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THERMALLY INDUCED FLOW INSTABILITIES IN TWO-PHASE MIXTURES IN THERMAL EQUILIBRIUM

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This thesis is dedicated to

my parents and my wife.

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NOMENCLATURE

Latin

Ao, Ac, Ae	cross sectional area at (A), (B), and (D)
Ai	coefficients defined in Appendix E
Bi	coefficients defined in Appendix E
c	mass concentration
C_k	kinematic wave velocity
$C_r{}^*$	ratio of C_k at $z = \lambda$ and ℓ
Cm, me	coefficient for friction factor in (C) and (D)
$C_1{}^*{}_{,2,3,4}$	function defined in Sect. VIII.1
D_o, D, D_e	hydraulic diameters at (A), (B), and (D)
D(n)	D-partition parameter
D_c	diffusion coefficient
E(z)	defined by Eq. (V.2.35)
$f_{m,\ me}$	mixture friction factor in (C) and (D)
f_σ	capillary (body) force
f_o, f_s	liquid friction factor in (A) and (B)
f(z)	dimensionless heat flux profile
F(z)	integral of f(z)
$f_{1,2,\ldots,9}$	various constitutive functional relations
g_o, g, g_e	axial gravitational forces in (A), (B), and (D)
$g(\eta, S)$	function defined by Eq. (V.1.32)
H(z, S)	function defined by Eq. (V.2.40)

NOMENCLATURE (Continued)

Latin

i	enthalpy
Δi_{12}	subcooling
Δi_s	maximum subcooling
Δi_{fg}	latent heat
j	volumetric flux
k_i, k_e	inlet and exit orifice coefficients
K	defined by Eq. (IX.4.4)
l_o, l, l_e	length of inlet, heated, and exit channels
$L^*(S^*)$	shifted characteristic function
M	defined by Eq. (IX.4.4)
N_{pch}	phase change number
N_d	drift number
N_ρ	density number (ratio)
N_f	friction number
N_{Fr}	Froude number
N_s	surface number
N_{st}	two-phase Stanton number
N_{Ec}	two-phase Eckert number
N_{sub}	subcooling number
N_{Res}	Reynolds number for the liquid at saturation (flow is entirely liquid)
n'	see Eq. (VII.3.12), factor related to two-phase friction factor

NOMENCLATURE (Continued)

Latin

P	pressure
P_s	system pressure
ΔP	pressure drop
ΔP_{ex}	system pressure drop
$Q(S)$	characteristic function
$q_w''(z)$	wall heat flux
q_o''	average heat flux
$S = a + j\omega$	perturbation variables ($j = \sqrt{-1}$)
a	amplification factor
ω	frequency
S_d	dynamic stability number given by Eq. (IX.4.24)
T	temperature
t	time
v	velocity
v_r	relative velocity
v_{fi}	velocity at the inlet of heated channel
V_{gm}, V_{fm}	diffusion velocity of each phase
V_{gj}, V_{fj}	drift velocity of each phase
x	vapor quality
x_e	exit quality
z	axial coordinate
$Z^*(S^*)$	shifted characteristic function

NOMENCLATURE (Continued)

Greek

α	vapor void fraction
Γ_g, Γ_f	mass generation for each phase
δv	inlet perturbation
ε	perturbation magnitude ($\ll \bar{v}_{fi}$)
θ	angle between (B) and (D)
λ	heated region non-boiling length
Λ_n(n=1, ,16)	various transfer function
$\Lambda_{A,B,C,D}$	transfer functions for (A), (B), (C), and (D)
μ	viscosity
ξ	heated perimeter
ρ	density
$\Delta\rho$	density difference between two phases
σ	surface tension
τ	particle coordinate
$\tau_{12,23,34,13}$	residence time in (B), (C), (D), and in the heated region
φ_{ce}	compressibility effect on enthalpy change
ϕ_{de}	dissipation effect on enthalpy change
ω	frequency of oscillation
Ω	local reaction frequency
Ω_o	average reaction frequency
Ω_∞	equivalent reaction frequency in the mixture

NOMENCLATURE (Continued)

Subscript

e	exit
f	liquid
fo	liquid in (A)
g	vapor
i	inlet
m	mixture
me	mixture in (D)
s	saturation
12	region (B)
23	region (C)
34	region (D)

Symbols and Operator

*	dimensionless
$\frac{Dg}{Dt}$	$\frac{\partial}{\partial t} + v_g \frac{\partial}{\partial z}$
$\frac{D_f}{Dt}$	$\frac{\partial}{\partial t} + v_f \frac{\partial}{\partial z}$
δG	perturbated part of variable G
$\overline{G}$	steady state part of variable G
PNS	name of computer program
TSP	name of computer program
(A)	upstream un-heated region
(B)	liquid region

NOMENCLATURE (Concluded)

Symbols and Operator

(C)	mixture heat region
(D)	downstream un-heated region

NOTE: Symbols which cannot be found here should be referred in the nomenclature for Appendices A, B, and C.

NOMENCLATURE (Appendices A, B, and C)

Latin

A	cross sectional area
$a^{\alpha\beta}$	surface matrix tensor ($\alpha, \beta = 1, 2$)
C_g	mass concentration of vapor
D	hydraulic diameter
f_σ	capillary (body) force (vector)
f_m	Darcy type mixture friction factor
G	function associated with intensive variable
g	body force field (vector)
$\bar{H}$	reference mean curvature of interface
i	enthalpy
J	flux (vector, tensor)
j	volumetric flux (vector)
K	flux from the line on the surface
$\bar{m}$	reference number of surface per length
$\bar{N}$	average number of surface passing the point in Δt
n	unit normal at the interface
P	pressure
q	heat flux (vector)
q_w''	wall heat flux
r	radial coordinate
$r_{i,\,o}$	radius of inner and outer pipes

NOMENCLATURE (Appendices A, B, and C) (Continued)

Latin

t	time
t_{α}^{j}	hybrid tensor of interface ($\alpha = 1,2$ and $j = 1,2,3$), see Ref. [58]
u	internal energy
V_{km}	diffusion velocity with respect to local v_m (vector)
$\tilde{V}_{km}$	diffusion velocity with respect to averaged $\langle\langle v_m \rangle\rangle$ (vector)
V_{kj}	drift velocity with respect to local j (vector)
$\tilde{V}_{kj}$	drift velocity with respect to averaged $\langle\langle j \rangle\rangle$ (vector)
v	velocity (vector)
v_i	velocity of interface (vector)
v_{in}	velocity of interface in normal direction
X	position vector
z	axial coordinate

Greek

α_k	time averaged probability of each phase ($k = g, f$)
α $(= \alpha g)$	local void fraction of vapor phase
Γ_k	mass generation due to interface transport ($k = g, f$)
δ	thickness associated with interface
ϵ	small time interval corresponds to δ
$\Pi_{i,o}$	perimeter of inner and outer pipes
θ	angular coordinate
ξ	shifted time coordinate
ξ_h	heated perimeter

NOMENCLATURE (Appendices A, B, and C) (Continued)

Greek

ρ	density
σ	surface tension
$\mathcal{T}$	stress tensor
τ	shear stress tensor
ϕ_b	generation in the volume
φ_s	surface generation
$\overline{\phi}_{sk}$	generation due to surface transport ($k = g, f$)
$\overline{\phi}_{de}$	energy dissipation for the mixture enthalpy equation
$\overline{\phi}_{ce}$	compressibility effect of the mixture enthalpy equation
ψ	function associated with extensive variable

Subscript and Superscript

D	diffusion transport
f	liquid phase
g	vapor phase
i	interface
j	three-dimensional field index
k (= g, f)	each phase
m	mixture
n	normal direction to the interface
o	reference
r, θ, z	cylindrical coordinates
T	turbulent transport
α, β	surface coordinate index

NOMENCLATURE (Appendices A, B, and C) (Concluded)

Symbols

$-$	time average
$\overline{G} = Gm$	mixture intensive variables
$\overline{\overline{G}}_k$	α_k weighted mean values
$\overline{\overline{\psi}} = \psi_m$	density weighted mixture property
$\overline{\overline{\psi}}_k$	density weighted property of each phase
$[\quad]k$	with $(k = s,g,f,T)$; sets of time intervals
$<\ >$	area average
$< Gm >$	mixture intensive variables
$<<G_k>>$	$<\alpha_k>$ weighted mean value
$<<\psi_m>>$	density weighted mixture property
$<<\psi_k>>$	density weighted property of each phase
$(\quad)_\alpha$	vapor phase weighted mixture property
$(\quad)_{,\beta}$	covariant derivative of surface
*	dimensionless properties
cov	covariant of area average
$\sum_i$	summation on the interfaces passing in Δt at X
$\sum_k$	summation on both phases

SUMMARY

An analysis is presented of thermally induced flow instabilities in two-phase mixture. The mixture field equations were obtained by means of a statistical average procedure applied to the two phases. Furthermore, by integrating the above mixture field equations over a cross sectional area, a one-dimensional mathematical model was developed.

The dynamic response of the system to inlet flow perturbations has been derived from the above model; thus obtaining the characteristic equation which predicts the onset of instabilities. Similarity groups which characterize the phenomenon were derived and discussed. The characteristic equation has been solved by computers, and stability maps in appropriate parametric domain have been obtained. It has been found that the Stability Plane, i.e., Subcooling-Phase Change number plane, is best suited for analyzing the problem. The effects of various parameters, such as the heat flux, subcooling, pressure, inlet velocity, inlet orificing, and exit orificing, on the stability boundary have been analyzed. Particular attention was directed to the effects of 1) the relative velocity between the phases, 2) the nonuniform heat flux profile, and 3) the static and dynamic influences of various friction factor models.

In addition to numerical solutions, some simple stability criteria under particular conditions have been obtained. Both results have been compared with the experimental results reported in this country and abroad.

CHAPTER I

INTRODUCTION

I-1. Relevance of the Problem

Thermally induced flow instabilities can introduce operational and safety problems to systems and components of interest to power generating utilities, chemical process industries, and aerospace industries. Examples of such systems and components include nuclear reactors, liquid rocket engines, heat exchangers, cryogenic equipment boilers, evaporators, and various chemical process units.

The existence of thermo-hydraulic excursions and/or oscillations has been known for some time, and the appearance of these instabilities either at subcritical or at supercritical pressures is highly undesirable. They may not only degrade the performance of the system but can also result in premature burnout and control problems which can become destructive.

I-2. Advances Required

It can be concluded from the discussion of the present state of the art in Chapter II that the following advances are required:

1) To derive from the appropriate mathematical model both stability maps and stability criteria which can be used to predict the onset of thermo-hydraulic excursion and oscillations in two-phase mixture with relative velocities.

2) To obtain correct and important similarity groups which can reduce the number of governing parameters.

3) To present the results in a parametric plane which is useful to researchers and designers.

4) To obtain simple stability criteria from the analysis which can be used for design purposes.

5) To corroborate the predicted results with experimental data reported in this country and abroad.

I-3. Thesis Objectives

The research program which has been carried out in the course of this investigation has been designed to provide the analytical results required for predicting reliably and realistically, the onset of thermally induced flow oscillations in two-phase mixtures.

The particular objectives of this research are:

1) To derive and discuss one-dimensional two-phase flow field equations based on statistical and area averaging together with the necessary constitutive relations.

2) To obtain correct and important similarity groups which govern the kinematics and dynamics of the system.

3) To derive the response functions of various parameters to an inlet flow perturbation by solving the above field equations analytically. Furthermore, the characteristic equation which describes the onset of the instabilities will be obtained from the dynamic response of the system.

4) To present the theoretical predictions of the instabilities in an appropriate parametric domain and to corroborate them with experi-

mental data reported in this country and abroad.

5) To study the effects of various parameters on the extent of stability regions, with special emphasis on the influence of the relative velocity.

I-4. Outline of the Thesis

From the review on the present understanding of thermally induced flow instabilities in Section II.1, it can be seen that several instability mechanisms exist. In this dissertation there is presented a quantitative formulation and solution of the stability problem concerned with a particular instability phenomenon. The particular mechanism which is analyzed in this research is based on the effects of density wave propagations. It is well known that the propagation phenomena induce time lag effects in the dynamic response of a system, hence they may lead to unstable flow, i.e., low frequency oscillations or flow excursion.

It may be beneficial to present here a brief outline of the dissertation so that a better understanding can be attained of the material presented herein.

Chapter II is devoted to a review of the present state of knowledge of thermally induced flow oscillations and two-phase flow dynamics. The various instability mechanisms as well as the existing theories explaining the phenomena are discussed.

The rest of the chapters fall into two parts. Part 1, consisting of Chapters III to VII, presents the development of a theoretical analysis concerned with the dynamics of the system. Part 2, consisting of Chapters VIII to XII, is devoted to the application of the theory derived in Part 1 to the stability analysis of the system, the solution of the characteristic equation, and the comparisons with existing experimental data. The outline of Parts 1 and 2 is as follows.

Outline of Part 1 (Theoretical Analysis)

In this part of the analysis there is derived the mathematical model which can describe the dynamics of a two-phase diabatic system, as well as the characteristic equation which predicts the onset of the thermally induced flow instabilities.

The present study starts from the rigorous derivation of the field equations based on the statistical averaging and then proceeds to the area averaging in order to obtain a one-dimensional model suitable to the system. The detailed mathematical procedures of the above analysis are given in Appendices A to C and the results are summarized in Chapter III.

The formulation of the problem is given in Chapter IV, where a number of simplifications are introduced. Eventually the momentum equation is decoupled from the energy and continuity equations. Subsequently the kinematic problem is solved first in Chapter V, then the dynamics of the system is analyzed in Chapter VI.

Throughout the integration of field equations, the perturbation method is used in order to linearize the system. Furthermore, the system is divided into four regions, i.e., the upstream un-heated section, the liquid heated section, the mixture heated region, and the downstream unheated region. The kinematic problem is solved by transforming the continuity and energy equations into the form of Lagrange's differential equations and integrating them from upstream to downstream along the characteristics.

The analysis includes effects such as nonuniform heat flux and the influences of various frictional pressure drop models on the characteristic equation. As far as the instabilities due to the density wave propagations are concerned, the present study is quite general. However, it should be noted here that both thermal non-equilibrium effects and the acoustical wave interactions with the density wave (kinematic wave) have been neglected in the analysis.

Outline of Part 2 (Application of the Theory)

In this part of the dissertation the applicability of the theory developed in Part 1 to the stability analysis is demonstrated. Since the system with uniform heat flux profile has the simplest characteristic equation and the parametric study on this model gives some general trends of the effects of various parameters on the stability, this is chosen as a basic model.

By taking the system with a uniform heat flux profile the characteristic equation is made dimensionless and thus the similarity parameters governing the stability of the system are obtained. The significance and the physical meaning of the above dimensionless groups are discussed in Chapter VIII.

In Chapter IX, a brief introduction is given of stability theorems such as the Mikhailov Criterion and the D-Partition Method which are important for the analysis of the system. Then the Stability Plane which is suitable for the presentation of the stability boundaries is discussed. Based on the characteristic equation and the stability theorems, some analytical conclusions on the effects of various parameters are obtained.

Furthermore, in Section IX.4, simple stability criteria which can be used for design purposes are developed.

In Section IX.5, the excursive stability criterion is derived and its relation with the characteristic equation is discussed.

Chapter X consists of the numerical analysis on the stability of the system. Two computer programs are developed for the mapping of the stability boundaries and for the test of the stability at any particular operational condition. The effects of various parameters on the stability boundaries are examined by means of computers.

In Chapter XI, the above results as well as the simple stability criterion are compared to experimental data reported in this country and abroad.

The final two chapters are devoted to a discussion and the conclusions.

CHAPTER II

STATE OF THE ART

II-1. Mechanism of Instabilities

Thermally induced flow instabilities may be divided into two main categories: excursive instabilities and oscillatory instabilities due to propagation phenomena.

