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THERE IS MORE TO DISCOURSE THAN MEETS THE EARS: LOOKING AT THINKING AS COMMUNICATING TO LEARN MORE ABOUT MATHEMATICAL LEARNING

ABSTRACT. Traditional approaches to research into mathematical thinking, such as the study of misconceptions and tacit models, have brought significant insight into the teaching and learning of mathematics, but have also left many important problems unresolved. In this paper, after taking a close look at two episodes that give rise to a number of difficult questions, I propose to base research on a metaphor of *thinking-as-communicating*. This conceptualization entails viewing learning mathematics as an initiation to a certain well defined *discourse*. Mathematical discourse is made special by two main factors: first, by its exceptional reliance on symbolic artifacts as its *communication-mediating* tools, and second, by the particular *meta-rules* that regulate this type of communication. The meta-rules are the observer's construct and they usually remain tacit for the participants of the discourse. In this paper I argue that by eliciting these special elements of mathematical communication, one has a better chance of accounting for at least some of the still puzzling phenomena. To show how it works, I revisit the episodes presented at the beginning of the paper, reformulate the ensuing questions in the language of thinking-as-communication, and re-address the old quandaries with the help of special analytic tools that help in combining analysis of mathematical content of classroom interaction with attention to meta-level concerns of the participants.

In the domain of mathematics education, the term *discourse* seems these days to be on everyone's lips. It features prominently in research papers, it can be heard in teacher preparation courses, and it appears time and again in a variety of programmatic documents that purport to establish instructional policies (see e.g. NCTM, 2000). All this could be interpreted as showing merely that we became as aware as ever of the importance of mathematical conversation for the success of mathematical learning. In this paper, I will try to show that there is more to discourse than meets the ears, and that putting communication in the heart of mathematics education is likely to change not only the way we teach but also the way we think about learning and about what is being learned. Above all, I will be arguing that communication should be viewed not as a mere aid to thinking, but as almost tantamount to the thinking itself. The *communicational approach to cognition*, which is under scrutiny in this paper, is built around this basic theoretical principle.

In what follows, I present the resulting vision of learning and explain why this conceptualization can be expected to make a significant con-

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tribution to both theory and practice of mathematics education. I begin with taking a close look at two episodes that give rise to a number of difficult questions. The intricacy of the problems serves as the immediate motivation for a critical look at traditional cognitive research, based on the metaphor of learning-as-acquisition, and for the introduction of an additional conceptual framework, grounded in the metaphor of learning-as-participation. In the last part of this article, in order to show how the proposed conceptualization works, I revisit the episodes presented at the beginning of the paper, reformulate the longstanding questions in the new language, and re-address the old quandaries with the help of specially designed analytic tools.

1. QUESTIONS WE HAVE ALWAYS BEEN ASKING ABOUT MATHEMATICAL THINKING AND ARE STILL WONDERING ABOUT

In spite of its being a relatively young discipline, the study of mathematical thinking has a rich and eventful history. Since its birth in the first half of the 20th century, it has been subject to quite a number of major shifts (Kilpatrick, 1992; Sfard, 1997). These days it may well be on its way toward yet another reincarnation. What is it that makes this new field of research so prone to change? Why is it that mathematics education researchers never seem truly satisfied with their own past achievements?

There is certainly more than one reason, and I shall deal with some of them later. For now, let me give a commonsensical answer, likely to be heard from anybody concerned with mathematics education – teachers, students, parents, mathematicians, and just ordinary citizens concerned about the well-being of their children and their society. The immediate suspect, it seems, is the visible gulf between research and practice, expressing itself in the lack of significant, lasting improvement in teaching and learning that the research is supposed to bring. It seems that there is little correlation between the intensity of research and research-based development in a given country and the average level of performance of mathematics students in this country (see e.g. Macnab, 2000; Schmidt et al., 1999; Stigler and Hiebert, 1999). This, in turn, means that as researchers we may have yet a long way to go before our solutions to the most basic problems asked by frustrated mathematics teachers and by desperate students become effective in the long run. The issues we are still puzzled about vary from most general questions regarding our basic assumptions about mathematical learning, to specific everyday queries occasioned by concrete classroom situations. Let me limit myself to just two brief examples of teachers' and researchers' dilemmas.

A function g(x) is partly represented by the table below. Answer the questions in the

x	$g(x)$
0	-5
1	0
2	5
3	10
4	15
5	20

(1) What is $g(6)$? _____
(2) What is $g(10)$? _____
(3) The students in grade 7 were asked to write an expression for the function g(x).
Evan wrote $g(x) = 5(x - 1)$
Amy wrote $g(x) = 3(x - 3) + 2(x - 2)$
Stuart wrote $g(x) = 5x - 5$
Who is right? Why?

Figure 1. *Slope* episode – The activity sheet.

Example 1: Why do children succeed or fail in mathematical tasks? What is the nature and the mechanism of the success and of the failure?

Or, better still, why does mathematics seem so very difficult to learn and why is this learning so prone to failure? This is probably the most obvious among the frequently asked questions, and it can be formulated at many different levels. The example that follows provides an opportunity to observe a 'failure in the making' – an unsuccessful attempt at learning that looks like a rather common everyday occurrence.

Figure 2 shows an excerpt from a conversation between two twelve year old boys, Ari and Gur, grappling together with one of a long series of problems supposed to usher them into algebraic thinking and to help them in learning the notion of function.[1] The boys are dealing with the first question on the worksheet presented in Figure 1. The question requires finding the value of the function $g(x)$, represented by a partial table, for the value of x that does not appear in the table ($g(6)$). Before proceeding, the reader is advised to take a good look at Ari and Gur's exchange and try to answer the most natural questions that come to mind in situation like this: What can be said about the boys' understanding from the way they go about the problem? Does the collaboration contribute in any visible way to their learning? If either of the students experiences difficulty, what is the nature of the problem? How could he be helped? What would be an effective way of overcoming – or preventing altogether – the difficulty he is facing?

While it is not too hard to answer some of these questions, some others seem surprisingly elusive. Indeed, a cursory glance at the transcript is enough to see that while Ari proceeds smoothly and effectively, Gur is unable to cope with the task. Moreover, in spite of Ari's apparently adequate algebraic skills, the conversation that accompanies the process of solving does not seem to help Gur. We can conclude by saying that while Ari's performance is fully satisfactory, Gur does not 'pass the test'.

WHAT IS DONE	WHAT IS SAID
	- 25:40 -
[1] A. is trying to get the expression from the table	[1] A.: [1a] Wait, how do we find out the slope again? [1b] No, no, no, no. Slope, no, wait, [1c] intercept is negative 5. [1d]Slope
	[2] G.: What are you talking about?
	[3] A.: I'm talking about this. It's 5.
	[4] G.: It doesn't matter if it's on (mumble)
	[5] A.: 5x. Right?
[6] A. has written 5x + -5	[6] G.: What's that?
	[7] A.: It's the formula, so you can figure it out.
	[8] G.: Oh. How'd you get that formula?
[9] to do the next task: find g(6)	[9] A.: and you replace the x by 6.
	[10] G.: Oh. Ok, I
	[11] A.: [11a] Look. Cause the, um the slope, is the zero. [11b] Ah, no, the intercept is the zero.
	[12] G.: Oh, yeah, yeah, yeah. So you got your
[13] "each": A. points to both columns, indicating that you have to check both "from zero to what": he points to the x column	[13] A.: [13a] And then you see how many is in between each, [13b] like from zero to what
[14] the left counterpart of the right-column 0 is 1	[14] G.: And the slope is, so the slope is 1.
[15] "zero": he circles the zero n the x column on G.'s sheet	[15] A.: [15a] Hum? No, the slope, [15b] see you look at zero,
[16] -5 is the f(x) value when x = 0	[16] G.: [16a] Oh that zero, ok. [16b] So the slope is minus 5
	[17] A.: yeah. And
	[18] G.: How are you supposed to get the other ones?
[19] A. first points to x column ("going down by ones"), then the f(x) column ("by fives"), and again to f(x) column ("look here")	[19] A.: [19a] You look how many times it's going down, like we did before. So it's going down by ones. [19b] So then it's easy. This is ah by fives. See, it's going down by ones, so you just look here
	[20] G.: Oh. So it's 5
	[21] A.: yeah. 5x plus
	[22] G.: Negative 5.
	[23] A.: Do you understand?
	[24] G.: [24a] Negative 5. Yeah, yeah, ok. [24b] So what is g 6?
	[25] A.: [25a] 5 times 6 is 30, plus negative 5 is 25. [25b] So we did get it right.
[26] "this column": he points to x column	[26] G.: No, but it's - in this column there?
	[27] A.: yeah
	[28] G.: [28a] Oh, then that makes sense. [28b] (writes) It's 30. [28c] What is g 10? ... 40
	[29] A.: 20, ah 40. No, 45.
	[30] G.: No,
	[31] A.: 45
	[32] G.: because 20
	[33] A.: 10 times 5 is 50, minus
[34] he points to the two entries in the last row	[34] G.: Well, 5 is 20, so 10 must have 40
[35] A. circles the 10 in g(10) on G.'s sheet	[35] A.: times 5
	[36] G.: Oh, we do that thing. Ok, just trying to find it.
	[37] A.: yeah
[38] again he points to the last row of the table	[38] G.: Cause I was thinking cause 5 is 20,
	[39] A.: It's 45. Yeah
	[40] G.: (mumble) So it's 45.
	- 27:42 -

Figure 2. *Slope* episode – the protocol.

So far so good: The basic question about the overall effectiveness of the students' problem-solving efforts does not pose any special difficulty. Our problem begins when we attempt a move beyond this crude evaluation and venture a quest for a deeper insight into the boys' thinking. Let us try, for example, to diagnose the nature of Gur's difficulty. The first thing to say would be "Gur does not understand the concept of function" or, more precisely, "He does not understand what the formula and the table are all about, what is their relation, and how they should be used in the present context". Although certainly true, this statement has little explanatory power. What Tolstoy said about unhappiness seems to be true also about the lack of understanding: Whoever lacks understanding fails to understand in his or her own way. We do not know much if we cannot say anything specific about the unique nature of Gur's incomprehension.

In tune with a long-standing tradition, many researchers are likely to approach the problem quite differently. As Davis (1988) pointed out, rather than asking *whether* a person understands, we should ask *how* he or she understands. Indeed, "students usually *do* deal with meanings", he says, except that they often "*create their own meanings*" (p. 9, emphases in the original). Thus, we could analyze the event in terms of students' idiosyncratic conceptual constructions. We could say, for example, that unlike his partner, Gur has not, as yet, developed an adequate conception of function. One look at the transcript now, and we identify the familiar nature of the inadequacy: The sequence [28]–[34] shows that Gur holds the ill-conceived idea of linearity, according to which the values of any function should be proportional to the argument (this belief is a variant of the well known misconception according to which any function should be linear; see e.g. Markovitz et al., 1986, Vinner and Dreyfus, 1989).[2] This is important information, no doubt, but is it enough to satisfy our need for explanation? Is it enough for us to say we have understood Gur's thinking? Is it sufficient to guide us as teachers who wish to help Gur in his learning?

