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Thermal Stresses in Functionally Graded Beams

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Thermoelastic equilibrium equations for a functionally graded beam are solved in closed-form to obtain the axial stress distribution. The thermoelastic constants of the beam and the temperature were assumed to vary exponentially through the thickness. The Poisson ratio was held constant. The exponential variation of the elastic constants and the temperature allow exact solution for the plane thermoelasticity equations. A simple Euler–Bernoulli-type beam theory is also developed based on the assumption that plane sections remain plane and normal to the beam axis. The stresses were calculated for cases for which the elastic constants vary in the same manner as the temperature and vice versa. The residual thermal stresses are greatly reduced, when the variation of thermoelastic constants are opposite to that of the temperature distribution. When both elastic constants and temperature increase through the thickness in the same direction, they cause a significant raise in thermal stresses. For the case of nearly uniform temperature along the length of the beam, beam theory is adequate in predicting thermal residual stresses.

Nomenclature

A, B, D	=	beam stiffness coefficients
A_{ij}	=	coefficients in the characteristic equation
a_i, b_i	=	arbitrary constants
c_{ij}	=	elastic constants
E	=	Young's modulus
$\bar{E}$	=	plane strain Young's modulus of the beam
G	=	shear modulus
h	=	beam thickness
M, N	=	force and moment resultants
M^T, N^T	=	thermal force and moment
r_i	=	ratio of arbitrary constants a_i and b_i
T	=	temperature
$U, W, U_c,$ U_p, W_c, W_p	=	displacement functions, complementary and particular solutions
u, w	=	displacements in the x and z directions
α_i	=	characteristic roots
α_x, α_z	=	coefficients of thermal expansion
β	=	thermoelastic coupling coefficients
γ	=	exponent for variation of thermoelastic constants
γ_{ij}	=	shear strain
ε	=	normal strains
θ	=	temperature distribution
κ	=	exponent for temperature variation
κ_x	=	beam curvature
λ	=	exponent for variation elastic constants
ν	=	Poisson's ratio
ξ	=	Fourier transform variable
σ_{ij}	=	normal stresses
τ	=	shear stresses
ω	=	$\gamma + \kappa - \lambda$

Introduction

FUNCTIONALLY graded materials (FGM) possess properties that vary gradually with respect to the spatial coordinates. For example, the insulating tile for a reentry vehicle can be designed such that the outside is made of a refractory material, the load carrying structure is made of a strong and tough metal, and the transition from the refractory material to the metal is gradual through the thickness. In traditional composite materials, the volume fraction of the fibers or the inclusions is uniform, whereas in FGMs they vary gradually. In laminated composites, the properties change abruptly across the interface between successive plies, which is again contrasted by FGMs by allowing smoother variation of properties. Although fabrication technology of FGMs is at infancy, there are many advantages to them. Suresh and Mortensen[1] provide an excellent introduction to the fundamentals of FGMs.

As the use of FGMs increases, for example, in aerospace, military, automotive, and biomedical applications, new methodologies have to be developed to characterize FGMs, and also to design and analyze structural components made of these materials. Simple but efficient and accurate analysis procedures are required for optimization studies also. One such problem is that of response of FGMs to thermomechanical loads. Although FGMs are highly heterogeneous, it will be useful to idealize them as continua with properties changing smoothly with respect to the spatial coordinates. This will enable obtaining closed-form solutions to some fundamental solid mechanics problems and also will help in developing finite element models of the structures made of FGMs. Aboudi et al.[2,3] developed a higher-order micromechanical theory for FGMs (HOTFGM) that explicitly couples the local and global effects. Later the theory was extended to free-edge problems.[4] Pindera and Dunn[5] evaluated the higher-order theory by performing a detailed finite element analysis of the FGM. They found that the HOTFGM results agreed well with the finite element results. Marrey and Sankar[6,7] studied the effects of stress gradients in textile composites consisting of unit cells large compared to the thickness of the composite. Their method resulted in direct computation of plate stiffness coefficients from the micromechanical models rather than using the homogeneous elastic constants of the composite and plate thickness. Some of the concepts in their analysis of stress gradient effects in heterogeneous material systems are applicable to functionally graded (FG) material also.