Excursive instabilities were first analyzed successfully by Ledinegg (2) in 1938. It has been shown that, under certain conditions, the steady state system pressure drop versus flow curve has a negative slope hence, as the flow rate is not a single valued function of the pressure drop, a flow excursion may occur. In his analysis, Ledinegg assumed that the heat flux was uniform, but later this criterion has been extended by a number of other investigators (3,4) for more general cases.

The oscillatory type instabilities are rather complicated dynamic phenomena which may be subdivided into four different mechanisms:

a) instabilities due to pressure wave propagation,

b) instabilities due to thermodynamic nonequilibrium,

c) instabilities due to flow regime change, and

d) instabilities due to kinematic wave propagation.

The most common oscillations (5-10) encountered in heated channels are low frequency, i.e., chugging oscillations. There is considerable evidence that some relationship exists between the residence time of the particle and the period of the chugging, i.e., low frequency oscillations.

Therefore, several analyses have been formulated and carried out by considering the propagation of kinematic waves and the attendant time lag effects.

Although high frequency oscillations, which are associated with the propagation of pressure waves, have been observed in some experiments (11), they may be of less importance for practical applications than the low frequency oscillations. On the other hand, Yadigaroglu and Bergles (37) observed frequent occurrences of higher mode oscillations which they explained by the presence of standing enthalpy waves in the single phase region. These two high frequency oscillations (11, 37) may or may not be related and provide an interesting problem.

The instabilities which have been observed by Jeglic and Grace (12) in experiments conducted with water at low pressures flowing through a smooth pipe were apparently due to thermodynamic nonequilibrium. Under these conditions and because of poor nucleation, the liquid can become highly superheated. However, once a bubble is nucleated, it grows explosively (because of the high liquid superheat) ejecting the liquid from the duct while interrupting the inlet flow. After the liquid is ejected from the duct, the pressure decreases, new liquid enters, becomes superheated, and the process repeats itself.

The disturbances which are created by flow regime changes can also produce oscillatory behavior. Wallis and Hensley (19) analyzed the slug flow in a long, large-diameter riser and concluded that cyclic variation of vapor content could produce periodic fluctuations of loop flow rate.

II-2. Previous Work

II-2.1. History of Analytical Work

There have been numerous analytical studies directed at obtaining a better understanding of thermally induced flow oscillations, at determining their mechanism and deriving stability criteria. Although most of the investigations have been done in the last decade, we cannot omit two outstanding studies of the early 50's (13,15).

The initial analysis concerned with the transient operation of boiler channels has been done in 1953 by the Russian scientists Teletov and Serov (13) who were able to obtain transfer functions for a distributed parameter system, which give the response of the fluid enthalpy and density to perturbations of the heat flux. The derivation of this transfer function was based on a simple but important transformation which is discussed further in Section II-2.2. Their analysis was limited, however, to a homogeneous flow (in which the effect of the relative velocity between phases is neglected) and a thermodynamic equilibrium condition.

Teletov's formulation of the problem became the foundation of the rather rigorous analysis of Zuber (1) based on the kinematic wave theory of M. J. Lighthill (14). The analysis of (1) is applicable to both the slip flow model (in which the effect of the relative velocity between the phases is accounted for) and thermodynamic nonequilibrium.

Although the excellent studies by Crocco and Cheng, which are summarized in their book (15) in 1956, are not directly related to the two-phase flow oscillations, since they were concerned with rocket engine combustion instabilities, their analysis, however, has offered very valuable information on the instability mechanisms of heated channels due to time lag effects. Furthermore, by comparing theoretical results with experimental data, Crocco and Cheng demonstrated the applicability and reliability of the linearized (small disturbance) theory to analyses and studies of combustion instabilities.

Besides these early studies by Teletov and Serov (13) and by Crocco and Cheng (15), numerous analytical investigations have been conducted in this field, although unfortunately many of them are less sophisticated than the pioneering studies mentioned above.

In general, two approaches have been followed. The first is based on phenomenological models which are obtained from the assumed similarity with a simple mechanical system or an electrical circuit having excitations. The second approach is to formulate the problem from the conservation laws for the mixture.

Because of its simplicity, many of the studies belong to the first group; however, the applicability of their results is severely restricted by its own nature. Needless to say, one must supply several experimental coefficients or correlation functions into these formulations, since, strictly speaking, the assumed models are not based on conservation laws.

Shortcomings of such analyses are evidenced by the fact that the values of the coefficients or the functions change with operating conditions and design configurations. We may point out two reasons for this deficiency:

1) the assumed models are inaccurate, and

2) the lack of knowledge of similarity groups which characterize phenomena.

This latter aspect constitutes one of the most important problems in the general analyses of two-phase flow systems. We shall close our discussion of the phenomenological approach without referring to individual models, because the models seem to be of limited value. Furthermore, the phenomenological approach can be looked upon as a transitional method of analysis which leads to the second approach based on the conservation laws.

In general, a theoretical approach must be based on the conservation equations, appropriate constitutive equations, and imposed boundary conditions, together with correct physical approximations. Particularly if the system has complicated characteristics such as existence of interfaces, boiling heat transfer, change of two-phase flow regimes, turbulent flow, and time dependent variables, the formulation of the model should be manageable, and, at the same time, it should state the basic physical nature of the phenomenon. From this point of view two models, a homogeneous flow and a slip flow model, have been used in various analyses. In the first model the relative velocity between the two phases is neglected, whereas in the slip flow model this important characteristic of two-phase flow systems is taken into account.

Besides this classification into homogeneous and slip flow models, generally two distinct methods have been applied to obtain a solution of the problem. In the first, the system of partial differential equations is linearized by assuming small disturbances about a steady state. The response of the system to various perturbations, as well as stability criteria, is then obtained by using standard techniques. The second method is based on a numerical solution whereby the set of nonlinear partial differential equations is solved by numerical methods.

Following this classification, we shall discuss in the sections below the more relevant and/or representative theoretical studies of thermally induced flow oscillations, particularly of the low frequency instability, since it has been chosen as the main subject of this dissertation.

II-2.2. Homogeneous Flow Model (Linearized Theory)

As previously mentioned, Teletov* and Serov (13) were the first to formulate the dynamic problem of two-phase flow systems, although the analyses of similar dynamic problems of a rocket engine combustion instabilities had already been studied in detail by M. Summerfield (16), H. S. Tsien (17), Crocco and Cheng (15), among others. The study of Teletov and Serov (13) is limited to low frequency oscillations and deals with the transient responses of enthalpy and density to a heat flux disturbance. It also takes into account the effect of the wall heat capacity. By considering only low frequency oscillations, Teletov and Serov (13) were apparently the first to realize that the density was a function of enthalpy only and not of both enthalpy and pressure. It should be noticed, however, that, in Ref. (13) Serov did not integrate the momentum equation, consequently, the characteristic equation was not derived. Furthermore, his analysis was limited to homogeneous flow and thermodynamic equilibrium.

The importance of Serov's first analysis (13) rests on

1) decoupling the momentum equation from the energy and continuity equations,

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*Unpublished paper which is discussed in Serov's paper (13).

2) deriving the transfer function for a distributed parameter system, and

3) finding a simple transformation (given below) which relates the divergence of the velocity to the heat flux,

$$\frac{\partial v_m}{\partial z} = \Omega \equiv \frac{\Delta\rho}{\rho_g \rho_f} \; \frac{1}{\Delta i_{fg}} \; \frac{q_w'' \xi}{A_c} \qquad \text{(II2.1)}$$

Here v_m is a velocity of the mixture and Ω is a reaction frequency of the phase change, which can be expressed by the densities of each phase ρ_g and ρ_f, the density difference $\Delta\rho$, the latent heat Δi_{fg}, and the heat input $q_w''\xi/A_c$.

An analysis quite similar to (13), but including the response of the pressure, was given by Terano (18). The system was divided into three parts, liquid, mixture, and superheated vapor region. Unfortunately he used a simple capacitance model for the pressure response, which reduces the generality of the solution for the same reasons discussed in connection with the phenomenological models.

Wallis and Heasley (19) used a model similar to that of Serov (13). Using Lagrangian coordinates, they integrated the energy and continuity equations with a disturbed inlet flow. However, in contrast to the analysis of Serov (13), they neglected the effect of a variable heat transfer coefficient. In (19), the characteristic equation was obtained from the momentum balance for a lumped parameter system, and the Nyquist criterion was used to discuss the stability. Wallis and Heasley also used $\rho_m = \rho_m(i_m)$ and rederived Eq. (II2.1). Like the analysis of Serov, their analysis is limited to the thermodynamic equilibrium, homogeneous flow, and low frequency oscillations.

About 10 years later, following his original paper (13), Serov, et al. (20,21) integrated the momentum equation and obtained the characteristic equation for a distributed parameter system. Their analysis takes into account the variation of inlet flow and heat transfer coefficient, but neglects the displacement of the boiling boundary. The characteristic equation, derived in (20,21) is a fifth order exponential polynomial with two time delays. It has been solved for the stability boundaries in a simplified form by the D-Partition method. Some qualitative success was shown in the ζ-plane (inlet orifice).

Bouré (22,23) used a model similar to that of Serov. Consequently, his analysis is only applicable to thermodynamic equilibrium, homogeneous flow, and low frequency oscillations. Bouré, like Serov, assumed that $\rho_m = \rho_m(i_m)$ and expressed the continuity equation in terms of Lagrange's differential equation and independently rederived the important relation given by Eq. (II2.1). In contrast to the analysis of Wallis and Heasley (19), Bouré integrated also the momentum equation and thus obtained a characteristic equation for a distributed parameter system. Thus, Bouré was apparently the first to integrate the entire set of equations (continuity, energy, and momentum equations for the mixture) for the homogeneous flow model. The characteristic equation of (22) is a fifth order exponential polynomial with two time delays. The coefficients of polynomials differ from those of Serov, et al. (20,21), because the analysis of (23) accounts for the variation of inlet flow and the displacement of a boiling boundary (which was neglected by Serov), but neglects the wall heat capacity (which was accounted for by Serov). Satisfactory agreement with experimental data is reported in (24) where the comparison is shown in the dimensionless subcooling and inlet velocity plane. A parametric study is also given in (24); however, the choice of the representative parameters and the domain of representation are not very satisfactory, since one of the parameters is not bounded as the inlet velocity increases.

The method of Bouré (24) was used by Zuber (25) to analyze flow instabilities in the near-critical and super-critical thermodynamic region. However, Zuber's analysis differs in two aspects from (24): 1) the constitutive equations are different, and 2) the resulting characteristic equations are different. The mean value theorem was applied to simplify the characteristic equation which finally took a form of third order exponential polynomials with two time delays. Under some conditions, he was able to obtain a simple algebraic criterion which was useful as a design criterion for the friction dominated flow at high pressure.

So far we have discussed the analyses based on 1) the homogeneous flow model, i.e., no slip condition, and 2) thermodynamic equilibrium condition. Such conditions can be attained with high flow rates at near-critical pressures. However, at lower pressures both the effects of relative velocity and of thermodynamic nonequilibrium become important and cannot be neglected. This is particularly true for alkali metals and natural circulation loops.

II-2.3. Slip Flow Model (Linear and Nonlinear Theory)

As is mentioned in (1,33), the traditional "slip" flow models are not formulated in terms of the center of gravity, consequently, they cannot be used to analyze and predict correctly dynamic phenomena in a system where the relative velocity between the two phases is important. In particular, since the traditional slip flow formulations were not expressed in terms of the center of mass of the mixture, the authors were forced to use three different expressions either for the density of the mixture or for the phase velocity ratio (see for example Ref. (35,7, etc.)), as well as two different expressions for the enthalpy of the mixture.

Besides this basic shortcoming of traditional formulations, almost all "slip" flow models have been solved directly by computers. Meyer (35) was an early user of this method applying a momentum integral and a finite difference method. The works by Jones (36) and by Carver (7), among others, are notable in their detailed treatments of various effects. These direct methods require expensive computer time and great care in the programming in order to avoid numerical instabilities. We may quote the words from Carver himself (7, p. 4) in this connection: "Such approaches require considerable expensive computer time, the expense increasing with the degree of sophistication. Again they depend heavily on void, pressure, drop, and heat transfer correlations used, and under certain conditions the validity of these is questionable." In their analysis, the set of nonlinear simultaneous partial differential equations has been solved step by step in the time domain with one parameter disturbed, but all others were held fixed. It should be noted that this approach, in addition to being expensive for parametric studies, does not provide an insight into the physical aspects of the problem. Consequently, it is not very helpful in advancing the understanding of the phenomenon. However, it also should be noted that a nonlinear theory, by its own nature, has an advantage over a linear theory <u>when</u> ultimate nonlinear responses in an unstable region or responses to large changes in steady state operational conditions are required.

The extreme care which must be taken when solving the set of nonlinear partial differential equations by computer and the difficulties which may be encountered are best exemplified by the fact that a Ph.D. thesis, an A.E.C. report, as well as several papers were published which claimed a "new theory" of oscillations and a "new criterion" for predicting the onset of oscillations (40). Two years later it was shown by Cornelius (39) that the oscillations predicted by Harden (40) were numerical instabilities of the computer programming. Here we quote the words of Cornelius (39, p. 62):

> Harden's solution was carefully examined. It was found that the solution depends upon the time step employed, and that the calculated enthalpy distribution was extremely erratic .... . This indicated that the sustained oscillations resulted from a numerical instability, and were not representative of the physical system.

These shortcomings have been removed in the analysis of Zuber (1) which was formulated in terms of the center of mass and takes into account both the effects of the relative velocity between the two phases and the effects of the thermodynamic nonequilibrium condition.

When the relative motions between phases exist, the field equations should be formulated with respect to the baricenter of the mixture. The idea that physical laws are invariant under a coordinate transformation and that it is necessary to write field equations with respect to the center of mass in order to treat the mixture as a whole is well known. This result first appeared in Maxwell's kinetic theory of gas mixtures (26), and was pointed out for heterogeneous chemically reacting substances by various authors such as von Karman*, Prigogine (27), Hirschfelder (28), and Trusedell (29).

Although two-phase flows are locally separated flow media, i.e., separated into the vapor and liquid regions bounded by the interfaces, a similar approach can be used by proper averaging in space and/or in time. This was done by Zuber (30,31,32,33) and applied to a dynamic problem (1) in 1967. The detailed averaging of two-phase mixture was given by Vernier and Delhaye (34). In (1), the problem was formulated in terms of four field equations (the continuity, energy, and momentum equations for the mixture and the continuity equation for the vapor expressed in terms of kinematic waves) and seven constitutive equations. By using kinematic wave velocities and small perturbations on the variables, the set of equations has been integrated analytically resulting in a characteristic equation for a system with distributed parameters.

The study of (1) is not only the first, but also the only analysis which formulates correctly the dynamics of the slip flow with respect to the baricenter of the mixture and gives the characteristic equation by means of an analytical method. However, the effects of both the nonuniform heat flux distribution and of a variable friction, i.e., the effects of static changes as well as of the dynamic responses of a friction factor, have not been examined in (1), although they are important for nuclear reactor applications. Consequently, it is of great interest to derive a characteristic equation which, in addition to the previously

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*See paper by Nachbar, et al. (38).

considered effect of relative motion of two phases, would take into account the nonuniform heat flux distribution and the variable friction factor. Furthermore, it is highly desirable to obtain a solution to this characteristic equation and thereby obtain criteria for the prediction of the onset of thermally induced flow oscillations.

Part 1

THEORETICAL ANALYSIS

CHAPTER III

FLUID DYNAMICS OF ONE-DIMENSIONAL TWO-PHASE FLOW

III-1. Governing Equations

It is well established in continuum mechanics that the conceptual models for single phase flow of a gas or of a liquid are formulated in terms of field equations which describe the conservation laws of mass, momentum, energy, charge, etc. These field equations are then complemented by appropriate constitutive equations such as the constitutive equations of state, stress, chemical reactions, etc., which specify the thermodynamic, transport, and chemical properties of a given constituent material, i.e., of a specified solid, liquid, or gas.

It is to be expected, therefore, that the conceptual models which describe the steady state and dynamic characteristics of structured multiphase or multi-component media, should be also formulated in terms of the appropriate field and constitutive equations. However, the derivation of such equations for the flow of structured media is considerably more complicated than for strictly continuous, i.e., homogeneous media, i.e., for single phase flow.

