Thermal Stresses Due to a Plane Crack in General Anisotropic Material

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A solution is given for the thermoelastic stress field due to the obstruction of a uniform heat flux by a plane crack in a generally anisotropic body. A Green's function formulation is used to reduce the problem to a set of singular integral equations which are solved in closed form. When the crack is assumed to be traction free, the crack opening displacement is found to be negative over one half of the crack unless a sufficiently large far field tensile stress is superposed. The problem is, therefore, reformulated assuming a contact zone at one crack tip. The extent of this zone and the stress intensity factors in all three modes at each crack tip are obtained as functions of the applied stress and heat flux.

Introduction

When the flow of heat in a solid is disturbed by some discontinuity such as a hole or a crack, the local temperature gradient around the discontinuity is increased. Thermal disturbances of this type can produce material failure through crack propagation. The problem is complicated by the fact that the thermal distortion may cause the crack to open or close, hence changing the thermal boundary conditions. A number of studies dealing with flaw-induced thermal stresses in infinite isotropic regions have been published by Florence and Goodier (1963) and Olesiak and Sneddon (1959). For the plane crack in an infinite isotropic homogeneous body, it can be argued from consideration of symmetry that the crack faces will not separate and there will be only a mode II stress intensity factor, unless heat is generated in the crack.

Anisotropic Materials

The widespread use of composite materials in structural applications has generated renewed interest in anisotropic material behavior. In particular, information on thermal stress concentrations around material discontinuities in anisotropic bodies will have application in high-temperature composite materials. Solutions have been published for the axisymmetric problem of the penny-shaped crack in a transversely isotropic material by Tsai (1983) and for the two-dimensional (plain strain) problem of the Griffith crack in a general anisotropic material by Atkinson and Clements (1977).

Atkinson and Clements (1977) give a solution for the two-dimensional Griffith crack obstructing a uniform heat flux in a general anisotropic medium. They show that modes I, II, and III stress intensity factors are obtained unless the material has certain symmetries, which suggests that a mixed mode of fracture may occur. It also indicates that the crack must close for at least one direction of the heat flow. Closure is also obtained in thermoelastic interface crack problems (Martin-Moran et al., 1983, Barber and Conminou, 1983) where the resulting change in the thermal boundary conditions at the crack faces leads to nonuniqueness of solution in certain cases.

Atkinson and Clements did not consider these questions. Their solution assumes that the crack is always open and hence can only apply for a restricted range of conditions. Furthermore, they did not give explicit expressions for the crack opening displacement from which the physical feasibility of their solution could be checked.

These authors considered the cases of specified temperature on the crack faces and also that of a specified heat flux across the crack. However, the boundary condition taken for this latter problem was $\partial T/\partial X_2 = -Q(X_1)$ across the crack faces and this can only be truly representative of the heat flux for certain symmetries of the thermal conductivity tensor, i.e., $K_{12} = 0$ for a crack lying on the X_1 plane. Furthermore, there are difficulties in adapting the solution method to the more general boundary condition, which will be addressed in a subsequent paper.

In this paper the problem is reconsidered using a Green's function formulation which will allow us to express the solution in terms of physical variables so that it is easier to determine at intermediate stages whether the solution is physically reasonable. The method is then extended to consider the case where the crack is partially closed.

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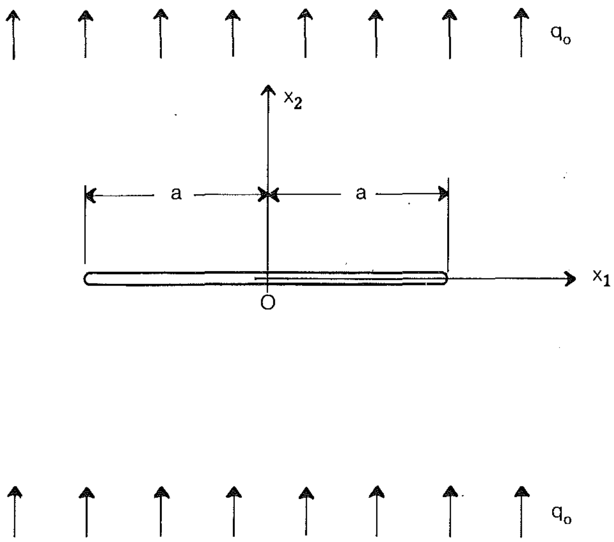

