Discrete Comput Geom 36:381–390 (2006)
DOI: 10.1007/s00454-006-1244-0

Discrete & Computational
Geometry

There Is No Face-to-Face Partition of R^5 into Acute Simplices*

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Dedicated to Prof. Miroslav Fiedler on the occasion of his 80th birthday

Abstract. We prove that a point in the Euclidean space R^5 cannot be surrounded by a finite number of acute simplices. This fact implies that there does not exist a face-to-face partition of R^5 into acute simplices. The existence of an acute simplicial partition of R^d for $d > 5$ is excluded by induction, but for $d = 4$ this is an open problem.

1. Introduction

There are many algorithms for generating partitions into acute triangles in R^2 (for instance, each acute triangle is a plane-filler). Until 2004, Eppstein et al. [2] showed that there exists a partition of R^3 into acute tetrahedra. Note that acute simplicial partitions (defined in Section 2 below) are very useful in numerical analysis, since they yield irreducible and diagonally dominant stiffness matrices, when solving the equation

$$-\Delta u + bu = f$$

by standard linear conforming finite elements in a bounded polytopic domain in R^d with some boundary conditions and $b \geq 0$ small enough. In this case the discrete maximum principle takes place, see [4] (and also [5] for nonlinear problems). The necessity of solving partial differential equations for dimensions $d > 3$ arises in statistical physics, financial mathematics, general relativity, particle physics, etc. Therefore, it would be useful to have an algorithm that produces acute partitions in higher-dimensional spaces. However, in this paper we prove that a point in the Euclidean space R^d cannot be

* This research was supported by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503, and Grant No. A 1019201 of the Grant Agency of the Academy of Sciences of the Czech Republic.

surrounded by a finite number of acute simplices for $d \geq 5$, i.e., even locally there are no acute partitions (see Theorem 6.2).

In two-dimensional acute partitions each vertex is obviously surrounded by at least five triangles. In Section 3 we prove that each vertex in acute partitions of R^3 has to be surrounded by at least 20 tetrahedra (and this number is attainable, since the regular icosahedron can be partitioned into 20 acute tetrahedra). In Section 4 we show that a point in R^4 can be surrounded by 600 acute simplices, but it is an open problem whether there exists an acute simplicial partition of R^4. In Section 5 we prove that there is no partition of R^5 into acute simplices. Section 6 is devoted to extensions for $d \geq 6$.

2. Basic Definitions

The convex hull of $d+1$ points in R^d for $d \in \{1, 2, 3, \ldots\}$, which are not contained in a hyperplane (of dimension $d-1$), is called a *simplex* or more precisely a *d-simplex*. Its $(d-1)$-dimensional faces are called *facets*. For $d > 1$ the inner angle α between two facets is defined by means of the scalar product of their unit outward normals n_1 and n_2,

$$\cos\alpha = -n_1 \cdot n_2, \tag{2.1}$$

and it is called a *dihedral angle*. A d-simplex is said to be *acute* if all its dihedral angles are less than $\pi/2$. Their number is $\binom{d+1}{2}$.

Definition 2.1. Let $\overline{\Omega} = R^d$ be a closed domain (i.e., the closure of a domain). If the boundry $\partial\overline{\Omega}$ is contained only in a finite number of hyperplanes, we say that $\overline{\Omega}$ is *polytopic*. Moreover, if $\overline{\Omega}$ is bounded, it is called a *polytope*.

Definition 2.2. A set of d-simplices is said to be a *partition* of a closed polytopic domain $\overline{\Omega}$ into simplices, if

(i) the union of all these simplices is $\overline{\Omega}$,
(ii) the interiors of these simplices are mutually disjoint,
(iii) any facet of any simplex in the partition is either a facet of another simplex in the partition, or a subset of the boundary $\partial\overline{\Omega}$,
(iv) the set of vertices of all simplices from the partition has no accumulation points in R^d.

Remark 2.3. The partition from the above definition is sometimes also called a face-to-face partition, due to condition (iii). Condition (iv) guarantees that each partition of a polytope is formed only by a finite number of simplices.

Definition 2.4. A partition is said to be *acute* if all its simplices are acute.

Definition 2.5. We say that simplices $S_1, \ldots, S_k$ *surround a point* A if A is a vertex of each S_i, A lies in the interior of $\overline{\Omega} = \bigcup_i S_i$, and $S_1, \ldots, S_k$ form a partition of $\overline{\Omega}$.