In order to appreciate the difficulties in deriving balance equations for structured, i.e., inhomogeneous media, we recall that, in continuum mechanics, the field theories are constructed on integral balances of mass, momentum, and energy. Thus, if the variables in the region of integration are continuously differentiable and the Jakobian transformation between material and spatial coordinates exists, then the Eulerian type differential balance can be obtained by using the Leibniz Rule, or more specifically, the Reynolds Transport Theorem which allows us to interchange differential and integral operations.

In multi-phase or multi-component flows, the presence of interfaces introduces great difficulties in the mathematical and physical formulation of the problem.

From the mathematical point of view, a multi-phase flow can be considered as a field which is subdivided into single phase regions with moving boundaries separating the constituent phases. The differential balance holds for <u>each</u> subregion; however, it cannot be applied to the <u>set</u> of these subregions in the normal sense without violating the above conditions of continuity.

From the point of view of physics, the difficulties which are encountered in deriving the field and constitutive equations appropriate to multi-phase flow systems stem from the presence of the interface and the fact that both the steady and dynamic characteristics of multi-phase flows depend upon the structure of the flow, see Table 1.* For example, the steady state and the dynamic characteristics of dispersed two-phase flow systems depend on the collective dynamics of solid particles, bubbles, or droplets interacting with each other and with the surrounding continuous phase; whereas, in the case of separated flows, these characteristics depend upon the structure and dynamics of the interface.

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*This classification of structured flow has been obtained from a personal communication with Dr. Zuber.

In order to determine the collective interaction of particles and the dynamics of the interface, it is necessary to describe first the local properties of the flow and then to obtain a macroscopic description by means of appropriate averaging procedures.

For dispersed flows, for example, it is necessary to determine the rates of nucleation, evaporation or condensation, motion, and disintegration of single droplets (bubbles) as well as the collisions and coalescence processes of several droplets (or bubbles).

For separated flows, the structure and the dynamics of the interface greatly influence the rates of mass, heat, and momentum transfer as well as the stability of the system. For example, the performance and flow stability of a condenser for space applications depend on the dynamics of the vapor interface. Similarly, the rate of droplet entrainment from a liquid film, and therefore the effectiveness of film cooling, depends on the stability of the vapor-liquid interface.

It can be concluded from this discussion that, in order to derive the field and constitutive equations appropriate to structured, multiphase media, it is necessary to describe the local characteristics of the flow from which the macroscopic properties should be obtained by means of an appropriate averaging procedure. It is evident also that the design, performance, and very often the safe operation of a great number of important technological systems depend on the availability of realistic and accurate field and constitutive equations.

In theory, the problem could be formulated in terms of field equations applicable to each continuous subregion with matching boundary conditions at the moving interface. Such a formulation would result in a multiboundary problem with the position and the condition at the boundary being unknown. It is evident that, unless the topography of the interface is simple, as it is for example in separated flows, such an approach would encounter unsurmountable mathematical difficulties. Consequently, different methods of analysis must be applied to the different classes of structured flows.

For separated two-phase flows, the problem can be formulated by considering two continua coupled by the appropriate jump conditions at the boundary, i.e., at the interface. It is evident that these jump conditions will play a most important role in such a formulation.

However, for dispersed flows in order to eliminate the mathematical difficulties caused by the discontinuities of the variables, it is useful to transform the entire field to a continuum. This can be accomplished by a time averaging procedure or by means of Boltzman's equation applied to the dispersed phase.

Since a formulation based on the two-fluid model is expressed in terms of two equations of continuity, two momentum equations, and two energy equations, an analysis based on this model may encounter mathematical difficulties. The model is therefore not well suited for analyses of system dynamics. Neither can it be used to determine the mixture properties (in particular the entropy) and the similarity groups.

The problem of thermally induced flow instability falls into the above category. Thus, it is better to formulate the field equations with respect to the center of mass by proper averaging procedures, as has been mentioned in Section II-2.

Basically, two distinct methods have been employed, the space average and the statistical average. Since in the two-phase flow field the volumes occupied by each phase are usually of macroscopic scale, the space averaged field equations become independent of one or two space coordinates. This is a significant difference from the equations for the chemically reacting mixtures on the molecular level where we can take and use the mixture volume element containing both components on a microscopic scale. Thus, the resulting field equations are three dimensional. On the other hand, if the statistical average, i.e., the time average, is employed, the mixture equations can be considered as strictly local. Therefore, the result is analogous to that of the molecular level mixtures.

In Appendix A the time averaged field equations have been obtained by a method similar to that used by Vernier and Delhaye (34), but including the effect of surface tension, which was not accounted for in the analysis of (34). Furthermore, their analysis was based on the two-fluid theory. Thus, they did not obtain the mixture field equations in terms of the mixture variables which have been developed in this analysis. These field equations are then one-dimensionalized by area averaging in Appendix B. Furthermore, the similarity parameters governing the system are obtained from the differential equations in Appendix C. In the following, the summary of Appendices A to C is given in the form useful for the present analysis.

A. Field Equation

We formulate the problem in terms of the time smoothed and area averaged field and constitutive equations. The variables appearing in the equations should be understood as averaged or weighted mean values defined in Appendices A and B. From the summary of field equations, i.e., Appendix C-1, we have

for the conservation of mass of the mixture:

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m v_m}{\partial z} = 0 \tag{III1.1}$$

for the conservation of mass of the vapor phase:

$$\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial \alpha \rho_g v_g}{\partial z} = \Gamma_g \tag{III1.2}$$

for the conservation of momentum of the mixture:

$$\rho_m \left\{ \frac{\partial v_m}{\partial t} + v_m \frac{\partial v_m}{\partial z} \right\} = - \frac{\partial P_m}{\partial z} - \frac{f_m}{2D} \rho_m v_m^2 + \tag{III1.3}$$

$$+ \left\{ g \rho_m + f_\sigma \right\} - \frac{\partial}{\partial z} \left\{ \frac{\alpha}{1-\alpha} \frac{\rho_g \rho_f}{\rho_m} V_{gj}^2 \right\}$$

for the conservation of energy of the mixture:

$$\rho_m \left\{ \frac{\partial i_m}{\partial t} + v_m \frac{\partial i_m}{\partial z} \right\} = \frac{q_w'' \xi}{A} + \phi_{ce} + \phi_{de} - \tag{III1.4}$$

$$- \frac{\partial}{\partial z} \left\{ \alpha \frac{\rho_g \rho_f}{\rho_m} V_{gj} \Delta i_{fg} \right\}$$

where the explicit form for the terms Γ_g, f_σ, ϕ_{ce}, and ϕ_{de} have been obtained from the interface conditions and are given in Appendix B. Hence, the mass generation term due to the interface mass transfer is given by

$$\Gamma_g = -\frac{1}{A}\int_A \sum_i \frac{1}{\Delta t}\left(\frac{1}{v_{in}}\right)\left\{ n_g \cdot \left[\rho_g (v_g - v_i)\right]\right\}_i dA \tag{B5.8}$$

the capillary forces f_σ by

$$f_\sigma = \frac{1}{A}\int_A \sum_i \frac{1}{\Delta t}\left(\frac{1}{v_{in}}\right)\left\{\left(t^j_\alpha \, \sigma \, a^{\alpha\beta}\right)_{,\beta}\right\}_i dA \tag{B6.5}$$

the compressibility effect ϕ_{ce} by

$$\begin{aligned} \phi_{ce} = \frac{1}{A}\int_A \frac{1}{\Delta t}\Big\{ \int_{[\Delta t]} \frac{DP}{Dt}\, dt \; - \\ -\sum_i \left(\frac{1}{v_{in}}\right)\Big[- n_g \cdot \left(\frac{P_g + P_f}{2}\right)(v_g - v_f) + \\ + \left(\frac{v_g + v_f}{2}\right)\cdot\left(t^j_\alpha \, \sigma a^{\alpha\beta}\right)_{,\beta} - \left(t^j_\alpha \, \sigma a^{\alpha\beta} v_i\right)_{,\beta}\Big]_i \Big\} dA \end{aligned} \tag{B7.8}$$

and the dissipation term ϕ_{de} by

$$\begin{aligned} \phi_{de} = \frac{1}{A}\int_A \frac{1}{\Delta t}\Big\{ \int_{[\Delta t]} \tau^{\mu} : \nabla v \, dt \; - \\ - \sum_i \left(\frac{1}{v_{in}}\right)\left[n_g \cdot \left(\frac{\tau_g^{\mu} + \tau_f^{\mu}}{2}\right)\cdot (v_g - v_f)\right]_i \Big\} dA \end{aligned} \tag{B7.9}$$

Here the variables appearing under the summation signs in the above four equations are the ones at the interfaces, and P, τ^{μ}, and v in the time integrals are the instant local values of the pressure, stress, and velocity, respectively. Furthermore, t^{j}_{α}, $a^{\alpha\beta}$, and $(\ )_{,\beta}$ stand for the hybrid tensor, the surface matrix tensor, and the covariant derivative of the surface coordinate (58).

We note that the various terms due to the effects of the surface tension and of dissipation which appear explicitly in Eq. (B6.5), (B7.8), and (B7.9) were not taken into account nor did they appear in any previous analysis or formulation of dispersed two-phase flow. We note also that, if these surface and dissipation effects are omitted, then the form of the equations reduces to those reported in (1).

Referring to the above four field equations, which have been obtained by using time and area averages, we note that mixture continuity, momentum, and energy equations, i.e., Eq. (III1.1), (III1.3), and (III1.4), respectively, are somewhat similar to those of the single phase one-dimensional model. Actually, the mixture continuity equation has exactly the same form as that for the continuum without internal discontinuities. However, the mixture momentum equation has two additional terms which do not appear in a single phase equation. One is the capillary (body) force f_{σ} which takes into account the surface tension effects and can be considered as momentum source or sink. The other is a drift stress term, shown as the last term on the right hand side of Eq. (III1.3), and it expresses the momentum drift or diffusion due to the relative motions between two phases in addition to the molecular and turbulent diffusions which has been taken into account by the friction factor f_m. In the mixture energy equation (III1.4), we have also an additional term, i.e., an energy drift or diffusion term, the last term of Eq. (III1.4), due to the transport of energy by relative motions of the phases. Recalling Eq. (B7.8) and (B7.9), the two-phase compressibility effect on enthalpy ϕ_{ce} and the dissipation term ϕ_{de} contain the effects of interfaces in addition to those of the single phase flow.

The continuity equation for vapor phase, Eq. (III1.2), has a vapor generation term Γ_g which appears after the continuity equation has been averaged over time. As we can see from Eq. (B5.8), the constitutive equation for Γ_g , it accounts for <u>the mass transfer at the interfaces</u>. Equation (III1.2) can be transformed in terms of mixture velocity v_m and the drift velocity V_{gj}, thus

$$\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial \alpha \rho_g v_m}{\partial z} = \Gamma_g - \frac{\partial}{\partial z}\left\{\frac{\alpha \rho_g \rho_f}{\rho_m} V_{gj}\right\} \qquad \text{(III1.2')}$$

Since the frame of reference follows the mixture velocity v_m, the drift or diffusion of vapor mass with respect to v_m appears on the right hand side of Eq. (III1.2). With Eq. (III1.1), (III1.2'), (III1.3), and (III1.4) we have formulated the problem in terms of the mixture velocity v_m, statistical center of mass velocity, and all the fluxes due to relative motions of phases are expressed as drift fluxes, i.e., expressed through the vapor drift velocity V_{gj}, since the expressions for V_{gj} have been proposed for certain two-phase flow regimes (32).

B. Constitutive Relations

In addition to four field equations, i.e., the continuity equations for the mixture and for the one-phase, mixture momentum, and energy equations, we need additional information to describe the system completely, as follows.

I) The Thermal Equation of State

The mixture density ρ_m from Eq. (B1.6)

$$\rho_m = \alpha \rho_g + (1-\alpha)\rho_f \tag{III1.5}$$

with the thermal equation of state for each phase

$$\rho_g = \rho_g(P_g, T_g) \tag{III1.6}$$

$$\rho_f = \rho_f(P_f, T_f) \tag{III1.7}$$

II) The Caloric Equation of State

The mixture enthalpy i_m from Eq. (B1.18)

$$i_m = \frac{\alpha \rho_g i_g + (1-\alpha)\rho_f i_f}{\rho_m} \tag{III1.8}$$

with the caloric equation of state for each phase

$$i_g = i_g(P_g, T_g) \tag{III1.9}$$

$$i_f = i_f(P_f, T_f) \tag{III1.10}$$

III) The Constitutive Equation for Phase Change from Eq. (B5.8)

$$\Gamma_g = f_1 \qquad \text{(III1.11)}$$

IV) The Kinematic Constitutive Equation, i.e., the Relation for the Relative Motion of Phases

(Relative Equation of Motion)

The vapor drift velocity V_{gj} from Eq. (B3.8)

$$V_{gj} \equiv (1-\alpha)(v_g - v_f) \qquad \text{(III1.12)}$$

and the drift constitutive equation from Eq. (B6.11)

$$V_{gj} = f_2 \qquad \text{(III1.13)}$$

V) The Rheological Constitutive Equation from Eq. (B6.6) Which Specifies the Friction Factor

$$f_m = f_3 \qquad \text{(III1.14)}$$

VI) The Capillary Force from Eq. (B6.5)

$$f_\sigma = f_4 \qquad \text{(III1.15)}$$

VII) Compressibility Effect on Enthalpy ϕ_{ce} from Eq. (B7.8)

$$\phi_{ce} = f_5 \qquad \text{(III1.16)}$$

VIII) Dissipation due to Irreversible Work from Eq. (B7.9)

$$\phi_{de} = f_6 \qquad \text{(III1.17)}$$

IX) Boundary Condition for Heat Flux from Eq. (B7.6)

$$q_w'' = f_7 \tag{III1.18}$$

X) The Definition of the Mixture Velocity v_m from Eq. (B1.16)

$$v_m = \frac{\alpha \rho_g v_g + (1-\alpha) \rho_f v_f}{\rho_m} \tag{III1.19}$$

XI) The Definition of the Mixture Pressure P_m from Eq. (B1.17)

$$P_m = \alpha P_g + (1-\alpha) P_f \tag{III1.20}$$

XII) Mechanical State between Two Phases from Eq. (B6.12)

$$P_g - P_f = f_8 \tag{III1.21}$$

XIII) Thermal State between Two Phases from Eq. (B7.11)

$$T_g - T_f = f_9 \tag{III1.22a}$$

or

$$i_g - i_f = \Delta i_{fg} = f_9' \tag{III1.22b}$$

XIV) Geometrical Parameters (given)

$$D = \text{constant} \tag{III1.23}$$

$$\xi/A = \text{constant} \tag{III1.24}$$

$$g = \text{constant or function of } t$$

Here it should be noted that we have obtained the explicit functional forms for Γ_g, f_σ, ϕ_{ce}, and ϕ_{de}, i.e., Eq. (B5.8), (B6.5), (B7.8), and (B7.9), respectively. However, they are expressed in terms of the instant local variables. Consequently, modifications on these relations should be made such that the arguments of the functions for Γ_g, f_σ, ϕ_{ce}, and ϕ_{de}, i.e., f_1, f_4, f_5, and f_6, in Eq. (III1.11, 15, 16, and 17), could be expressed by averaged variables.

In the case of thermal and mechanical equilibrium (60), the constitutive equations (B1.11, 21, and 22) reduce to

$$P_g = P_f = P_s$$

$$T_g = T_f = T_s \qquad \text{or} \qquad \Delta i_{fg} = \Delta i_{fg(T_s)}$$

and $$P_s = f\,(T_s)$$

From the field equations, we have two continuity equations, i.e., one for the mixture, one for the vapor or liquid phase, the mixture momentum, and energy equations. In addition to the above equations, we generally need eighteen constitutive relations and three geometrical parameters, D, ξ/A_c, and gravitational field force g. Accordingly, the total number of equations is twenty-five. The variables appearing in these equations are, ρ_m, v_m, P_m, i_m, α, ρ_g, v_g, P_g, T_g, i_g, ρ_f, v_f, P_f, T_f, i_f, Γ_g, V_{gj}, f_m, f_σ, ϕ_{ce}, ϕ_{de}, q_w'', D, ξ/A_c, and g. By assuming that we can express the functions appearing in the constitutive relations, i.e., from f_1 to f_9, by the above variables and/or by independent variables z and t, the total number of dependent variables is also twenty five.