There are other approximations that can be used to model the variation of properties in an FGM. One such variation is the exponential variation, where the elastic constants vary according to formulas of the type $c_{ij} = c^0_{ij} e^{\lambda z}$. Many researchers have found this functional form of property variation to be convenient in solving elasticity problems. For example, Delale and Erdogan[8] derived the crack-tip stress fields for an inhomogeneous cracked body with constant Poisson ratio and with a shear modulus variation given by

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$\mu = \mu_0 e^{(\alpha x + \beta y)}$. Sankar[9] solved the plane elasticity problem of an FGM beam subjected to transverse loading using a Fourier series technique. It was found that for slowly varying loads beam theory solutions are adequate. However, when the loading occurs over a small area as in contact problems, elasticity solutions are needed.

Although elasticity equations can provide exact solutions, they are limited to simple geometries, specific boundary conditions, and special types of loadings. Hence, it will be useful to develop simple beam/plate theories for structures made of FGMs. The validity of the beam/plate theories can be checked by comparison with the elasticity solutions. In this paper we analyze a FGM beam subjected to thermal loading. No external loads are applied on the beam, but a thermal gradient is assumed to exist across the thickness of the beam. The plane thermoelasticity equations are solved exactly to obtain displacement and stress fields. A beam theory similar to the Euler–Bernoulli beam theory is developed, and the beam theory results are compared with elasticity solutions. It is found that the beam theory results agree quite well with the elasticity solution when the temperatures do not vary along the beam axis and only through the thickness variation exists.

Elasticity Analysis

The dimensions of the FGM beam and the coordinate system are shown in Fig. 1. Note that the x axis is along the bottom of the beam, not in the midplane. The length of the beam is L and thickness is h. The beam is assumed to be in a state of plane strain normal to the xz plane, and the width in the y direction is taken as unity. The boundary conditions are similar to those of a simply supported beam, but the exact boundary conditions will become apparent later. The top and bottom surfaces of the beam ($z = 0$ and h) are assumed to be free of tractions. The temperature distribution θ in the beam is assumed to be of the following form:

$$\theta(x, z) = T(x)e^{\kappa z} \tag{1}$$

where κ is a constant. We will tacitly assume the reference temperature (temperature at which stresses and strains vanish) as $\theta = 0$. The function $T(x)$ can be expressed in the form of a Fourier series as

$$T(x) = \sum_{n=1}^{\infty} T_n \sin \xi x \tag{2}$$

where $\xi = n\pi/L$ and $n = 1, 2, 3 \ldots$. We will develop the thermal stress analysis procedures for the temperature distribution $T_n e^{\kappa z} \sin \xi x$. The solution for an arbitrary temperature distribution can be obtained by superposition, as in Eq. (2).

The differential equations of equilibrium are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xz}}{\partial z} = 0, \qquad \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z} = 0 \tag{3}$$

Assuming that the material is orthotropic at every point and also that the principal material directions coincide with the x and z axes, the constitutive relations are

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & c_{55} \end{bmatrix} \left\{ \begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{pmatrix} - \theta \begin{pmatrix} \alpha_x \\ \alpha_z \\ 0 \end{pmatrix} \right\} \tag{4}$$

where $[c]$ is the elasticity matrix and α are the coefficients of thermal expansion. We will introduce the thermomechanical coupling coefficients β such that

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} & 0 \\ c_{13} & c_{33} & 0 \\ 0 & 0 & c_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix} - \theta \begin{Bmatrix} \beta_x \\ \beta_z \\ 0 \end{Bmatrix} \tag{5}$$

where the β are defined by

$$\begin{Bmatrix} \beta_x \\ \beta_z \end{Bmatrix} = \begin{bmatrix} c_{11} & c_{13} \\ c_{13} & c_{33} \end{bmatrix} \begin{Bmatrix} \alpha_x \\ \alpha_z \end{Bmatrix} \tag{6}$$