Fig. 1 The thermoelastic plane crack

Statement of the Problem

Let $x_1\ x_2\ x_3$, denote Cartesian coordinates and suppose that a homogeneous generally anistropic material occupies the entire space except for the region $x_2=0$, $|x_1|<a$, $-\infty<x_3<\infty$, where there is a crack. The crack obstructs the heat flow as shown in Fig. 1. The crack is initially assumed to remain open and hence be free of tractions, and to prevent the transfer of heat between its faces. We, therefore, have the boundary conditions

$$q_2=0 \quad ; \quad -a<x_1<a \ ; \ x_2=0 \tag{1}$$

$$\sigma_{i2}=0 \quad ; \quad -a<x_1<a \ ; \ x_2=0 \tag{2}$$

$$q_2\to q_0 \quad ; \quad \sqrt{x_1^2+x_2^2}\to\infty \tag{3}$$

$$\sigma_{ij}\to 0 \quad ; \quad \sqrt{x_1^2+x_2^2}\to\infty \tag{4}$$

The boundary conditions, and hence the displacement, stress, and temperature fields, are independent of the coordinate x_3, but we emphasize that the solution is three-dimensional in the sense that the stress components σ_{i3} and displacement component u_3 are nonzero because of the general anisotropy of the material.

It is convenient to represent the solution as the sum of a uniform heat flux in an unflawed solid (which involves no thermal stress) and a corrective solution, for which the boundary conditions are

$$q_2=-q_0 \quad ; \quad a<x_1<a \ ; \ x_2=0 \tag{5}$$

$$\sigma_{i2}=0 \quad ; \quad -a<x_1<a \ ; \ x_2=0 \tag{6}$$

$$q_2\to 0 \quad ; \quad \sqrt{x_1^2+x_2^2}\to\infty \tag{7}$$

$$\sigma_{ij}\to 0 \quad ; \quad \sqrt{x_1^2+x_2^2}\to\infty \tag{8}$$

Mathematical Formulation

The problem will be formulated in terms of the thermoelastic Green's function derived in a previous paper (Sturla and Barber, 1988), corresponding to a temperature discontinuity of magnitude T_0 on the half-line $x_2=0$, $x_1>0$. The appropriate heat flux and tractions on the surface $x_2=0$ are

$$q_2=-\frac{K_{22}(\tau-\bar{\tau})T_0}{4\pi i x_1} \quad ; \ x_2=0 \tag{9}$$

$$\sigma_{i2}=G_i T_0 \log|x_1| \quad ; \ x_2=0 \tag{10}$$

where τ is the root with positive imaginary part of the equation

$$K_{22}\tau^2+2K_{12}\tau+K_{11}=0, \tag{11}$$

K_{ij} is the thermal conductivity tensor and G_i is a function of the thermoelastic constants for the material, defined by equation (42) of Sturla and Barber (1988).

The thermal boundary conditions (5), (7), can be satisfied by distributing Green's functions of the above form with weight $\Omega(\xi)$ in the interval $-a<x_1<a$. From equations (5) and (9) we then have

$$-K\int_{-a}^{a}\frac{\Omega(\xi)d\xi}{x_1-\xi}=q_2(x_1,0)=-q_0 \ ; \ -a<x_1<a \tag{12}$$

where $K=K_{22}(\tau-\bar{\tau})/4\pi i$ is a real constant.

This distribution will generally produce a discontinuity in temperature in the region $x_2=0$, $x_1>-a$, but temperature continuity outside the crack (i.e., $x_1>a$) can be imposed by enforcing the auxiliary condition

$$\int_{-a}^{a}\Omega(\xi)d\xi=0. \tag{13}$$

Equations (12) and (13) have the well known solution

$$\Omega(\xi)=-\frac{q_0\xi}{\pi K\sqrt{a^2-\xi^2}} \quad -a<\xi<a \tag{14}$$

and the stresses on the plane $x_2=0$ can now be found from equation (10) in the form

$$\sigma_{i2}=G_i\int_{-a}^{a}\Omega(\xi)\log|x_1-\xi|d\xi \tag{15}$$

$$=(q_0/K)G_i x_1 \quad ; \quad -a<x_1<a \tag{16}$$

$$=(q_0/K)G_i\{x_1-\sqrt{x_1^2-a^2}\} \ ; \ x_1>a \tag{17}$$

after substituting for $\Omega(\xi)$ from equation (14) and performing the integration.

