) \\ &\quad + z_2^{-1} T_2(z_1^2, z_1 z_2^2) + z_1^{-1} z_2^{-1} T_3(z_1^2, z_1 z_2^2)] \\ &\quad \cdot (1 \quad z_1 \quad z_2 \quad z_1 z_2). \end{aligned}$$

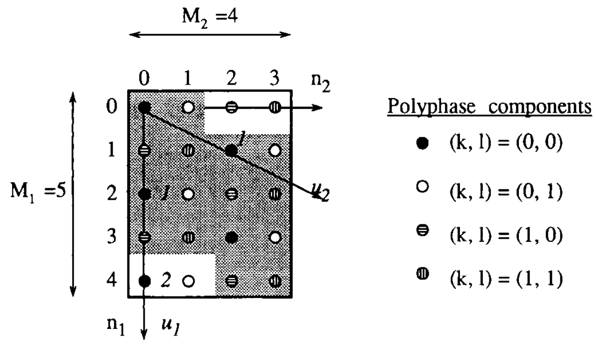


Fig. 4. Region of support for the impulse response of Example 2 with hexagonal subsampling. $M_1 \times M_2 = 5 \times 4$. The indices (k, l) refer to the four polyphase components.

Example 4

For hexagonal subsampling, choose $w_1 = 2$ and $w_2 = 2$ in (35), which gives $M_1 \times M_2 = 5 \times 4$. From (33) we get $k' = 1 - k$ and $l' = 1 - l$. This impulse response is illustrated in Fig. 4. If we assume that the first two filters are symmetric and the other two are antisymmetric, the linear phase condition becomes

$$H_{p0,0,0}(z_1, z_2) = z_1^{-1} z_2^{-1} H_{p0,1,1}(z_1^{-1}, z_2^{-1})$$

$$H_{p0,0,1}(z_1, z_2) = z_1^{-1} z_2^{-1} H_{p0,1,0}(z_1^{-1}, z_2^{-1})$$

$$\vdots$$

$$H_{p3,0,0}(z_1, z_2) = -z_1^{-1} z_2^{-1} H_{p3,1,1}(z_1^{-1}, z_2^{-1})$$

$$H_{p3,0,1}(z_1, z_2) = -z_1^{-1} z_2^{-1} H_{p3,1,0}(z_1^{-1}, z_2^{-1}).$$

Example 5

A four-subband system would be designed by matrices of the forms

$$\boldsymbol{E} = \begin{pmatrix} e_{00} & e_{01} & e_{02} & e_{03} \\ e_{10} & e_{11} & e_{12} & e_{13} \\ e_{13} & e_{12} & e_{11} & e_{10} \\ e_{03} & e_{02} & e_{01} & e_{00} \end{pmatrix}$$

$$\boldsymbol{\Delta}(z_1, z_2) = \begin{pmatrix} z_1^{-\alpha} z_2^{-\beta} & 0 & 0 & 0 \\ 0 & z_1^{-\gamma} z_2^{-\delta} & 0 & 0 \\ 0 & 0 & z_1^{-\zeta} z_2^{-\tau} & 0 \\ 0 & 0 & 0 & z_1^{-\eta} z_2^{-\xi} \end{pmatrix}$$

where $\eta = -\alpha + (\mu_1 - w_1)$, $\zeta = -\gamma + (\mu_1 - w_1)$, $\tau = -\delta + (\mu_2 - w_2)$, and $\xi = -\beta + (\mu_2 - w_2)$. The powers α and γ can be in the range $0 \cdots (\mu_1 - w_1)$, and β and δ can be in the range $0 \cdots (\mu_2 - w_2)$ (see (40b)).