We assume that all elastic stiffness coefficients c_{ij} and β vary exponentially in the z direction:

$$[c(z)] = e^{\lambda z} \begin{bmatrix} c_{11}^0 & c_{13}^0 & 0 \\ c_{13}^0 & c_{33}^0 & 0 \\ 0 & 0 & c_{55}^0 \end{bmatrix}, \qquad \begin{Bmatrix} \beta_x \\ \beta_z \end{Bmatrix} = e^{\gamma z} \begin{Bmatrix} \beta_x^0 \\ \beta_z^0 \end{Bmatrix} \tag{7}$$

where λ and γ are constants that define the gradation of the thermoelastic properties, $c_{ij}^0 = c_{ij}(0)$ and $\beta_i^0 = \beta_i(0)$. Substituting from Eq. (5) into Eqs. (3), and using strain-displacement relations ($\varepsilon_{xx} = \partial u/\partial x$, etc.), we obtain the following two equations in $u(x, z)$ and $w(x, z)$:

$$\frac{\partial}{\partial x}\left(c_{11}\frac{\partial u}{\partial x} + c_{13}\frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial z}\left(c_{55}\frac{\partial u}{\partial z} + c_{55}\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}(\beta_x \theta)$$

$$\frac{\partial}{\partial x}\left(c_{55}\frac{\partial u}{\partial z} + c_{55}\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(c_{13}\frac{\partial u}{\partial x} + c_{33}\frac{\partial w}{\partial z}\right) = \frac{\partial}{\partial z}(\beta_z \theta) \tag{8}$$

We will assume solutions of the form

$$u(x, z) = U(z)\cos \xi x, \qquad w(x, z) = W(z)\sin \xi x \tag{9}$$

From the forms of the displacements, one can note that the boundary conditions at the left- and right-end faces of the beam are given by

$$w(0, z) = w(L, z) = 0, \qquad \sigma_{xx}(0, z) = \sigma_{xx}(L, z) = 0 \tag{10}$$

which is typical of simply supported beams. Substituting from Eqs. (9) into Eq. (8) and also using $\theta = T_n e^{\kappa z} \sin \xi x$, we obtain a pair of ordinary differential equations for $U(z)$ and $W(z)$:

$$-c_{11}^0 \xi^2 U + c_{13}^0 \xi W' + c_{55}^0 U'' + c_{55}^0 \lambda U' + c_{55}^0 \xi W' + c_{55}^0 \lambda \xi W = T_n \xi \beta_x^0 e^{\omega z}$$

$$-c_{55}^0 \xi U' - c_{55}^0 \xi^2 W - c_{13}^0 \xi U' - c_{13}^0 \lambda \xi U + c_{33}^0 W'' + c_{33}^0 \lambda W' = T_n(\gamma + \kappa)\beta_z^0 e^{\omega z} \tag{11}$$

where $(\cdot)' \equiv \mathrm{d}(\cdot)/\mathrm{d}z$ and $\omega = (\gamma + \kappa - \lambda)$.

To simplify the calculations, we will assume that the FGM is isotropic at every point. Further, we will assume that Poisson's ratio is a constant through the thickness. Then the variation of Young's modulus is given by $E(z) = E_0 e^{\lambda z}$ and we will assume $\beta_x^0 = \beta_z^0 = \beta_0$. The elasticity matrix $[c]$ is related to the Young's modulus and Poisson's ratio by

$$[c] = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix} \tag{12}$$

The solution of Eqs. (11) consists of complementary functions U_c and W_c and particular integrals U_p and W_p. The complementary functions can be derived as[9]

$$U_c(z) = \sum_{i=1}^{4} a_i e^{\alpha_i z}, \qquad W_c(z) = \sum_{i=1}^{4} b_i e^{\alpha_i z} \tag{13}$$

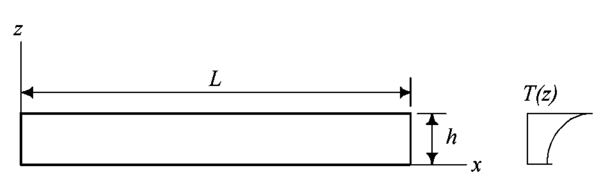

Fig. 1 FG beam subjected to a temperature variation $T(z)$ in the thickness direction; note that the x axis is along the bottom surface of the beam.

