Thermal Resistances of Circular Source on Finite Circular Cylinder With Side and End Cooling

M. M. Yovanovich
Distinguished Professor Emeritus,
Principal Scientific Advisor,
Microelectronics Heat Transfer Laboratory,
Department of Mechanical Engineering,
University of Waterloo,
Waterloo, Ontario N2L 3G1, Canada
e-mail: mmyov@mhtlab.uwaterloo.ca

General solution for thermal spreading and system resistances of a circular source on a finite circular cylinder with uniform side and end cooling is presented. The solution is applicable for a general axisymmetric heat flux distribution which reduces to three important distributions including isoflux and equivalent isothermal flux distributions. The dimensionless system resistance depends on four dimensionless system parameters. It is shown that several special cases presented by many researchers arise directly from the general solution. Tabulated values and correlation equations are presented for several cases where the system resistance depends on one system parameter only. When the cylinder sides are adiabatic, the system resistance is equal to the one-dimensional resistance plus the spreading resistance. When the cylinder is very long and side cooling is small, the general relationship reduces to the case of an extended surface (pin fin) with end cooling and spreading resistance at the base. The special case of an equivalent isothermal circular source on a very thin infinite circular disk is presented.
[DOI: 10.1115/1.1568124]

1 Introduction

Many researchers over several decades have studied different aspects of spreading/constriction resistance of a circular area subjected to different boundary conditions. Spreading/constriction resistance solutions have been reported for steady heat conduction or current flow into the four regions shown in Fig. 1. The four regions are defined as: (i) an isotropic half-space, (ii) a semi-infinite circular flux tube, (iii) a very thin disk of infinite extent, and (iv) a finite length circular cylinder with different film coefficients imposed on the side and end surfaces. The mathematical problems of current flow and heat conduction from an isopotential/isothermal circular area into a half-space are mathematically analogous to the classical capacitance problem for an isopotential thin circular disk in free space.

Mathematicians and physicists [1–3] applied different analytical methods to obtain the capacitance of an isopotential circular disk in free space. Since the capacitance is based on the ratio of the total charge to its potential, it was necessary to find the axisymmetric charge density distribution on the surface of the disk. This is an example of an inverse problem which requires the solution of a complex mixed boundary-value problem [2]. Solutions and spreading/constriction results for the analogous conduction problem are summarized and discussed in [4].

Electrical and mechanical engineers have used different analytical and numerical methods to find the spreading/constriction resistances for the other regions shown in Fig. 1. Their results are frequently reported in tables and plots, and a few correlation equations have been given for spreading/constriction resistance as a function of one parameter such as the relative size of the heat source for different flux distributions.

The main objective of this paper is to present a new general analytical solution for the system depicted in Fig. 1(*d*). The general solution will give the dimensionless system resistance which depends on four dimensionless system parameters and the heat flux distribution parameter, and when applicable the spreading/constriction resistances. The second objective is to show how the general solution goes directly to several special cases, previously examined by other researchers, depending on the limiting values of some of the system parameters. The third objective is to report solutions for some special cases. The numerical values will be presented in tables, and whenever possible, correlation equations will be given.

Review of Previous Work

The studies on the thermal spreading/constriction resistance of a circular heat source have a long history. Carslaw and Jaeger [4] reported the analytical results, obtained by several investigators, for the circular source of radius a situated on an isotropic substrate of thermal conductivity k whose dimensions are much larger than the source radius as shown in Fig. 1(*a*). Analytical solutions were reported for two boundary conditions: i) an isoflux circular source and ii) an isothermal source. The spreading resistances were defined for the two boundary conditions.

Roess [5] in an extensive unpublished work found the solution for spreading resistance of an quasi-isothermal circular source of radius a placed on one end of a semi-infinite circular flux tube of radius b and isotropic thermal conductivity k as shown in Fig. 1(*b*). The mixed boundary condition problem was resolved by the use of an equivalent isothermal flux distribution. The analysis is highly mathematical; however, Roess reported the results for the dimensionless spreading resistance $4kaR_s$ as a function of the relative source radius $\epsilon = a/b$ in the form of a power-series.

Smythe [6] solved the mixed boundary value problem arising from the steady flow of a current into a right circular cylinder of radius b. The current enters the cylinder through a coaxial, perfectly conducting, circular disk of radius a. The solution was based on the superposition of the equivalent isothermal flux distribution solution and the isoflux solution. This was accomplished by combining flux distributions to give an approximate flux distribution corresponding to an isothermal source.

Kennedy [7] found through analysis the temperature distributions for steady conduction within a finite length flux tube of radius b, thickness t, and thermal conductivity k due to an isoflux circular source of radius a placed on one end of the flux tube, shown in Fig. 2. Three cases were examined corresponding to the boundary conditions specified along the lateral boundary $r=b$ and the end $z=t$. The three cases were: (i) along $r=b$, $0<z<t$, $q(b,z)=0$ and along $z=t$, $0<r<b$, $T(r,t)=0$; (ii) along $r=b$, $0<z<t$, $T(b,z)=0$ and along $z=t$, $0<r<b$, $q(r,t)=0$; and (iii) along $r=b$, $0<z<t$, $T(b,z)=0$ and along

Contributed by the Electronic and Photonic Packaging Division for publication in the JOURNAL OF ELECTRONIC PACKAGING. Manuscript received by the EPPD Division February 25, 2002. Guest Editors: Y. Muzychka and R. Culham.

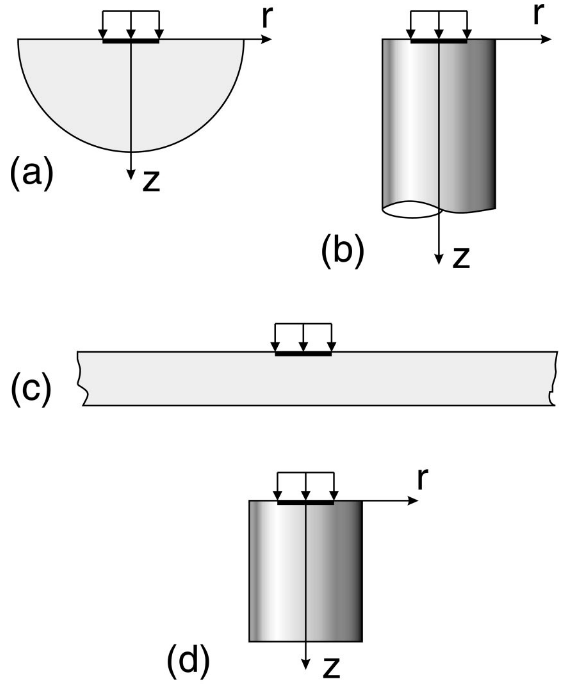

Fig. 1 Schematic of solution domains—(*a*) half-space, (*b*) flux tube, (*c*) infinite plate, (*d*) finite length cylinder