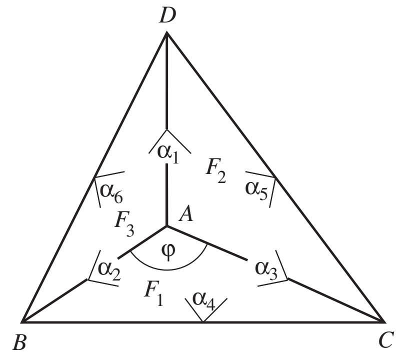

Fig. 1

3. Preliminary Considerations

First we recall an elementary result for a tetrahedron (i.e., 3-simplex) which will be used in Section 5.

Lemma 3.1. *The sum of all dihedral angles in a tetrahedron is greater than* 2π.

Proof. Let $ABCD$ be a tetrahedron. Denote by α_1, α_2, and α_3 its dihedral angles at edges passing through the vertex A and let α_4, α_5, and α_6 be dihedral angles at corresponding opposite edges (see Fig. 1). Consider four spherical triangles that arise by intersecting the tetrahedron $ABCD$ by four sufficiently small spheres centred at the vertices A, B, C, and D. Then from the Riemannian geometry we get

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &> \pi, \\ \alpha_2 + \alpha_4 + \alpha_6 &> \pi, \\ \alpha_3 + \alpha_4 + \alpha_5 &> \pi, \\ \alpha_1 + \alpha_5 + \alpha_6 &> \pi. \end{aligned}$$

Summing these inequalities, we obtain the desired result. □

Remark 3.2. The lower bound 2π in Lemma 3.1 cannot be enlarged for acute tetrahedra, since the sum of all dihedral angles of the acute tetrahedron with vertices $A = (-1, 0, -\varepsilon)$, $B = (-1, 0, \varepsilon)$, $C = (1, -\varepsilon, 0)$, and $D = (1, \varepsilon, 0)$ tends to 2π for $\varepsilon \to 0$.

Now let $d \geq 3$ and let F_1, F_2, and F_3 be arbitrary facets of a d-simplex S. Since F_1 is a $(d-1)$-simplex, its inner angle φ between its $(d-2)$-dimensional faces $F_1 \cap F_2$ and $F_1 \cap F_3$ is defined similarly to (2.1), but in the hyperplane containing F_1. The angle φ is called a *solid angle*.

The intersection $I = F_1 \cap F_2 \cap F_3$ has dimension $d-3$. Let L be a three-dimensional space orthogonal to I (for $d = 3$, let $L = R^3$). Then $S \cap L$ is a tetrahedron. Applying the Cosine theorem from spherical trigonometry to a sufficiently small sphere centred at one of its vertices contained in I, we get (see p. 465 of [3] for details)

$$\cos\alpha_1 = -\cos\alpha_2\cos\alpha_3 + \sin\alpha_2\sin\alpha_3\cos\varphi,$$

where $\alpha_1 = \angle F_2F_3$, $\alpha_2 = \angle F_1F_3$, and $\alpha_3 = \angle F_1F_2$ (see Fig. 1 for $d = 3$). In Lemma 3.3 below we prove that the solid angle φ of an acute simplex is always less than the associated dihedral angle α_1.

Lemma 3.3. *Let $d \geq 3$. If a d-simplex is acute, then under the above notation we have*

$$\varphi < \alpha_1.$$

Proof. Since all dihedral angles α_i are less than $\pi/2$, we find by the above Cosine theorem that

$$\cos\varphi = \frac{\cos\alpha_1 + \cos\alpha_2\cos\alpha_3}{\sin\alpha_2\sin\alpha_3} > \cos\alpha_1 + \cos\alpha_2\cos\alpha_3 > \cos\alpha_1. \qquad \square$$

By induction we get a well-known characteristic property of all acute simplices (see, e.g., Satz 4 of [3]):

Corollary 3.4. *If a d-simplex is acute, then each of its m-dimensional faces is also an acute simplex for $m \in \{2, \ldots, d-1\}$.*

Remark 3.5. The converse implication is not true. For instance, all faces of the tetrahedron with vertices $A = (-1, 0, 0)$, $B = (1, 0, 0)$, $C = (0, -1, \varepsilon)$, and $D = (0, 1, \varepsilon)$ for an $\varepsilon \in (0, 1)$ are acute, whereas its dihedral angles at the edges AB and CD are obtuse.