where a_i and b_i are arbitrary constants to be determined from the traction boundary conditions on the top and bottom surfaces and α_i are the roots of the characteristic equation for α:

$$\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix} = 0 \tag{14a}$$

where

$$A_{11} = [(1-2\nu)/2]\alpha^2 + [(1-2\nu)/2]\lambda\alpha - (1-\nu)\xi^2$$

$$A_{12} = \xi\alpha/2 + [(1-2\nu)/2]\lambda\xi, \qquad A_{21} = -\xi\alpha/2 - \nu\lambda\xi$$

$$A_{22} = (1-\nu)\alpha^2 + (1-\nu)\lambda\alpha - [(1-2\nu)/2]\xi^2 \tag{14b}$$

Note that the characteristic equation (14a) is a quartic equation in α and will result in four roots, and that is why we have four terms in the complementary solution given in Eq. (13). The arbitrary constants a_i and b_i are related by

$$r_i = \frac{b_i}{a_i} = -\frac{(1-2\nu)\alpha_i(\lambda+\alpha_i) - 2(1-\nu)\xi^2}{\xi\alpha + (1-2\nu)\lambda\xi} \tag{15}$$

The details of the derivation of the complementary functions may be found in Ref. 9. The particular integrals will be of the form

$$U_p(z) = c_U e^{\omega z}, \qquad W_p(z) = c_W e^{\omega z} \tag{16}$$

The constants c_U and c_W can be found by substituting the assumed solution in Eq. (16) in the governing differential equations (11), which results in the following pair of equations for the constants c_U and c_W:

$$\begin{bmatrix} -\xi^2 c_{11}^0 + c_{55}^0\omega(\omega+\lambda) & c_{55}^0\xi(\omega+\lambda) + c_{13}^0\xi\omega \\ -c_{13}^0\xi(\omega+\lambda) - c_{55}^0\xi\omega & -\xi^2 c_{55}^0 + c_{33}^0\omega(\omega+\lambda) \end{bmatrix} \begin{Bmatrix} c_U \\ c_W \end{Bmatrix} = \begin{Bmatrix} \xi T_n\beta_0 \\ (\gamma+\kappa)T_n\beta_0 \end{Bmatrix} \tag{17}$$

The four arbitrary constants a_i can be found from the traction boundary conditions on the top and bottom surface of the beam. In the present thermal stress problem, we assume the top and bottom surfaces of the beam are traction free:

$$\tau_{xz}(x,0) = G_0\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\bigg|_{z=0} = 0$$

$$\tau_{xz}(x,h) = G_h\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\bigg|_{z=h} = 0$$

$$\sigma_{zz}(x,0) = c_{13}^0\frac{\partial u}{\partial x}\bigg|_{z=0} + c_{33}^0\frac{\partial w}{\partial z}\bigg|_{z=0} - T_n\beta_0\sin\xi x = 0$$

$$\sigma_{zz}(x,h) = c_{13}^h\frac{\partial u}{\partial x}\bigg|_{z=h} + c_{33}^h\frac{\partial w}{\partial z}\bigg|_{z=h} - T_n\beta_0 e^{(\gamma+\kappa)h}\sin\xi x = 0 \tag{18}$$

where G is the shear modulus, $G_0 = G(0)$, $G_h = G(h)$, and $C_{ij}^h = C_{ij}(h)$. Substituting for u and w from Eq. (9) into Eq. (18), we obtain the boundary conditions in terms of U and W:

$$U'(0) + \xi W(0) = 0, \qquad U'(h) + \xi W(h) = 0$$

$$-c_{13}^0\xi U(0) + c_{33}^0 W'(0) = T_n\beta_0$$

$$-c_{13}^h\xi U(h) + c_{33}^h W'(h) = T_n\beta_0 e^{(\gamma+\kappa)h} \tag{19}$$