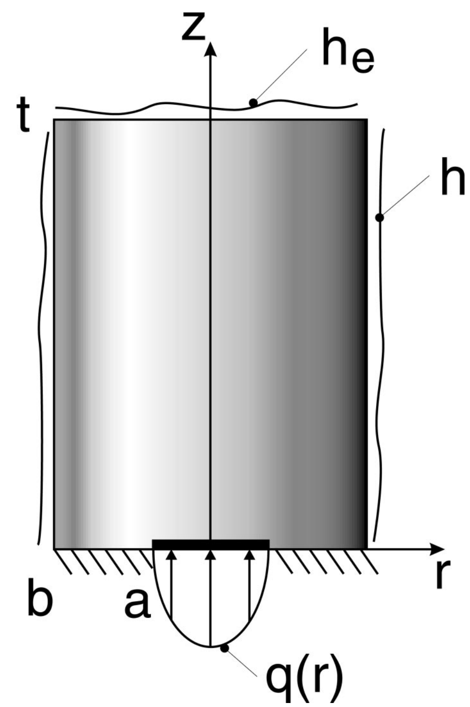

Fig. 2 Finite length cylinder with side and end cooling

$z=t$, $0<r<b$, $T(r,t)=0$. Kennedy [4] reported the solutions for the temperature $T(r,z)$ within the flux tube, the centroid temperature $T(0,0)$, and he presented plots for the normalized temperature $T(r,z)/T(0,0)$.

Kennedy [7] did not present explicit relations for spreading and system resistances. Since the temperature plots were nondimensional, $T(0,0)k/(qa)$, its easy to find the dimensionless system resistance based on the centroid temperature by dividing by the factor πka.

The system resistance is equal to the spreading resistance when $T(b,z)=0$; otherwise, when $q(b,z)=0$, the total resistance is equal to the spreading resistance plus the one-dimensional conduction resistance

$$R_{\rm sys}=R_s+\frac{t}{k\pi b^2} \tag{1}$$

The dimensionless spreading resistance $4kaR_s$ depends on the two geometric parameters: $\epsilon=a/b$, the relative source size and $\tau=t/b$, the relative thickness of the flux tube.

Mikic and Rohsenow [8] obtained analytical relations for spreading resistances for a circular heat source on one end of a semi-infinite flux tube and a finite length finite flux tube as shown in Figs. 1(*b*) and (*d*). They gave solutions for the isoflux source and the quasi-isothermal source based on the equivalent isothermal flux distribution for the $q(b,z)=0$ boundary condition. They presented plots of the dimensionless spreading resistance $4kaR_s$ as a function of ϵ for the semi-infinite flux tube.

It was shown through analysis that the spreading resistance can be obtained by means of the alternative definition:

$$QR_s=\frac{2}{a^2}\int_0^a T(r,0)rdr-\frac{2}{b^2}\int_0^b T(r,0)rdr \tag{2}$$

where $T(r,0)$ represents the temperature rise of points in the plane of the heat source $z=0$. Simple, approximate correlation equations were presented for small values of ϵ.

Hunter and Williams [9] presented an approximate analytical solution for the isothermal circular source on a semi-infinite flux tube. The dimensionless spreading resistance $4kaR_s$ was plotted against ϵ and compared against several approximate solutions which are valid for small values of ϵ only. They did not give a correlation equation for their numerical values.

Strong et al. [10] used finite difference methods, formulated in oblate spheroidal coordinates, to obtain source temperatures and dimensionless spreading resistances for several source heat flux distributions for circular sources situated on the surface of isotropic half-spaces. The analytical solutions presented in [4] were used to validate the numerical scheme which was found to provide accurate values of temperature and dimensionless spreading resistance. They gave plots of the source temperature distribution and tabulated values of the corresponding spreading resistance for each heat flux distribution.

Gibson [11] reported an exact method of approach to the complex mixed boundary value problem (isothermal source) for the semi-infinite flux tube. The problem was reduced to a Fredholm integral equation. The integral was solved by standard methods to obtain an expression for the dimensionless spreading resistance. A plot of $4kaR_s$ versus ϵ for a wide range of values was presented along with a correlation equation.

Yovanovich [12] obtained a general analytical relation for the dimensionless spreading resistance for a general, axisymmetric heat flux distribution $q(r)$ over the circular source on a semi-infinite flux tube. He used the alternative definition of spreading resistance recommended by Mikic and Rohsenow [8]. From the general relation, another general relation was found for $q(r)=q_0[1-(r/a)^2]^\mu$ where q_0 is the flux level at the centroid of the source and μ is the heat flux distribution parameter. By setting $\mu=-1/2$, the equivalent heat flux distribution, and setting $\mu=0$, the isoflux distribution, he was able to get the spreading resistance relations reported by others. He also gave the spreading resistance relation for the heat flux distribution corresponding to $\mu=1/2$. Tabulated values of the three dimensionless spreading resistances for $0\leqslant\epsilon\leqslant0.8$ were given. Several simple correlation equations were given for the three flux distributions.

Yovanovich [13] developed an integral method for finding the spreading resistance of single, planar, isoflux sources of arbitrary shape placed on isotropic half-spaces. The integral method was used to find the dimensionless spreading resistance of several shapes such as regular polygons (which include the square and circle), rectangles, and triangles, as well as other singly connected source areas.

Yovanovich and Burde [14] used the integral method [13] to find the dimensionless spreading resistance of several nonsymmetric, isoflux sources based on the centroid and average temperature basis.

Yovanovich et al. [15] also used the integral method [13] to obtain values of the dimensionless spreading resistances of arbitrary, planar, isoflux sources. They used the square root of the source area for the nondimensionalization, and reported that sources that had the same area and aspect ratio (e.g., a circle and a square) had spreading resistances which differed by less than approximately 1–2%.

Martin et al. [16] used the method of moments to find spreading resistances for several sources such as circle, square and equilateral triangle. They obtained numerical results for isothermal and isoflux boundary conditions. They also examined the effect of the boundary condition of the third kind which can be used to model the effect of a thin film such as an oxide. They presented their numerical results in tabulated form and by several correlation equations.