Although endowed with an extensive knowledge of students' typical misconceptions, we may still be in the dark about many aspects of this conversation and, more specifically, about the reasons for Gur's choices and responses. Thus, for example, what has been said so far does not give us a clue about the sources either of Gur's lasting confusion with the equation of linear function, or of his inability to follow Ari's explanations. The misconception that certainly plays a role in the last part of the exchange does not account for Gur's earlier responses to the notion of formula. These responses seem as unexpected as they are unhelpful. Moreover, although it is obvious that Gur does struggle for understanding, and although the ideas he wishes to understand do not appear to be very complex (indeed, what

could be more straightforward than the need to substitute a number into the formula in order to calculate the value of the function for this number?), all his efforts prove strangely ineffective – they do not seem to take him one step closer to the understanding of the solution explained time and again by Ari. It is not easy to decide what kind of action on the part of the 'more capable peer' (Vygotsky, 1978, p. 86) could be of help.

At this point one may claim that the difficulty we are facing as interpreters stems mainly from the scarcity of data at hand. The episode we are looking at does not provide enough information for any decisive statement on Ari's and Gur's mathematical thinking, some people are likely to say. Although certainly true, this claim does not undermine the former complaint: Although it would certainly be better to have more information, the episode at hand should also be understood on its own terms. What we need in order to make sense of the things the two boys are saying in the given situation are not just additional data, but also, and above all, better developed ways of looking, organized into more penetrating theories of mathematical thinking and learning. Before we turn to the story of the current quest after such theories, let us look at another case of mathematical learning.

Example 2: What should count as 'learning with understanding'?

The notion of understanding, so central to our present deliberations, turns out to be an inexhaustible source of difficulty for both theorists and practitioners. I will now illustrate this difficulty with yet another example related, this time, to the famous call for *meaningful learning* or *learning-with-understanding* that has been guiding our instructional policies for many years. This call was a landmark in the history of educational research in that it signaled the end of the behaviorist era and the beginning of the new direction in the study of human cognition. When more than six decades ago Brownell (1935) issued the exhortation for "full recognition of the value of children's experiences" and for making "arithmetic less a challenge to pupil's memory and more a challenge to his intelligence" (p. 31), his words sounded innovative, and even defiant. Eventually, these words helped to lift the behaviorist ban on the inquiry into the 'black box' of mind. Once the permission to look 'inside human head' was given, the issue of understanding turned into one of the central topics of research.

In spite of the impressive advances of this research, most educators agree today that finding ways to make the principle of learning-with-understanding operative is an extremely difficult task. Methods of 'meaningful' teaching "are still not well known, and most mathematics teachers probably must rely on a set of intuitions about quantitative thinking that

[1]	Rada, the teacher:	Can you count to 10?
[2]	Noa:	Yes. One, two, three, four, five, six, seven, eight, nine, ten.
[3]	Teacher:	Do you know more than ten?
[4]	Noa:	Yes. One, two, 3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20.
[5]	Teacher:	What is the biggest number you can think of?
[6]	Noa:	Million.
[7]	Teacher:	What happens when we add one to million?
[8]	Noa:	Million and one.
[9]	Teacher:	Is it bigger than million?
[10]	Noa:	Yes.
[11]	Teacher:	So what is the biggest number?
[12]	Noa:	Two millions.
[13]	Teacher:	And if we add one to two millions?
[14]	Noa:	It's more than two millions.
[15]	Teacher:	So can one arrive at the biggest number?
[16]	Noa:	Yes.
[17]	Teacher:	Let's assume that *googol* is the biggest number. Can we add one to googol?
[18]	Noa:	Yes. There are numbers bigger than googol.
[19]	Teacher:	So what is the biggest number?
[20]	Noa:	There is no such number!
[21]	Teacher:	Why there is no biggest number?
[22]	Noa:	Because there is always a number which is bigger than that?

Figure 3. Conversation between a pre-service teacher and a 7 year old girl, Noa (first grade).

involves both the importance of meaning – however defined – and computation," complains Mayer (1983, p. 77). Hiebert and Carpenter echo this concern when saying that promoting learning with understanding "has been like searching for the Holy Grail." "There is a persistent belief in the merits of the goal, but designing school learning environments that successfully promote learning with understanding has been difficult," they add (Hiebert and Carpenter, 1992, p. 65). The conversation between pre-service teacher Rada and the 7 year old girl Noa about the concept of 'the biggest number' (see Figure 3) highlights a certain aspect of the difficulty.

Clearly, for Noa, this very brief conversation becomes an opportunity for learning. The girl begins the dialogue convinced that there is a number that can be called 'the biggest' and she ends emphatically stating the opposite: "There is no such number!". The question is whether this learning may be regarded as learning-with-understanding, and whether it is therefore the desirable kind of learning.

To answer this question, one has to look at the way in which the learning occurs. The seemingly most natural thing to say if one approaches the task from the traditional perspective, already mentioned in the former example, is that the teacher leads the girl to realize the contradiction in her conception of number: Noa views the number set as finite, but she also seems aware of the fact that adding one to any number leads to an even bigger number. These two facts, put together, lead to what is called in the

literature 'a cognitive conflict' (see e.g. Tall and Schwartzenberger, 1978), and thus call for revision and modification of her number schema. This is what the girl eventually does. On the face of it, the change occurs as a result of rational considerations, and may thus count as an instance of learning with understanding.

And yet, something seems to be missing in this explanation. Why is it that Noa stays quite unimpressed by the contradiction the first time she is asked about the number obtained by adding one? Why doesn't she modify her answer when exposed to it for the second time? Why is it that when she eventually puts together the two contradicting claims – the claim that adding one leads to a bigger number and the claim that there is such thing as *the* biggest number – her conclusion ends with a question mark rather than with a firm assertion (see [22])? Isn't the girl aware of the logical necessity of this conclusion?

Another possibility, one I will discuss in detail later in this paper, is that Noa's change of mind has less to do with her understanding of the concepts than with her spontaneous use of mostly involuntary cues about the appropriateness of her answers found in the teacher's reactions. In this case, the decision to say, in the end, that "there is no biggest number" cannot be regarded as an evidence of 'learning-with-understanding', at least not according to how the term 'understanding' is usually interpreted in this context. If so, the adherents of meaningful learning are likely to criticize the teacher for the instructional strategy she used. And yet, from my numerous encounters with teachers, I do know, that for the great majority of them, the way Rada proceeded in the present example would be the natural choice. Teacher's intuitions are not anything to be easily dismissed by the researcher. We seem to be facing yet another dilemma likely to challenge teachers and researchers.

Summary: On the learning-as-acquisition metaphor, its advantages and its shortcomings

After having had a look at a number of questions spawned by the two brief episodes, it is time now to say a few words about research in mathematics education in general. The ways researchers have been looking at the studied phenomena may be diverse and many, but all the known approaches were, until recently, unified by the same basic vision of learning. Influenced by folk models of learning implicit in our everyday ways of talking, and further encouraged by numerous scientific theories of mind that conceptualize learning as storing information in the form of mental representations, the students of mathematical thinking and problem-solving tacitly adopted the metaphor of *learning as the acquisition* of

knowledge. The emphasis here is on the term *acquisition*, which underlines the individual nature of the endeavor. The acquisition may take place either by passive reception or by active construction, resulting in a personalized version of concepts and procedures. More often than not, these individual constructions have been termed *misconceptions* rather than simply conceptions. This suggestive label implies that one should expect a disparity between learners' private versions and the 'official', 'correct' edition of mathematical concepts. Terms such as *concept image* (as opposed to concept definition; Tall and Vinner, 1981) or *tacit models* (Fischbein, 1989; Fischbein et al., 1985), which began to appear in parallel to the notion of misconception may be regarded as very close in meaning, as they imply the same basic idea of discrepancy between individual and public conceptual constructions.

The theories of conceptual development to which all these notions are somehow related draw on the idea of internal representation and on the Kantian/Piagetian concept of schemes – organizing mental structures everyone supposedly constructs for oneself from the elementary building blocks called conceptions. It is through these mental schemes that our conceptions purportedly get their meaning. Cognitive psychology equated understanding with perfecting mental representations and defined learning-with-understanding as one that effectively relates new knowledge to knowledge already possessed. Within the acquisitionist framework, therefore, understanding is a mode of knowledge, whereas knowledge itself is conceptualized as a certain object which a person either possesses or not, and learning is regarded as a process of acquiring this object (cf. Sfard, 1998). Once acquired, the knowledge is carried from one situation to another and used whenever appropriate. To put it into Jean Lave's words, within this long-standing tradition,

> mind and its contents have been treated rather like a well-filled toolbox. Knowledge is conceived as a set of tools stored in memory, carried around by individuals who take the tools (e.g. 'foolproof' arithmetic algorithms) out and use them, the more often and appropriately the better, after which they are stowed away again without change at any time during the process. (Lave, 1988, p. 24)

With its many branches in the quickly developing new science of cognition, this approach had been flourishing for a few decades, spawning a massive flow of research (see e.g. Hiebert and Carpenter, 1992).

At this point, it must be emphasized that with all the above criticism, it was by no means my intention to disparage either the acquisition metaphor or the theories that grow out of it. The idea of students' idiosyncratic conceptions and the notion of learning-with-understanding have done much good to both the theory and the practice of mathematics education, and

right now seem particularly useful to those who try to bridge the science of the mind with the science of the brain. My only point is that whether we act as researchers or as practitioners, the notions grounded in the acquisition metaphor may be too crude an instrument for some of our present more advanced needs. Acquisition-based theories 'distill' cognitive activities from their context and thus tell us only a restricted part of the story of learning. The elements that they leave out of sight are often indispensable for the kind of understanding that should underlie any sensible practical decision. In the former paragraphs I illustrated this claim with two examples, and in the last part of this paper I will be arguing that these missing elements may, in fact, be significant enough to change the picture in a radical way. The conclusion I am opting for is that rather than rejecting the long-standing acquisition metaphor, we should supplement it with theories grounded in alternative metaphors. The communicational approach, deeply rooted in one such metaphor, is to be regarded as complementary rather than incompatible with the more traditional outlooks. In the next section, I precede the introduction of the communicational approach with the presentation of a complementary metaphor.

2. COMMUNICATIONAL APPROACH TO COGNITION

Participationist approach to cognition

The complementary conceptualization of learning I wish to introduce in this article grows from the sociocultural tradition. As emphasized by the editors in the introduction to this volume, the central feature of this latter trend, one that renders it its unique identity and puts it quite apart from the former approaches to human cognition, is its deeply suspicious attitude toward the long-standing sweeping claims about various cognitive invariants – whether those supposed to cross cultural borders, those expected to remain unaffected by historical changes, or those that are merely believed to be transferred by an individual from one situation to another. All this said, please note that the emphasis in this last sentence is on the word *sweeping*. While sociocultural theories issue an admonition against ungrounded assumptions about universality and alert us to the conceptual difficulty inherent in the notion, they do not claim the total non-existence of cognitive invariants (see e.g. sociocultural account of the phylo- and ontogenesis of language in Bruner, 1986; see also Cole, 1996; Tomasello, 1999; Mantovani, 2000).

Disillusioned with the explanatory power of theories that speak of context-independent traits of the individual, sociocultural psychologists prefer

to view learning as *becoming a participant in certain distinct activities* rather than as becoming a possessor of generalized, context-independent conceptual schemes. Representatives of different variants of the sociocultural framework speak of learning as "peripheral participation in a community of practice" (Lave and Wenger, 1991), as "an improved participation in an interactive system" (Greeno, 1997), as "initiation to a discourse" (Edwards, 1993; Harre and Gillett, 1995) or as "a reorganization of an activity" (Cobb, 1998). There is an ontological gulf between the old and the new metaphors, and because of this deep disparity the conceptions of learning engendered by these metaphors diverge along many dimensions.