A specific design example of a nonparaunitary linear phase filter bank is given here for $w_1 = w_2 = 0$ and $\mu_1 = \mu_2 = 1$:

$$H_p(z_1, z_2) = \underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}}_{W} \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z_1^{-1} & 0 & 0 \\ 0 & 0 & z_2^{-1} & 0 \\ 0 & 0 & 0 & z_1^{-1}z_2^{-1} \end{pmatrix}}_{\Delta} \cdot \underbrace{\begin{pmatrix} a_0 & a_2 & a_1 & a_3 \\ a_2 & a_0 & a_3 & a_1 \\ a_1 & a_3 & a_0 & a_2 \\ a_3 & a_1 & a_2 & a_0 \end{pmatrix}}_{E}.$$

As given by (6), the four polyphase components of the first filter become

$$H_{p0}(z_1, z_2) = \begin{pmatrix} a_0 + a_2z_1^{-1} + a_1z_2^{-1} + a_3z_1^{-1}z_2^{-1} & a_1 + a_3z_1^{-1} + a_0z_2^{-1} + a_2z_1^{-1}z_2^{-1} \\ a_2 + a_0z_1^{-1} + a_3z_2^{-1} + a_1z_1^{-1}z_2^{-1} & a_3 + a_1z_1^{-1} + a_2z_2^{-1} + a_0z_1^{-1}z_2^{-1} \end{pmatrix}.$$

The other three filters have polyphase components which are similar; only sign changes make the polynomials different. If we evaluate the polyphase matrix for a separable subsampling pattern with $N = 4$ and $D = \text{diag}(2 \;\; 2)$, then we get the filters

$$H_0 = \begin{pmatrix} a_0 & a_1 & a_1 & a_0 \\ a_2 & a_3 & a_3 & a_2 \\ a_2 & a_3 & a_3 & a_2 \\ a_0 & a_1 & a_1 & a_0 \end{pmatrix}$$

$$H_1 = \begin{pmatrix} a_0 & a_1 & a_1 & a_0 \\ a_2 & a_3 & a_3 & a_2 \\ -a_2 & -a_3 & -a_3 & -a_2 \\ -a_0 & -a_1 & -a_1 & -a_0 \end{pmatrix}$$

$$H_2 = \begin{pmatrix} a_0 & a_1 & -a_1 & -a_0 \\ a_2 & a_3 & -a_3 & -a_2 \\ a_2 & a_3 & -a_3 & -a_2 \\ a_0 & a_1 & -a_1 & -a_0 \end{pmatrix}$$

$$H_3 = \begin{pmatrix} a_0 & a_1 & -a_1 & -a_0 \\ a_2 & a_3 & -a_3 & -a_2 \\ -a_2 & -a_3 & a_3 & a_2 \\ -a_0 & -a_1 & a_1 & a_0 \end{pmatrix}.$$

Note that the filters are nonseparable when $a_3 \neq a_1a_2$. The matrix E, which is not orthogonal, was chosen to obtain the symmetries in the filters. The synthesis filters are given by

$$G_p(z_1, z_2) = \tfrac{1}{4}z_1^{-1}z_2^{-1}W\Delta(z_1^{-1}, z_2^{-1})E^{-T}.$$

E^{-1} retains the structure of E so that the synthesis filters, $G_0 \cdots G_3$, have the same symmetries as the analysis filters above. In Fig. 5, the frequency-amplitude response of H_0 is plotted for three different sets of coefficients. In (a) and (b) the impulse responses are separable and in (c) nonseparable. While the separable cases have either good stopband rejection or a smooth amplitude response, the nonseparable case gives a reasonably good compromise between stopband rejection and smoothness.

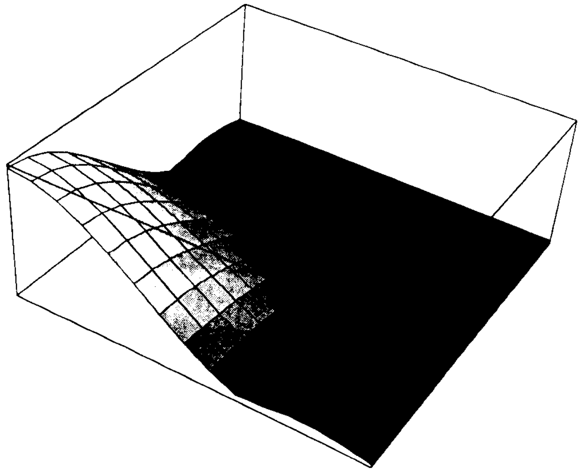
A paraunitary system can be obtained in this example if the E matrix is chosen as

$$E = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ a_1 & -a_0 & -a_3 & a_2 \\ a_2 & -a_3 & -a_0 & a_1 \\ a_3 & a_2 & a_1 & a_0 \end{pmatrix}$$