Negus and Yovanovich [17] obtained an accurate solution for the dimensionless spreading resistance for the true isothermal circular source on a semi-infinite flux. The solution was obtained by the linear superposition of Neumann solutions, and the results were presented in graphical form and in the form of an accurate correlation equation.

Negus and Yovanovich [18] applied the method of optimized images to find the spreading resistance of an isothermal circular source on a semi-infinite flux tube having a square cross section. The numerical results were presented as an accurate correlation equation.

Negus et al. [19] showed that when the square root of the source area is used to nondimensionalize the spreading resistance and the relative size is replaced by $\epsilon=\sqrt{A_s/A_t}$ where A_s is the source area and A_t is the flux tube cross-sectional area, then the dimensionless spreading resistance defined as $k\sqrt{A_s}R_s$ for circle/circle, circle/square and square/square source-to-flux tube ratios have very close values for $0<\epsilon<0.6$.

For each system tabulated values were presented. Also, a simple correlation equation was presented for the three systems. For the range $0\leqslant\epsilon\leqslant 0.5$, the maximum error is about 2%, and the maximum error becomes about 4% when the range is extended to $0\leqslant\epsilon\leqslant 0.7$.

Song et al. [20] and Lee et al. [21] presented analytical and approximate solutions for the isoflux circular source on a finite thickness circular disk with adiabatic sides $q(b,z)=0$ and with cooling over the entire end $z=t$ through a uniform convective coefficient or contact conductance denoted as h_e as shown in Fig. 2. The dimensionless spreading resistance and the relative source size were defined as in [16]. The dimensionless spreading resistance for this system depends on three dimensionless parameters: $\epsilon=a/b$, $\tau=t/b$, $\mathrm{Bi}_e=h_e b/k$. They presented geometric relations for conversion of a rectangular source on a rectangular plate to an equivalent circular source on an equivalent thin circular disk. They presented a relationship for the equivalent h_e for the case where the entire end surface is cooled by a heat sink.

The final example of an electrical spreading resistance problem is due to Foxall and Lewis [22] which is discussed in [3]. In this case an electrical current enters an infinite thin disk (see Fig. 1(*c*)) of thickness *t* and electrical resistivity ρ, through a circular contact area of radius *a*, and leaves through the bottom surface of the disk which is in perfect contact with an ideal conductor. Experimental values were reported [22] for five values of the relative disk thickness $t/a=10$, 5, 2, 1, 0.5 [3]. The experimental results were normalized with respect to the half-space theoretical value and were reported as $4aR_s/\rho=0.96$, 0.90, 0.80, 0.64, 0.43 for the five relative thicknesses respectively [3,22].

This relatively brief review of the pertinent literature shows that many researchers over six decades have examined many systems consisting of a single circular source on substrates which were modeled as regions shown in Fig. 1. The sources were isoflux, isothermal or had other axisymmetric flux distributions. The numerical results were presented in graphical and tabular form, and accurate or approximate correlation equations were also given.

The system shown in Fig. 2 has a circular source of radius *a* on one end of a finite length circular cylinder of radius *b*, thickness *t*, and thermal conductivity *k*. The sides $r=b$ and the end $z=t$ are cooled through uniform, but different heat transfer coefficients *h* and h_e, respectively. The spreading and total resistances for this system are presently not available. The dimensionless spreading and total resistances of this general problem will depend on several system parameters such as a/b, t/b, hb/k, $h_e b/k$ and the source heat flux distribution μ. The general relationships developed for this system will give the particular cases discussed in the foregoing review.

General Problem Statement and Mathematical Formulation

Consider a finite length circular cylinder of radius *b*, thickness *t*, and thermal conductivity *k* which is isotropic. There is a circular heat source of radius *a* located in the surface $z=0$ of the cylinder and it is coaxial. The heat flux over the source area is assumed to be axisymmetric, and the remainder of the surface is adiabatic. The lateral side $r=b$ and the opposite end $z=t$ are cooled by a fluid at fixed temperature T_f through uniform film coefficients *h* and h_e, respectively. The system is depicted in Fig. 2.

The heat which enters the system through the source area is conducted through the system and leaves through the side and end surfaces. The maximum temperature rise occurs at the center of the source area. The system thermal resistance is defined with respect to the average temperature rise of the source area.

The governing differential equation for the axisymmetric temperature rise $\theta(r,z)=T(r,z)-T_f$ is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\theta}{\partial r}\right)+\frac{\partial^2\theta}{\partial z^2}=0,\quad 0<r<b,\quad 0<z<t \tag{3}$$

The general boundary conditions are

$$\begin{aligned}
&r=0,\quad \frac{\partial\theta}{\partial r}=0\\
&r=b,\quad \frac{\partial\theta}{\partial r}=-\frac{h}{k}\theta\\
&z=t,\quad \frac{\partial\theta}{\partial z}=-\frac{h_e}{k}\theta\\
&z=0,\quad \frac{\partial\theta}{\partial z}=\begin{cases}-\dfrac{q(r)}{k} & \text{for } 0<r<a\\ 0 & \text{for } a<r<b\end{cases}
\end{aligned} \tag{4}$$

The axisymmetric heat flux over the source area has the general form

$$q(r)=\frac{Q(1+\mu)}{\pi a^2}\left[1-\left(\frac{r}{a}\right)^2\right]^{\mu}\quad \text{for } 0<\frac{r}{a}<1 \tag{5}$$

where Q is the total heat transfer rate from the source into the system, and μ is the heat flux distribution parameter. Three interesting distributions are obtained when $\mu=-1/2$, 0, 1/2. The flux distribution corresponding to $\mu=-1/2$ is minimum at the center and unbounded at the edge of the source area. This flux distribution is frequently used to approximate an isothermal source area. The second flux distribution corresponding to $\mu=0$ is clearly an

Table 1 Fourier coefficients for three flux distributions

$\mu=-\frac{1}{2}$,	$E_n=\frac{Q}{\pi bk}\cdot\frac{\sin(\delta_n\epsilon)}{\phi_n\delta_n^2\epsilon[J_0^2(\delta_n)+J_1^2(\delta_n)]}$
$\mu=0$,	$E_n=\frac{Q}{\pi bk}\cdot\frac{2J_1(\delta_n\epsilon)}{\phi_n\delta_n^2\epsilon[J_0^2(\delta_n)+J_1^2(\delta_n)]}$
$\mu=\frac{1}{2}$,	$E_n=\frac{Q}{\pi bk}\cdot\frac{3[\sin(\delta_n\epsilon)-\cos(\delta_n\epsilon)(\delta_n\epsilon)]}{\phi_n\delta_n^4\epsilon^3[J_0^2(\delta_n)+J_1^2(\delta_n)]}$

isoflux distribution. The third flux distribution corresponding to $\mu=1/2$ is parabolic; its maximum at the center and goes to zero at the edge of the source area.