Before I survey the most immediate entailments of the participation metaphor, two cautionary remarks are in order. First, no theory is built on a single metaphor. However, of those metaphors that can be identified, one is usually the most prominent and influential. Also, not all of the differences between the different approaches are necessitated by the respective metaphors. Some of the entailments are optional and sustained by a mere habit. Both types, however, deserve attention as both of them have a considerable impact on theory and on practice. Second, dichotomy between acquisition and participation should not be mistaken for any of the well-known theoretical distinctions. As was stressed above, even if the acquisition metaphor is more common in the traditional cognitivist approach than in sociocultural theorizing, it is not altogether absent from the latter. Sometimes, it may even be quite prominent. This is certainly the case when one speaks, with Vygotsky – a thinker generally recognized as one of the founders of the sociocultural trend in psychology – about "interiorization of higher mental functions" by their transmission from "interpsychological" to "intrapsychological plane" (Vygotsky, 1931/1981, p. 163). Neither is the acquisition/participation dichotomy equivalent to the distinction between individualist and social perspectives on learning. Whereas the social dimension is salient in the participation metaphor, it is not necessarily absent from the theories dominated by the acquisition metaphor. It is important to understand that the two distinctions have been made according to different criteria: while acquisition/participation division is ontological in nature and draws on two radically different answers to the fundamental question "What is this thing called learning?", the individual/social dichotomy does not imply a controversy as to the definition of learning, but rather rests on differing visions of the mechanism of learning.

As was already said, for participationists learning is first and foremost about the development of ways in which an individual participates in well-established communal activities. The participationist researcher is therefore attuned to the ongoing interactions that spur this development, rather

than to those properties of the individual that can be held responsible for the constancy of this person's behavior. This vision implies that we should be less interested in explanations based on such unobservables as mental schemes, than in descriptions of the processes of learning, their patterns and mechanisms. The descriptions may be drawn with a special attention to those hitherto ignored dimensions of a learning situation that underlie the learner's increasing ability to create and sustain the "relation of mutual accountability" with other members of the community (Wenger, 1998, p. 81). In simpler words, the participationist researcher focuses on the growth of mutual understanding and coordination between the learner and the rest of the community. All this means that while acqusitionists are mainly interested in pinpointing cross-contextual invariants of learning, participationists shift the focus to the activity itself and to its changing, context-sensitive dimensions. In the case of Ari and Gur in our first example, this means analyzing the conversation with an eye to all those elements and circumstances of the boys' joint activity which make their exchange ineffective. In the case of Noa and Rada, it means asking the parallel question about the mechanisms of interaction that led to the student's alignment with the teacher. In both cases the shift of focus to the interactional aspects of learning implies attention to many factors that, so far, were deemed irrelevant to the issue of cognitive development.

Indeed, the inclusion of the community in the picture of learning affects the scope of things that must be considered when the change in the newcomer's ways of acting is studied. When regarded not as an isolated entity but as a part of a larger whole, the learner becomes but an aspect of a new, much broader unit of analysis,[3] many elements of which must be brought into the account even if the ultimate focus of the study is the change in the individual. In the two episodes above, this means that describing all that happens between the interlocutors exclusively in terms of stand-alone cognition, that is, of the actors' abilities and the contents of their minds (whatever the sense of the last two terms), means overlooking a great many aspects and factors of change. In the final account, this is bound to lead to an impoverished, if not distorted, unhelpful picture of learning.

Not only does success in problem solving prove highly sensitive to the context of the activity, say participationists, but also the ways people act would change from one situation to another.[4] Thus, abstract scholarly learning may have the theoretical advantage of a broader scope, but in reality it would often prove much less effective than apprentice-like participation in the restricted repertoire of specific activities for which the person wishes to prepare herself. Obviously, this belief has many implications for both educational practice and research. Participationists advocate

'cognitive apprenticeship' (Brown et al., 1989) as a preferred mode of learning, and as researchers they are at least as interested in the informal and workplace "legitimate peripheral participation",[5] as in institutionalized scholarly learning.

Yet another time-honored question likely to incite passionate debates between acquisitionists and participationists regards the nature and sources of human knowing. Acquisitionist interest in universal factors with which to account for those aspects of learning that seem relatively insensitive to social, cultural, historical, and situational context implies an emphasis on human-independent circumstances of learning, such as the direct encounter between the individual and the world, and a range of biological determinants, from inheritance to physiological growth and to the structure of human brain. Participationists, who view learning as entering a certain human practice, obviously shift the emphasis to the society as the setting that produces and sustains this practice. Indeed, participationists' deep skepticism about cross-cultural invariants is fueled by their view of learning as beginning and ending in society – as spurred by the need for interaction and communication and geared towards its continual growth. Since our very survival depends on our being a part of community, it is this need for communication that seems to be inscribed in humans. High sensitivity of our ways of acting to social, cultural, historical and situational contexts is an inevitable derivative of the fact that the activities themselves, rather than being dictated by an external non-human world, have their roots in our cultural heritage and are constantly shaped and re-shaped by successive generations of practitioners. This discussion between acquisitionists and participationists clearly echoes the centuries long nature-or-nurture controversy and may thus be read as its modern version.[6] In our examples, the way participationists propose to approach the dilemma suggests that, in an attempt to explain Gur or Noa's performance, much attention should be given to a variety of contextual factors before one decides to account for children's performance in terms of permanent traits, such as their 'mathematical ability' or the lack thereof.

Conceptualizing thinking as communicating

Although the participation metaphor may now appear pretty well defined, most attempts at turning its entailments into a sound basis for research and for practical decision-making are still in their initial stages. As stated by Cole:

> Nowhere are these ideas so highly developed that it is possible to refer to them as a mature scientific paradigm with generally accepted theoretical foundations,

a methodology, and a well-delineated set of prescriptions for relating theory to practice. (Cole, 1995, p. 187)

The words 'these ideas' in the quote refer to the principles underlying the sociocultural approach to cognition, and the statement itself, made nearly a decade ago,[7] seems to be still pretty much in force. And yet, if not the situation itself, then at least the chances for finding what is still missing do seem better, these days. In this last decade, quite a few significant attempts have been made at constructing frameworks that would meet the standards of a 'mature research paradigm' while respecting the basic sociocultural principles. The *communicational approach* presented in the rest of this paper is one of the currently available products of these attempts. With its roots in Vygotskian writings and with its branches in contemporary philosophical-sociological thought (e.g. Wittgenstein and French postmodern thinkers) and in recent advances in linguistics, this outlook seems to stand a good chance for turning into a full-fledged research framework fulfilling in a reasonable way the requirements specified by Cole.

The basic tenet of the communicational approach to the study of human cognition is that *thinking may be conceptualized as a case of communication*, that is, as one's communication with oneself. Indeed, our thinking is clearly a dialogical endeavor, where we inform ourselves, we argue, we ask questions, and we wait for our own response. The conceptualization of thinking as communication is an almost inescapable implication of the thesis on the inherently social origins of all human activities. Anyone who believes, as Vygotsky did, in the developmental priority of communicational public speech over inner private speech (e.g. Vygotsky, 1987) must also admit that whether phylogenesis or ontogenesis is considered, thinking arises as a modified private version of interpersonal communication. All this amounts to the claim that thinking is nothing but our communicating with ourselves, not necessarily inner, and not necessarily verbal. At this point, it is important to stress the crucial difference between this statement and the long-standing hypothesis that equates thinking with internalized speech: the word *communication* is used here in a very broad sense and is not confined to interactions mediated by language. This conceptualization of cognition, even if not stated explicitly, seems to be finding its way into today's psychological thinking. Harre and Gillett (1995) go so far as to declare the emergence of a new kind of psychology, one that they call *discursive. Discursive psychology* has been described by these authors as one that rests on the assumption similar to the one just stated above: "Individual and private uses of symbolic systems, which . . . constitute thinking, are derived from interpersonal discursive processes that are

the main feature of the human environment." (p. 27). The reason why I describe the present approach with the term *communicational* rather than discursive in spite of its clear similarity to the position taken by Harre and Gillett (and possibly shared with others; see e.g. Edwards, 1997), is that the former differs from the latter in its epistemological underpinnings and this difference proves highly consequential in terms of theoretical and methodological entailments. This difference will be explained in one of the following paragraphs.

A number of immediate entailments of this conceptualization should now be pointed out. First, since communication may be defined as a person's attempt to make an interlocutor act, think or feel according to her intentions (c.f. Levinson, 1983; Sfard, 2000a,b), research that looks at cognition as a communicational activity focuses, in fact, on the phenomenon of mutual regulation and of self-regulation. It is exactly this phenomenon which was singled out by Leont'ev as the hallmark of being human: "[W]e do not meet in the animal world any special forms of action having as their sole and special end the mastery of the behavior of other individuals by attracting their attention" (Leont'ev, 1930, p. 59, quoted in Cole, 1988). Thus, when one is looking at cognition as a form of communication, an individual becomes automatically a nexus in the web of social relations – both a reason for, and the result of, these relations. This is true whether this individual is in a real-time interaction with others or acts alone. Whatever attempt at understanding human beings is made, it must now take into account that all human actions and deeds are guided, in one way or another, by forces of social cohesion, that is by the fact that, just like different organs in our body, the individual does not exist except as parts of a larger whole.

Further, from the proposed vision of cognition it follows that thinking is subordinated to, and informed by, the demand of making communication effective. When harnessing this fact to the analysis of cognitive mechanisms, the first point to remember is that the basic driving forces, and thus basic mechanisms, are likely to be almost the same whether one considers communicating with oneself or with others. Second, in this approach the dichotomy/thought communication practically disappears and speech is no longer considered as a mere 'window to the mind' – as an activity secondary to thinking and coming just to 'express' a ready-made thought. Although there is still room for the talk about thought and speech as two different things, these two 'things' are to be understood as inseparable aspects of basically one and the same phenomenon, with none of them being prior to the other.

Learning as initiation to a discourse

Within this conceptual framework, the focus of study is on discourse. In our research, the term *discourse* will be used to denote any specific instance of communicating, whether diachronic or synchronic, whether with others or with oneself, whether predominantly verbal or with the help of any other symbolic system. The particularly broad meaning of the term in the present context implies inclusion of instances that would probably be excluded from the category of discourse by everyday users of the term. For example, the production of a written or spoken text, often considered as the defining feature of discourse, is not a necessary ingredient of what will counts for us as 'discursive'. I shall use only one rule for deciding whether a given aspect of an observed situation should count as a component of the discourse or not: Since discourses are analyzed as acts of communicating, anything that goes into communication and influences its effectiveness – body movements, situational clues, interlocutors' histories, etc. – must be included in the analysis.

Learning mathematics may now be defined as an initiation to mathematical discourse, that is, initiation to a special form of communication known as mathematical. Let us look at those factors that are automatically included in the study of thinking as communicating and which dictate what must be learned if a person is to become a skillful participant of a given discourse. Two types of such factors deserve particular attention: the *mediating tools* (or simply *mediators*) that people use as the means of communication, and the *meta-discursive rules* that regulate the communicative effort. While tools are the shapers of the content, that is, of the object-level aspects of discourse (cf. Sfard, 2000b; Sfard and Kieran, 2001a), meta-discursive rules are the molders, enablers and navigators of the communicational activities (Sfard, 2000c). The more detailed description that follows explains why both mediators and meta-discursive rules can be regarded as principal carriers of cultural heritage.