As mentioned earlier, the solutions for U and W consist of complementary functions U_c and W_c and particular solutions U_p and W_p. Substituting for U and W from Eqs. (13) and (16) into Eq. (19), we obtain four equations for a_i:

$$\sum_{i=1}^{4}(\alpha_i + \xi r_i)a_i = -\omega c_U - \xi c_W$$

$$\sum_{i=1}^{4} e^{\alpha_i h}(\alpha_i + \xi r_i)a_i = (-\omega c_U - \xi c_W)e^{\omega h}$$

$$\sum_{i=1}^{4}\left(-c_{13}^0\xi + c_{33}^0 r_i\alpha_i\right)a_i = T_n\beta_0 + \xi c_{13}^0 c_U - \omega c_{33}^0 c_W$$

$$\sum_{i=1}^{4} e^{\alpha_i h}\left(-c_{13}^h\xi + c_{33}^h r_i\alpha_i\right)a_i = T_n\beta_0 e^{(\gamma+\kappa)h} + \left(\xi c_{13}^h c_U - \omega c_{33}^h c_W\right)e^{\omega h} \tag{20}$$

where r_i are the ratio between the arbitrary constants a_i and b_i [see Eq. (15)]. Solving for a_i, we obtain the complete solution for $U(z)$ and $W(z)$ and, hence, for $u(x,z)$ and $w(x,z)$. The stresses at any point in the beam can be evaluated in a straightforward manner using Eqs. (5).

Euler–Bernoulli Beam Theory for FGM Beams

We will follow the Euler–Bernoulli beam theory assumption that plane sections normal to the beam axis (x axis) remain plane and normal after deformation. Furthermore, we will assume that there is no thickness change, that is, w displacements are independent of z. Then the displacements can be written as

$$w(x,z) = w_b(x), \qquad u_b(x,z) = u_0(x) - z\frac{\mathrm{d}w_b}{\mathrm{d}x} \tag{21}$$

where the subscripts b in Eq. (21) denote beam theory displacements. Note that u_0 denotes the displacements of points on the bottom surface of the beam and not points in the beam midplane. We assume that the normal stresses σ_{zz} are negligible, that is, $\sigma_{zz} = 0$. Then the stress–strain relations take the simple form

$$\sigma_x = \bar{E}(z)\varepsilon_{xx} - \bar{\beta}\theta, \qquad \tau_{xz} = G(z)\gamma_{xz} \tag{22}$$

where the plane strain Young's modulus is given by $\bar{E} = E/(1-\nu^2)$ and $\bar{\beta} = \beta[(1-2\nu)/(1-\nu)]$.

Note that the relations $\bar{E}(z) = \bar{E}_0 e^{\lambda z}$ and $\bar{\beta}(z) = \bar{\beta}_0 e^{\gamma z}$ hold good because ν is constant. From Eq. (21), expressions for axial strain and stress can be derived as

$$\varepsilon_{xx} = \frac{\mathrm{d}u_0}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w_b}{\mathrm{d}x^2} = \varepsilon_{x0} + z\kappa$$

$$\sigma_{xx} = \bar{E}\varepsilon_x - \bar{\beta}\theta = \bar{E}\varepsilon_{x0} + z\bar{E}\kappa - \bar{\beta}\theta \tag{22a}$$

One can readily recognize the reference plane strain ε_{x0} and the beam curvature κ in Eq. (22a). The axial force and bending moment resultants N and M are defined as in the Euler–Bernoulli beam theory:

$$(N, M) = \int_0^h \sigma_{xx}(1, z)\,\mathrm{d}z \tag{23}$$

Note that the limits of integration in the definition of force and moment resultants in Eq. (23) are 0 and h. When σ_{xx} is substituted from Eq. (22a) into Eq. (23), a relation between the force and moment resultants and the beam deformations can be derived as follows:

$$\begin{Bmatrix} N \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}\begin{Bmatrix} \varepsilon_{x0} \\ \kappa \end{Bmatrix} - \begin{Bmatrix} N^T \\ M^T \end{Bmatrix} \tag{24}$$