General Solution

The general axisymmetric solution is

$$\theta(r,z)=\sum_{n=1}^{\infty}[E_n\cosh(\lambda_n z)+F_n\sinh(\lambda_n z)]J_0(\lambda_n r) \qquad (6)$$

It satisfies the boundedness condition $\theta(0,z)\neq\infty$ on the axis $r=0$, and the eigenvalues λ_n are related to the positive roots of the characteristic equation

$$\delta_n J_1(\delta_n)=\mathrm{Bi}J_0(\delta_n) \qquad (7)$$

with $\delta_n=\lambda_n b$, $\mathrm{Bi}=hb/k$ and $J_0(\cdot)$ and $J_1(\cdot)$ are Bessel functions of the first kind of order zero and one, respectively [23]. The parameter range is $0\leqslant\mathrm{Bi}<\infty$.

The boundary condition at $z=t$ is satisfied if the Fourier-Bessel coefficients are related such that

$$F_n=-E_n\phi_n \qquad (8)$$

with the function

$$\phi_n=\frac{\mathrm{Bi}_e+\delta_n\tanh(\delta_n\tau)}{\delta_n+\mathrm{Bi}_e\tanh(\delta_n\tau)} \qquad (9)$$

where $0<\tau=t/b<\infty$ is the relative cylinder thickness; and with the boundary parameter: $0\leqslant\mathrm{Bi}_e=h_eb/k<\infty$. The function ϕ_n has two asymptotes corresponding to the limits on the end cooling parameter Bi_e. They are

$$\mathrm{Bi}_e\to0,\quad \phi_n\to\tanh(\delta_n\tau)\qquad\text{and}\qquad \mathrm{Bi}_e\to\infty,\quad \phi_n\to\coth(\delta_n\tau)$$

and if $\delta_n\tau>2.70$, then $\phi_n\approx1$ for $n\geqslant1$. The remaining Fourier-Bessel coefficients are found from the boundary conditions in $z=0$. These conditions require

$$E_n=\frac{\int_0^a q(r)rJ_0(\lambda_n r)dr}{-\phi_n\lambda_n\int_0^b rJ_0^2(\lambda_n r)dr} \qquad (10)$$

After substitution for $q(r)$ and evaluation of the integrals, with application of the orthogonality property of Bessel functions, one finds for the Fourier coefficients the general relationship

$$E_n=\frac{2Q}{\pi bk}\left(\frac{2}{\delta_n\epsilon}\right)^{\mu}\frac{\Gamma(2+\mu)J_{1+\mu}(\delta_n\epsilon)}{\phi_n\delta_n^2[J_0^2(\delta_n)+J_1^2(\delta_n)]},\quad n=1,2,3\ldots \qquad (11)$$

where $\Gamma(\cdot)$ is the Gamma function [23].

The Fourier coefficients for the three heat flux distributions: $\mu=-1/2$, 0, 1/2 are given in Table 1.

Maximum (Centroid) Temperature Rise

The maximum temperature rise which occurs at the center of the source area is given by

$$\theta_{\max}=\theta(0,0)=\sum_{n=1}^{\infty}E_n \qquad (12)$$

This general relation applies to the three heat flux distributions.

System Thermal Resistance

The total resistance of the system is defined with respect to the mean temperature rise of the source area

$$\theta_{\mathrm{ave}}=\frac{2}{a^2}\int_0^a r\theta(r,0)dr$$

and

$$R_{\mathrm{sys}}=\frac{\theta_{\mathrm{ave}}}{Q}$$

Since the temperature excess in the surface $z=0$ is

$$\theta(r,0)=\sum_{n=1}^{\infty}E_nJ_0(\lambda_n r) \qquad (13)$$

one finds for the system resistance the general relationship

$$R_{\mathrm{sys}}=\frac{2}{Q}\sum_{n=1}^{\infty}E_n\frac{J_1(\delta_n\epsilon)}{\delta_n\epsilon} \qquad (14)$$

The dimensionless system resistance $\Psi=4akR_{\mathrm{sys}}$ is given by the general relationship

$$\Psi=\frac{16}{\pi\epsilon}\sum_{n=1}^{\infty}\left(\frac{2}{\delta_n\epsilon}\right)^{\mu}\frac{\Gamma(2+\mu)J_{1+\mu}(\delta_n\epsilon)J_1(\delta_n\epsilon)}{\phi_n\delta_n^3[J_0^2(\delta_n)+J_1^2(\delta_n)]},\quad n=1,2,3\ldots \qquad (15)$$

where $\Gamma(2+\mu)$ is the Gamma function [23]. The general relation reduces to the following three relations for $\mu=-1/2$, the equivalent isothermal source provided $\epsilon<0.6$, and $\tau>1$, and $\mu=0$, the isoflux source, and $\mu=1/2$, the parabolic flux distribution

$$\text{for }\mu=-\frac{1}{2},\quad \Psi=\frac{8}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{J_1(\delta_n\epsilon)\sin(\delta_n\epsilon)\varphi_n}{\delta_n^3[J_0^2(\delta_n)+J_1^2(\delta_n)]} \qquad (16)$$

and

$$\text{for }\mu=0,\quad \Psi=\frac{16}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{J_1^2(\delta_n\epsilon)\varphi_n}{\delta_n^3[J_0^2(\delta_n)+J_1^2(\delta_n)]} \qquad (17)$$

and for $\mu=1/2$

$$\Psi=\frac{24}{\pi\epsilon}\sum_{n=1}^{\infty}\frac{J_1(\delta_n\epsilon)\sin(\delta_n\epsilon)\varphi_n}{\delta_n^3[J_0^2(\delta_n)+J_1^2(\delta_n)]}\left[\frac{1}{(\delta_n\epsilon)^2}-\frac{1}{(\delta_n\epsilon)\tan(\delta_n\epsilon)}\right] \qquad (18)$$

where for convenience we introduce the end cooling function $\varphi_n=1/\phi_n$

$$\varphi_n=\frac{\delta_n+\mathrm{Bi}_e\tanh(\delta_n\tau)}{\mathrm{Bi}_e+\delta_n\tanh(\delta_n\tau)} \qquad (19)$$

The general solution for the total system resistance depends on four dimensionless system parameters

$$\Psi=f(\epsilon,\tau,\mathrm{Bi},\mathrm{Bi}_e) \qquad (20)$$

with parameter ranges $0<\epsilon\leqslant1$, $0<\tau<\infty$, $0\leqslant\mathrm{Bi}<\infty$, and $0\leqslant\mathrm{Bi}_e<\infty$.