Factors that render discourses their distinct identities: Mediating tools and meta-discursive rules

Let us turn first to the *mediating tools*. "Man differs from animals in that he can make and use tools", says Luria (1928, p. 493). Communication, either inter-personal or self-orientated (thinking) would not be possible without symbolic tools, with language being the most prominent among them. In my opening examples, additional symbolic tools used by the children are the numerical notation, graphs, tables, and algebraic formulas. The tightness of the relation between the ways we conceptualize and the ways we symbolize can be seen, for example, from the fact that all our verbal

references to numbers (see e.g. those of Noa in Figure 3) bear distinct marks of the decimal notation, whether the decimal numerals are actually displayed or not (think, for example, about the way we perform mentally any calculation, notably multiplication by ten).

This last statement, referring to the role of symbols in thinking, is central enough to the present discussion to deserve further elaboration. Contrary to what is implied by a common understanding of a tool in general and of symbolic tools in particular, within the communicational framework one does not conceive of artifacts used in communication as mere auxiliary means that come to provide expression to pre-existing, pre-formed thought. Rather, one thinks about them as part and parcel of the act of communication and thus of cognition (for detailed argument see Sfard, 2000a). There is therefore no sense in which one could talk about thought as having an existence independent of the symbolic tools used in the process of communication. This means, among others, that we should regard as rather senseless such statements as "the same thought has been conveyed by different means" (which, however, does not mean we cannot *interpret* two expressions in the same way, with *interpretation* and *thought* being two different things). In other words, there is no 'cognitive essence' or 'pure thought' that could be extracted from one symbolic embodiment and put into another.

Let me now say a few words about *meta-discursive rules*. While tools play a central role in shaping the visible, object-level (content-related) aspects of discourse, meta-discursive rules are what guides the general course of communicational activities. It is noteworthy that meta-discursive rules are mostly invisible and act 'from behind the scene'. Because of their implicit nature, and in spite of their ubiquity, they have not been given any direct attention in the past. These days, the situation is changing quite rapidly, as the general interest in participationist framework and in discursive activities of 'mathematically-speaking' communities begins to spread (see e.g. Voigt, 1985, 1996; Bauersfeld, 1995; Lampert, 1990; Lampert and Blunk, 1998; Forman, 1996; Forman and Larreamendy-Joerns, 1998; Cobb, Wood and Yackel, 1993; Yackel and Cobb, 1996; O'Connor, 1998; Morgan, 1996; Sfard, 2000a,b,c; for a survey see Lampert and Cobb, in press).[8]

It is important to state right away that the term *meta-rules* is very broad and that, because of certain subtleties of its intended meaning, it is prone to misinterpretations. The first thing to note is that the idea is close to many other discourse-related concepts known from philosophical, sociological, and anthropological literature. Thus, for example, it is not altogether different from what Wittgenstein (1953) calls *language games* and what Bordieu

(1999) names *dispositions* (the latter, taken together, constitute *habitus*). It is also related to what Goffman (1974) refers to as interaction *frames* (see also Bateson, 1973), and what Bruner (1983) includes in the idea of *format.* The search for family resemblance must also lead, inevitably, to the fundamental work in sociology by Schutz (1967) and in ethnomethodology by Garfinkel (1967). In the domain of mathematics education, the term *socio-mathematical norms* used by some authors (e.g. Yackel and Cobb, 1996) may be viewed as describing a certain subset of meta-discursive rules, even though there is a subtle difference between the notions *rule* and *norm* (see discussion of this difference in Sfard, 2000c). This is to say that the term *meta-discursive rule* used in this article does not come as an entirely new construct but rather as an almost self-explanatory term supposed to encompass all the phenomena signaled by the notions listed above.

It is important to stress that in concert with Wittgenstein's idea of language games and with Bordieu's approach to the issue of social regulations, meta-rule should be understood as "an explanatory hypothesis constructed by the theorist in order to explain what he sees" (Bouveresse, 1999) rather than anything that is 'really there'. That is, meta-rules are usually not anything the interlocutors would be fully aware of, or would follow consciously. What a discourse analyst views as a meta-discursive rule can be compared to what a physicist considers to be a law of nature: the regularity that is seen by those who observe, but not necessarily by those who are seen as 'implementing' the rule.[9] Taking the interpretive status of the meta-discursive rule as a point of departure, I can now be a little more specific about this concept, while trying to illustrate it with a few examples (for a much more detailed treatment see Sfard, 2000c).

Within the communicational framework, meta-discursive rules should be understood as expressing themselves in regularities observed in those aspects of communicational activities that are not directly related to the particular content of the exchange (which does not mean the rules do not have an impact on the interlocutors' grasp of the content or that they do not change when the contents change). In concert with meta-discursive rules, people undertake actions that count as appropriate in a given context and refrain from behaviors that would look out of place. In the case of mathematical discourse, this category of rules includes those that underlie the uniquely mathematical ways of defining and proving. Further, it is thanks to spontaneously, non-reflectively observed meta-rules that interlocutors are able to navigate inter-personal exchange and regulate self-communication. It is within the system of meta-rules that people's culturally-specific norms, values, and beliefs are encoded. The way symbolic tools

should be used in the given type of communication is yet another aspect where a distinct category of meta-rules may be identified. There are also special sets of meta-rules involved in regulating interlocutors' mutual positioning and shaping their identities.

The variety of meta-rules navigating and molding a particular discourse is obviously very broad and heterogeneous, and, along with the meta-rules specific to this particular discourse, usually contains a sizable bulk of implicit regulations related to more general aspects of communication, and probably common to a wide range of discourses (Cazden, 1988). It is important to stress that meta-discursive rules are responsible not only for the ways people communicate, but also for the very fact that they are able to do so in the first place. These rules have an enabling effect in that they eliminate an infinity of possible discursive moves and leave the interlocutors with only a manageable number of choices.

Since meta-rules are tacit, they are usually taught and learned 'on the run', with teachers and students quite unaware of this learning. Some of the meta-rules that are included in this hidden curriculum are truly indispensable, some others may enter the scene as if against the teacher's better judgement. Close analysis that aims at eliciting these tacit ingredients of learning may lead to re-appreciation of certain educational principles. As I will argue below while revisiting the opening examples, such analysis would often show that even those 'unwanted' meta-rules may be an effective, sometimes irreplaceable, means for significant learning.

On the methodological aspects of the communicational research framework

The claim that the communicational approach has a chance to grow into a fully-fledged research framework cannot look fully convincing unless we can be certain of the possibility of supplementing it with a strong methodology. Although the efforts to build such methodology are still under way, it is quite clear that the proposed conceptualization of thinking implies a wide range of data-collecting strategies and can be expected to produce a rich and greatly diversified family of analytical methods. In addition to the already existing discourse and conversation analyses, those who work within the communicational approach to cognition have yet to construct and test their own methods of handling data, tailored according to their specific needs. Such methods seem to be on their way (see e.g. Steinbring et al., 1998; Lampert and Blunk, 1998). Above all, thanks to the disappearance of cognition/communication dichotomy, the present object of study, that is discursive processes, is much more accessible than the more traditional one – the cognitive processes 'in the mind'.

Let me add a word of caution. A few decades ago Wittgenstein (1953) issued a powerful argument against mentalism, requiring that psychological discourse be purified from any reference to 'mental states' and to the inherently unobservable entities 'in the mind'. In the now developing approaches to cognition, this exhortation is being interpreted and operationalized in more than one way. While discursivist psychologists are ready to follow Wittgenstein's call all way down (Harre and Gillett, 1995; Edwards, 1997), extreme logical behaviorism is not the outlook promoted in this paper. References to such 'unobservables' as people's *intentions* are made in the definition of communication underlying the communicational approach, and will often, if not always, feature prominently in analyses carried out within this framework. More generally, the leading assumption here is that our experiences, feelings, and intentions are central to all our decisions, and thus cannot be omitted in any serious attempt at understanding human actions. And yet, in the light of Wittgenstein's well substantiated caveat, even those who agree with this assumption may still wonder how such mentalist ideas as 'human experience' can be made researchable. Let me then remind ourselves, that when Wittgenstein was warning against mentalist language, he was doing this out of a concern about the possible circularity of the resulting definitions. It can be shown, however, that the danger of circularity is obviated if one refrains from comparisons between mental states of different people. Indeed, the use of such terms as *intentions* is safe as long as it is understood that the status of any claim about other people's intentions the researcher can make is *interpretive*, and thus any comparison that is being made is between the *researcher's own interpretations* of other people's intentions (for a more complete argument see Sfard and Kieran, 2001a).

The ultimate conclusion from these last remarks is that the only viable possibility for the researcher is to provide a *convincing interpretation* of the observed phenomena, as opposed to their definitive explanation. The interpretation should try to be as compelling, cogent, and trustworthy as possible, but it will nevertheless always remain subject to questioning and modifications. As interpreters, we should not make any claims either to exclusivity or completeness: tentativeness is the endemic property of interpretation, and the coexistence of alternative (or complementary) interpretations is part and parcel of the interpretive framework.

3. How does the communicational approach change the picture? Initial questions revisited

It is time now to demonstrate how the communicational approach, as presented in the former section, can possibly add to our understanding of the initial questions. Let me return to these questions, then, and try to look at them through the conceptual lens that equates thinking with communicating.

Why do children succeed or fail in mathematical tasks? What is the nature and the mechanism of the failure?

Let us return to the *Slope* episode, presented in Figures 1 and 2. We are now going to engage in an activity not unlike that of archeologists who use scarce remnants of an ancient vessel to reconstruct the original whole. If thinking is communicating, then a conversation between two persons is a complex combination of several tightly interrelated, partially overlapping attempts at communication, only some of which are accessible to observers, but all of which influence all the others. What is actually heard is like those available remnants of the ancient vessel and what is added through interpretation are the replacements of the missing parts. The reconstructed elements, although but a product of the archeologist's imagination, turn the scattered pieces into an integrated whole.

Within our present framework, Gur's failure is understood as a failure to communicate. In fact, within the communicational approach this failure should no longer be called 'Gur's'. Although it is true that the boy proves unable to lead an effective dialogue either with his partner or with himself, it is probably also true that this inability is not his inherent property but rather the property, and possibly the product, of the interaction between the two boys. In order to understand this point better, I will have to take a close, detailed look at the way the communication evolves. Scrutinizing the way the mathematical content enfolds will be the first thing to do, but it will not be the only one. In the preceding paragraphs I was talking about tacit factors that may have a considerable impact on the course and effectiveness of discursive interactions. In the attempt to understand the reasons for the lasting ineffectiveness of the communication I will thus have to look at these hidden factors as well. With this goal in mind, I will now use two types of analysis which complement each other, as one of them deals with object-level aspects of communication while the other aims at the meta-level factors. These two methods, called *focal* and *preoccupational* analysis respectively, join the quickly growing set of analytic tools that are being constructed these days by those who believe, like I do, that answers to many stubborn questions about human ways of being in the world can

be found in the 'discursive trace' the humans leave behind them. The two specific types of analysis presented below have been developed by Carolyn Kieran and myself while we were grappling with issues such as those that have been raised in this paper.[10]

Focal analysis. Let me first probe deeper in the issue of the effectiveness of communication that comes to the fore the moment cognition is conceptualized in communicational terms. This latter notion, *effectiveness of communication*, may be presented as dependent on the degree of clarity of the *discursive focus* – the communication will not be regarded as effective unless, at any given moment, all the participants seem to know what they are talking about and feel confident that all the parties involved refer to the same things when using the same words. The word *focus* requires explanation. While trying to define this term in our Montreal project, we first thought of it as *the expression used by an interlocutor to identify the object of her or his attention.* Later, because of our awareness of the importance of communication mediating tools, we thought that it would be important to include some indication of *what and how one is attending to* – looking at, listening to, etc. – when speaking or thinking. We decided, therefore, to consider two focal ingredients, *pronounced* and *attended* (for example, in Ari's utterance "Ah, no, the intercept is the zero" ([11b]) the pronounced focus is the words 'the intercept' and the attended focus is the scanning procedure he uses to locate the intercept in the table). We knew, however, that there is more to communication than the pronounced and attended aspects. Whatever is pronounced or seen evokes a whole cluster of experiences, and relates the person to an assortment of statements he or she is now able to make on the entity identified by the pronounced focus. We decided to give the name *intended focus* to this collection of experiences and discursive potentials (in the case of Ari's utterance quoted above, the intended focus is all the statements the boy is likely to make, and all the attended foci he is likely to enact, while using 'the intercept' as a pronounced focus). We can now use these terms to say that the difficulty of human communication stems from the fact that intended focus, which seems to be the crux of the matter, is an essentially private dynamic entity that changes from one utterance to another. This difficulty, however, may often have a straightforward solution: The attended focus can be used as a public exponent of the intended focus, and thus plays a cardinal role in the success of communication.