To satisfy the traction-free boundary condition (6), we must superpose a solution of the corresponding isothermal problem with tractions equal and opposite to those of equation (16) in the range $-a<x_1<a$. This solution is conveniently represented by a distribution of dislocations of strength $B_i(\xi)$ in the same range. The solution for a single dislocation of strength B_i was obtained by Stroh (1958), the corresponding stresses and displacement on $x_2=0$ being

$$\sigma_{i2}=\frac{d_i}{2\pi x_1} \tag{18}$$

$$u_k=\frac{1}{4\pi}\sum_{\alpha}[A_{k\alpha}M_{\alpha j}+\bar{A}_{k\alpha}\bar{M}_{\alpha j}]d_j\log|x_1| \tag{19}$$

in the notation of Stroh (1958). In particular,

$$B_i=b_{ij}d_j, \text{ where } b_{ij}=\frac{i}{2}\sum_{\alpha}[A_{i\alpha}M_{\alpha j}-\bar{A}_{i\alpha}\bar{M}_{\alpha j}],$$

$A_{i\alpha}$ and $M_{\alpha j}$ are functions of the material constants, defined by equations (7) and (38) of Stroh (1958), and the summation on α is over the three roots with positive imaginary part of the equation

$$|c_{i1k1}+p_\alpha(c_{i1k2}+c_{i2k1})+p_\alpha^2 c_{i2k2}|=0 \tag{20}$$

Combining (16) and (18) the boundary condition (6) is satisfied if

$$\frac{1}{2\pi}\int_{-a}^{a}\frac{d_i(\xi)d\xi}{x_1-\xi}+q_0G_ix_1/K=0 \quad -a<x_1<a \tag{21}$$

It is also necessary to impose the closure condition

$$\int_{-a}^{a}d_i(\xi)d\xi=0 \tag{22}$$

to enforce continuity of displacements in the region $x_2=0$, and $x_1>a$. Solving for $d_i(\xi)$ we have

$$d_i(\xi)=\frac{q_0G_i}{K}\frac{(2\xi^2-a^2)}{\sqrt{a^2-\xi^2}} \tag{23}$$

and hence the stresses σ_{i2} on the plane $x_2=0$, $|x_1|>a$ are given by

$$\sigma_{i2}=\frac{q_0G_i}{K}[x_1\pm\sqrt{x_1^2-a^2}+\frac{q_0G_i}{2\pi K}\int_{-a}^{a}\frac{(2\xi^2-a^2)}{(x_1-\xi)\sqrt{a^2-\xi^2}} \tag{24}$$

$$=\pm\frac{q_0G_i}{2K}\frac{a^2}{\sqrt{x_1^2-a^2}} \quad \begin{matrix}+;\ x_1>a\\ -;\ x_1<-a\end{matrix} \tag{25}$$

from equations (17), (18), and (23). We note that stress intensity factors in all three modes are obtained at $x_1=a+$of magnitude:

$$K_{\rm I}=\lim_{x_1\to a}\sqrt{x_1-a}\,\sigma_{22}=\frac{q_0G_2}{K}\left(\frac{a}{2}\right)^{3/2} \tag{26}$$

$$K_{\rm II}=\lim_{x_1\to a}\sqrt{x_1-a}\,\sigma_{12}=\frac{q_0G_1}{K}\left(\frac{a}{2}\right)^{3/2} \tag{27}$$

$$K_{\rm III}=\lim_{x_1\to a}\sqrt{x_1-a}\,\sigma_{32}=\frac{q_0G_3}{K}\left(\frac{a}{2}\right)^{3/2} \tag{28}$$