Characteristics of the General Relations

If the lateral sides of the cylinder are adiabatic, i.e., $\mathrm{Bi}=0$, then δ_n are the roots of $J_1(\delta_n)=0$, and the one-dimensional resistance of the system is

$$R_{1D}=\frac{t}{k\pi b^2}+\frac{1}{h_e\pi b^2} \qquad (21)$$

The dimensionless one-dimensional resistance for $\mathrm{Bi}=0$ is defined as

$$R_{1D}^{*} \equiv 4akR_{1D} = \frac{4\epsilon}{\pi}\left[\tau + \frac{1}{\mathrm{Bi}_e}\right] \tag{22}$$

The dimensionless spreading resistance, defined as $\psi = 4kaR_s$, is therefore

$$\psi = \Psi - R_{1D}^{*}. \tag{23}$$

As $\mathrm{Bi}\to 0$ the first term of the summation goes to the one-dimensional total resistance and the sum of the remaining terms goes to the spreading resistance of the system

$$\psi = \frac{16}{\pi\epsilon}\sum_{n=2}^{\infty}\frac{J_1^2(\delta_n\epsilon)\varphi_n}{\delta_n^3 J_0^2(\delta_n)}, \quad \text{for } \mu = 0 \tag{24}$$

Also, the first eigenvalue approaches the asymptote $\delta_1 \to \sqrt{2\mathrm{Bi}}$ as $\mathrm{Bi}\to 0$. The functions which appear have the asymptotes [23]

$$\text{as } x\to 0, \quad \tanh(x)\to x, \quad J_0(x)\to 1, \quad J_1(x)\to \frac{x}{2}$$

Substituting these limits into the first term, then letting $\mathrm{Bi}\to 0$ gives the relation for the dimensionless total one-dimensional resistance.

The general relationship for the total system resistance gives several special relationships depending on the values of the system parameters: ϵ, τ, Bi, Bi_e.

Calculation of the Eigenvalues

The foregoing general relationships require accurate values of the eigenvalues which must be calculated by means of the characteristic equation. The Bessel functions $J_0(x)$, $J_1(x)$ can be computed accurately by means of polynomial approximations or with a computer algebra system. The roots of the characteristic equation are located in the intervals

$$(n-1)\pi < \delta_n < n\pi, \quad n = 1,2,3,\ldots$$

and they have the following properties

$$\text{for } \mathrm{Bi}\to\infty, \quad \delta_n \text{ are roots of } J_0(\delta_n) = 0$$

The first three roots are $\delta_1 = 2.404825558$, $\delta_2 = 5.520078110$, $\delta_3 = 8.653727913$ [23], and

$$\text{for } \mathrm{Bi}=0, \quad \delta_n \text{ are roots of } J_1(\delta_n) = 0$$

and the first four roots are $\delta_1 = 0$, $\delta_2 = 3.831705970$, $\delta_3 = 7.015586670$, $\delta_4 = 10.17346814$ [23]. Also, as $\mathrm{Bi}\to 0$, $\delta_1 \to \sqrt{2\mathrm{Bi}} \to 0$. The Newton-Raphson iterative method is recommended for calculation of the roots for any value of Bi. The relation for the nth root is

$$\delta_{n,j+1} = \delta_{n,j} - \frac{\delta_{n,j}J_1(\delta_{n,j}) - \mathrm{Bi}J_0(\delta_{n,j})}{\delta_{n,j}J_0(\delta_{n,j}) + \mathrm{Bi}J_1(\delta_{n,j})}, \quad j = 1,2,3,\ldots \tag{25}$$

The calculations converge rapidly. Eight decimal place values are found after 5 to 6 iterations.

Special Cases Arising From the General Relationship

Special cases arise naturally from the general relationship depending on the magnitude and/or range of the system parameters. Several special cases are presented below.

Spreading Resistance in Flux Tubes With Adiabatic Sides. For this important case, $\mathrm{Bi}=0$, δ_n are the roots of $J_1(\delta_n) = 0$ which are computed by means of the following modified Stokes approximation:

$$\delta_n = \frac{\beta_1}{4}\left[1 - \frac{6}{\beta_1^2} + \frac{6}{\beta_1^4} - \frac{4716}{5\beta_1^6} + \frac{3902918}{70\beta_1^8}\right] \tag{26}$$

with $\beta_1 = \pi(4n+1)$ and $n = 1,2,3\ldots$

Table 2 Coefficients for spreading resistance [5]

r_1	r_3	r_5	r_7	r_9	r_{11}
1.40925	.295910	.0525419	.0210419	.0110752	.00631188

Spreading Resistance: Isothermal Contact Area. Several analytical studies [5,6,8,9,11,12,17,18] have considered the problem of spreading resistance of an isothermal circular contact area located on the end of a very long circular flux tube. The mixed boundary conditions in the contact plane $z=0$, are (i) constant temperature over the contact area $0 \le r < a$, and (ii) zero temperature gradient over the noncontact portion of the contact plane $a < r \le b$ making the problem considerably more difficult analytically. This mixed boundary-value problem is discussed in some detail in [2].

Various analytical and numerical techniques were used to produce an isothermal contact area for large relative contact radius $\epsilon < 0.8$. Roess [5] in an extensive unpublished work found the spreading resistance for the equivalent isothermal flux distribution corresponding to $\mu = -1/2$. The analysis is highly mathematical; however, results in the form of a power-series [5] were reported

$$4akR_s = 1 - r_1\epsilon + r_3\epsilon^3 + r_5\epsilon^5 + r_7\epsilon^7 + r_9\epsilon^9 + r_{11}\epsilon^{11} \tag{27}$$

where the coefficients $r_1, \ldots, r_{11}$ are given in Table 1. It was found [5] that when $\epsilon < 0.3$, the heat flux distribution produces a contact area temperature distribution which for most practical applications can be considered to be isothermal. It was reported [5] that when $\epsilon = 0.4$ and $\epsilon = 0.5$, the temperature along the edge $r = a$, exceeded the centroid temperature by approximately 3.86 and 9.90%, respectively. For $\epsilon > 0.5$, the temperature distribution is nonuniform.