Let me now apply the focal analysis to the *Slope* episode. It is useful to begin with a closer look at Ari's utterances so as to prepare a contrasting background for Gur's case. The flow of Ari's tripartite focus has been

Utterances	Pronounced	Attended	Intended
[1a], [1b], [11a] [1c], [11b] [1c] [11b]	"the slope" "the intercept" "negative five" "The zero"	**Table intercept**[I]: 1. Find 0 in left column of the table 2. Find the number in the right column of the table corresponding to that 0	The intercept
[3],[5]	"Slope"	**Formula slope**[II]: The coefficient of x in the formula 5x+-5	The slope
Writes: 5x+-5		**Formula intercept**[III]: The free coefficient in the formula 5x+-5	The intercept
[1d], [13],[15], [19]	"Slope"	**Table slope**[IV]: 1. go to the 0 in the left column of the table 2. check the size of the increase between successive numbers in the left column 3. If the increase is 1, then simply find the difference between a number and the one just above it in the right column	The slope

Please note: While looking at the figure, one has to keep in mind that the words *slope* and *intercept* are used in the focal analysis as referring to abstract features of abstract mathematical objects (linear functions), rather than to any kind of symbols. For example, slope is that characteristic of the linear function which finds its 'material' expressions in the coefficient of x in the formula ax+b, in the 'jump' in the y value corresponding to the jump of 1 in the x value, and in the slant of the graph of the function. We say that the slope is represented by all these symbolical means, but is not any of them in particular.

[I] "Table intercept" is an attending procedure for identifying the intercept of a function with the help of a table; It can also be described in structural terms as "The right-column counterpart of the left-column zero".
[II] "Formula slope" is an attending procedure for identifying the slope of a function with the help of a formula
[III] "Formula intercept" is an attending procedure for identifying the intercept with the help of a formula
[IV] "Table slope" is an attending procedure for identifying the slope of a function with the help of a table

Figure 4. *Slope* episode – analysis of Ari's tripartite focus.

charted in Figure 4. Probably the most salient feature of the boy's talk is its being tightly integrated by the intended focus. While the different utterances are built around different pronounced foci, and imply differing attended foci, they all seem to speak either about slope or intercept of the same linear function.

This stability of intended focus justifies comparing Ari's discourse on function *g* to what I once called "actual reality discourses" (Sfard, 2000a), the main characteristic of which is their being about material objects, and their being guided and navigated by actual or imagined pictures of these objects. Indeed, the way the boy uses the function and related notions (such as slope, intercept, specific values of the function) reminds one, in many respects, of the way people speak of, say, trees, chairs, and persons. In Ari's discourse on functions, like in discourses on material things, the object under consideration seems to preserve its identity while its image and its attended aspects are changing from one utterance to another. It is as if Ari was performing a sequence of zoom-ins and zoom-outs from this object (the function) to its particular part (the slope), then to the whole function again, and then to its other particular ingredient (e.g. the intercept). What makes this metaphor of zooming convincing is the ease and confidence with which Ari makes the transitions from one function-related element to another. Another noteworthy phenomenon is the agility with which he moves between different representations: from his well-formed attending procedures for, say, finding the slope in the table, to the one which involves

Ari			Gur		
Pronounced	**Attended**	**Intended**	**Pronounced**	**Attended**	**Intended**
[11] [11a] "the slope" [11b] "the intercept" "the zero[II]"	**Table intercept**	**The intercept**			
			[12] "your.."	?	?
[13] [13a] "how many... in between each" [13b]"from zero[II] to"	**Table slope**	**The slope**			
			[14] "slope", "1"	**The reverse of table intercept[I]**	?
[15] [15a] "slope" [15b] "zero[II]"	**Table intercept**	**The intercept**			
			[16] [16a] "slope" [16b] "that zero[II]" "-5"	**Table intercept**	?

[I]We called this attended focus "The reverse of table intercept" because what Gur is looking at may be described as "The left-column counterpart of the right-column zero", and since "The right-column counterpart of the left-column zero" is the table intercept.

[II] The recurring appearance of the word zero is evidently one of the sources of confusion, since there are two zeros in the table (in the first row in the left-hand column and in the second row in the right-hand column). In particular, it is not clear which of the two zeros is referred to by Ari in [15b], since any of them could be used to appreciate the increase in the y-value corresponding to the increase of 1 in the x-value which Ari is looking for. In [14], Gur is evidently looking at the right-hand column zero, and this is why he points to 1, which is this zero's left-hand counterpart.

Figure 5. *Slope* episode – analysis of focus flow.

the formula, and then back to the first one. See, for example, how in [1d], [3], [5], [13], [15], and [19] he shifts his glance back and forth between the expression and the table, while the intended focus, the slope, remains the same. Ari's exclamation in [25] "So we *did* get it right!", made after he extrapolated *g(6)* from the table and compared it to the *g(6)* computed with the just constructed formula justifies the claim that in his discourse the word 'function' does not signify either the table or the expression, but rather something that unifies the two. The boy evidently knows quite well what features of the table and what kinds of calculations with the formula correspond to each other. His good sense of the isomorphism between the different symbolic systems makes him able to arrive at the same goal in many different ways, just like having a good sense of a physical object makes one able to imagine many different ways of transferring this object from one position to another. Finally, it is remarkable how Ari keeps confounding the words 'slope' and 'intercept', and how in spite of that he confidently moves on, knowing what to look at and what calculations to make. Once again, this relative immunity of the discursive flow to inadvertent verbal confusions, and Ari's ability to correct himself, are yet another feature characteristic of discourses on things that can be seen or imagined.

For the sake of brevity I will say that Ari's discourse is *objectified* (see also *object-mediated* discourse in Sfard 2000a, 2000b). From now on, I will call this name all those discourses that display features similar to

those of Ari's discourse, described above. One look at Gur's part in the conversation suffices to realize that the description does not apply to his discourse. Indeed, the majority of defining features of objectified discourse are missing from Gur's talk. As has been shown by the detailed analysis of focus flow (see Figure 5), the boy cannot move with ease between table and formula, and his attended focus is extremely sensitive to the change of pronounced focus. In fact, most of the time Gur does not show any initiative of his own and while apparently following Ari, he seems to be lacking the consistent intended focus that would keep different utterances together. His interpretations of Ari's statements are ad hoc and rather unrelated to each other. As a result, various symbolic tools – the table, the formula 5x + (–5), the expressions $g(x)$, $g(6)$, and $g(10)$ – function in his talk as self-sustained independent objects, with no evidence whatsoever for a joint intended focus that would turn them, discursively speaking, into 'representations' of different aspects of one thing. The evidence for this disintegration is more than ample. First, Gur openly wonders about the reasons for Ari's attempt to find the formula before calculating $g(6)$. He does it not only at the outset, when Ari sets out to construct the formula, but also much later, when he is watching Ari doing the job ([2], [6], [8]). When the formula for $g(x)$ is already there, Gur still wonders what it has to do with $g(6)$ ([24]). Not only is he ignorant of the connection between the just calculated 5x + (–5) and $g(6)$, but he also cannot make sense of Ari's substitution (he puts down on his worksheet as an answer the intermediate result 30 instead of the final 25). The most telling evidence for the lack of object mediation is the way Gur approaches the subsequent task of calculating $g(10)$. As if the formula was never there, he assumes the table is governed by a simple proportion: he decides that if '5 is 20, then 10 must be 40' ([34]). One can say that he is unable to zoom-out from $g(6)$ or $g(10)$ to the function $g(x)$. Neither is he able to zoom-in to the intercept and the slope from the formula ([6], [8]) or from the table ([14], [16], [18]).

To sum up, Gur does not have a sustained intended focus of his own. In each of his turns, he is building a new intended focus by constructing his interpretation to what Ari just said. Since he is not guided by his own pre-existing intended focus, his interpretations are highly depended on environmental clues, such as those that expose some fit with the words used by Ari (e.g., when Ari uses the word 'zero', Gur looks at the first zero he can find in the inscription). As a result, the involuntary slips of pronounced focus, which for Ari are but easily correctable momentary lapses, for Gur have rather grave consequences. First, it is Ari who says 'slope' by mistake, but then immediately corrects himself ([11]).[11] Gur continues with the 'slope', even though he applies Ari's explanation for finding the in-

tercept ([14]–[16]). Moreover, as a result of another misunderstanding, he does not choose the correct attended focus. This time, the apparent reason for the error is Gur's mistaken interpretation of Ari's synecdochic use of the expression 'the zero' as a pronounced focus (compare [11], [14], [15], [16]). All this is a compelling evidence of the feebleness of his intended focus and his difficulty with becoming a skillful participant in fully-fledged objectified mathematical discourse.

While Ari's private channel seems perfectly focused and continuous, the discourse between the two boys is incoherent. Ari does make several attempts to overcome the incoherence by explicitly pointing to his attended foci.[12] This is what he does, for example, in [13], [15], and [19]. The gesturing, however, does not work. Pointing to the attended focus, *per se*, is not enough to create an adequate intended focus. We can see two possible reasons for the ineffectiveness of Ari's intervention. First, Ari does not really try to coordinate intended and attended foci – he just points to the table or to the formula without specifying the attending procedures. Second, Ari does not probe Gur's understanding. He seems uninterested in Gur's thinking to such extent that he does not even notice Gur's slips of pronounced focus or his erroneous answers. From this point of view, the situation may be quite different for Gur who, so it seems, is keen on keeping the conversation going.