Thus, the total number of unknowns and of equations is the same. Consequently, our description of systems is consistent and complete in a mathematical sense.

Although our present one-dimensional model is quite general, it should be remembered that

1. we have neglected the normal stress and the velocity covariant terms in the mixture momentum equation (see Eq. (B6.8)), and

2. we have neglected the axial conduction and the enthalpy-velocity covariant in the energy equation (see Eq. (B7.7)).

For two-phase flow boiling in a constant area duct, the normal stress and the axial conduction can be neglected almost always. On the other hand, the covariant term may not be neglected if the flow is laminar. In such a case, we should supply additional information relating the covariants to the mixture convective terms.

C. Velocity Field

In general, two-phase flow systems with transport of mass, momentum, and energy are characterized by the existence of two different densities and velocities. To say nothing of the importance of the difference in the densities, it is necessary to introduce two velocity fields in order to take into account the effects of the relative motions of the phases, i.e., the diffusion of mass, momentum, and energy. In our analysis, the problem was formulated in terms of the mixture velocity v_m and the vapor drift velocity V_{gj}. However, there are several velocity fields which are useful in analyzing various aspects of a two-phase flow problem.

In what follows, we shall summarize the definitions and the interrelations of various velocity fields which have been developed in Appendix B. First, we recall that v_g and v_f are the area averaged statistical center of mass velocities of the vapor and liquid phase, respectively. Then the velocity of the center of mass of the mixture is defined by

$$v_m = \frac{\alpha \rho_g v_g + (1-\alpha)\rho_f v_f}{\rho_m} \qquad \text{(III1.26)}$$

and the relative velocity v_r by

$$v_r = v_g - v_f \qquad \text{(III1.27)}$$

The volumetric fluxes of each phase are given by

$$j_g = \alpha v_g \quad , \quad j_f = (1-\alpha) v_f \qquad \text{(III1.28)}$$

which can be considered as the velocity when one of the phases superficially occupies the total flow area.

The velocity of the center of volume, i.e., volumetric flux of the mixture is thus defined by

$$j = j_g + j_f = \alpha v_g + (1-\alpha) v_f \qquad \text{(III1.29)}$$

If the relative velocity between the phases exists, the velocities v_m and j are not equal due to the difference in the densities of the two phases.

The diffusion velocities of each phase, i.e., the velocity with respect to the mass center of the mixture, are defined by

$$V_{gm} = v_g - v_m = (1-\alpha)\frac{\rho_f}{\rho_m} v_r \qquad \text{(III1.30)}$$

$$V_{fm} = v_f - v_m = -\alpha\frac{\rho_g}{\rho_m} v_r \qquad \text{(III1.31)}$$

which are frequently used in the analysis of a heterogeneous chemically reacting system. In a two-phase flow system, the drift velocities of each phase, i.e., the velocities with respect to center of volume are important (30), are defined by

$$V_{gj} = v_g - j = (1-\alpha) v_r \qquad \text{(III1.32)}$$

$$V_{fj} = v_f - j = -\alpha v_r \qquad \text{(III1.33)}$$

Several important relations between the above velocities can be obtained directly from the definitions. For example, from Eq. (III1.26) and Eq. (III1.29) we get

$$j = v_m + \alpha(1-\alpha)\frac{\Delta\rho}{\rho_m} v_r \qquad \text{(III1.34)}$$

or

$$= v_m + \alpha\frac{\Delta\rho}{\rho_m} V_{gj} \qquad \text{(III1.35)}$$

From Eq. (III1.30) and (III1.31)

$$V_{fj} = -\frac{\alpha}{1-\alpha} V_{gj} \tag{III1.36}$$

Finally, we note, if the relative velocity is zero, then

$$V_{gm} = V_{fm} = V_{gj} = V_{fj} = v_r = 0 \tag{III1.37}$$

and

$$v_g = v_f = v_m = j \tag{III1.38}$$

which characterizes the homogeneous velocity field.

D. Alternative Form of Continuity Equations

In order to specify the conservation of mass in two-phase mixtures, it is necessary to employ two continuity equations (33). This has been done in the previous section by introducing the vapor continuity equation in addition to the mixture continuity equation, i.e., Eq. (III1.2) or Eq. (III1.2'), and Eq. (III1.1), respectively. It is of interest now to consider the alternative forms of the equation of continuity by taking several different frames of reference and/or introducing transport coefficients for the diffusion of masses.

1. In Terms of Center of Mass Velocities of Each Phase

The most elementary form of the conservation of masses for each phase can be obtained directly from the time and area average of the single phase field equations, i.e., Appendices A and B. Thus we have

$$\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial \alpha \rho_g v_g}{\partial z} = \Gamma_g \tag{III1.39}$$

$$\frac{\partial(1-\alpha)\rho_f}{\partial t} + \frac{\partial(1-\alpha)\rho_f v_f}{\partial z} = \Gamma_f = -\Gamma_g \qquad \text{(III1.40)}$$

which are expressed in terms of the center of mass velocities of each phase v_g and v_f. The right hand sides of Eq. (III1.39) and (III1.40) are the mass generation or phase change terms of the vapor and liquid phase, respectively. It should be noted that these terms do not appear in original continuity equations before averaging, since they are the terms expressing the interfacial transport of masses which can appear in the differential balance only after the time averaging.

Furthermore, in this formulation the diffusion term does not appear explicitly due to the fact that the frames of reference have been taken as the component velocities. Finally, the mixture continuity equation (III1.1) can be obtained by adding the above two equations, i.e., Eq. (III1.39) and (III1.40), and using the definitions of mixture density ρ_m, Eq. (III1.5) and of mixture velocity v_m, Eq. (III1.12). Hence,

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m v_m}{\partial z} = 0 \qquad \text{(III1.41)}$$

2. In Terms of Mixture Velocity

By taking the frame of reference on the mixture center of mass and expressing the relative motions of phases with respect to it by the diffusion velocities defined by Eq. (III1.30) and (III1.31), we have

$$\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial}{\partial z}(\alpha \rho_g v_m) = \Gamma_g - \frac{\partial}{\partial z}\left\{\alpha \rho_g V_{gm}\right\} \qquad \text{(III1.42)}$$

and

$$\frac{\partial(1-\alpha)\rho_f}{\partial t} + \frac{\partial(1-\alpha)\rho_f v_m}{\partial z} = -\Gamma_g - \frac{\partial}{\partial z}\left\{(1-\alpha)\rho_f V_{fm}\right\} \qquad \text{(III1.43)}$$

In this formulation, the diffusion fluxes which represent the transport of mass with respect to the mass center of the mixture appear explicitly in the right hand side of the equations.

Instead of using the volumetric concentration α we may also express the above equations in terms of mass concentration of vapor c defined by

$$c = \frac{\alpha \rho_g}{\rho_m} \qquad \text{(III1.44)}$$

Substituting Eq. (III1.44) into Eq. (III1.42) and recalling Eq. (III1.41), we get

$$\frac{\partial c}{\partial t} + v_m \frac{\partial c}{\partial z} = \frac{\Gamma_g}{\rho_m} - \frac{1}{\rho_m}\frac{\partial}{\partial z}\left\{c\,\rho_m V_{gm}\right\} \qquad \text{(III1.45)}$$

A similar equation can also be obtained for the liquid phase.

Furthermore, the diffusion coefficient D_c can be defined from the diffusion flux of Eq. (III1.42) in analogy with a chemically reacting system, hence

$$\alpha \rho_g V_{gm} = c\,\rho_m V_{gm} \equiv -\rho_m D_c \frac{\partial c}{\partial z} \qquad \text{(III1.46)}$$

Substituting Eq. (III1.46) into Eq. (III1.45), we obtain the following familiar diffusion equation.

$$\frac{\partial c}{\partial t} + v_m \frac{\partial c}{\partial z} = \frac{\Gamma_g}{\rho_m} + \frac{1}{\rho_m}\frac{\partial}{\partial z}\left(\rho_m D_c \frac{\partial c}{\partial z}\right) \qquad \text{(III1.47)}$$

3. In Terms of the Volumetric Fluxes

The continuity relation also can be considered from the frame moving with the volume center of the mixture. By introducing the drift velocities of each phase, i.e., Eq. (III1.32) and (III1.33), in Eq. (III1.39) and (III1.40), we get

$$\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial (\alpha \rho_g j)}{\partial z} = \Gamma_g - \frac{\partial}{\partial z}\left\{\alpha \rho_g V_{gj}\right\} \qquad \text{(III1.48)}$$

and

$$\frac{\partial (1-\alpha)\rho_f}{\partial t} + \frac{\partial (1-\alpha)\rho_f j}{\partial z} = -\Gamma_g - \frac{\partial}{\partial z}\left\{(1-\alpha)\rho_f V_{fj}\right\} \qquad \text{(III1.49)}$$

The last terms in the right hand side of each of the above equations represent the drift of vapor and liquid mass with respect to the mixture volume center.

On the other hand, by carrying out the differentiations of the left hand side of Eq. (III1.39) and (III1.40), we get

$$\frac{\partial \alpha}{\partial t} + \frac{\partial \alpha v_g}{\partial z} = \frac{\Gamma_g}{\rho_g} - \frac{\alpha}{\rho_g}\frac{D_g \rho_g}{Dt} \qquad \text{(III1.50)}$$

and

$$\frac{\partial (1-\alpha)}{\partial t} + \frac{\partial (1-\alpha) v_f}{\partial z} = -\frac{\Gamma_f}{\rho_f} - \frac{(1-\alpha)}{\rho_f}\frac{D_f \rho_f}{Dt} \qquad \text{(III1.51)}$$

Here the substantial derivatives are taken from the vapor and liquid frames of reference. These equations can be considered as the continuity equations in terms of void fraction. Thus, from the volume point of view, each continuity equation has a source term due to the mass transfer and a sink due to the compressibility of each phase. By adding the above two equations, we obtain

$$\frac{\partial j}{\partial z} = \Gamma_g \frac{\Delta \rho}{\rho_g \rho_f} - \left[\frac{\alpha}{\rho_g} \frac{D_g \rho_g}{Dt} + \frac{(1-\alpha)}{\rho_f} \frac{D_f \rho_f}{Dt} \right] \qquad \text{(III1.52)}$$

which describes the center of volume velocity. The first term of the right hand side is the volume source due to the phase change and the second term is the volume sink due to the compressibility. This form of the continuity balance has already been derived and discussed in detail in (31)(32).

It should be noted here that, if each phase undergoes the isochoric process (60), the compressibility effect drops. Hence,

$$\frac{\partial j}{\partial z} = \Gamma_g \frac{\Delta \rho}{\rho_g \rho_f} \qquad \text{(III1.53)}$$

4. In Terms of Kinematic Wave Velocity

Under the same assumption, i.e., the isochoric process for each phase, Eq. (III1.50) can be expressed by j and V_{gj} in the following form

$$\frac{\partial \alpha}{\partial t} + j \frac{\partial \alpha}{\partial z} + \frac{\partial}{\partial z}(\alpha V_{gj}) = \frac{\rho_m \Gamma_g}{\rho_f \rho_g} \qquad \text{(III1.54)}$$

Furthermore, if the vapor drift velocity can be considered as a function of the void fraction only (32), then we have, from Eq. (III1.54),

$$\frac{\partial \alpha}{\partial t} + \left\{ j + V_{gj} + \alpha \frac{\partial V_{gj}}{\partial \alpha} \right\} \frac{\partial \alpha}{\partial z} = \frac{\rho_m \Gamma_g}{\rho_f \ \rho_g} \qquad \text{(III1.55)}$$

Following the analysis of (32), we define the kinematic wave velocity C_k by

$$C_k \equiv j + V_{gj} + \alpha \frac{\partial V_{gj}}{\partial \alpha} \qquad \text{(III1.56)}$$

Hence, upon substitution of Eq. (III1.56) into Eq. (III1.55), we obtain the void propagation equation

$$\frac{\partial \alpha}{\partial t} + C_k \frac{\partial \alpha}{\partial z} = \frac{\rho_m \Gamma_g}{\rho_f \ \rho_g} \qquad \text{(III1.57)}$$

Under the condition of constant ρ_g and ρ_f, we can express Eq. (III1.57) in terms of the mixture density, as follows

$$\frac{\partial \rho_m}{\partial t} + C_k \frac{\partial \rho_m}{\partial z} = - \Gamma_g \frac{\Delta \rho \rho_m}{\rho_g \rho_f} \qquad \text{(III1.58)}$$

which is called the density propagation equation.

In most two-phase flow systems, the change of the drift velocity with respect to α is relatively small (32), thus, if we neglect the term $\alpha \frac{\partial V_{gj}}{\partial \alpha}$ in Eq. (III1.56), we get

$$C_k \approx j + V_{gj} \qquad \text{(III1.59)}$$

under the condition

$$j + V_{gj} \gg \frac{\partial V_{gj}}{\partial \alpha} \qquad \text{(III1.60)}$$

With Eq. (III1.59) the density propagation equation becomes

$$\frac{\partial \rho_m}{\partial t} + \left(j + V_{gj} \right) \frac{\partial \rho_m}{\partial z} \doteq - \Gamma_g \frac{\Delta\rho \, \rho_m}{\rho_g \rho_f} \qquad \text{(III1.61)}^*$$

III-2. Dimensionless Groups and Similarity Criteria

In Appendix C-2 we developed the similarity groups based on the one-dimensional two-phase flow model, i.e., Eq. (III1.1), (III1.2'), (III1.3), and (III1.4).

Under the assumption that all of the constitutive relations have similar functional forms for two different systems, the requirements for their similarity are:

I) That each of the eight dimensionless groups, i.e., N_{pch}, N_d, N_ρ, N_f, N_{Fr}, N_s, N_{St}, and N_{Ec}, has the same value in system I and II.

II) The dimensionless form of the boundary and initial conditions be identical in system I and II.