Smythe [6] solved the mixed boundary value problem arising from the steady flow of a current into a right circular cylinder of radius b. The current enters the cylinder through a coaxial, perfectly conducting, circular disk of radius a. The solution was based on the superposition of the equivalent isothermal flux distribution solution and the isoflux solution. This was accomplished by the following combined flux distribution:

$$q = \frac{Q}{\pi b^2}\left\{\epsilon^{1.5} + \frac{1}{2\epsilon\sqrt{1-u^2}}\left[\frac{1}{\epsilon} - \epsilon^{2.5}\right]\right\}$$

with $\epsilon = a/b$ and $u = r/a$. This flux distribution has two limits as $\epsilon\to 0$ and $\epsilon\to 1$. For the first limit, the flux distribution goes to the flux distribution corresponding to an isothermal circular contact area on a half-space, i.e., $q = Q/(2\pi a^2\sqrt{1-u^2})$. For the second limit, the flux distribution goes to the uniform flux distribution, $q = Q/(\pi b^2)$.

The approximate relationship of [6] can be cast into the following form for the dimensionless spreading resistance:

$$4akR_s = (1-\epsilon^{3.5})\psi(\mu = -1/2) + \epsilon^{3.5}\psi(\mu = 0) \tag{28}$$

where $\psi(\mu = -1/2)$ and $\psi(\mu = 0)$ are the dimensionless spreading resistances defined as $\psi \equiv 4akR_s$ for the equivalent isothermal flux and the isoflux distributions, respectively.

The numerical values of the dimensionless spreading resistance for the three flux distributions $(\mu = 1/2, 0, -1/2)$ are given in Table 3. Four hundred terms of the series solutions were used to compute the four decimal place values shown. The values given for $\epsilon = 0$ are the half-space values [4]. The dimensionless spreading resistance depends on the flux distribution such that the minimum values correspond to the equivalent isothermal flux and the maximum values correspond to the parabolic flux distribution. The isoflux values lie between these values over the full range of ϵ. The isothermal values computed by means of the solutions of [5,6,11,17] are also presented in Table 3. For relative contact radii in the range: $0 \le \epsilon \le 0.5$, the differences between the various models is less than approximately 2% which occurs between the Roess

Table 3 Dimensionless spreading resistance, $4akR_s$

ϵ	μ 1/2	μ 0	μ −1/2	Roess [5]	Gibson [11]	Negus and Yovanovich [17]	Smythe [6]
0	1.125	1.0808	1	1	1	1	1
.1	.9843	.9401	.8592	.8594	.8594	.8594	.8593
.2	.8450	.8009	.7205	.7205	.7209	.7208	.7207
.3	.7085	.6649	.5853	.5853	.5865	.5865	.5865
.4	.5763	.5337	.4558	.4558	.4587	.4586	.4590
.5	.4500	.4092	.3342	.3342	.3398	.3396	.3408
.6	.3316	.2936	.2232	.2232	.2328	.2324	.2349
.7	.2235	.1896	.1262	.1262	.1409	.1403	.1444
.8	.1284	.1008	.0483	.0478	.0680	.0672	.0723

[5] and Smythe [6] solutions. The correlation of [5] gives the largest differences for large values of ϵ. The differences between the values calculated by the correlation equations of Gibson [11], and Negus and Yovanovich [17] are less than approximately 1% for all values of ϵ. The approximate solution of Smythe [6] is seen to be very good up to $\epsilon=0.7$ where the values are approximately 2.4% greater than those obtained by the correlation equations of [11] and [17].

Correlation Equations

The analytical solutions presented in the foregoing and the analytical/numerical isothermal solution of Gibson [11] and the isothermal and isoflux solutions of Negus and Yovanovich [17] are correlated in the same manner by the polynomial

$$4akR_s=a_0+a_1\epsilon+a_3\epsilon^3+a_5\epsilon^5+a_7\epsilon^7 \tag{29}$$

The correlation coefficients for the three analytical solutions are based on fitting the numerical values found in Table 4. The correlation coefficients for the isothermal solution are those reported by [11] and [17].

Dimensionless Spreading Resistances in Half-Space. For $\epsilon\to 0$ and $\tau>1$, the system resistance approaches the spreading resistance in a half-space. The heat flux distributions and the corresponding spreading resistances are given in Table 5. The analytical results are: $\psi(\mu=-1/2)=1$, $\psi(\mu=0)=1.0808$ and $\psi(\mu=1/2)=1.1252$.

Perfect Thermal Contact Along Sides. If there is perfect thermal contact along the sides of the cylinder, then $\mathrm{Bi}\to\infty$, and then δ_n are the positive roots of $J_0(\delta_n)=0$ which can be computed by means of the following modified Stokes approximation:

$$\delta_n=\frac{\beta_0}{4}\left[1+\frac{2}{\beta_0^2}-\frac{62}{3\beta_0^4}+\frac{15116}{30\beta_0^6}\right] \tag{30}$$

with $\beta_0=\pi(4n-1)$ and $n=1,2,3\ldots$

Very Thin Cylinders. For very thin cylinders where the thickness to source radius is much less than one, i.e., $t/a<0.1$ or $\tau/\epsilon<0.1$, the heat dissipated by the source flows directly to the opposite face in a one-dimensional manner with negligible spreading. The system resistance approaches the one-dimensional resistance for an isothermal source

$$\text{for } \frac{t}{a}\to 0,\quad R_{\text{sys}}\to R_{1d}=\frac{t}{k\pi a^2}+\frac{1}{h_e\pi a^2} \tag{31}$$

The dimensionless system resistance has the following limit:

$$\text{for } \frac{\tau}{\epsilon}\to 0,\quad 4akR_{\text{sys}}\to R_{1d}^*=\frac{4}{\pi}\left[\frac{\tau}{\epsilon}+\frac{1}{\epsilon \mathrm{Bi}_e}\right] \tag{32}$$

This asymptote is valid for the parameter ranges: $0\le \mathrm{Bi}<\infty$ and $0<\mathrm{Bi}_e<\infty$.

Spreading Resistance of Isothermal Circular Source on Thin Infinite Disk. Consider an isothermal circular source of radius a with heat flux parameter $\mu=-1/2$ in contact with a thin infinite disk $\epsilon\to 0$ of thickness t and thermal conductivity k as shown in Fig. 3. Since the bottom surface is assumed to be isothermal, then $h_e\to\infty$. The solution for a closely related problem is given in [1]. The solution for the temperature is given in terms of an infinite integral.