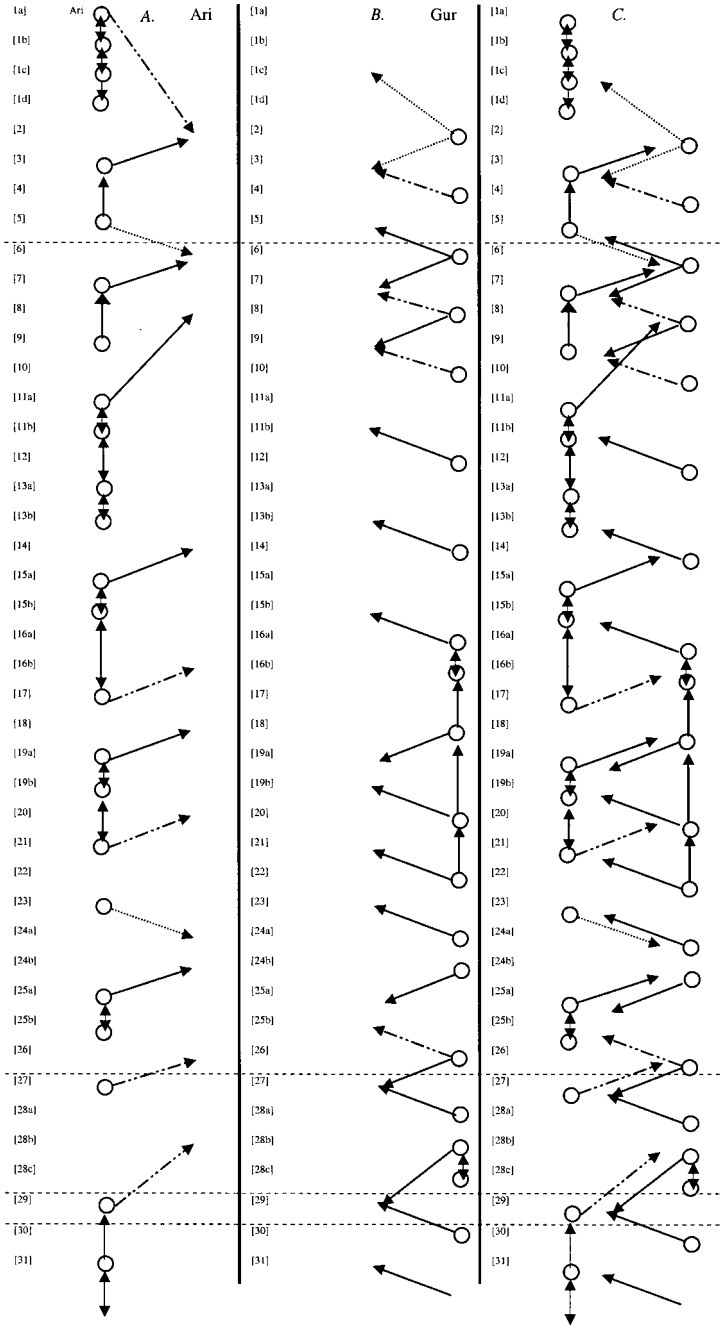
These last claims on the boys' differing attitudes toward the interaction, although plausible, are not yet grounded in a systematic analysis. Carrying out such analysis does seem a worthwhile endeavor, though. Indeed, we may be touching here upon the hidden reasons for the observed communication breaches: The boys' disparate expectations and wishes with respect to the interaction, as well as some interpersonal, mathematics-unrelated, goals and desires that may be preoccupying them as they are talking one to the other – all this is quite likely to interfere with the object-level effectiveness of the exchange. To check this conjecture, let me turn then to *preoccupation analysis*.

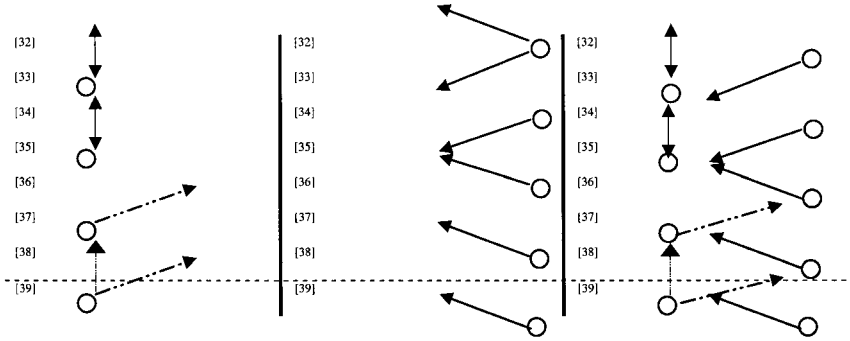
Preoccupation analysis. To have a better grasp of what is meant here by this last term, let us recall that interpersonal communication was defined as an attempt to make other people act or feel according to one's intentions. It is important to stress now that there are two types of intentions which may be conveyed through communicative actions. First, there are overt object-level (cognitive) intentions, related to the declared goal of a given activity. In the case of school mathematical discourse, a student may have an immediate object-level goal of solving a mathematical problem that, in turn, is embedded into the long-term goal of learning some

new mathematics. In the 'Slope' episode some aspects of these object-level intentions have already been taken care of with the help of focal analysis. The other type of discursively conveyed intentions, which are usually less visible even if not less influential, is related to various aspects of the interaction, and thus has the discourse itself as its object. This latter category, which may be called meta-discursive or meta-level, is wide and multifarious, and it includes, on the one hand, interlocutors' concerns about the way the interaction is being managed and, on the other hand, the weighty, and sometimes quite charged, issues of the relationship between interlocutors. After all, every instance of communication is an occasion for re-negotiating interlocutors' mutual positioning and their respective identities. Different means are usually used by participants for communicating the object-level and meta-level intentions. While the former are best expressed in explicit ways, the latter are likely to reside in forms of utterances and in mechanisms of interaction rather than in their explicit contents. Because of the predominantly covert nature of inter-personal messages, the meta-level intentions conveyed through discourse often remain invisible even to those whom they affect (some interlocutors are more reflective and some other much less so, and thus people may be aware of their own meta-discursive intentions to varying degree; still, the concern about the meta-level is always present and we are always witnessing this coexistence of two agendas: the one related to content, and the other to the way the discourse evolves).

The two categories of discursive intentions, object- and meta-level, seem unrelated and, on the face of it, the latter could be left aside when the cognitive aspect of learning interaction is being investigated. In fact, there is a constant tension between the two types of intentions, if only because of the simple fact that they compete for being the focus. Interpersonal communication is a particularly complex phenomenon in that at any given moment each participant is simultaneously involved in a number of object-level and meta-level activities: in trying to understand the explicit contents of previous utterances and to produce new ones, in monitoring the interaction, in presenting oneself to others the way the person would like to be seen, in engineering one's position within the given group, and so on. Since all these different concerns must be attended to at the same time, it seems a miracle that people are ever up to the task of communicating at all.

Our principal tool in the preoccupational analysis is the *interactivity flowchart* that helps to evaluate the interlocutors' interest in activating different channels and in creating a real dialogue with their partners. We can look upon consecutive utterances in a discourse as endowed with invisible


A.
Ari
B.
Gur
C.



Legend. In the flowchart, the two *personal channels*, namely the respective 'parts' of the two boys, are shown in separate columns, *a* and *b*. The numbers marking the little circles correspond to the numbers of the utterances in the episode transcript.
There are two types of arrows that originate in the different utterances.

- *Reactive* arrow (an arrow which points vertically or diagonally backward/upward): this type of arrow expresses the fact that the source utterance is a reaction to the target utterance;
- *Proactive* arrow (an arrow pointing vertically or diagonally forward/downward): this type of arrow symbolizes the fact that the source utterance *invites* a response, so that the following utterance is expected to be a reaction.

Figure 6. *Slope* episode – Interactivity flowchart.

arrows that relate them to other utterances – those which have already been pronounced and those which are yet to come. These arrows express the participants' meta-discursive wishes: the wish to react to a previous contribution of a partner or the wish to evoke a response in another interlocutor (see an additional explanation in the legend of Figure 6). The conversational organization of these *reactive* and *proactive* arrows would often reveal certain regularities. The recurring forms of reactive and proactive behaviors, in their turn, may help in deciding whether interlocutors are really addressing and interpreting their partners or, in fact, are concentrating on a 'conversation with themselves'. In our Montreal study, interaction analysis has been done with the help of a diagram in which the imaginary arrows mentioned above are made visible.

The interaction flowchart of the *Slope* episode is presented and explained in Figure 6. From this graph one may learn quite a lot about Ari's and Gur's attitudes toward the interpersonal communication. A detailed analysis shows that Ari may be not genuinely interested in the interaction. He does not initiate any of the exchanges and he does not respond to many of Gur's proactive utterances. During the whole two-minute long episode he makes only two or three proactive statements (see [1], [5], [23]), all of which are meta-discursive (in comparison, Gur makes nine proactive

statements, eight of them formulated as object-level questions). In fact, these utterances do not even seem to be genuinely proactive: After asking his questions Ari does not wait for an answer and makes it clear that he is eager to finish the job of explaining as quickly as possible. It is quite obvious that he never gives much thought to Gur's answers and does not really care whether his partner is sincere when he says he does understand. So much for his uninitiating attitude. Ari's unresponsiveness expresses itself in his indifference toward Gur's attempts to create an exchange. Rather than answering Gur's proactive utterances, he continues his own line of talk. All along the way he ignores Gur's questions and requests for explanations (see [4]–[6], [11]–[13] and, above all, [29]–[40], where Gur tries to explain his thinking), letting signs of Gur's incomprehension and distress go unnoticed.[13]

One look at the flowchart 6*B* reveals that Gur is still very much interested in interaction, and truly dependent on it. This is evidenced by the profusion of both reactive and proactive utterances, revealing his *initiating* and *responsive* attitude. As a result, the contrast between the two boys' discursive behaviors is now even more pronounced than before.

All this means that while Ari is keen on keeping his thinking from distractions, Gur is preoccupied with the exchange of ideas. Indeed, many of Ari's utterances take the form of a dialogue with himself (see, e.g., [1], [11], [19], [25]), whereas Gur is obviously addressing his partner. In this situation, it is not surprising that while there are long stretches of continuity along Ari's private channel (see 6*a;* in particular, notice [1]-[21] and [29]–[35]), Gur's private channel is practically non-existent (see 6*B*). It is also interesting to note that Ari's private channel has a distinct argumentative structure: Even when talking to Gur, he often sounds as if he is arguing with himself. Constant self-monitoring is one of the distinctive features of Ari's discursive actions. This is how he is able to correct his own mistakes and double-check his own solutions.[14] It is obvious that there is a *hidden part* to his discourse, in which Ari quickly performs the recursive computation mentally (see, in particular, utterance [25] which seems to sum-up such inner computation).[15]

It is also noteworthy that Ari tries to curb the discourse and, at the same time, to conceal this fact with different camouflaging techniques. Keen to protect his private channel from distractions, and aware of the fact that he is not playing according to expectations, Ari tries to soften his unsociable image by lip-service utterances. Thus, from time to time he acknowledges Gur's contributions ([15]: "Hum?", [17], [21], [27]: "Yeah"), but it is obvious that his curt "Yes"es and "Hum?"s are only ceremonial and, in fact, do not express a genuine concern for what Gur is saying. Indeed, in all

the above cases it turns out that Gur's utterances to which Ari said "Yeah" were in fact either incorrect or unfocused, and they were wrong in such a way that it should have been immediately obvious to Ari, had he really listened. Gur, in his turn, has a wide assortment of *discourse-spurring* and *face-saving* techniques. Thus, for example, he uses them to mask his misunderstandings rather than deal with them (see, e.g., his "yeah, yeah" in [12] and in [24], and his "Oh, that makes sense" in [28] when, in fact, nothing seems to make sense to him). The fact that in the first sub-episode he begins questioning Ari without even trying to solve the problem himself shows that he puts up with his partner's superiority and does not really trust his own mathematical capacities.