Equal and opposite stress intensity factors are obtained at $x_1=-a$. The displacement in the entire space is given by

$$u_k=2\mathrm{Re}\Big[\frac{E_kq_0}{4\pi i}\int_{-a}^{a}\frac{[\log(z_t-\xi)-1][z_t-\xi]\xi}{\sqrt{a^2-\xi^2}}d\xi$$
$$+\sum_\alpha A_{k\alpha}F_\alpha q_0\int_{-a}^{a}\frac{[\log(z_\alpha-\xi)-1][z_\alpha-\xi]\xi}{\sqrt{a^2-\xi^2}}d\xi$$
$$+\frac{q_0}{2\pi K}\sum_\alpha A_{k\alpha}M_{\alpha j}G_j\int_{-a}^{a}\frac{(\xi^2-a^2/2)\log(z_\alpha-\xi)}{\sqrt{a^2-\xi^2}}d\xi\Big] \tag{29}$$

$$=2\mathrm{Re}\Big[\frac{q_0}{2k}\sum_\alpha A_{k\alpha}M_{\alpha j}G_j\left\{\frac{z_\alpha}{2}\sqrt{z_\alpha^2-a^2}-\frac{z_\alpha^2}{2}\right\}$$
$$+\frac{E_kq_0}{4i}\left\{\frac{a^2}{2}\log(z_t+i\sqrt{a^2-z_t^2})+\frac{iz_t}{2}\sqrt{a^2-z_t^2}-\frac{z_t^2}{2}+a^2\right\}$$
$$+\sum_\alpha A_{k\alpha}F_\alpha q_0\Big\{\frac{a^2}{2}\log(z_\alpha+i\sqrt{a^2-z_\alpha^2})$$
$$+\frac{iz_\alpha}{2}\sqrt{a^2-z_\alpha^2}-\frac{z_\alpha^2}{2}+a^2\Big\}\Big] \tag{30}$$

where E_k and F_α are functions of the material constants, defined by equations (22) and (38) of Sturla and Barber (1987), also $z_t=x_1+\tau x_2$ and $z_\alpha=x_1+p_\alpha x_2$, where p_α is defined by equation (20). The temperature distribution and hence the strains behave like $1/z_\alpha$ as $|z_\alpha|\to\infty$, therefore the displacement vector $u_k\sim\log|z_\alpha|$ when $|z_\alpha|\to\infty$.

The crack opening displacement is given by

$$\Delta u_2=u_2(x_1,0^+)-u_2(x_1,0^-) \tag{31}$$

$$=(q_0b_{2j}G_j/2K)x_1\sqrt{a^2-x_1^2} \tag{32}$$

using equation (36) of Sturla and Barber (1987).

From equation (32) it can be seen that Δu_2 must be negative in either $-a<x_1<0$ or $0<x_1<a$, depending on the sign of $b_{2j}G_j$ (except in the special case $b_{2j}G_j=0$) and therefore the initial assumption of a fully open crack is invalid for either direction of the heat flow unless the crack has some initial separation between its faces or is opened by an applied tensile stress. If $b_{2j}G_j=0$, $\Delta u_2=0$ for all x_1 within the crack and hence there is no tendency for the crack to open or close. There is, however, a relative *tangential* displacement between the crack faces. This case is obtained if the material is symmetrical about the plane $x_1=0$.

For the latter case, the boundary condition (4) is modified to include

$$\sigma_{22}\to\sigma_0\ ;\ \sqrt{x_1^2+x_2^2}\to\infty \tag{33}$$

and this new problem can be treated by superposing the solution due to Stroh (1958) for the isothermal problem of a Griffith crack in a general anisotropic medium opened by a uniform tensile stress, σ_0.

In particular, we find that the crack opening displacement (equation (32)) is increased to

$$\Delta u_2=\left(\frac{q_0b_{2j}G_j}{K}x_1+2b_{22}\sigma_0\right)\sqrt{a^2-x_1^2},\ |x_1|<a \tag{34}$$

and the opening mode stress intensity factor becomes

$$K_{\rm I}=\pm\frac{q_0G_2}{K}\left(\frac{a}{2}\right)^{3/2}+\sqrt{\frac{a}{2}}\,\sigma_0 \quad \begin{matrix}+;\ x_1>a\\ -;\ x_1<-a\end{matrix} \tag{35}$$

The other stress intensity factors, $K_{\rm II}$, $K_{\rm III}$, are unaffected by the applied tensile stress.