The dimensionless spreading resistance, $\psi=4kaR_s$, is given by

$$\psi=\frac{4}{\pi}\int_0^\infty \tanh(\beta\chi)\sin(\beta)J_1(\beta)\,\frac{d\beta}{\beta^2} \tag{33}$$

where $\chi=t/a$ is the relative disk thickness. The hyperbolic tangent can be expressed as

$$\tanh(\beta\chi)=1-\frac{2e^{-2\beta\chi}}{1+e^{2\beta\chi}}$$

The dimensionless spreading resistance can be written in the alternative form

$$\psi=1-\frac{8}{\pi}\int_0^\infty \frac{e^{-2\beta\chi}\sin(\beta)J_1(\beta)}{1+e^{-2\beta\chi}}\,\frac{d\beta}{\beta^2} \tag{34}$$

which shows the very thick disk asymptote

$$\text{as }\ \chi\to\infty,\quad \psi\to 1$$

For quick numerical integrations, the first form is recommended for $\chi<2$ and the second form is recommended for $\chi>2$. Accurate values of ψ were computed by means of computer algebra systems

Table 4 Correlation coefficients for spreading resistance solutions, $4akR_s$

	μ 1/2	μ 0	μ −1/2	Gibson [11] Isothermal	N-Y [17] Isothermal	N-Y [17] Isoflux
a_0	1.12517	1.08085	0.999961	1	1	1.08076
$-a_1$	1.41038	1.41002	1.40981	1.409183	1.40978	1.41042
a_3	.235387	.259714	.303641	.338010	.34406	.26604
a_5	.0117527	.0188631	.0218272	.067902	.04305	−.00016
a_7	.0343458	.0420278	.0644683	⋯	.02271	.058266

Table 5 Flux distributions and spreading resistances in half-space

μ	$-1/2$	0	$1/2$
$\dfrac{a^2 q}{Q}$	$\dfrac{1}{2\pi\sqrt{1-u^2}}$	$\dfrac{1}{\pi}$	$\dfrac{3\sqrt{1-u^2}}{2\pi}$
$4akR_s$	1	$\dfrac{32}{3\pi^2}$	1.1252

such as Maple and Mathematica. They are reported in Table 6 for selected values of the relative plate thickness χ. The plate can be modeled as *infinitely* thick when $\chi \geqslant 100$.

Correlation Equation for Spreading Resistance in Infinite Disk. A correlation equation based on the entries in Table 6 is

$$\psi = \frac{a_1 a_2 + a_3 \chi^{a_4}}{a_2 + \chi^{a_4}}, \quad 0 \leqslant \chi \leqslant 10 \tag{35}$$

with correlation coefficients given in Table 7. The correlation equation gives values of ψ with a maximum difference of about 0.4% relative to the tabulated values. The electrical experimental values [3,22] discussed above are in good agreement with the analytical values given in Table 6.

Extended Surface (Pin Fin) With End Cooling and Spreading Resistance. The general solution approaches the solution for an extended surface (pin fin) if $\mathrm{Bi} \to 0$ and $\tau > 1$, $\mu = 0$ for $0 \leqslant \mathrm{Bi}_e < \infty$ and $0 < \epsilon \leqslant 1$. If $\mathrm{Bi} < 0.2$, then the dimensionless system resistance approaches the first term of the summation

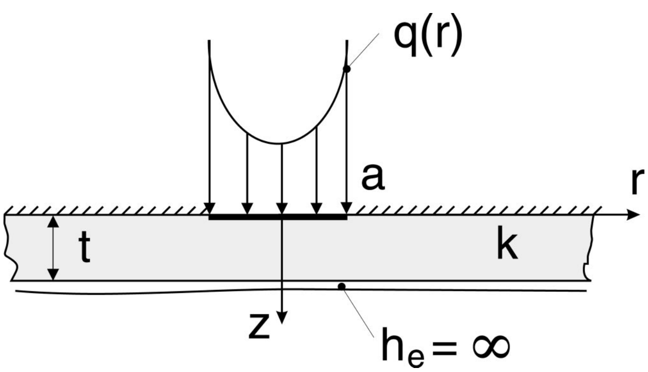

Fig. 3 Equivalent isothermal circular source on thin infinite disk

Table 6 Dimensionless spreading resistances in infinite disk

χ	ψ	χ	ψ
0	0	2.0	0.7889
.10	0.1089	3.0	0.8559
.20	0.2015	4.0	0.8910
.30	0.2824	5.0	0.9124
.40	0.3532	6.0	0.9268
.50	0.4149	7.0	0.9372
.60	0.4684	8.0	0.9450
.70	0.5148	9.0	0.9511
.80	0.5551	10	0.9560
.90	0.5901	20	0.9779
1.0	0.6206	100	0.9956

Table 7 Correlation coefficients for infinite disk

a_1	a_2	a_3	a_4
0.002915	0.61378	0.99617	1.1783

$$\Psi \to \frac{16}{\pi\epsilon} \frac{J_1^2(\delta_1 \epsilon)\varphi_1}{\delta_1^3 J_0^2(\delta_1)} \tag{36}$$

with

$$\varphi_1 = \frac{\delta_1 + \mathrm{Bi}_e \tanh(\delta_1 \tau)}{\delta_1 \tanh(\delta_1 \tau) + \mathrm{Bi}_e} \tag{37}$$

and the first root of the characteristic equation is approximated by the correlation

$$\delta_1 = \frac{2.404826}{\left[1 + \left(\dfrac{2.404826}{\sqrt{2\mathrm{Bi}}}\right)^{2.238}\right]^{1/2.238}} \tag{38}$$

which gives values with a maximum error less than approximately 2% for the full range: $0 \leqslant \mathrm{Bi} < \infty$. For $\mathrm{Bi} \to 0$, since $\delta_1 \to \sqrt{2\mathrm{Bi}} \to 0$, then $J_0(\delta_1) \to 1$, $J_1(\delta_1) \to (\delta_1)/2$ and $J_1(\delta_1 \epsilon) \to (\delta_1 \epsilon)/2$, and the system resistance goes to the first term which approaches the fin resistance with spreading resistance and end cooling

$$\Psi \to \frac{4}{\pi} \frac{\varphi_1}{\epsilon \delta_1} \to R_{\mathrm{fin}}^* = \frac{4}{\pi \epsilon \sqrt{2\mathrm{Bi}}} \left[\frac{\sqrt{2\mathrm{Bi}} + \mathrm{Bi}_e \tanh(\sqrt{2\mathrm{Bi}}\,\tau)}{\mathrm{Bi}_e + \sqrt{2\mathrm{Bi}} \tanh(\sqrt{2\mathrm{Bi}}\,\tau)} \right] \tag{39}$$

This relation shows the dependence of the general dimensionless fin resistance on the system parameters: ϵ, τ, Bi, Bi_e.