Let me now put together the focal and preoccupational analyses in an attempt to see what this combined outlook tells us about Ari's and Gur's learning. The first thing to say is that once thinking has been conceptualized as communicating, the dynamic, ever changing and extremely context-sensitive dimension of thinking comes to the fore. Gur's ineffective actions are no longer seen as a direct result of some stable, context independent problem-solving 'scenarios' stored in his mind and likely to repeat themselves in any other situation involving a similar task; rather, they are regarded as a chain of momentary decisions made in immediate spontaneous reaction to his partner's utterances. Since Gur does not seem to have either his own clear way of proceeding or a coherent interpretation of Ari's discursive actions, his responses are globally incoherent even if they sometimes make an impression of being locally appropriate. This uncontrolled spontaneity also accounts, at least partially, for the communication breaches that plague this conversation.

The detailed picture of the incoherent conversation provided by the focal analysis is to be understood as containing an observer's interpretive reconstruction of the participants' thinking, that is, of the 'dialogues' that take place along interlocutors' private channels. On the basis of this analysis, and in tune with what has been said above, there is a substantial difference between Ari's and Gur's thinking: While Ari is focused on self-communication and follows his own discursive line on the expense of inter-personal communication, Gur privileges his interaction with Ari to the almost total neglect of his private channel. Figuratively speaking, Gur gives up his own thinking in the attempt to interpret Ari's dialogue with himself.

If so, why does this seemingly serious effort have such an unhappy, unsatisfactory ending? A plausible answer to this question comes with the results of the preoccupational analysis. While Gur's interest in Ari's thinking is unquestionable, it is counterpoised by another, not less pervas-

ive concern: Gur's concern about his positioning and about face-saving. It may well be that the fear of appearing as unable and unworthy prevents the boy from pursuing his wish to interpret his partner in a consistent and eventually successful way.

This two-dimensional analysis brings about a rather consequential change in our understanding of the mechanism of failure. What was seen so far as an almost direct derivative of one's personal skills and, more often than not, as an outcome of the person's given 'mathematical potential', is now regarded as a product of a collective action. The analysis has shown that when two people are engaged in communication, it takes the two to produce a failure. Ari did not contribute to Gur's predicament deliberately, but he did contribute nevertheless, if only by his presence and his insensitivity to Gur's needs. Were Gur working alone, or were he assisted by a different partner, he might have acted in a different and much more successful way. All this leads to a reasonable doubt about the soundness of research in which cognition and cognitive skills are treated as stand-alone factors that can be studied in isolation from other aspects of the situation. Not less importantly, it makes us suspicious of the common practice of trying to establish children's 'mathematical potential' on the basis of isolated, superficially evaluated, incidents of learning.

The consequences of the alternative theoretical interpretation do not end in words. First and foremost, the participationist framework that stresses change and distrusts permanent labeling brings a more hopeful picture of learning. It says, among others, that the teacher should not be too eager to project from a student's past success or failure into his or her future performance. Since permanent labeling has the quality of a self-fulfilling prophecy, the importance of this caveat can hardly be overestimated. Further, the analysis of the *Slope* episode made it abundantly clear that the beneficial effects of students' collaborative problem solving cannot be taken for granted. If students' interactions are to enhance learning, the communicative skills of the students must be taught. Careful analyses of diverse classroom episodes can be trusted to provide a good idea about what could be done in order to make mathematical communication, and thus mathematical learning, more effective. From this single episode we can already tentatively conclude that interlocutors should probably learn to make their attended foci explicit, that learning alone may sometimes be more effective than learning with others, and that one should be very careful while deciding who should be a given child's partner in collaborative learning.

What should count as 'learning with understanding'?

Let us return now to Noa's case (see Figure 3). The former attempt to interpret and explain the brief exchange raised questions about the meaning of the term *learning-with-understanding* and left us uncertain as to whether Noa's apparent change of mind with respect to the existence of 'the biggest number' was a case of meaningful learning. I will claim now that an alternative interpretation may be provided by putting the analysis of the episode in terms of discursive uses of words, and by a close inspection of the discursive mechanism that compels the girl to change this use.

Before I do this, however, let me elaborate on the idea of *objectified* discourse that appeared in the analysis of the *Slope* episode and may now prove helpful in the case of Noa and Rada's conversation on numbers. In the *Slope* episode, I described Ari's discourse on function as *objectified* because the boy was talking about functions as if these were some real, self-sustained objects. Looking at the way Ari spoke about functions, it is reasonable to say that he experienced the word function as referring to a well-defined, self-sustained entity, existing independently of the discourse itself. This property of his intended focus can be induced from the fact that all along the way Ari is making swift, smooth back-and-forth transitions from one attended focus (the table) to another (the expression 5x+(–5)) while preserving the same pronounced focus (function *g*). He is thus using the different symbols – the table, the expression, and probably also a graph, which is not presented but can be imagined – as if all these symbols were *representations* of one specific object. The special property of this objectified discourse is that it subsumes several independently created discourses, turning them into discourses 'about the same thing' and making it possible to express in the new language everything that can be said, alas in a different way, in any of the subsumed discourses. For instance, discourse about functions subsumes discourses about graphs and the discourse about algebraic expressions. In this subsuming discourse, the sentence "The intercept of this function is –5" replaces, simultaneously, the sentence "This straight line crosses the y-axis at y = 5" in the discourse on graphs, and the sentence "The free coefficient in this formula is –5" in the discourse on expressions. This subsuming effect is clearly visible in Ari's discursive actions, but can hardly be found in what Gur is saying. Being but a beginner in the discourse on functions, Gur has a visible difficulty with finding parallels between graphs and expressions. What for Ari constitutes "two different representations of the same function", for Gur remains a pair of unrelated marks on paper. As long as Gur's use of the different symbols remains unobjectified, his difficulty with following Ari's swift discursive moves is quite understandable.

Back to the *Biggest Number* episode, I will now argue that much of what is happening between Noa and Rada may be explained by the fact that unlike the teacher, the girl uses the number-related words in an unobjectified way. The term 'number' functions in Noa's discourse as an equivalent of the term 'number-word', and such words as *hundred* or *million* are things in themselves rather than mere pointers to some intangible entities. If so, Noa's initial claim that there is the biggest number is perfectly rational. Or, conversely, the claim that there is no biggest number is inconsistent with her unobjectified use of the word 'number': After all, there are only so many number-words, and one of them must therefore be the biggest, that is, must be the last one in the well ordered sequence of numbers (with the order of the sequence determining the relations 'bigger than' or 'smaller than' among its elements). Moreover, since within this type of use the expression 'million and one' cannot count as a number (but rather as a concatenation of numbers), the possibility of adding one to any number does not necessitate the non-existence of the biggest number.

Like in the case of Ari and Gur, the communication between Noa and Rada is obstructed by the fact that one of the interlocutors uses central notions in the objectified way while the other fails to do so. Unlike in the former case, however, the meta-discursive behavior of the interlocutors is now quite different, and their efforts to improve the communication are genuine enough to be ultimately quite successful. Indeed, this time, *both* interlocutors seem interested in aligning their positions. The teacher keeps repeating her question about the existence of 'the biggest number', thus issuing meta-level cue signaling that the girl's response failed to meet expectations. In order to go on, Noa tries to adjust her answers to these expectations, and she does it in spite of the fact that what she is supposed to say evidently does not fit with her use of the words *the biggest number*. The requirement of the change exposes the girl to possibilities she has not considered. Moreover, at the present stage, she does not have means to deal with the problem. Although it must be quite obvious to her that the required change has to be somehow related to the fact that one can always add one to any number, the relation between this fact and the claim about the non-existence of the biggest number cannot possibly be clear. In spite of this, the girl is evidently willing to comply with the rules of the game imposed by the more experienced interlocutor.

Thus, it looks like Noa's principal effort is to fulfill the teacher's discursive expectations. Her focus is at the communication rather than on trying to figure out for herself what is wrong with her use of numbers. One may say that she is trying to understand 'through the other' before she is going to build her own understanding. Without questioning, she looks

upon the teacher's discourse as superior to her own, and as the 'correct' one. Her lack of confidence in her own discursive ways expresses itself in the last question: she already gave a satisfactory answer, now she tries to relate this answer to the other things that have been said in the encounter, thus attempting to reconstruct the teacher's reasoning.

Concerned about the issue of learning with understanding some people may say that the girl's modifications of her answers were made for all the wrong reasons: She was simply keen to please the teacher and was guessing her intentions. To attain this goal, she was playing a rational game, but her rationality was not of the kind traditional teachers would like to see. It was the rationality of guessing from meta-discursive cues rather than inferring from object-level relations. Adherents of the principle of learning-with-understanding are thus likely to join Cazden (1988) in criticizing this kind of situation as one in which established patterns of communication give but "the illusion that learning is actually occurring" (p. 48). This implies that the true learning – the one they use to call "with understanding" – should have followed a different path.

The question, however, is whether such alternative path is always possible. In Noa's case one can hardly think of any other, exclusively rational object-level route toward the eventual objectification of her discourse on numbers. To put it in a more traditional language, it is difficult to see how the child could take a more 'meaningful' path toward re-conceptualization of the notion of number. In order to change her discursive habits and dispositions, she had to undergo an experience of incomprehension – of seeing alternatives to the only possibility of which she was aware when starting the conversation. The meta-level means employed by the teacher to show her such possibilities could not be replaced with any direct object-level considerations. Indeed, Noa's discourse was perfectly coherent, and there were no contradictions between her use of number as a designated word and her claim that one of the numbers must therefore be the biggest. Thus, contrary to the traditional cognitivist analysis I have presented in the beginning, Noa's case cannot count as one of cognitive conflict stemming from holding several incompatible beliefs about number. Noa's eventual dilemma had its roots in an *inter*-discursive clash, not in *intra*-discursive contradiction. In a case like this, one has no chance to modify one's discursive habits on her own. In order to change them, one has to be led outside her own discourse by others. Only then can the conflict necessary to create the learning-engendering experience of incomprehension eventually arise.[16]

More generally, what we have seen in Noa's case may well be one of the principal forms of learning we all employ throughout our lives. It is

thanks to the intricate combination of object-level and meta-level tuning to our interlocutors that we make our way toward better communication and perfect our participation in specialized discourses. Participants come to discourses with their own, possibly idiosyncratic, uses of words and their own expectations with regard to the rules of the game that is to be played. The actual shape of the exchange will be the resultant of the interaction between the expectations of all the interocutors. Of course, not all of them would influence the rules of the game to the same extent. In any specialized discourse there is usually a dominant, authoritative, voice that informs the rules more than all the others. In the classroom, the lead belongs to the teacher. Only too often, in order to learn, students have to follow the teacher before having a firm grasp of the new discourse into which they are thus led. This kind of learning is not likely to be valued by the followers of the traditionally understood principle of learning-with-understanding. And yet, this kind of learning cannot be replaced with any other. This impossibility is inherent in the claim that all our thinking is essentially social, and this is the deep meaning of Vygotsky's famous statement:

> Any function in child's cultural development appears twice, on two planes. First, it appears on the social plane, and then on the psychological plane. First it appears between people as an interpsychological category, and then within the child as an intrapsychological category. (Vygotsky, 1931/1981, p. 163)

There is yet another, more general, implication of the present example. The learning that occurred in the just analyzed episode is no longer viewed as a result of *cognitive* conflict. If anything, the situation we have been witnessing can be described as one of *discursive* conflict, an occurrence quite different from that of being exposed to what looks like well-justified mind independent facts that contradict each other. Indeed, while the concept of cognitive conflict implies one's ability to *rationally* justify two colliding claims-about-the-world, the notion of discursive conflict stresses the clash of habitual uses of words, which is an inherently discursive phenomenon. In our present case, we could observe a conflict between the two interlocutors' discursive uses of the words 'number' and 'bigger number'. While aware of the fact that the teacher was applying these terms in a way quite different from her own, Noa was ignorant of the reasons for this incompatibility. In this case, therefore, the girl had to *presume* the superiority of her teacher's use in order to have any motivation at all to start thinking of rational justification for a change in her own discursive habits.