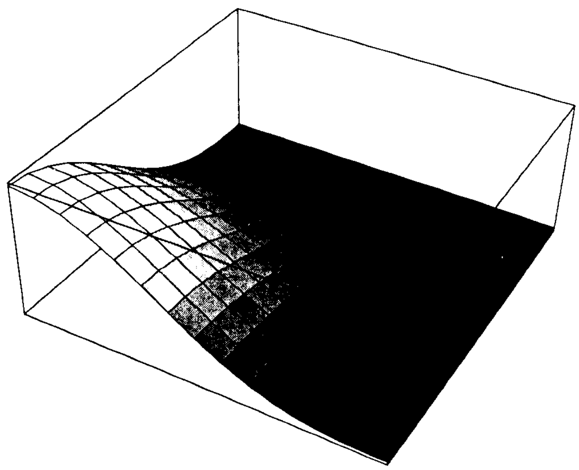
with

$$a_2 = \sqrt{a_0^2 \frac{1 - a_0^2 - a_1^2}{a_0^2 + a_1^2}} \qquad a_3 = -\sqrt{a_1^2 \frac{1 - a_0^2 - a_1^2}{a_0^2 + a_1^2}}.$$

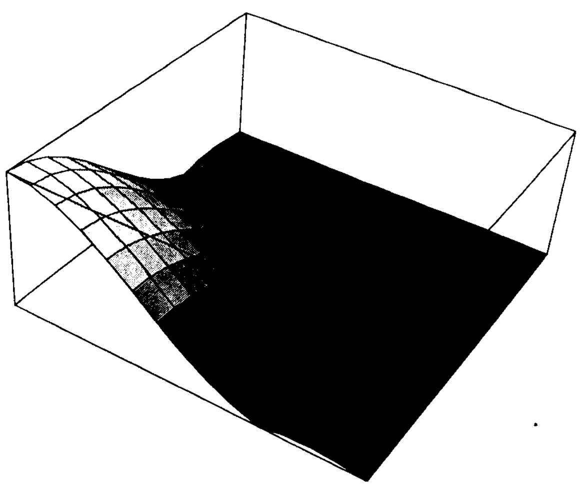
There are no paraunitary linear phase filter banks in the two-channel 1-D case [24]; thus, no separable solution exists in the four-channel 2-D case. However, the structure above gives four-channel paraunitary linear phase filter banks and thereby demonstrates the greater freedom offered by nonseparable systems.



(a)



(b)



(c)

Fig. 5. Two-dimensional amplitude-frequency response of separable versus nonseparable perfect reconstruction linear phase filters for separable subsampling by 2 in each dimension. Comparison of the low-pass filters, $\boldsymbol{H}_0$. (a) Separable filter with good out-of-band rejection but nonsmooth frequency response. The filter coefficients are $a_0 = 1$, $a_1 = a_2 = 3$, and $a_3 = a_1 \cdot a_2 = 9$. (b) Separable filter with smooth frequency response but poor out-of-band rejection. The coefficients are $a_0 = 1$, $a_1 = a_2 = 2$, and $a_3 = a_1 \cdot a_2 = 4$. (c) Nonseparable filter with smooth frequency response and out-of-band rejection comparable to (a). In this case, the filter coefficients are $a_0 = 1$, $a_1 = a_2 = 3$, and $a_3 = 6$.

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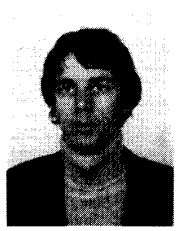

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