Following the scaling of Appendix C-2, i.e., Eq. (C2.1), the four

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[*] This form can also be obtained from the mixture continuity equation (III1.41) with a different assumption. By substituting Eq. (III1.34) into (III1.41), we have

$$\frac{\partial \rho_m}{\partial t} + \left(j + V_{gj} \right) \frac{\partial \rho_m}{\partial z} = - \Gamma_g \frac{\Delta\rho \, \rho_m}{\rho_g \rho_f} + \left(\rho_f - \rho_m \right) \frac{\partial V_{gj}}{\partial z} \qquad \text{(III1.62)}$$

Under the condition that the vapor velocity is much larger than V_{gj} or $\frac{\partial V_{gj}}{\partial z} \approx 0$, we can neglect the last term of the right hand side of Eq. (III1.62).

field equations in III-1 can be expressed by the dimensionless variables with the above similarity groups as follows:

From the mixture continuity equation

$$\frac{\partial \rho_m^*}{\partial t^*} + \frac{\partial \rho_m^* v_m^*}{\partial z^*} = 0 \qquad \text{(III2.1)}$$

From the continuity equation for the vapor phase

$$\frac{\partial \alpha \rho_g^*}{\partial t^*} + \frac{\partial \alpha \rho_g^* v_m^*}{\partial z^*} = N_{pch} \Gamma_g^* - N_d \frac{\partial}{\partial z^*} \left\{ \frac{\alpha \rho_g^* \rho_f^*}{\rho_m^*} V_{gj}^* \right\} \qquad \text{(III2.2)}$$

From the equation of motion for the mixture

$$\rho_m^* \left\{ \frac{\partial v_m^*}{\partial t^*} + v_m^* \frac{\partial v_m^*}{\partial z^*} \right\} = -\frac{\partial P_m^*}{\partial z^*} - N_f \rho_m^* v_m^{*2} + \qquad \text{(III2.3)}$$

$$+ \frac{1}{N_{Fr}} \rho_m^* + N_s f_\sigma^* - N_\rho N_d^2 \frac{\partial}{\partial z^*} \left\{ \frac{\alpha}{1-\alpha} \frac{\rho_g^* \rho_f^*}{\rho_m^*} V_{gj}^{*2} \right\}$$

From the energy equation for the mixture

$$\rho_m^* \left\{ \frac{\partial i_m^*}{\partial t^*} + v_m^* \frac{\partial i_m^*}{\partial z^*} \right\} = N_{St}\, q_w^* + N_{Ec} \left[\Phi_{ce}^* + \frac{N_f}{4} \Phi_{de}^* \right] \qquad \text{(III2.4)}$$

$$- \left\{ \frac{N_\rho}{N_\rho \alpha_0 + (1-\alpha_0)} \right\} N_d \frac{\partial}{\partial z^*} \left\{ \frac{\alpha \rho_g^* \rho_f^*}{\rho_m^*} \Delta i_{fg}^* V_{gj}^* \right\}$$

Here $N_{pch} = \dfrac{\Gamma_{g0}\, \ell}{\rho_{g0} v_{m0}}$ phase change number (III2.5)

$N_d = \dfrac{\rho_{f0} V_{gj0}}{\rho_{m0} v_{m0}}$ drift number

$$N_\rho = \frac{\rho_{gc}}{\rho_{fo}}$$ density ratio

$$N_f = f_m/2D^*$$ friction number

$$N_{Fr} = \frac{v_{mo}^2}{g\ell}$$ Froude number

$$N_s = \frac{\bar{m}\,\sigma_i\,\bar{H}\,\ell}{\rho_{mc}\,v_{mc}^2}$$ surface number

$$N_{St} = \frac{q''_{wo}\,\xi\,\ell\,/A}{\rho_{mc}\,v_{mc}\,\Delta i_{fgc}}$$ Stanton number

$$N_{Ec} = \frac{v_{mo}^2}{\Delta i_{fgc}}$$ Eckert number

Thus the requirements for two different systems I and II to be similar are:

a) The kinematic, dynamic, and energy similarity conditions require

$$N_{pchI} = N_{pchII} \tag{III2.6}$$

$$N_{dI} = N_{dII} \tag{III2.7}$$

$$N_{\rho I} = N_{\rho II} \tag{III2.8}$$

$$N_{fI} = N_{fII} \tag{III2.9}$$

$$N_{FrI} = N_{FrII} \tag{III2.10}$$

$$N_{sI} = N_{sII} \tag{III2.11}$$

$$N_{StI} = N_{StII} \tag{III2.12}$$

$$N_{EcI} = N_{EcII} \tag{III2.13}$$

b) The initial and boundary conditions which, for example, can be stated by

$$\text{at} \quad z^* = 0 \qquad t^* \geq 0 \tag{III2.14}$$

$$\alpha_{I} = \alpha_{II} \sim \alpha_0$$

$$v^*_{m\,I} = v^*_{m\,II}$$

$$P^*_{m\,I} = P^*_{m\,II}$$

$$i^*_{m\,I} = i^*_{m\,II}$$

c) All of the constitutive functional forms including the equations of state are the same between system I and system II.

In practice, the above requirements usually cannot be satisfied completely due to the large number of conditions imposed. This difficulty is similar to that frequently exhibited in the case of the modeling problems of ships, rocket engines, etc. Thus, it is important to know under what conditions some of the similarity criteria can be relaxed. Let us discuss the cases which occur frequently and are useful for both modeling and analysis.

Homogeneous Flow Model

If the drift number N_d is much smaller than the phase change number N_{pch}

$$N_d = \frac{\rho_{fo} V_{gjo}}{\rho_{mo} v_{mo}} \ll N_{pch} = \frac{\Gamma_{go} l}{\rho_{go} v_{mo}} \tag{III2.15}$$

then the system is reaction (phase change) controlled and the drift or diffusion of the mass is negligible in view of the continuity equation for the vapor phase, Eq. (III2.2). On the other hand, if the drift number N_d is much smaller than unity, then all of the drift terms can be neglected, i.e., the last terms of Eq. (III2.2), (III2.3), and (III2.4) can be dropped from the field equations.

Thus, the condition for the homogeneous flow model is

$$N_d \ll 1 \tag{III2.16}$$

Drift Stress Tensor

If Eq. (III2.16) does not hold, the effect of the relative motions should be included in the analysis. The mass and the energy drift, the last terms of Eq. (III2.2) and (III2.4), respectively, are the order of N_d; therefore, these two terms cannot be neglected unless Eq. (III2.16) is satisfied. In contrast to the above two terms, the drift stress term in Eq. (III2.3) is the order of $N_\rho N_d^2$. The density ratio N_ρ is always smaller and frequently much smaller than unity. Hence, under the condition that

$$N_\rho N_d^2 \ll 1 \tag{III2.17}$$

we can neglect the drift stress term in Eq. (III2.3)

The above order of magnitude analysis only holds where $\alpha/1-\alpha \sim 1$, which excludes the flow with high void fraction, although the similarity of the dynamic effect of the relative motion in two different systems is governed by this group.

In order to examine the magnitude of this term accurately, it should be replaced by $\frac{\alpha_c}{1-\alpha_c} N_\rho N_d$. This is a consequence of the fact that α is a dimensionless variable. It should be noted here that this group is important,* because it scales the dynamic effect of relative motion on the interface.

Capillary Force

In many analyses of boiling two-phase flow, the capillary (body) force has been neglected heretofore. The scaling of the capillary (body) force is given by the dimensionless group N_s, the surface number of Eq. (C3.7). If the order of N_s is much smaller than unity, this effect may be neglected. Thus, the capillary force can be neglected if

$$N_s = \frac{\bar{m}\,\sigma_c\,\bar{H}}{\rho_{mc}\,v_{mc}^2/\ell} \ll 1 \tag{III2.18}$$

In view of Eq. (C2.11), (C2.12), and (C3.7), such is the case when

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*Using the constitutive equations for V_{gj} of (32), we have for a churn turbulent bubbly flow regime

$$\frac{\alpha_o}{1-\alpha_o} N_\rho N_d^2 = \frac{\rho_g}{\rho_f}\,\frac{2\sqrt{\sigma\Delta\rho g}}{\rho_f v_{fi}^2}\,\frac{\alpha_o}{1-\alpha_o} \sim 10^{-3} \qquad (\alpha_o \sim 0.5)$$

on the other hand, for an annular flow

$$\frac{\alpha_o}{1-\alpha_o} N_\rho N_d^2 = \frac{560(1-x)}{N_{Res}}\,\frac{\alpha_o}{1-\alpha_o} \sim 0.1 \qquad (\alpha_o \sim 0.05)$$

where these numbers are obtained for the water at 1000 psi, v_{fi} = 3 ft/sec, D = 0.02 ft. The above results show the importance of the relative motion on the dynamics in an annular flow regime in comparison with a churn turbulent bubbly flow regime.

the system pressure approaches the near critical value or the number of the bubbles in the forced convection system is relatively small. On the other hand, if the velocity of the mixture is small and the bubble density is large, the capillary force may be important and it cannot be neglected.

Compressibility Effect

The compressibility effect (or pressure wave), i.e., the second term of Eq. (III2.4), has an order of magnitude of N_{Ec}, the two-phase Eckert number. Physical meaning of this number is that it represents the ratio of the kinetic energy to the latent energy of phase change. Thus, if the velocity of mixture is small when compared to the sound velocity, the effect of the compressibility can be neglected. The mathematical condition for this is

$$N_{Ec} = \frac{v_{mc}^{2}}{\Delta i_{fgo}} \ll 1 \qquad \text{(III2.19)}$$

With Eq. (III2.19) satisfied, the constant fluid property condition in the mixture leads to the decoupling of the energy and continuity equation from the momentum equation. This is one of the most important and widely used tools to attack two-phase flow problems.

On the other hand, if the mixture velocity approaches the acoustic wave velocity, or if a system with a long duct operates under a low, reduced pressure, this compressibility effect may not be neglected.

The Energy Dissipation

In view of the mixture energy equation (III2.4), we can see that the order of the magnitude of the energy dissipation term is $N_{Ec}N_f/4$. The order of the friction number N_f for most two-phase flows can be considered as unity; therefore, the dissipation effect can be treated in a way similar to the compressibility effect. Furthermore, in forced convection boiling systems, the energy input is quite large; thus, under the condition that

$$\frac{N_{Ec}}{N_{St}} = \frac{(\rho_{mo} v_{mo} A)\, v_{mo}^2}{q_w'' \xi \ell} \ll 1 \qquad \text{(III2.20)}$$

both the compressibility and the dissipation effects can be neglected.

CHAPTER IV

FORMULATION OF THE DYNAMIC PROBLEM

IV-1. Description of the System and Approximations

A. Thermodynamic Process

In order to understand the mechanism of the thermally induced flow oscillations and to formulate the mathematical model which describes the real physical system, it is necessary to examine the thermodynamic processes and the flow characteristics of the system. The typical components of the system of interest are shown in Fig. 1. They consist basically of four different regions (A), (B), (C), and (D).

(A)	Upstream Un-heated Region	Single-phase
(B)	Heated Liquid Region	
(C)	Heated Mixture Region	Two-phase
(D)	Downstream Un-heated Region	

The system of interest extends to the components where the thermally induced flow instabilities can be affected in a systematic manner. If the system consists of a single heated channel without any bypass, Fig. 2, a large number of the components in the loop should be considered. This is because any disturbance propagating in the loop has a definite functional relation in terms of time and space lag to the heated section and, therefore, it can influence the stability of the system. On the other hand, if the heated section consists of multiple channels which converge

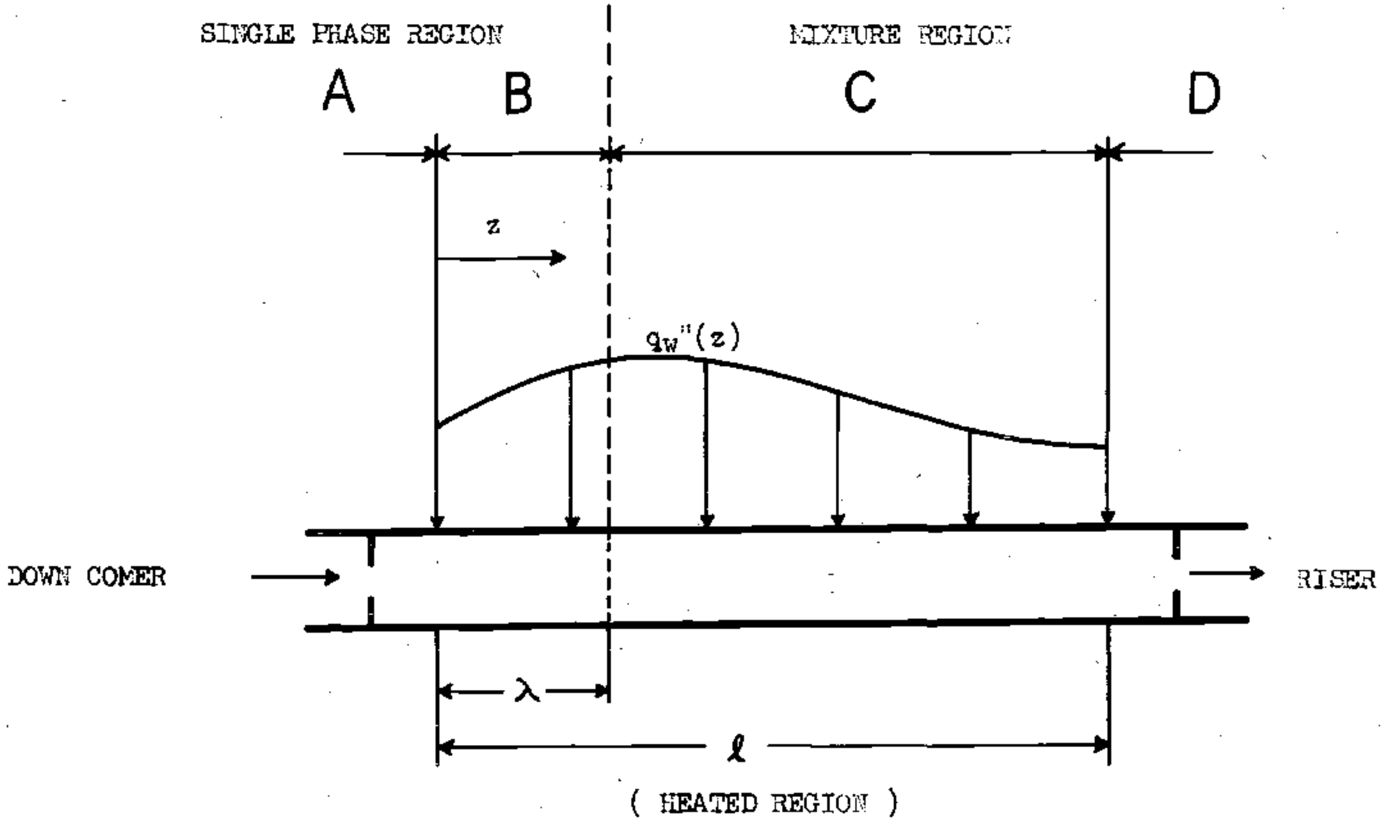

Figure 1. The System

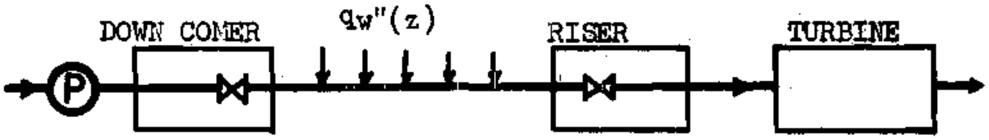

Figure 2. Single Heated Channel System

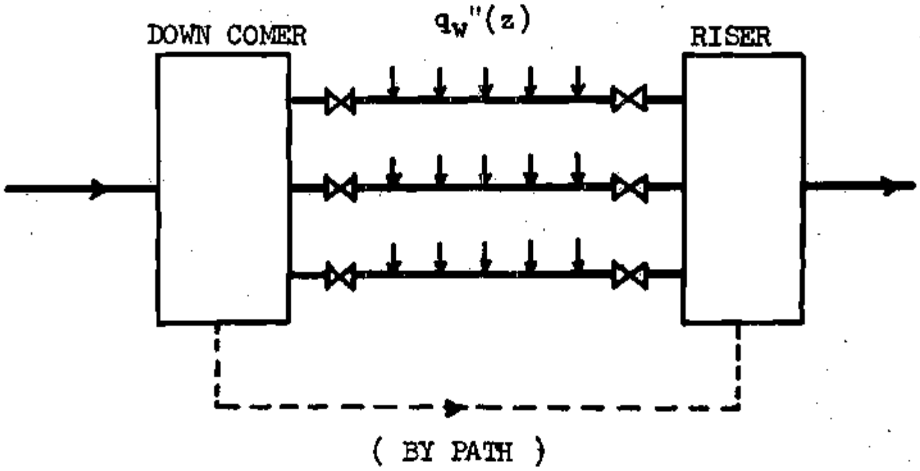

Figure 3. Multiple Heated Channel System

to one channel at the riser and down comer with sufficiently large volume capacitance, Fig. 3., it is sufficient to consider the system between these two volume reservoirs. In this case, the systematic form of disturbance propagation is effectively insulated from the remaining portion of the system, i.e., from the pump and the turbine, etc.

The first case of a single heated channel system is, therefore, far more complicated than the second case of a multi-channel system and requires information on the dynamic responses of the pump, turbine, etc. However, in view of practical applications such as nuclear reactors, conventional power plants with boiling, etc., it is sufficient to consider the second case.