Thus, perhaps the most dramatic difference between the cognitivist and communicational interpretations of the *Biggest Number* episode lies in their respective visions of breaches-in-understanding that motivate learn-

ing. The concept of cognitive conflict assumes that the learner is in a constant quest after the truth about the world, and whatever new knowledge is acquired, it is the result of this learner's attempts at adjusting her understanding to the externally given, mind independent aggregates of facts and ideas. Clearly, this kind of endeavor could be pursued, at least in theory, without the mediation of other people. In contrast, the idea of discursive conflict stresses the need for communication as a principal drive for our cognitive actions, and points to the wish to adjust one's discursive uses of words to that of other people as one of the main motives for learning.[17]

4. AFTERWORD: CHANGING WORLDS WITH WORDS

In this article I tried to demonstrate the power of the idea of thinking-as-communicating to bring a valuable change in our vision of learning in general, and mathematical learning in particular. This change, it seems, is not just a change in words. Together with the new words come new ideas about what goes into learning and what should be done to promote this learning.

In the analyses above I did my best to show that the communicational approach, based on the learning-as-participation metaphor, does much more than add new information. What I hope to have shown is that this special outlook would often change the picture in such a way that even the 'old' parts of the image – the parts that could be seen before – acquire a new meaning. The overall transformation that occurred in our vision of the two classroom scenes as a result of communicational re-interpretation was quite remarkable. What until now was seen as a function of stable or semi-stable 'possessions' and dispositions of the individual became a dynamic property of human interactions, one that does not have an existence beyond these interactions. Teacher's decisions that, so far, were likely to be seen as somehow out of tune with the principle of learning-with-understanding, have been rehabilitated and promoted to the rank of helpful and valuable, if not outright indispensable. Above all, the hitherto ignored aspects of learning have been elicited and ascribed principal importance.

All this said, let me stress once again that the communicational approach should be seen as a complement rather than as a replacement for the more traditional outlooks. My present preference for the communicational framework and for the underlying participation metaphor does not mean rejection of the other metaphor, nor an attempt to undermine this other metaphor's valuable contribution to our understanding of learning in general, and learning mathematics in particular. In my opinion, the only reasonable conclusion from the recent criticism of the more traditional

cognitivist approaches is that the manner the acquisition metaphor finds its way into scientific concepts has to be refined, and its entailments must be carefully rethought. Rather than rejecting the metaphor as such, one should cleanse the discourse on learning from its unhelpful, undesirable entailments.

This 'reconciliatory' declaration may, of course, raise some eyebrows. In the light of the rather far-reaching changes in the vision of learning entailed by the change of metaphor, how can one keep saying that the different metaphors are 'complementary' rather than incompatible? Of course, the claim about complementarity cannot be true unless the ontological and epistemological foundations of the traditional framework undergo a certain revision, and the basic notions are reconceptualized. The kind of change I am talking about is, in a sense, analogous to the one that was necessary in mathematics in order to enable the co-existence of Euclidean and non-Euclidean geometries within one consistent and surprisingly useful system; or to the change in physics, that enabled two seemingly incompatible visions of sub-atomic phenomena to be used intermittently, depending on the questions asked. In the study of human mind, like in geometry, it must be understood that the basic assumptions on which this whole framework rests are not about what the world 'really is' but only about how the world may be thought of, in certain situations. Of course, incommensurable outlooks cannot be applied to the same phenomenon at the same time, just like wave and corpuscular theories of light cannot be combined in one answer to the same question. And yet, I can think of many situations where it would be reasonable to try both these approaches in an attempt to find which one would provide a more helpful solution of the problem at hand. In this paper I was trying to show that in such conceptual 'competition', the communicational approach should be considered as a serious candidate. Our wish to model life in scientifically simplified ways with models that, nevertheless, look like life itself, is inherently insatiable; and yet, along with the numerous frameworks available to the students of the human mind these days, the one that equates thinking with communicating may have something important to offer.

NOTES

1. The episode is taken from the research project in Montreal, directed by Carolyn Kieran and myself, since 1993. The aim of the 30 session long teaching sequence produced for the sake of the study was to introduce the students to algebra while investigating their ways of constructing algebraic concepts and testing certain hypotheses about possible ways of spurring these constructions. The present episode is taken from the

21st meeting. More information on the study, as well as another outlook at the present episode, may be found in Kieran and Sfard (1999), Sfard and Kieran (2001a, 2001b).

2. That the expectation of proportionality is a well-known phenomenon has been evidenced lately by the following episode in the popular TV series addressed to a young audience, *Friends*. A person tries to prevent an 18-year-old boy from marrying a 44-year-old woman. He says: "She is so much older than you are. And think about the future: when you are 36, she will be 88". "Yeah, I know", says the boy.
3. Quite a number of units of analysis among those proposed by representatives of different sociocultural schools seem to be good candidates for the type of study required by participationists. Among the most widely known and applied are *activity*, the unit proposed by activity theorists, *discourse* or its segments, the unit suggested by discursive psychologists, and *practice*, introduced by those among participationists who are most strongly oriented toward sociological issues.
4. It is interesting to note that this seemingly 'factual' statement is an object of fierce debate between the traditional cognitivists and the adherents of the sociocultural approach. The controversy is often framed in the language of transfer of learning: While acqusitionists' belief in the possibility of far-reaching transfer remains firm in spite of rather meager empirical evidence, participationists either deny such possibility or simply say, as Lave (1988) did, that the concept of transfer is fundamentally misconceived. Indeed, a consistent follower of a participationist framework must realize, sooner or later, that the idea of transfer, which implies 'displacement' of certain mental entities, simply does not fit with the participationist conceptualization of learning (see also the ongoing debate on transfer in *Educational Researcher*, e.g. Brown et al., 1989; Anderson et al., 1996; Greeno, 1997; Sfard, 1998; Cobb and Bowers, 1999).
5. This definition of learning was proposed by Lave and Wenger, 1991. 'Cognition at work' began to attract researchers' attention already in the 1970s and it has been turning recently into a favorite theme of study for those interested in learning (see e.g. Engstrom and Middleton, 1996; http://www.helsinki.fi/~jengestr/activity/1.htm). Many of these studies may appear as simply recording what and how people do in workplaces. However, if one accepts Wenger's (1998) definition of practices as histories of learning, then doing and learning become practically indistinguishable.
6. As noted by many, Bruner (1990) among them, the question 'nature-or-nurture' is probably ill-posed. It has its roots in what may be called 'the hardware-first fallacy', the conviction that whenever a physiological difference is found between two groups of people, this difference may be held responsible for the differences in these two groups' behavior. These days it is already clear that the uni-directional causal vision of the relation between biological and cognitive factors has little grounding. Recent findings have shown that human activities, rather than being determined by a pre-formed neural system, are partly responsible for this system's structure and functioning. It seems, therefore, that human evolving culture perpetuates itself not only by affecting human minds but also by changing their brains, with the processes of change happening on both phylogenetic and ontogenetic levels.
7. The quote is taken from a text that was first presented as a conference talk in 1992.
8. One can distinguish between two different trends in the research focusing on discourse, only one of which views communication as truly central, if not outright the same as, cognitive processes. The other, less radical trend reflects the interest in communication as an *aid* to learning rather than as an object of learning in itself.
9. To bar an undesirable entailment of the metaphor, let me immediately add that unlike unanimated physical objects, people can – and usually do – play the double role of

actors and observers. If so, meta-discursive rules often become an object of reflection and thus also of regulation.

10. For the scarcity of space, focal and preoccupational analyses will be presented here very briefly. Of necessity, the all-important discussion of their epistemological foundations will be completely omitted. For more details see Sfard and Kieran, 2001a, Sfard, 2000b.
11. Later, for a moment, Ari does show a sign of absentmindedness when he overlooks Gur's mistake and repeats 'slope' when he really means intercept – see in [15] and [17] his unjustified confirmation of what Gur has said in [14]. The question can rightly be asked, how we know that Ari's intended focus was intercept, and not slope. We can be quite sure of it for at least two reasons. First, the number both boys point to is the intercept. Second, when Gur asks Ari later ([18]) how they are "supposed to get the other ones", Ari immediately answers with a prescription for finding the slope, showing, therefore, that slope is 'the other one' – namely different from what they have just found.
12. It is noteworthy that unlike Gur, who is pointing all the time in both episodes, Ari only points for the sake of interpersonal communication, and he never makes any movements when conversing 'with himself'.
13. For example, in [24], after responding to Ari's explicit question with an assurance that he had understood his partner's former explanations, Gur asks: "So what is g(6)?". From here it is immediately obvious that he can see no connection between looking for the formula of a function and calculating its particular value, g(6) (this interpretation finds its further confirmation in the way Gur proceeds to find out g(10)). In this way, Gur makes it abundantly clear that his "yeah, yeah" in response to Ari's "Do you understand?" is but a face-saving device, and that in fact he has no inkling of what has been going on in the discourse up to this point. Ari, however, does not seem to notice his partner's predicament.
14. In [11] he 'undoes' his own slip of the tongue because he sees that what he said does not fit with what he attends to. Similarly, he corrects himself in [29]. He also verifies his own solutions, and he does it spontaneously. The most telling example, from this point of view, is his statement "So we *did* get it right" in [25], which he enthusiastically makes after computing $g(6)$. Obviously, he has criteria according to which to judge the correctness of the result. Although he doesn't say so, the only possible way for him to assess the result is to compare it to the one which may be obtained in another way; this other way can only be the recursive pattern he detected earlier in the table – the increase of the values in the right-hand column by 5 each time.
15. The interesting thing is that if he went to such trouble to find the formula, finding the value of the function with the formula must be for him the 'canonical method'. Erna Yackel and Paul Cobb (Yackel and Cobb, 1996) would say that answering the question about a concrete value of a function by using the formula is a socio-mathematical norm Ari accepted.
16. This is the way in which Noa might ever be able to overcome the learning paradox: She had to know what she was supposed to understand in order to understand this. Indeed, how could the girl understand new discursive use of the words 'number', 'bigger than', 'million and one', 'the biggest number' without being first exposed to this new use? She was not going to invent this use herself! Here is where the expecting/verifying zigzag came into play. It is thanks to the fact that her communication with others broke down, exposing the inappropriateness of her discursive habits, that she was compelled to revise her use of the word 'number'.

17. This vision of the way in which communication breaches spur learning provides an answer to the dilemma posed by Smith, diSessa, and Rochelle (1993) who were perhaps the first writers to cast doubt on the idea of cognitive conflict. The authors wondered how resolution of a cognitive conflict could be possible at all:

> As cognitive competition, [this idea] cannot explain why expert ideas win out over misconceptions. The rational replacement of one conception with another requires criteria for judgment. As knowledge, those criteria must be constructed by the learner, and neither confrontation nor replacement explain the origins of such principles for choosing concepts, crucial data, or theories. (p. 126)

When the idea of cognitive conflict is replaced by the notion of discursive conflict, the dilemma seems to disappear. If the need for good communication is seen as a principal drive for learning and change, then student's basic readiness to yield to what they regard as expert use of words is the functional counterpart of the 'criteria for judgement' mentioned by Smith, diSessa, and Rochelle.

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