Remark 3.6. The regular tetrahedron is not a space-filler, since each of its dihedral angles has the value

$$\alpha = \arccos\tfrac{1}{3} \approx 71^\circ.$$

To a given face of the regular tetrahedron we may adjoin another regular tetrahedron of the same size in a unique manner. In this way, a given edge can be shared by five regular tetrahedra, but a small gap appears (see Fig. 2).

In [2] Eppstein et al. give a constructive proof of the following assertion:

Theorem 3.7. *There is an acute partition of R^3 into tetrahedra.*

The main idea of the proof is to distribute congruent regular icosahedra in R^3 so that each two neighbours have one common edge. Each icosahedron is then partitioned into 20 acute tetrahedra and the remaining gaps are partitioned into four different kinds of acute tetrahedra.

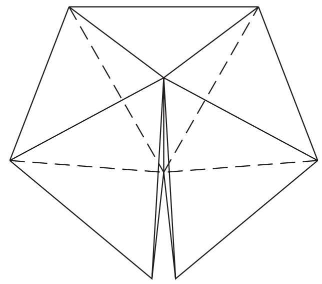

Fig. 2

Theorem 3.8. *Each vertex in an acute partition of* R^3 *is surrounded by at least* 20 *tetrahedra.*

Proof. Let $S_1, \dots, S_t$ be all tetrahedra that surround an arbitrary vertex A in an acute partition of R^3. Then $P = \bigcup_{i=1}^t S_i$ is a convex polyhedron with t triangles on its boundary ∂P. By the classical Euler formula

$$v + t = e + 2, \tag{3.1}$$

where v is the number of vertices and e is the number of edges on ∂P. Clearly,

$$2e = 3t. \tag{3.2}$$

Since S_i are acute, each interior edge containing the point A is shared by at least five tetrahedra. This means that any vertex on ∂P is the intersection of at least five edges from ∂P. Since each edge has two vertices, we obtain

$$5v \le 2e. \tag{3.3}$$

From (3.2), (3.3), and (3.1) we find that

$$t = 4e - 5t \ge 10v + 10t - 10e = 20.$$ □

4. Two Conjectures in R^4

It is not clear whether the constructive proof of Theorem 3.7 (see [2]) can be generalized to the four-dimensional space. By [7] (see also [1]) there exists a regular polytope in R^4, called the 600-*cell*, whose three-dimensional surface is formed by 600 regular tetrahedra. It has 120 vertices (since its dual is another regular polytope called the 120-*cell*). In this case the famous Euler–Poincaré formula [6] has the form

$$v + t = e + c, \tag{4.1}$$

where v, e, t, and c, respectively, are the number of vertices, edges, triangles, and tetrahedra on the surface. Since each triangular face is shared by two tetrahedra and each tetrahedron has four triangular faces, we get

$$2t = 4c \tag{4.2}$$

(see the table below, where we intentionally do not reduce by two to indicate that two simplices always share a common $(d-2)$-dimensional face). This means that the regular 600-cell has 1200 triangular faces and from (4.1) we find that it has 720 edges. Since each tetrahedron has six dihedral angles of value α, the total sum of all dihedral angles is $600 \cdot 6\alpha$. Therefore, the sum Σ of all dihedral angles of regular tetrahedra sharing a given edge is

$$\Sigma = 3600\alpha/720 = 5\alpha \approx 1.959\pi < 2\pi.$$

The fact that Σ is less than 2π is a consequence of Lemma 3.3. Each edge is shared by exactly five tetrahedra and the small gap from Fig. 2 does not appear for the 600-cell, since each tetrahedral cell is in a different hyperplane.

From the above we observe that in four-dimensional space a point can be surrounded by 600 acute 4-simplices. Each of them can be defined as the convex hull of the centre of gravity of the 600-cell and a particular regular tetrahedron from its boundary.

Conjecture 4.1. *A vertex in R^4 cannot be surrounded by less than* 600 *acute simplices*.

Conjecture 4.2. *There is no acute partition of R^4 into simplices*.