The definition of beam stiffness coefficients A, B, and D is

$$(A, B, D) = \int_0^h \bar{E}(1, z, z^2)\,\mathrm{d}z \tag{25}$$

and the thermal force and moment are defined as

$$(N^T, M^T) = \int_0^h (1, z)\bar{\beta}\theta\,\mathrm{d}z \tag{26}$$

Explicit expressions for the beam stiffness coefficients can be derived using $\bar{E}(z) = \bar{E}_0 e^{\lambda z}$:

$$A = \frac{\bar{E}_h - \bar{E}_0}{\lambda}, \qquad B = \frac{h\bar{E}_h - A}{\lambda}, \qquad D = \frac{h^2\bar{E}_h - 2B}{\lambda}$$

$$N^T = \frac{\left[e^{(\gamma+\kappa)h} - 1\right]}{(\gamma+\kappa)}, \qquad M^T = \frac{1 + e^{(\gamma+\kappa)h}[(\gamma+\kappa)h - 1]}{(\gamma+\kappa)^2}$$

$$\bar{E}_0 = \bar{E}(0), \qquad \bar{E}_h = \bar{E}(h) \tag{27}$$

The inverse relations corresponding to those in Eq. (24) are

$$\begin{Bmatrix} \varepsilon_{x0} \\ \kappa \end{Bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix}^{-1} \begin{Bmatrix} N + N^T \\ M + M^T \end{Bmatrix} = \begin{bmatrix} A^* & B^* \\ B^* & D^* \end{bmatrix} \begin{Bmatrix} N + N^T \\ M + M^T \end{Bmatrix} \tag{28}$$

Because there are no external forces applied to the beam, $N \equiv 0$ and $M \equiv 0$. Equation (28) can be solved to obtain the deformations ϵ_{x0} and κ. When substituted back in Eq. (22), the stresses at any point in the beam can be obtained.

Results and Discussion

The procedures described in preceding sections were applied to an FGM beam with the properties shown in Table 1. The length of the beam was taken as 100 mm and the thickness as 10 mm. In all examples, the temperature variation was assumed to be of the form $\theta(x, z) = \Delta T_0 e^{\kappa z}$, where $\Delta T_0 = 100$, and κ was such that the ratio $\theta(x, h)/\theta(x, 0) = 10$. For the thickness of 10×10^{-3} m, the temperature distribution resulted in $\kappa = 230\ \mathrm{m}^{-1}$. To perform the Fourier series summation in Eq. (2), 51 terms were used. The thermoelastic coupling coefficient was assumed to be of the form $\beta(z) = \beta_0 e^{\gamma z}$, with $\beta_0 = E_0/10^4$. The values of γ were varied, but they were related to λ as shown in Table 1.

The stresses were normalized with respect to the thermal stress term $\beta_0 \Delta T_0$. The through the thickness axial stress distribution for the five beams are plotted in Figs. 2–6. In all cases, the elasticity solutions agreed very well with beam theory solutions, and the two

Table 1 Properties of FG beams used, $\kappa = 230\ \mathrm{m}^{-1}$

Beam number	E_0, GPa	E_h, GPa	λ, m^{-1}	γ/λ
1	1	1	0	0
2	10	1	$-\kappa$	1
3	1	10	$+\kappa$	1
4	10	1	$-\kappa$	1.5
5	1	10	$+\kappa$	1.5

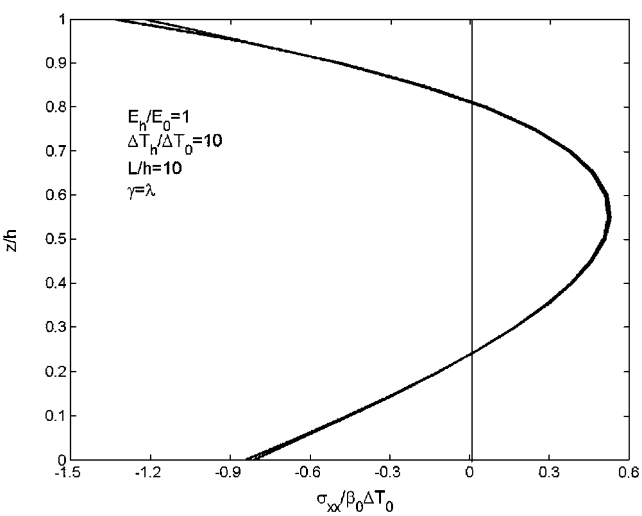

Fig. 2 Thermal stress distribution in a homogeneous beam; elasticity solution and beam theory solution are almost indistinguishable.