We can always choose the coordinate system such that $q_0b_{2j}G_j<0$, corresponding to the case where the crack tends to close at the tip $x_1=a$. In this case, equation (34) defines a positive crack opening displacement for all x_1 provided

$$\sigma_0>-q_0b_{2j}G_ja/2b_{22}K \tag{36}$$

Solution With Partial Contact

If the inequality (36) is not satisfied, a negative crack opening displacement is predicted near $x_1=a$ and we anticipate contact between the crack faces, as shown in Fig. 2. The contact is assumed to be frictionless and to afford no resistance to heat flow. There is, therefore, no temperature discontinuity between the faces, except in the open region $-a<x_1<c$. The heat conduction problem is, therefore, identical to that for an insulating crack of extent $-a<x_1<c$ and can be formulated in terms of a distribution $\Omega(\xi)$ of Green's function in this range. Enforcing the condition (5) over this range we obtain

$$K\int_{-a}^{c}\frac{\Omega(\xi)d\xi}{(x_1-\xi)}=q_0\quad;\ -a<x_1<a \tag{37}$$

with the auxiliary condition

$$\int_{-a}^{c}\Omega(\xi)d\xi=0 \tag{38}$$

whose solution is

$$\Omega(\xi)=-\frac{q_0}{\pi K}\frac{\xi+(a-c)/2}{\sqrt{(a+\xi)(c-\xi)}} \tag{39}$$

The mechanical boundary conditions at the crack faces can be stated in the form

$$\sigma_{12}=\sigma_{23}=0 \qquad -a<x_1<a \tag{40}$$

$$\sigma_{22}=0 \qquad -a<x_1<c \tag{41}$$

$$u_2(x_1,0^+)=u_2(x_1,0^-) \qquad c<x_1<a \tag{42}$$

$$\sigma_{22}\to\sigma_0\ ;\ \sigma_{12},\ \sigma_{23}\to 0 \qquad \sqrt{x_1^2+x_2^2}\to\infty. \tag{43}$$

We subtract the unperturbed uniform tension solution, $\sigma(x_1,x_2,x_3)=\sigma_0$, thereby modifying the boundary conditions (41), (43) to

$$\sigma_{22}=-\sigma_0\quad;\quad -a<x_1<c \tag{44}$$

$$\sigma_{22}\to 0\quad;\quad\sqrt{x_1^2+x_2^2}\to\infty. \tag{45}$$

As for the case of a fully opened crack, we solve this boundary value problem by constructing the appropriate stress and displacement fields in terms of a distribution of dislocations $B_i(\xi)$ in the range $-a<x_1<a$. In view of the boundary condition (42), the distribution $B_2(\xi)$ must be nonzero only in the range $-a<x_1<c$, but B_1, B_3 extend over the entire crack length since slip is permitted in the contact region $c<x_1<a$.

The total traction on the crack faces due to the thermoelastic solution and the dislocation distributions can then be written

$$\sigma_{i2} = -\frac{q_0 G_i}{\pi K}\int_{-a}^{c} \frac{\xi+(a-c)/2}{\sqrt{(a+\xi)(c-\xi)}} \log|x_1-\xi|\,d\xi + \frac{1}{2\pi}\int_{-a}^{a} \frac{d_i(\xi)d\xi}{(x_1-\xi)} \quad (46)$$

$$= 0\ , \quad i=1,3; \quad -a<x_1<a \quad (47)$$

$$= -\sigma_0, \quad i=2\ ; \quad -a<x_1<c \quad (48)$$

The integral in the first term in equation (46) can be evaluated to give

$$\int_{-a}^{c} \frac{\xi+(a-c)/2}{\sqrt{(a+\xi)(c-\xi)}} \log|x_1-\xi|\,d\xi = -\pi\left(x_1-\frac{c-a}{2}\right) \quad ; \ -a<x_1<c \quad (49)$$

$$= -\pi\left\{x_1-\frac{c-a}{2}+\sqrt{(x_1+a)(x_1-c)}\right\}; \quad x_1>c$$

and hence the boundary conditions (47), (48) are meet if $d_i(\xi)$ $(i=1,2,3)$ satisfies the integral equations

$$\frac{2q_0G_i}{K}\left\{x_1+\frac{a-c}{2}-H(x_1-c)\sqrt{(x_1+a)(x_1-c)}\right\} + \frac{1}{2\pi}\int_{-a}^{a}\frac{d_i(\xi)d\xi}{(x_1-\xi)}=0\ ;\ i=1,3\ ;\ -a<x_1<a \quad (50)$$

$$\frac{2q_0G_2}{K}\left\{x_1+\frac{a-c}{2}\right\}+\frac{1}{2\pi}\int_{-a}^{a}\frac{d_2(\xi)d\xi}{(x_1-\xi)}=-\sigma_0; \quad -a<x_1<c \quad (51)$$

It is also necessary to impose the conditions

$$\int_{-a}^{a} B_i(\xi)=0; \quad i=1,2,3 \quad (52)$$

$$B_2(\xi)=0 \qquad c<x_1<a \quad (53)$$

to ensure that there is continuity of displacement on the plane $x_2=0$, $x_1>a$ and that the crack closes in $c<x_1<a$.