Approximation Relations for Spreading Resistance in Heat Sink Base Plate

The general solution can be used to calculate the spreading resistance in the rectangular base plate of a heat sink due to a rectangular heat source as described in [20] and [21]. For this case we set the heat flux parameter to $\mu = 0$ and $\mathrm{Bi} = 0$. The equivalent source radius is obtained from $a = \sqrt{A_s/\pi}$ and the cylinder radius is obtained from $b = \sqrt{A_p/\pi}$ where A_s and A_p represent the plan areas of the source and base plate respectively. The relative source size was defined as $\epsilon = a/b$, and the remaining system parameters were defined as $\mathrm{Bi}_e = h_e b/k$ and $\tau = t/b$ where t is the thickness of the base plate.

They nondimensionalized the constriction resistance based on the centroid and area-average temperatures using the square root of the contact area as recommended by Yovanovich in several papers. They compared the analytical results against some experimental results and numerical results over a range of the independent parameters: ϵ, τ, Bi. The agreement between the analytical and numerical results were reported to be in very good agreement.

Lee et al. [21] recommended a simple closed-form expression for the dimensionless constriction resistance based on the area-average and centroid temperatures. They defined the dimensionless spreading resistance parameter as $\psi = \sqrt{\pi} k a R_s$, and they recommended the following approximations:

for the area-average temperature basis

$$\psi_{\mathrm{ave}} = \frac{1}{2}(1 - \epsilon)^{3/2} \phi_c \tag{40}$$

and for the centroid temperature basis

$$\psi_{\max} = \frac{1}{\sqrt{\pi}}(1 - \epsilon)\varphi_c \tag{41}$$

where

$$\varphi_c = \frac{\mathrm{Bi} \tanh(\delta_c \tau) + \delta_c}{\mathrm{Bi} + \delta_c \tanh(\delta_c \tau)} \quad \text{with} \quad \delta_c = \pi + \frac{1}{\sqrt{\pi}\epsilon}$$

The foregoing approximations are stated to be within ±10% of the analytical and numerical results [20,21]. They did not, however, indicate were the maximum errors occur.

Summary and Concluding Remarks

A general solution has been presented for spreading/constriction and system resistances for a circular source on a finite circular cylinder with side and end cooling. The dimensionless resistances: $\psi = 4kaR_s$, and $\Psi = 4kaR_{\text{sys}}$ depend on four dimensionless system parameters: ϵ, τ, Bi, Bi_e, and selected values of the heat flux parameter $\mu = -1/2$, 0, 1/2. Special cases which have been examined by several researchers arise directly from the general relationships presented in this paper. Some important special cases are:

a. spreading resistance in a half-space: $\epsilon \to 0$, $\psi = f(\mu)$;
b. spreading resistance in a flux tube: $\text{Bi}=0$, $\tau > 1$, $\psi = f(\epsilon, \mu)$;
c. spreading resistance in a thin infinite disk: $t/a < 0.1$, $\epsilon \to 0$, $\mu = -1/2$, $h_e \to \infty$, $\psi = f(t/a)$;
d. extended surface (pin fin) with end cooling: $\text{Bi} < 0.2$, $\tau > 1$, $\epsilon = 1$, $\mu = 0$, $4kbR_{\text{fin}} = f(\tau, \text{Bi}, \text{Bi}_e)$.

Selected values of the dimensionless spreading resistances in a flux tube and an infinite plate are presented in tables. Correlation equations are presented for dimensionless spreading resistances in a flux tube and an infinite plate.

The dimensionless resistances can be computed quickly and accurately by means of Computer Algebra Systems for all values of the system parameters and the three heat flux parameter values.

The general relationships given in this paper can be used to model many thermal problems encountered in microelectronic and telecommunication applications.

Acknowledgments

The continued financial support of the Natural Sciences and Engineering Research Council of Canada for over thirty years is appreciated. The students (Burde, Martin, Negus) who participated in the early work contributed much. The UW faculty (L. Y. Chow, G. E. Schneider, A. B. Strong, J. C. Thompson) and others (J. V. Beck, J. R. Dryden, B. B. Mikic) added to the quality of the work. The assistance of Rick Culham and Yuri Muzychka in the preparation of figures is greatly appreciated.

Nomenclature

A_s, A_t = source and flux tube area; m^2
Bi = side surface Biot no.; $\text{Bi} = hb/k$
Bi_e = end surface Biot no.; $\text{Bi}_e = h_e b/k$
a, b = source and flux tube radii; m
a_i = correlation coefficients
E_n, F_n = Fourier coefficients
h, h_e = side and end heat transfer coefficients; $\text{W/m}^2 \cdot \text{K}$
$J_0(\cdot)$, $J_1(\cdot)$ = Bessel functions of order 0 and 1
$J_v(\cdot)$ = Bessel function of arbitrary order v
k = isotropic thermal conductivity; W/m·K
Q = heat transfer rate; W
q = source mean heat flux; W/m^2
R_{fin} = fin resistance; K/W
R_s, R_{sys} = spreading and system resistances; K/W
R_{1D} = one-dimensional system resistance; K/W
R_{1d} = one-dimensional resistance; K/W
r_i = Roess correlation coefficients
r, z = cylindrical polar coordinates; m
$T(r,z)$ = temperature in cylinder; K
t = cylinder and plate thickness; m
u = dimensionless position in source area; $= r/a$
β = dummy variable; $= \lambda a$
β_0 = parameter in modified Stokes approximation of roots of $J_0(x) = 0$
$= \pi(4n-1)$, $n = 1,2,3 \ldots$
β_1 = parameter in modified Stokes approximation of roots of $J_1(x) = 0$
$= \pi(4n+1)$, $n = 1,2,3 \ldots$
$\Gamma(\cdot)$ = Gamma function
δ_n = dimensionless eigenvalues; $= \lambda_n b$
ϵ = relative source size; $= a/b$
θ = temperature difference; $= T(r,z) - T_f$
λ_n = eigenvalues; m^{-1}
μ = heat flux distribution parameter
ρ = electrical resistivity; Ω m
τ = relative cylinder thickness; $= t/b$
ϕ = function defined in Eq. (9)
φ = function defined in Eq. (19)
χ = relative plate thickness; $= t/a$
Ψ = dimensionless system resistance; $= 4kaR_{\text{sys}}$
ψ = dimensionless spreading resistance; $= 4kaR_s$

Subscripts

ave = average value
e = cylinder end heat transfer coefficient
f = fluid temperature
fin = fin or extended surface
max = maximum value
s = spreading, source
sys = system
t = flux tube

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