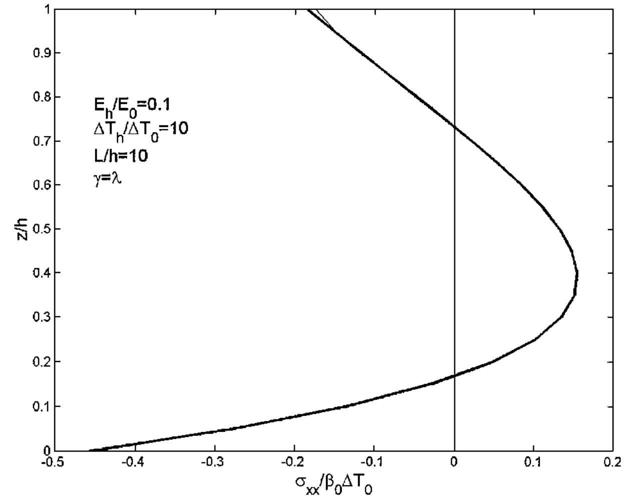

Fig. 3 Thermal stresses in an FG beam; thermoelastic constants and the temperature have opposite type of distribution through the thickness ($\lambda = -\kappa$), and this reduces the thermal stresses.

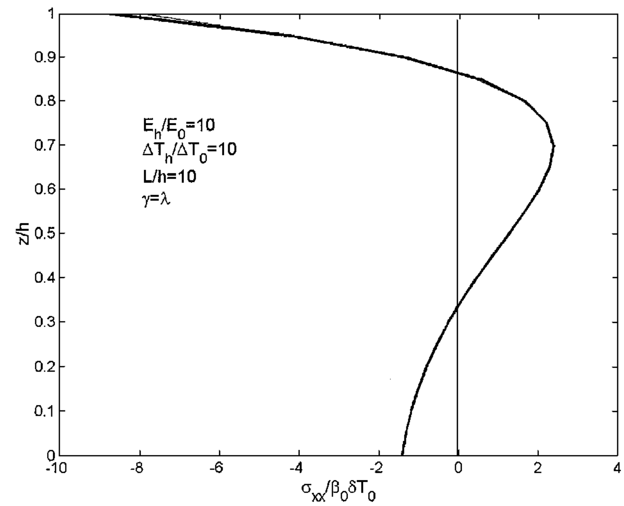

Fig. 4 Thermal stresses in an FG beam wherein the thermoelastic constants and the temperature vary in a similar manner through the thickness, that is, $\lambda = \kappa$; this increases the thermal stresses significantly.

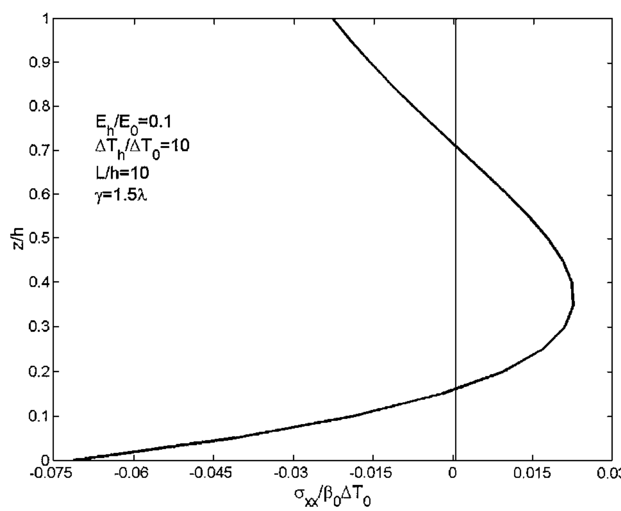

Fig. 5 Thermal stresses in an FG beam with $\gamma = 1.5\lambda$, but the thermoelastic constants and the temperature vary in an opposite manner through the thickness, that is, $\gamma = -\kappa$.