Since $B_i=b_{ij}d_j$ where b_{ij} is a nonsingular matrix, we have

$$\int_{-a}^{a} d_i(\xi)d\xi=0 \qquad i=1,3 \quad (54)$$

$$d_2(x_1)=-\frac{b_{21}d_1(x_1)+b_{23}d_3(x_1)}{b_{22}} \qquad c<x_1<a \quad (55)$$

from equations (52), (53) and hence we can solve equations (50), (54) obtaining

$$d_i(\xi)=\frac{q_0G_i}{4K}\left[\frac{(a+c)^2}{\sqrt{a^2-\xi^2}}-8(\xi+a)H(c-\xi)\sqrt{\frac{c-\xi}{\xi+a}}\right]; \quad i=1,3 \quad (56)$$

We now write equation (51) and the corresponding closure condition (54) in the form

$$\frac{1}{2\pi}\int_{-a}^{c}\frac{d_2(\xi)d\xi}{(x_1-\xi)}=-\sigma_0-\frac{2q_0G_2}{(a+c)K}\left\{x_1+\frac{(a-c)}{2}\right\}$$

$$-\frac{1}{2\pi}\int_{c}^{a}\frac{d_2(\xi)d\xi}{(x_1-\xi)}\ ; \ -a<x_1<c \quad (57)$$

$$\int_{-a}^{c} d_2(\xi)d\xi=-\int_{c}^{a} d_2(\xi)d\xi \quad (58)$$

where the expression involving $d_2(\xi)$ in the range $c<\xi<a$ can be evaluated using equations (55), (56).

Equations (55), (56), (57), and (58) have the solution

$$d_2(\xi)=\frac{q_0b_{2j}G_j}{b_{22}K}\frac{\left[2\left\{\xi-\left(\frac{c-a}{2}\right)\right\}^2-\left(\frac{a+c}{2}\right)^2\right]}{\sqrt{(\xi+a)(c-\xi)}} + 2\sigma_0\frac{\left[\xi-\left(\frac{c-a}{2}\right)\right]}{\sqrt{(\xi+a)(c-\xi)}} - \frac{q_0}{4K}\frac{(b_{21}G_1+b_{22}G_2)}{b_{22}}\left[\frac{(a+c)^2}{\sqrt{a^2-\xi^2}} - 8(\xi+a)H(c-\xi)\sqrt{\frac{c-\xi}{\xi+a}}\right] \quad (59)$$

The stresses on $x_2=0$ can now be obtained from equations (46), (56), and (59) in the form

$$\sigma_{i2}=\frac{q_0(a+c)^2}{8K}\frac{\mathrm{sgn}(x_1)}{\sqrt{x_1^2-a^2}} \qquad |x_1|>a \quad i=1,3 \quad (60)$$

$$\sigma_{22}=\left\{\frac{q_0b_{2j}G_j}{b_{22}K}\left(\frac{a+c}{2}\right)^2+\sigma_0(2x_1-c+a)\right\}\frac{\mathrm{sgn}(x_1)}{\sqrt{(x_1+a)(x_1-c)}} + \left(\frac{a+c}{2}\right)^2\frac{(b_{21}G_1+b_{23}G_3)q_0}{2b_{22}K}\frac{[H(x_1-a)-H(-a-x_1)]}{\sqrt{x_1^2-a^2}} \quad x_1>c; \quad x_1<-a; \quad (61)$$