Equalities (4.1) and (4.2) hold for more general clusters of 4-simplices, in particular, for any convex polytope in R^4 whose three-dimensional surface is formed by tetrahedra. Moreover, for an acute partition of such a polytope we get the acuteness inequality

$$5e \le 3t, \tag{4.3}$$

since each edge has to be shared by at least five triangular faces (each having three edges).

The following table surveys relations similar to (4.1), (4.2), and (4.3) for $d = 2, 3, 4, 5$:

d	Euler–Poincaré formula	Simplicial equality	Acuteness inequality
2	$v = e$	$2v = 2e$	$5 \le v$
3	$v + t = e + 2$	$2e = 3t$	$5v \le 2e$
4	$v + t = e + c$	$2t = 4c$	$5e \le 3t$
5	$v + t + f = e + c + 2$	$2c = 5f$	$5t \le 4c$

5. The Nonexistence of Acute Partitions in R^5

In this section we prove that Theorem 3.7 cannot be generalized to R^5. The key idea of the proof is based on the fact that a point in R^5 cannot be surrounded by a finite number of acute simplices.

Theorem 5.1. *There is no acute partition of R^5 into* 5*-simplices*.

Proof. Assume, to the contrary, that such an acute partition exists and choose an arbitrary vertex $A \in R^5$ of simplices. Set

$$P = \bigcup_{i=1}^{f} S_i,$$

where $S_1, \dots, S_f$ are all 5-simplices containing the given vertex A. We see that P is a convex polytope, since it can be represented as the intersection

$$P = \bigcap_{i=1}^{f} H_i,$$

where H_i are closed half-spaces such that $S_i \subset H_i$ and ∂H_i contains that facet of S_i which is opposite to A. Hence, for the polytope P we may apply the Euler–Poincaré formula [6]

$$v + t + f = e + c + 2, \tag{5.1}$$

where v, e, t, c, and f, respectively, are the number of vertices, edges, triangles, tetrahedra, and facets on the boundary ∂P. Since each facet is a 4-simplex, it has five tetrahedral faces, and since each tetrahedral face belongs to exactly two adjacent facets, we get the equality

$$2c = 5f. \tag{5.2}$$

Thus, the Euler–Poincaré formula (5.1) can be reduced to the form

$$5v + 5t = 5e + 3c + 10, \tag{5.1$'$}$$

which contains only the numbers of "lower-dimensional" simplices.

Since each S_i is acute, all its tetrahedral faces are also acute due to Corollary 3.4. Hence, each triangular face from ∂P has to be shared by at least five tetrahedra from ∂P (each having four triangular faces), i.e.,

$$5t \le 4c. \tag{5.3}$$

Denote by $\alpha_1^T, \dots, \alpha_6^T$ all dihedral angles of a given tetrahedron T. Then by Lemma 3.1,

$$2\pi < \sum_{i=1}^{6} \alpha_i^T.$$

Moreover, the sum of all dihedral angles $\alpha_1^E, \dots, \alpha_{n_E}^E$ of tetrahedra around a given edge E from ∂P cannot be greater than 2π (it is, in fact, less than 2π by Lemma 3.3). Therefore,

$$2\pi c < \sum_T \sum_{i=1}^{6} \alpha_i^T = \sum_E \sum_{j=1}^{n_E} \alpha_j^E \le 2\pi e,$$

where the sums $\sum_T$ and $\sum_E$ are taken over all tetrahedral faces T and edges E from ∂P, respectively. Consequently,

$$c < e. \tag{5.4}$$

Since each tetrahedron has four vertices and each vertex is shared by at least five tetrahedra from ∂P, we have (see Remark 5.2 below)

$$5v \leq 4c. \tag{5.5}$$

Gathering (5.4), (5.1′), (5.3), and (5.5) together, we find that

$$8c < 5e + 3c + 10 = 5v + 5t \leq 8c,$$

which is a contradiction. □

Remark 5.2. Estimate (5.5) is, in fact, very pessimistic. For $d = 5$ it is easy to find that for partitions into nonobtuse 5-simplices, whose dihedral angles are all less than of equal to $\pi/2$, each vertex $B \in \partial P$ is shared by at least $16 = 2^{d-1}$ facets. Each facet has five tetrahedral faces (one of them is opposite to B, the other four contain B). Thus, there are at least $64/2 = 32$ tetrahedral faces containing B. Since each tetrahedron has four vertices, we have $8v \leq c$ (see (5.5)). This estimate can still be improved for acute partitions.