For generality, we imagine two volume capacitances which can insulate any systematic propagation of disturbances. Then the system between these capacitances will be analyzed. This system may or may not include such accessories as the down-comer, riser, pump, and turbine depending on the positions of the volume reservoirs. The subdivision of the system will follow the previous four region approximations, namely, upstream unheated (A), heated liquid (B), heated mixture (C), and downstream unheated (D) regions. The thermodynamic process starts with the subcooled fluid having enthalpy i_1, entering the heated duct (B) with the velocity v_{fi}. As the energy is being transferred from the heated wall to the fluid, the temperature and enthalpy, i_f, will increase. Due to the developments of the thermal boundary layer or the superheat capacitance of the liquid, the boiling boundary may not coincide with the point where the bulk liquid enthalpy reaches the saturation value i_{fs}. In our analysis, we shall assume thermal equilibrium between the phases, which is a reasonable assumption at high pressures. Thus we take the boundary between (B) and (C) at $z = \lambda$ where $i_f = i_{fs}$.

In region (C), the phase change takes place and the mixture enthalpy i_m, void fraction α increases toward the end of (C) as more heat is added to the fluid. For a number of systems of practical interest, particularly for systems at high reduced pressures, it is reasonable to assume that this process takes place at an approximately constant pressure, since the pressure drop is relatively small compared with the absolute pressure of the system.

The processes, in both (A) and (D), i.e., the un-heated regions, can be considered as isenthalpic. Furthermore, the complete liquid phase occupies the upstream un-heated region (A). Thus, the assumption of incompressibility is appropriate. On the other hand, in the downstream unheated region (D), the mixture of vapor and liquid enters either into the steam separator or into the volume capacitance directly. From the previous discussion, either of the system components can be considered as the end of the system of interest. Therefore, from this point any systematic wave propagation will be neglected. Furthermore, we assume that the pressure drop effect on the mixture properties in region (D) is negligibly small, thus we treat the mixture in (D) as isocholic (60).

The assumption that the effect of pressure variation on the thermodynamic properties of each phase can be neglected is implied by the assumption that the density is a function of enthalpy only and not of both enthalpy and pressure. This condition will be valid if the two-phase

Eckert number is smaller than unity. From this limitation it can be seen that the assumption holds only if the rate of propagation of disturbance is much slower than the velocity of the pressure wave, otherwise these two waves interact with each other. Thus, under this assumption, only low frequency oscillation can be analyzed.

B. Transport Process

The transport processes of mass, momentum, and energy are best understood from the field equations with appropriate constitutive relations (61). In our analysis the field equations given in III-1 will be used and some further simplifications on this one-dimensional model will be made. Now, let us examine the transport process in the four different regions separately.

In the upstream un-heated region (A), the liquid flows incompressibily and isenthalpically, hence, only the momentum transport is important. The kinematics of the fluid (velocity field) can be obtained directly from the continuity equation. Knowing the velocity field, the acceleration, gravity, frictional, valve, and orifice pressure drops can be obtained.

Since, in the liquid heated region (B) the fluid is still considered to be incompressible, the velocity field can be immediately obtained from the conservation of mass. The boundary between (B) and (C) will be calculated from the energy equation under thermal equilibrium conditions and by neglecting the pressure and dissipation effect on the liquid enthalpy in comparison with the heat input. The pressure drops can be calculated by the same method as used in (A).

In the heated mixture region (C), the transport processes are complicated. Following the discussion of the thermodynamic process, we assume that the density is a function only of the enthalpy and not both the enthalpy and pressure. This important conclusion first obtained by Teletov and Serov (13) permits us to decouple the momentum equation from the continuity and energy equation. Thus, again we can proceed with the integration of the latter two equations and obtain the kinematics of the fluid independent of the dynamic conditions imposed by the momentum equation. By knowing the density and velocity of the mixture, the various pressure drops may be calculated. For simplicity and because of the lack of knowledge on the constitutive equation, we also assume that the capillary body force is negligible which implies that $N_\sigma \ll 1$.

From the previous discussion, we assume that the mixture in the downstream un-heated region (D) is isochoric and isenthalpic. It should be noted that this does not imply constant density, since the mixture entering the region (D) may have different densities depending on the upstream disturbances. The kinematics of the fluid can be obtained from the mass conservation law whereas the pressure drop will be calculated in a way similar to that used in the boiling region. Furthermore, the pressure drop at the orifice or valve will be calculated through the orifice coefficients.

C. Time Lag and Space Lag

It is useful to consider the process from the Eulerian and Lagrangian points of view. If we follow a fluid particle entering section (A) at τ_0 and study the change of its properties from the mass center of that particle in the time coordinate, our observation is from the Lagrangian or particle coordinate, Fig. 4. The center of mass enters in regions (B), (C), and (D) at time τ_1, τ_2, and τ_3, respectively, and leaves region (D) at τ_4. The transition from (B) to (C), i.e., the inception of bulk boiling, occurs when the liquid enthalpy increases due to the heat input and reaches the saturation liquid enthalpy. In region (C), the enthalpy of the mixture continually increases due to the evaporation. The residence time of the particle in region (B) is denoted by τ_{12}, and the total residence time in the heated section by τ_{13}. In a similar manner, we can also define τ_{01} and τ_{34}. Since all of these residence times express the time necessary to bring about specific changes in the position or properties, it will be called the time lag. Except in the case when the inlet enthalpy is disturbed, the time lag τ_{01} has no significant physical meaning other than that it represents the residence time, since the disturbances of the velocity, pressure, etc. propagate with infinite velocities. On the other hand, the time lags τ_{12}, τ_{13}, and τ_{34} have significant relations with the propagation of disturbances, and hence they are important for the stability analysis.

It is of interest also to consider the spatial or Eulerian description of the process, Fig. 5. In this case the time lags are replaced by the space lags and hence they indicate the boundaries of the various operational regions. Among the indicated four space lags, the one corresponding to the time lag τ_{12} and denoted by λ is particularly important, since it defines the boundary between the liquid region and the mixture region, and it is only obtainable by solving the conservation equations. Other space lags are given by the geometrical description of the system.

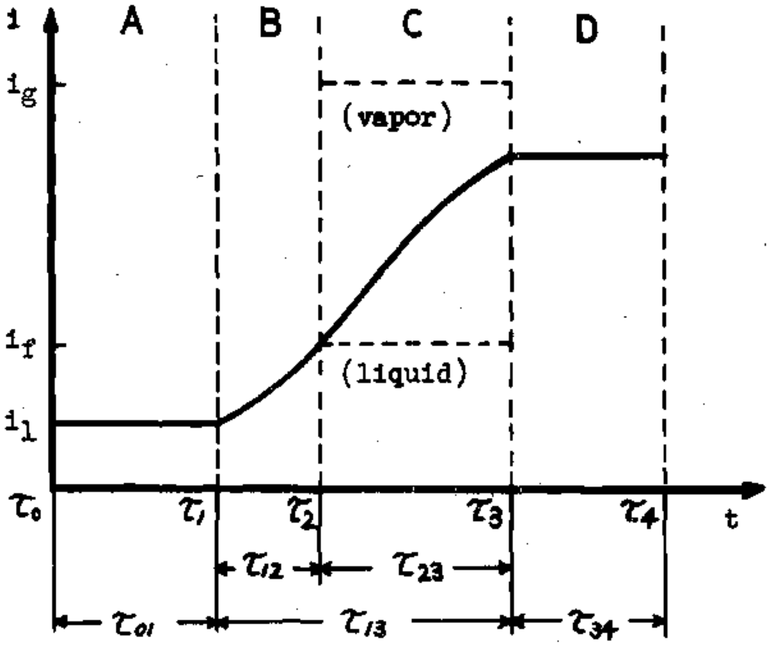

Figure 4. Lagrangian Description of Enthalpy

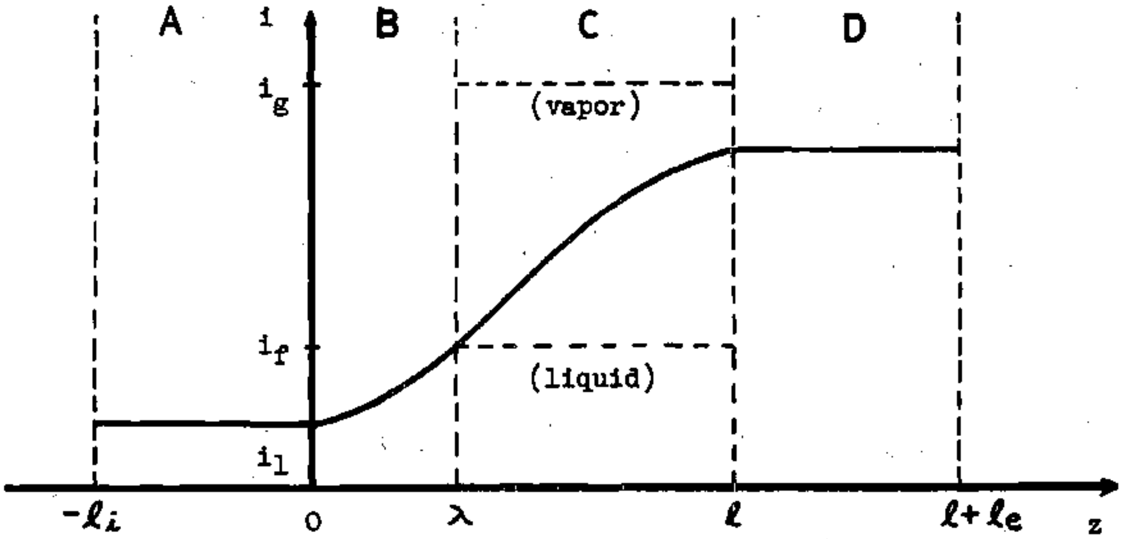

Figure 5. Eulerian Description of Enthalpy

In a dynamic analysis, the fluctuations of space lag λ are important, since they can be considered as the source of a fluid undergoing a phase change. By changing the position of this source, the mixture enthalpy, density, and the pressure drop will fluctuate, thus we have a generation and propagation of waves created at λ due to its fluctuation.

IV-2. Governing Equations for the Upstream Un-heated Region (A)

Following the discussion in IV-1, we assume that the liquid density is a function only of the system pressure P_s which can be considered to be uniform throughout the system. Thus

$$\rho_f = \rho_f (P_s) = \text{const.} \tag{IV2.1}$$

Without a loss of generality, we can assume that this region (A) consists of a constant area duct and a pressure drop device such as an orifice. In this case, from the continuity equation and Eq. (IV2.1), we have

$$v_{fc} = \left(\frac{Ac}{Ao}\right) v_{fi}(t) \tag{IV2.2}$$

Here Ac and Ao denote the flow area for the heated section and upstream un-heated region (A).

The equation of motion for the duct is

$$-\frac{dP}{dz} = \rho_f \left\{ \frac{\partial v_{fo}}{\partial t} + v_{fc} \frac{\partial v_{fc}}{\partial z} + g_c + \frac{f_c}{2D_o} v_{fo}^2 \right\} \tag{IV2.3}$$

Assuming that the position of the orifice or valve is at the entrance to the heated region, we have

$$\Delta P_i = k_i \rho_f v_{fi}^2 \tag{IV2.4}$$

The friction factor fo can be obtained from the Reynolds number and the roughness parameters of the duct. Therefore,

$$f_o = f_o (\rho_f , v_{fo} , \mu_f , \varepsilon_o , D_o) \tag{IV2.5}$$

In addition to the above equations, here we impose the isenthalpic condition, thus

$$\frac{Di_{fo}}{Dt} = 0 \tag{IV2.6}$$

IV-3. Governing Equations for the Heated Liquid Region (B)

By neglecting the velocity and enthalpy covariant terms, axial conduction, normal stress, pressure, and the dissipation effects on the enthalpy, we obtain the following three conservation equations.

$$\frac{\partial \rho_f}{\partial t} + \frac{\partial \rho_f v_f}{\partial z} = 0 \tag{IV3.1}$$

$$\frac{\partial i_f}{\partial t} + v_f \frac{\partial i_f}{\partial z} = \frac{q_w' \xi}{\rho_f A_c} \tag{IV3.2}$$

and

$$-\frac{\partial P}{\partial z} = \rho_f \left\{ \frac{\partial v_f}{\partial t} + v_f \frac{\partial v_f}{\partial z} + g + \frac{f_s}{2D} v_f^2 \right\} \tag{IV3.3}$$

The constitutive equation of state is given by

$$\rho_f = \rho_f\ (P_s) = \text{const.} \tag{IV3.4}$$

Equations (IV3.1,2,3, and 4) specify the four variables P, ρ_f, v_f, and i_f in the heated liquid region (B). The friction factor f_s should be given by the constitutive relation,

$$f_s = f_s\ (P_s,\ v_f,\ \mu_f,\ \epsilon_s,\ D) \tag{IV3.5}$$

These four equations can be solved, if the heat flux q_w'' is a known function of independent variables z and t. Although the wall has some heat capacity which can influence the heat input to the fluid, we assume that it is small compared to the heat capacity of the fluid. This enables us to neglect the entire wall effect on the heat transfer, if the heat source generates a steady flux. Thus we take

$$q_w'' = q_w''(z) \tag{IV3.6}$$

Assuming that P_s, ξ/Ac, ϵ, and D are known and constant, we have six dependent variables, ρ_f, v_f, i_f, q_w'', P, and f_s, whereas the equations describing the system are three field equations (IV3.1,2,3) and three constitutive equations (IV3.4,5,6). Thus the total number of unknowns and of equations is the same. Consequently, the formulation is mathematically consistent.

IV-4. Governing Equations for the Heated Mixture Region (C)

Field Equations

In order to take into account the relative motions of each phase, it is useful to formulate the problem in terms of mixture field equations with some diffusional equations. This has been done in Chapter III. By neglecting the velocity and enthalpy covariant terms, axial conduction, normal stress, capillary force, pressure, and dissipation effects on the enthalpy, Eq. (III1.1, 2', 3, and 4) reduce to those reported in (1). Thus we obtain

the continuity equation of the mixture

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m v_m}{\partial z} = 0 \tag{IV4.1}$$

the continuity equation for the vapor

$$\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial}{\partial z}(\alpha \rho_g v_m) = \Gamma_g - \frac{\partial}{\partial z}\left\{\frac{\alpha \rho_g \rho_f}{\rho_m} V_{gj}\right\} \tag{IV4.2}$$

the energy equation for the mixture

$$\rho_m \left\{\frac{\partial i_m}{\partial t} + v_m \frac{\partial i_m}{\partial z}\right\} = \frac{q_w'' \xi}{A_c} - \frac{\partial}{\partial z}\left\{\frac{\alpha \rho_g \rho_f}{\rho_m} V_{gj}\, \Delta i_{fg}\right\} \tag{IV4.3}$$

and the momentum equation for the mixture

$$-\frac{dP}{dz} = \rho_m \left\{\frac{\partial v_m}{\partial t} + v_m \frac{\partial v_m}{\partial z}\right\} + g \rho_m + \frac{f_m}{2D} \rho_m v_m^2 + \frac{\partial}{\partial z}\left\{\frac{\rho_f - \rho_m}{\rho_m - \rho_g}\, \frac{\rho_g \rho_f}{\rho_m} V_{gj}^2\right\} \tag{IV4.4}$$

Here we assumed that

$$N_s \text{ (surface number)} \ll 1 \tag{IV4.5}$$

$$N_{Ec} \text{ (two-phase Eckert number)} \ll 1 \tag{IV4.6}$$

Constitutive Equations

In addition to the above four field equations (IV4.1,2,3, and 4), we should specify the constitutive relations appearing in Chapter III. Following the discussion of the thermodynamic process (see IV-1), we assume thermal equilibrium between two phases. Furthermore, the fluid properties of both vapor and liquid phases can be assumed to be a function only of the system pressure P_s which is uniform throughout the system. This implies that enthalpy and density of each phase are given constant once the system pressure P_s is specified. Hence, the thermal and calorical equation of state can be described as

$$\rho_f = \rho_f(P_s) \tag{IV4.7}$$

$$\rho_g = \rho_g(P_s) \tag{IV4.8}$$

and

$$\rho_m = \rho_m(P_s, \alpha) = \alpha \rho_g + (1-\alpha)\rho_f \tag{IV4.9}$$

$$i_f = i_f(P_s) \tag{IV4.10}$$

and

$$i_g = i_g(P) \tag{IV4.11}$$

$$i_m = i_m(P_s, \alpha) = \frac{\alpha \rho_g i_g + (1-\alpha)\rho_f i_f}{\rho_m} \tag{IV4.12}$$

In addition to the above relations, we need

the constitutive relation for the relative motion (32)

$$V_{gj} = f_2 \tag{IV4.13}$$

the reological constitutive equation

$$f_m = f_z \tag{IV4.14}$$

and the boundary condition for heat flux

$$q_w'' = q_w''(z) \tag{IV4.15}$$

Assuming that P_s, ξ/Ac, D, and g are known, the dependent variables are ρ_m, v_m, α, P, f_m, ρ_g, Γ_g, ρ_f, V_{gj}, i_m, q_w'', i_{gs}, and i_{fs}. The equations describing the system are the four field equations (IV4.1,2,3, and 4) and the nine constitutive equations (IV4.7,8,9,10,11,12,13,14, and 15). Thus, the total number of unknowns and equations is the same.