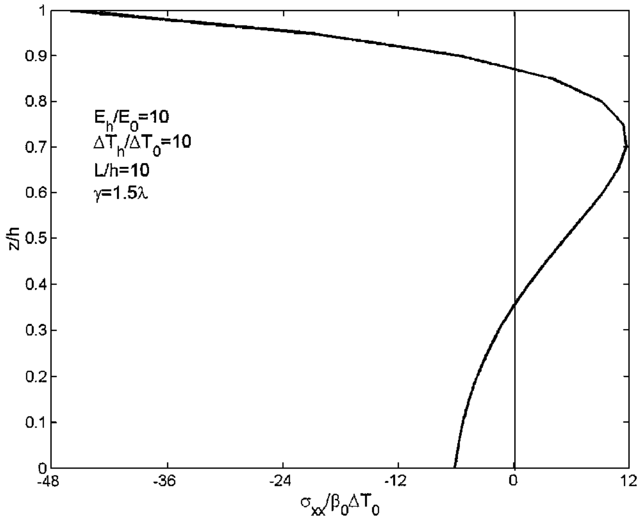

Fig. 6 Thermal stresses in an FG beam wherein thermoelastic constants and temperature vary in a similar manner through the thickness, that is, $\lambda = \kappa$ and $\gamma = 1.5\lambda$; this increases the thermal stresses significantly.

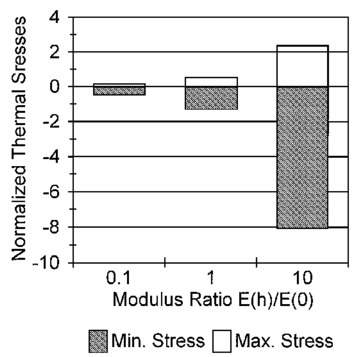

Fig. 7 Maximum and minimum values of normalized thermal stresses ($\sigma_x/\beta_0\Delta T_0$) for $\gamma = \lambda$.

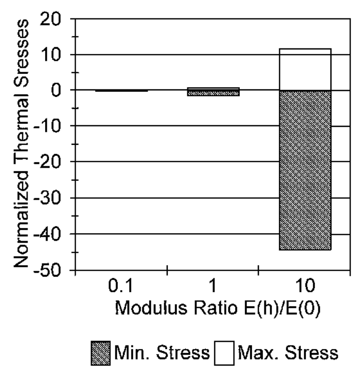

Fig. 8 Maximum and minimum values of normalized thermal stresses ($\sigma_x/\beta_0\Delta T_0$) for $\gamma = 1.5\lambda$.

curves were mostly indistinguishable. The maximum tensile stress always occurred in the vicinity of the neutral axis of the beam. When the variation of the thermomechanical properties, E and β, were in opposite sense to the temperature variation, that is, $\lambda = -\kappa$, the thermal stresses were greatly reduced. This is the case with beams 2 and 4. On the other hand, when the variation of E and β were in the same sense as the temperature, that is, $\lambda = +\kappa$, the thermal stresses increased tremendously (beams 3 and 5). Increase in γ also resulted in increased stresses when other parameters were kept constant. The maximum (tensile) and minimum (compressive) stresses in various beams are compared in bar charts presented in Figs. 7 and 8, where the values are shown for $E_h/E_0 = 0.1$, 1, and 10. As shown in Fig. 8, the stresses are negligibly small for $E_h/E_0 = 0.1$ compared to values corresponding to $E_h/E_0 = 1$ and 10.

Conclusions

An elasticity solution is obtained for FG beams subjected to temperature gradients. Poisson's ratio is assumed to be a constant, and Young's modulus is assumed to vary in an exponential fashion through the thickness. The thermoelastic coupling coefficient and also the temperature were assumed to vary exponentially through the thickness. A simple Euler–Bernoulli-type beam theory is also developed, based on the assumption that plane sections remain plane. The results indicate that the thermoelastic properties of the beam can be tailored to reduce the thermal residual stresses for a given temperature distribution. This can be accomplished by varying the thermoelastic constants in a manner opposite to the gradation of temperature through the thickness. In the present examples, the temperature did not vary along the beam axis, and hence, the beam results also agreed well with the elasticity solutions. If the temperature variation along the x axis is drastic, then significant differences between beam theory and elasticity solutions are expected. The present method of analysis will be useful in the design and optimization of thermal barrier coatings, thermal insulation tiles, and other thermal protection systems.

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