6. Extension to Higher Dimensions

Definition 6.1. Let A be a vertex of a d-simplex S for $d > 1$. Dihedral angles between any two facets of S which both contain A are called *adjacent to* A.

The acuteness assumption on simplices from Theorem 5.1 can be weakened as follows.

Theorem 6.2. *For $d \geq 5$ there are no d-simplices surrounding a point $A \in R^d$ whose dihedral angles adjacent to A are all acute.*

Proof. We proceed by induction. Let $d = 5$ and let $A \in R^5$. Assume, to the contrary, that there exist d-simplices $S_1, \dots, S_f$ surrounding A whose dihedral angles all adjacent to A are acute. Let $A_1, \dots, A_n$ all be vertices of all S_i different from A. Consider a sphere $\mathcal{S}$ centred at A with radius smaller than $\min_j |AA_j|$. Let P be a convex hull of the points $\mathcal{S} \cap AA_j$ for $j = 1, \dots, n$. By (2.1) it is easy to verify that $P \cap S_i$ are acute simplices that partition P for $i = 1, \dots, f$. Obviously they surround the point A, which by Theorem 5.1 is impossible.

Further, let $d > 5$ be given, let $A \in R^d$, and let Theorem 6.2 be valid for $d - 1$. Suppose again that there exist d-simplices $S_1, \dots, S_k$ surrounding A whose dihedral angles all adjacent to A are acute. Set

$$P = \bigcup_{i=1}^{k} S_i,$$

and let B be an arbitrary vertex on the boundary ∂P. We may assume that $S_1, \dots, S_m$ ($m < k$) are those simplices that share the edge AB. Since their dihedral angles adjacent

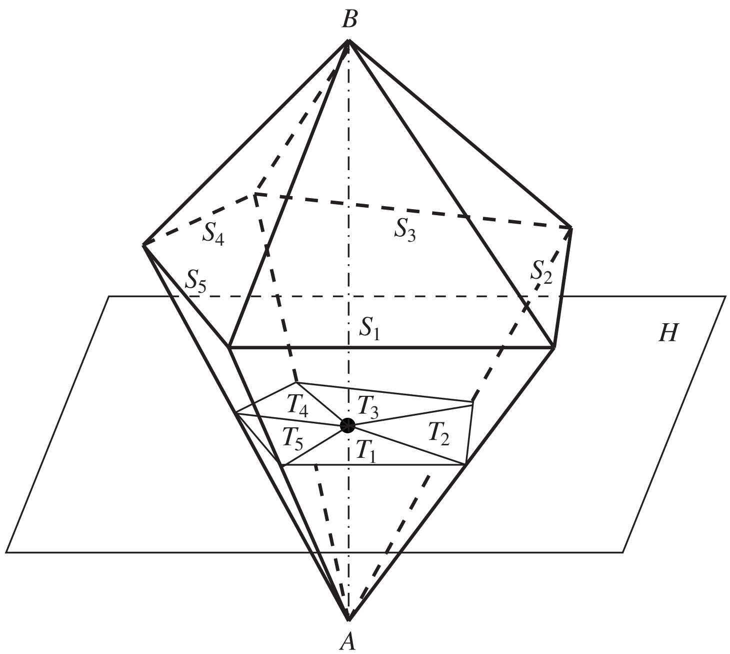

Fig. 3

to A are acute, the set

$$\bigcup_{i=1}^{m} S_i \setminus \{A\}$$

lies in the open half-space whose boundary is orthogonal to AB and passes through A. Hence, there exists a hyperplane H orthogonal to AB that separates A from the other vertices of $S_1, \ldots, S_m$ (see Fig. 3 for $d = 3$).

Then $T_i = S_i \cap H$ are $(d-1)$-simplices which surround the point $AB \cap H$ in the hyperplane H. Since H is orthogonal to AB, the acute dihedral angle of S_i at the edge AB is the same as the associated angle at the vertex $AB \cap H$ of T_i, and therefore, it is also acute. However, by the induction hypothesis this is impossible. □

Corollary 6.3. *For $d \geq 5$ there is no acute partition of R^d into d-simplices.*

Acknowledgement

The author is indebted to Jan Brandts and Jakub Šolc for fruitful discussions.

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Received May 3, 2005, *and in revised form January* 9, 2006. *Online publication June* 5, 2006.