It should be noted here that velocities of each phase v_g and v_f do not appear in this formulation explicitly. However, they are related to the mixture velocity and the vapor drift velocity by Eq. (III1.12) and Eq. (III1.3), as it has been explained in Section III-1.C.

IV-5. Governing Equations for the Downstream Un-heated Region (D)

In this region the mixture is treated as an isochoric fluid. Furthermore, without loosing generality, we can assume that this region (D) consists of a constant area duct and a pressure drop device such as an orifice. The thermal equation of state is given by

$$\frac{D\rho_{me}}{Dt} = 0 \tag{IV5.1}$$

And thus the continuity equation becomes

$$v_{me} = \left(\frac{A_c}{A_e}\right) v_m(\ell, t) \tag{IV5.2}$$

Here A_e is the flow area of the duct in the region (D).

The equation of motion for the duce is given by

$$-\frac{dP}{dz} = \rho_{me}\left\{\frac{\partial v_{me}}{\partial t} + v_{me}\frac{\partial v_{me}}{\partial z}\right\} + g_e\rho_{me} + \frac{f_{me}}{2D_e}\rho_{me}v_{me}^2 \tag{IV5.3}$$

$$+\frac{\partial}{\partial z}\left\{\frac{\rho_f - \rho_{me}}{\rho_{me} - \rho_g}\,\frac{\rho_f\rho_g}{\rho_{me}}\,V_{gj}^2\right\}$$

For the orifice, assuming that it is on the exit of the heated region, we have

$$\Delta P_e = k_e\,\rho_m(\ell, t)\,v_m^2(\ell, t) \tag{IV5.4}$$

The friction factor f_{me} can be given by the equation similar to Eq. (IV4.14)

$$f_{me} = f_3 \tag{IV5.5}$$

And the vapor drift velocity from Eq. (IV4.13)

$$V_{gj} = f_2 \tag{IV5.6}$$

In addition to the above equation, we have an isenthalpic condition, thus

$$\frac{Di_{me}}{Dt} = 0 \qquad \text{(IV5.7)}$$

IV-6. Method of Solution

The dynamic problem of our system of interest has been formulated by considering four different regions, i.e., the upstream un-heated, heated liquid, heated mixture, and downstream un-heated regions. In order to obtain solutions for the system, boundary and/or initial conditions should be given. Recalling the objectives of the present analysis, i.e., to obtain the transient response and the stability criteria of the system, there are several methods to approach the problem. For example, linear and nonlinear theories are available and could be used to obtain the solution. Furthermore, we have a choice of the disturbance which should be imposed on the system. As we can see from our formulation, the governing differential equations are nonlinear; thus, it will be expected that in case of an unstable operation the nonlinearity becomes important for large departures from the equilibrium state. Due to the presently limited knowledge on nonlinear partial differential equations, the general solution for our system by analytical means is almost inaccessible.

However, in most practical cases, the information on the stability boundary is much more important than the ultimate unstable dynamical response beyond the stability boundary. According to the Liapunov theorem (50), the stability of the linearized system corresponds to the stability of the nonlinear system which operates under quasi-equilibrium conditions. Thus, taking the advantage of this theorem, we will proceed by using the linear theory. In our analysis the disturbance will be given in the form of an inlet velocity perturbation following (19,22,25), though, for example, the perturbation of the inlet subcooling or of the heat flux is also permissible. Consequently, we impose the following boundary and initial conditions on the density, pressure, enthalpy, and velocity. By taking the origin of the z coordinate at the boundary between (A) and (B), the inlet of the heated region, we have

$$\rho_f = \rho_f\,(P_s) \qquad \text{at} \quad z = 0 \quad t \geqq 0 \qquad \text{(IV6.1)}$$

$$P = P_s = \text{const.} \qquad \text{at} \quad z = 0 \quad t \geqq 0 \qquad \text{(IV6.2)}$$

$$i_f = i_1 = \text{const.} \qquad \text{at} \quad z = 0 \quad t \geqq 0 \qquad \text{(IV6.3)}$$

$$v_f = v_{fi}(t) = \overline{v}_{fi} + \delta t(t) \qquad \text{at} \quad z = 0 \quad t \geqq 0 \qquad \text{(IV6.4)}$$

$$i_f = i_s(P_s) \qquad \text{at} \quad z = \lambda(t) \qquad \text{(IV6.5)}$$

Hence, the density ρ_f and the saturation liquid enthalpy i_{fs}, which are functions only of P_s, can be treated as constants. The steady state inlet velocity is denoted by $\overline{v}_{fi}$ and the perturbation on the velocity is given by $\delta v(t)$. In our analysis we use a frequency response method; therefore, the perturbation $\delta v(t)$ can be given by an exponential function.

$$\delta v(t) = \varepsilon e^{St} \qquad \text{(IV6.6)}$$

and

$$S = a + j\omega \qquad (j = \sqrt{-1}) \qquad \text{(IV6.7)}$$

Thus, S is a complex number, the real part gives the amplification coefficient of the particular oscillation mode, whereas the imaginary part represents the angular frequency ω. For the purpose of the linear perturbation analysis, we assume that $\varepsilon/\overline{v}_{fi}$ is much smaller than unity; therefore, ε is infinitesimal compared with finite $\overline{v}_{fi}$. In the following analysis, we shall retain the first-order terms in ε and neglect second and higher order terms.

Substitution of Eq. (IV6.5) into (IV6.4) yields

$$v_{fi} = \overline{v}_{fi} + \varepsilon e^{St} \qquad \text{at } z = 0 \qquad \text{(IV6.8)}$$

The schematic procedure for the analytic solution is given in Fig. 6. This diagram indicates the dramatic effects of decoupling the momentum equation from the continuity and energy equations. The kinematics of the fluid, i.e., the velocity field and the density variation, can be solved independently of the dynamics of the system. Recalling single phase potential flow theory, we find an analogy in attacking the two problems. As will be clarified in what follows, the only difference is that, in the potential flow analysis, the divergence of the velocity of the center of mass is zero, whereas in a two-phase boiling system, the divergence of the center of volume equals the volume source due to evaporation.

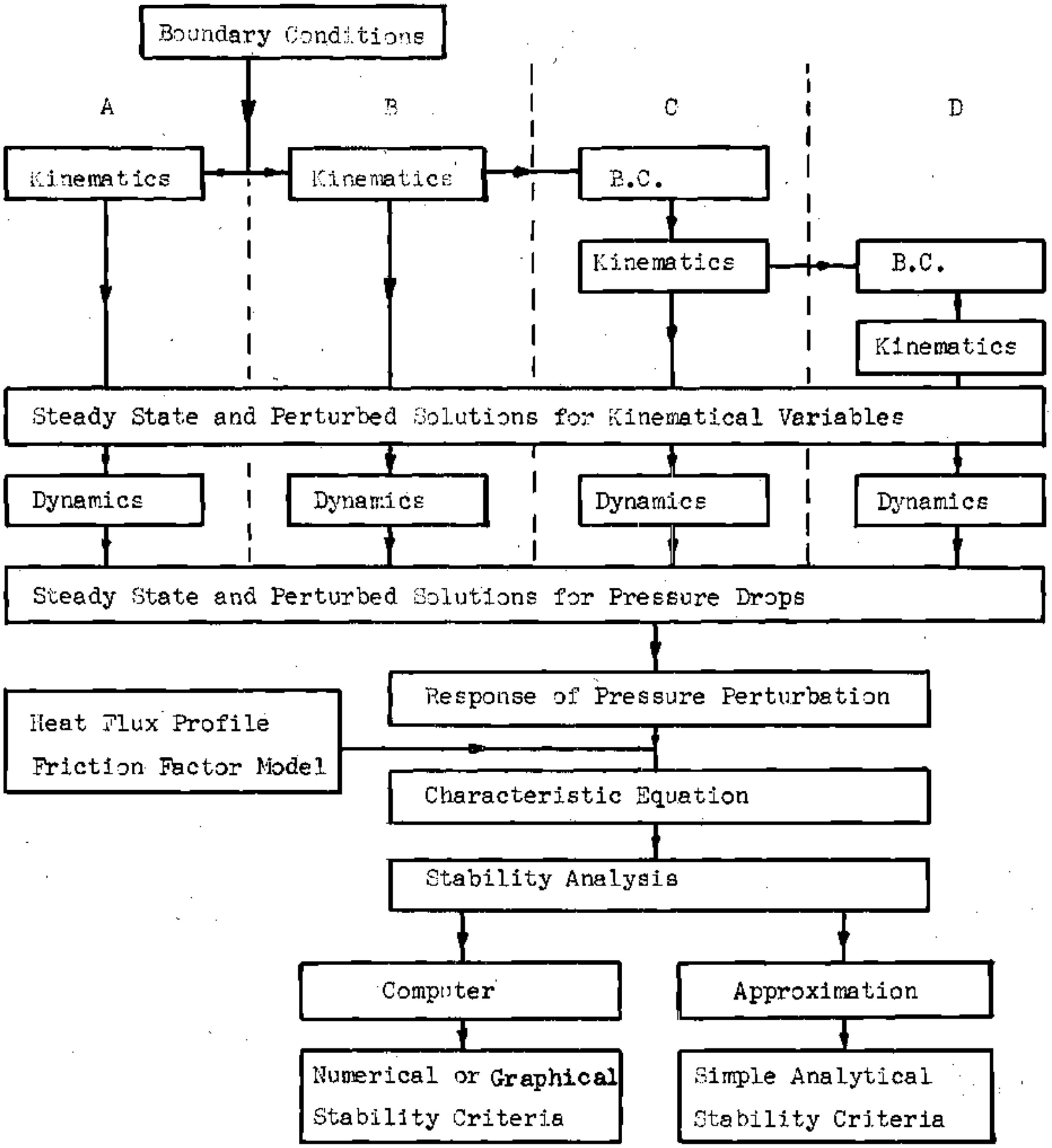

Figure 6. Schematic Procedure for Solution

CHAPTER V

KINEMATICS OF FLUID

V-1. Kinematics of Liquid Regions (A) and (B)

In the liquid region the density is constant, thus the velocity field can be solved immediately from the conservation of mass.

For region (A), from Eq. (IV2.2) and (IV6.7), we obtain

$$v_{fo}(t) = \left(\frac{A_c}{A_o}\right)\left[\,\bar{v}_{fi} + \varepsilon e^{st}\right] \qquad \text{(V1.1)}$$

Thus the steady state velocity $\bar{v}_{fo}$ is given by

$$v_{fo} = \left(\frac{A_c}{A_o}\right)\bar{v}_{fi} \qquad \text{(V1.2)}$$

and the perturbed part of v_{fo} by

$$\delta v_{fo}(t) = \left(\frac{A_c}{A_o}\right)\varepsilon e^{st} = \left(\frac{A_c}{A_o}\right)\delta v(t) \qquad \text{(V1.3)}$$

The density of the fluid is constant, hence

$$\rho_f = \rho_f\ (P_s) = \text{constant} \qquad \text{(V1.4)}$$

The enthalpy can be obtained from Eq. (IV2.6) and (IV6.3)

$$i_{fo} = i_1 \qquad \text{(V1.5)}$$

The results for the upstream un-heated region (A) show that the density and enthalpy are constant and the velocity v_{fo} is independent of position, it is a function of time only.

When solving for the kinematic variables in the heated liquid region (B), we recall that the position of the boundary between (B) and (C), i.e., the point $z = \lambda(t)$ where bulk boiling starts, is also unknown. The space lag λ will be found from the boundary condition (IV6.5) and the conservation equations of mass and energy. This can be done, because the differential equations describing the kinematics have the form of an initial value problem.

The continuity equation (IV3.1) with constant density reduces to

$$\frac{\partial v_f}{\partial z} = 0 \tag{V1.6}$$

Hence, upon integration with the boundary condition (IV6.7), we obtain

$$v_f(t) = v_{fi}(t) = \bar{v}_{fi} + \delta v(t) = \bar{v}_{fi} + \varepsilon e^{St} \tag{V1.7}$$

The velocity in region (B) is therefore a function only of time.

Knowing the velocity field, the energy equation (IV3.2) becomes

$$\frac{\partial i_f}{\partial t} + v_f(t)\frac{\partial i_f}{\partial z} = \frac{q_w'' \xi}{\rho_f A_c} \tag{V1.8}$$

This is a first-order partial differential equation whose solution can be obtained by means of characteristics. In order to proceed with the calculation, we define the average heat flux q_o'' by

$$q_o'' \equiv \frac{1}{\ell}\int_0^{\ell} q_w''(z)\,dz \tag{V1.9}$$

In view of the above equation

$$q_w''(z) = q_o''\, f(z) \tag{V1.10}$$

which defines the dimensionless heat flux distribution f(z). We assume that <u>f(z) is a continuously differentiable and positive function</u>. We also define the integral of f(z) by F(z), thus

$$F(z) \equiv \int_0^{z} f(z)\,dz \tag{V1.11}$$

The function F(z) has the dimension of length and gives the equivalent distance for the total amount of heat, if the heat flux is uniform q_o'' along the channel.