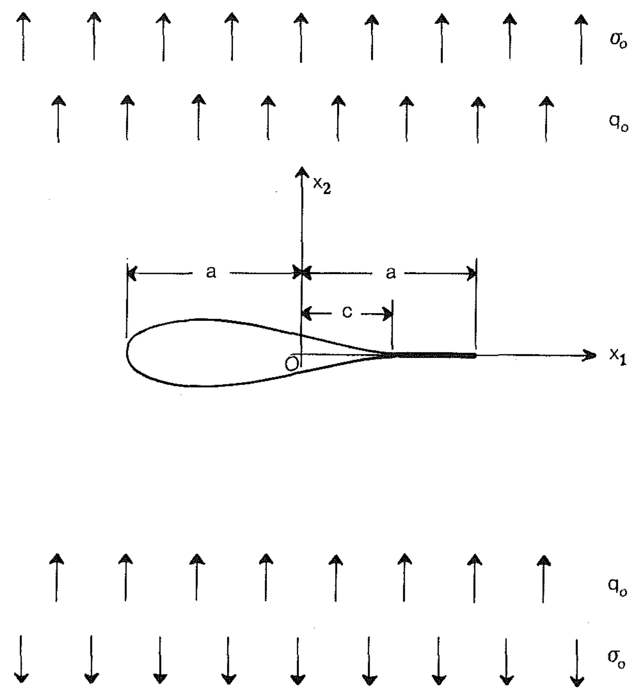

Fig. 2 The plane crack with partial contact

To complete the solution, we must determine the length of the separation zone from the condition that the stress σ_{22} tend to zero at $x_1 \to c$. Thus, from equation (61) we have

$$c = -\frac{4Kb_{22}\sigma_0}{q_0 b_{2j} G_j} - a \tag{62}$$

Using equation (62), equations (60) and (61) can then be written

$$\sigma_{i2} = 2K \left[\frac{b_{22}}{b_{2j}G_j}\right]^2 G_i \frac{\sigma_0^2}{q_0} \frac{\mathrm{sgn}(x_1)}{\sqrt{x_1^2 - a^2}} \quad |x_1| > a \quad i = 1,3 \tag{63}$$

$$\sigma_{22} = \mp\sigma_0 \sqrt{\frac{x_1 - c}{x_1 + a}} - \frac{2b_{22}K\sigma_0^2(b_{21}G_1 + b_{23}G_3)}{q_0(b_{2j}G_j)^2} \times \frac{[H(x_1 - a) - H(-a - x_1)]}{\sqrt{x_1^2 - a^2}} \quad \begin{matrix} -;\ x_1 > c \\ +;\ x_1 < -a \end{matrix} \tag{64}$$

Crack Opening Displacement and Stress Intensity Factors. By the definition of a dislocation, we have that the crack opening displacement can be obtained as

$$\Delta u_2 = -\int_{-a}^{x_1} B_2(\xi)d\xi = -b_{2j}\int_{-a}^{x_1} d_j(\xi)d\xi \tag{65}$$

$$= \frac{q_0 b_{2j} G_j}{K} (x_1 - c) \sqrt{(x_1 + a)(c - x_1)} \tag{66}$$

using equations (56), (59), and (62). Thus the crack opening displacement will be positive for $-a < x_1 < c$ if $q_0 b_{2j} G_j < 0$, as assumed. We note that the derivative of Δu_2 is also zero at the transition from contact to separation ($x_1 = c$), as in conventional contact problems.

From equations (63) and (64) it can be seen that all components σ_{i2} ($i = 1, 2, 3$) of the stress tensor are singular at both ends of the crack (i.e., $x_1 \pm a$). The stress intensity factors are given by:

$$K_I = -\frac{2b_{22}(b_{21}G_1 + b_{23}G_3)K}{\sqrt{2a}(b_{2j}G_j)^2} \frac{\sigma_0^2}{q_0} \quad ; \quad x_1 = a+ \tag{67}$$

$$= 2\left[-\frac{b_{22}K\sigma_0^3}{b_{2j}G_j q_0}\right]^{1/2} + \frac{2b_{22}(b_{21}G_1 + b_{23}G_3)K}{\sqrt{2a}(b_{2j}G_j)^2} \frac{\sigma_0^2}{q_0} ; \quad x_1 = -a- \tag{68}$$

$$K_{II,III} = \pm K\sqrt{\frac{2}{a}} G_i \left[\frac{b_{22}}{b_{2j}G_j}\right]^2 \frac{\sigma_0^2}{q_0} ; \ i = 1,3; \quad \begin{matrix} +;\ x_1 = a+ \\ -;\ x_1 = -a- \end{matrix} \tag{69}$$

Notice that stress intensity factors in all three modes are obtained at both ends of the crack, including a nonzero K_I at the closed tip, $x_1 = a$. This results from coupling between the tangential displacements (slip) in the contact zone and the stress component σ_{22}, on $x_2 = 0$.

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