Substituting the normalized heat flux, i.e., Eq. (V1.9), into the energy equation (V1.8), we obtain

$$\frac{\partial i_f}{\partial t} + v_f(t)\,\frac{\partial i_f}{\partial z} = \frac{q_o''\,\xi}{A_c\,\rho_f}\, f(z) \tag{V1.12}$$

Equation (V1.12) is a first-order linear partial differential equation which can be transformed to Lagrange form (59).

$$dt = \frac{dz}{v_f(t)} = \frac{di_f}{\frac{q_o''\,\xi}{A_c\,\rho_f}\, f(z)} \tag{V1.13}$$

Equation (V1.13 can be solved through a parameter with a boundary condition

(IV6.3). Now the problem can be converted into two initial value problems by introducing the particle coordinate or the method of characteristics. By taking the first equality of Eq. (V1.13), we have

$$dt = \frac{dz}{v_f(t)} \tag{V1.14}$$

which specifies the particle path. The equality between the first and third terms gives

$$di_f = \frac{q_0'' \xi}{A_c \rho_f} f(z)\, dt \tag{V1.15}$$

The initial condition will be specified by letting a particle enter region (B) with enthalpy i_1, at time τ_1, thus,

$$\text{at} \quad t = \tau_1 \quad \text{and} \quad z = 0 \quad i_f = i_1 \tag{V1.16}$$

With this condition, one obtains, after integrating Eq. (V1.14) with Eq. (V1.7), the following relation

$$z = \bar{v}_{fi}(t-\tau_1) + \varepsilon e^{st}\frac{[1 - e^{-s(t-\tau_1)}]}{s} \tag{V1.17}$$

or in a perturbed form

$$z = \bar{z} + \delta z \tag{V1.18}$$

Here $\bar{z}$ corresponds to the steady state particle path and δz to the perturbed part. Using Eq. (V1.18) and expanding f(z) in a Taylor series and neglecting the second and higher order powers of ϵ we get,

$$f(z) = f(\bar{z} + \delta z) \approx f(\bar{z}) + \frac{df(\bar{z})}{d\bar{z}}\,\delta z \qquad \text{(V1.19)}$$

Upon substitution of Eq. (V1.19) into Eq. (V1.15), one obtains

$$di_f = \left[\frac{q_0''\xi}{A_c\rho_f}\right]\left\{ f(\bar{v}_{fi}[t-\tau_1]) + f'(\bar{v}_{fi}[t-\tau_1]) \times \epsilon e^{st}\left[\frac{1-e^{-s(t-\tau_1)}}{s}\right]\right\} dt \qquad \text{(V1.20)}$$

or simply

$$di_f = \left[\frac{q_0''\xi}{A_c\rho_f}\right] f(z(t,\tau_1))\,dt \qquad \text{(V1.21)}$$

The Time Lag and the Space Lag

Before solving for the enthalpy, i.e., the integration of Eq. (V1.20), let us examine the time lag τ_{12} and the space lag λ defined in Section IV-1. The time required to increase the enthalpy from i_1 to i_{fs} in the liquid heated region was τ_{12} and the corresponding distance was λ, thus we set at

$$t = \tau_2,\ z = \lambda(t) \text{ and } i_f = i_{fs}(P_s) = \text{constant} \qquad \text{(V1.22)}$$

The integration of Eq. (V1.21) with Eq. (V1.22) yields

$$\left[\frac{\Delta i_{12}\,A_c\,\rho_f}{q_0''\xi}\right] = \int_{\tau_1}^{\tau_2} f(z(t,\tau_1))\,dt \qquad \text{(V1.23)}$$

Here Δi_{12} is a subcooling of the liquid and τ_{12} is the residence time of the particle entering the inlet at $t = \tau_1$, hence

$$\Delta i_{12} = i_{fs} - i_1 \tag{V1.24}$$

$$\tau_2 = \tau_1 + \tau_{12}\ (\tau_1) \tag{V1.25}$$

The steady state solution for Eq. (V1.23) can be found by setting $\epsilon = 0$ in the original differential equation (V1.12) with Eq. (V1.22), thus one gets

$$\left[\frac{\Delta i_{12} A_c \rho_f \bar{v}_{fi}}{q_o'' \xi}\right] = \int_c^{\bar{\lambda}} f(z)\,dz \quad = F(\bar{\lambda}) \tag{V1.26}$$

From Eq. (V1.26), the steady state space lag $\bar{\lambda}$ is obtained by taking the inverse of function F

$$\bar{\lambda} = F^{-1}\left(\frac{\Delta i_{12} A_c \rho_f \bar{v}_{fi}}{q_o'' \xi}\right) \tag{V1.27}$$

and the steady state residence time τ_{12} is given by

$$\bar{\tau}_{12} = \frac{\bar{\lambda}}{\bar{v}_{fi}} \tag{V1.28}$$

On the other hand, the unsteady solution is given by

$$\frac{\Delta i_{12} A_c \rho_f}{q_o'' \xi} = \int_{\tau_1}^{\tau_1+\tau_{12}} f(\bar{v}_{fi}[t-\tau_1])\,dt + \int_{\tau_1}^{\tau_1+\tau_{12}} f'(\bar{v}_{fi}[t-\tau_1]\,\frac{\varepsilon[e^{st}-e^{s\tau_1}]}{s}\,dt \tag{V1.29}$$

By defining

$$\tau_{12}(\tau_1) = \bar{\tau}_{12} + \delta\tau_{12}(\tau_1) \qquad \text{(V1.30)}$$

and using the Taylor expansion, we can express the first integral of Eq. (V1.29) as

$$I_1 = \int_{\tau_1}^{\tau_1+\tau_{12}} f(\bar{v}_{fi}[t-\tau_1])\,dt = \frac{1}{\bar{v}_{fi}} F(\bar{v}_{fi}\tau_{12}) \qquad \text{(V1.31)}$$

$$= \frac{F(\bar{\lambda})}{\bar{v}_{fi}} + f(\bar{\lambda})\,\delta\tau_{12}(\tau_1)$$

By defining a new function g as

$$g(\eta, s) \equiv \int_0^{\eta} f'(\eta)\, e^{\frac{s\eta}{\bar{v}_{fi}}}\, d\eta \qquad \text{(V1.32)}$$

the second integration of Eq. (V1.29) can be expressed as

$$I_2 = \frac{\varepsilon e^{s\tau_1}}{s\,\bar{v}_{fi}} \left\{ g(\bar{v}_{fi}\tau_{12}, s) - \left[f(\bar{v}_{fi}\tau_{12}) - f(0) \right] \right\} \qquad \text{(V1.33)}$$

Neglecting the second and higher order of ε and using Eq. (V1.28), we get

$$I_2 = \frac{\varepsilon e^{s\tau_1}}{s\,\bar{v}_{fi}} \left\{ g(\bar{\lambda}, s) - \left[f(\bar{\lambda}) - f(0) \right] \right\} \qquad \text{(V1.34)}$$

Now $\delta\tau_{12}$ can be obtained by substituting Eq. (V1.31) and Eq. (V1.34) into (V1.29) thus

$$\delta\tau_{12}(\tau_1) = -\frac{1}{f(\bar\lambda)\,\bar v_{fi}}\left\{g(\bar\lambda, s) - [f(\bar\lambda) - f(0)]\right\}\frac{\varepsilon e^{s\tau_1}}{s} \quad \text{(V1.35)}$$

but at

$$t = \tau_2 \quad , \quad \tau_1 = t - \tau_{12} = t - \bar\tau_{12} - \delta\tau_{12}(\tau_1) \quad \text{(V1.36)}$$

Hence, in the real time scale, t, $\delta\tau_{12}$ becomes

$$\delta\tau_{12}(t) = -\frac{1}{f(\bar\lambda)\,\bar v_{fi}}\left\{g(\bar\lambda, s) - [f(\bar\lambda) - f(0)]\right\}\frac{\varepsilon e^{st} e^{-s\tau_{12}(\tau_1)}}{s} \quad \text{(V1.37)}$$

Retaining only the first order of ε, finally we obtain

$$\delta\tau_{12}(t) = -\varepsilon e^{st}\frac{1}{f(\bar\lambda)\,\bar v_{fi}}\left\{g(\bar\lambda, s) - [f(\bar\lambda) - f(0)]\right\}\frac{e^{-s\bar\tau_{12}}}{s} \quad \text{(V1.38)}$$

For simplicity, let us define

$$\Lambda_1(s) \equiv -\frac{1}{f(\bar\lambda)\,\bar v_{fi}}\left\{g(\bar\lambda, s) - [f(\bar\lambda) - f(0)]\right\}\frac{e^{-s\bar\tau_{12}}}{s} \quad \text{(V1.39)}$$

Upon substitution of Eq. (V1.39) into (V1.38), the perturbation of particle residence time in the heated liquid region (B) becomes

$$\delta\tau_{12}(t) = \varepsilon e^{st}\Lambda_1(s) = \Lambda(s)\,\delta v(t) \quad \text{(V1.40)}$$

Knowing τ_{12}, the perturbation of space lag $\lambda(t)$ can be found from Eq. (V1.17) by setting at $t = \tau_2$, $z = \lambda(t)$

$$\lambda(t) = \bar{v}_{fi}\,\tau_{12} + \varepsilon e^{st}\,\frac{[1 - e^{-s\tau_{12}}]}{s} \tag{V1.41}$$

Hence

$$\lambda(t) = \bar{v}_{fi}\,\bar{\tau}_{12} + \varepsilon e^{st}\left\{\frac{1 - e^{-s\bar{\tau}_{12}}}{s} + \bar{v}_{fi}\,\Lambda_1(s)\right\} \tag{V1.42}$$

By defining $\Lambda_2(s)$ as

$$\Lambda_2(s) \equiv \frac{1 - e^{-s\bar{\tau}_{12}}}{s} + \bar{v}_{fi}\,\Lambda_1(s) \tag{V1.43}$$

the perturbation of λ, the heated liquid region, becomes

$$\delta\lambda(t) = \varepsilon e^{st}\,\Lambda_2(s) \tag{V1.44}$$

For example, the response of the boiling boundary to the sine heat flux profile, $q_w'' = q_o'' \frac{\pi}{2} \text{ sine } \frac{z\pi}{\ell}$, is given by

$$\Lambda_2 = \frac{1}{s^2 + A^2}\left\{ s + \frac{A}{\sin\left(\frac{\bar{\lambda}\pi}{\ell}\right)}\left[e^{-s\bar{\tau}_{12}} - \cos\left(\frac{\bar{\lambda}}{\ell}\pi\right)\right]\right\}$$

where A is given by

$$A = \frac{\bar{v}_{fi}\,\pi}{\ell}$$

The Enthalpy Response

Although the response of the liquid enthalpy for the dynamical problem is not directly necessary, it is important to know the energy wave propagation which can be applied in a single phase heat exchanger analysis. Here we integrate Eq. (V1.20) from τ_1 to t and then eliminate τ_1 in view of Eq. (V1.17), keeping in mind that z is an independent variable. We get

$$i_f(z,t) - i_1 = \frac{q_o''\xi}{A_c \rho_f \bar{v}_{fi}} \Big\{ \bar{F}(z) + \varepsilon e^{st} \Big[- f(z) \frac{\left(1 - e^{-\frac{sz}{\bar{v}_{fi}}}\right)}{s} + \tag{V1.45}$$

$$+ \left(g(z,s) - f(z) + f(0) \right) e^{-\frac{sz}{\bar{v}_{fi}}} \Big] \Big\}$$

V-2. Kinematics of the Heated Mixture Region (C)

A. Volumetric Flux Equation and Density Propagation Equation

The set of governing equations for the heated mixture region (C) has been derived in Section IV-4. We recall the condition of constant properties for the liquid and vapor phase in the mixture region, which is quite valid for the relatively high pressure system with thermodynamic equilibrium assumption. Furthermore, by taking the vapor drift velocity V_{gj} as a constant (see Section III-1 D), we can transform the two continuity equations (IV4.1) and (IV4.2) into the following form

$$\frac{\partial j}{\partial z} = \Gamma_g \frac{\Delta\rho}{\rho_g \rho_f} \tag{III1.53}$$

and

and

$$\frac{\partial \rho_m}{\partial t} + C_k \frac{\partial \rho_m}{\partial z} = - \Gamma_g \frac{\Delta \rho \, \rho_m}{\rho_g \rho_f} \tag{III1.61}$$

here

$$C_k \doteq j + V_{gj} \tag{III1.59}$$

The first equation can be called the volumetric flux equation which describes the increase of j due to the phase change. The second equation has a form of the wave equation and thus can be called as a density propagation equation. Following the analysis of Lighthill (14), we call C_k by "kinematic" wave velocity, since it describes the velocity of the density wave propagation.

The above two continuity relations are not sufficient to solve the kinematics of the mixture in (C), since we have three unknowns, j, ρ_m, and Γ_g in two equations. However, the additional information can be obtained from the enthalpy energy equation using the equations of state. Recalling equations (IV4.7,8,9,10,11, and 12), we have

$$d i_m = - \left| \frac{\Delta i_{fg} \rho_g \rho_f}{\Delta \rho} \right| \frac{d \rho_m}{\rho_m^2} \tag{V2.1}$$

Substituting Eq. (V2.1) into (IV4.3), i.e., the enthalpy energy equation, we obtain

$$\frac{\partial \rho_m}{\partial t} + C_k \frac{\partial \rho_m}{\partial z} = - \left[\frac{q_w'' \xi}{A_c} \frac{1}{\Delta i_{fg}} \right] \frac{\Delta \rho}{\rho_g \rho_f} \rho_m \tag{V2.2*}$$

---

[*] Here we used the same assumption concerning the drift velocity change as the one we applied in Section III-1 D. However, in order to obtain the vapor generation Γ_g in terms of the heat flux, these assumptions

Hence, from Eq. (V2.2) and Eq. (II1.16), we have, for the case of thermal equilibrium

$$\Gamma_g = \frac{q_w'' \xi}{A_c} \frac{1}{\Delta i_{fg}} \tag{V2.3}$$

which relates the vapor generation Γ_g to the heat flux and the latent heat. As was mentioned in the footnote on page 80, this result is not the consequence of the formulation in terms of the kinematic wave velocity, but of the constants i_f, i_g, ρ_g, and ρ_f with thermal equilibrium condition.

The characteristic frequency Ω of the phase change can be defined now by

$$\Omega(z) \equiv \frac{\Gamma_g \Delta\rho}{\rho_g \rho_f} = \frac{q_w''(z)\xi}{A_c \Delta i_{fg}} \frac{\Delta\rho}{\rho_g \rho_f} \tag{V2.4}$$

In terms of the normalized heat flux f(z), Eq. (V2.4) can be expressed as

$$\Omega(z) = \Omega_c f(z) \tag{V2.5}$$

Here

$$\Omega_c \equiv \frac{q_0''\xi}{A_c \Delta i_{fg}} \frac{\Delta\rho}{\rho_g \rho_f} \tag{V2.6}$$

which is the characteristic frequency of phase change if the heat flux is

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are not necessary, since it can be obtained from the energy and continuity equations directly.

uniform. Using the above definitions, the governing equations for the kinematics of the heated mixture take the following forms.

From the volumetric flux equation (III1.53)

$$\frac{\partial j}{\partial z} = \Omega_o f(z) \tag{V2.7}$$

From the density propagation equation (III1.61)

$$\frac{\partial \rho_m}{\partial t} + C_k \frac{\partial \rho_m}{\partial z} = - \Omega_o f(z) \rho_m \tag{V2.8}$$

The constitutive equation for C_k from Eq. (III1.59)

$$C_k = j + V_{gj} \tag{V2.9}$$

The constitutive equation for j from Eq. (III1.35)

$$j = v_m + \left(\frac{\rho_f}{\rho_m} - 1 \right) V_{gj} \tag{V2.10}$$

and V_{gj} will be treated as a constant. Thus

$$V_{gj} = \overline{V}_{gj} = \text{constant} \tag{V2.11*}$$

The equations from (V2.7) to (V2.10) with the constitutive equation for V_{gj} describe the kinematic behavior of the heated mixture region. In our analysis, it is necessary to assume that the drift velocity is a

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[*] Note that this is supported by experimental data (32) and is valid for several flow regimes, i.e., bubbly, slug, churn turbulent, etc.

function only of kinematic variables in order to decouple the momentum equation from the kinematic of the mixture. Furthermore, for simplicity we assumed that V_{gj} = constant, i.e., Eq. (IV2.11), since from this condition we can linearize the system of governing equations. The analytical and experimental verifications for this assumption are given in (30, 32).

B. Kinematic Wave Velocity C_k

The integration of the volumetric flux equation (V2.7) yields

$$j(z,t) = v_{fi}(t) + \Omega_0 \int_{\lambda(t)}^{z} f(z)\,dz \tag{V2.12}$$

Using the definition of F(z), Eq. (V1.11), we get

$$j(z,t) = v_{fi}(t) + \Omega_0 \left[F(z) - F(\bar{\lambda}) \right] \tag{V2.13}$$

which expresses the increase of j due to the evaporation.

By substituting Eq. (V2.13) into Eq. (V2.9), C_k becomes

$$C_k(z,t) = v_{fi}(t) + V_{gj} + \Omega_0 \left[F(z) - F(\bar{\lambda}) \right] \tag{V2.14}$$

Using the expressions for v_{fi} and λ, i.e., Eq. (V1.7) and (V1.42), in the above equation and expanding the last term in a Taylor series, then keeping the terms up to the order of ϵ, we get

$$\begin{aligned} C_k(z,t) &= \bar{C}_k(z) + \delta C_k(t) \\ &= \bar{v}_{fi} + V_{gj} + \Omega_0 \left[F(z) - F(\bar{\lambda}) \right] + \left[\delta v - \Omega_0 f(\bar{\lambda}) \delta\lambda \right] \end{aligned} \tag{V2.15}$$

Thus the steady state and perturbed parts of the kinematic wave velocity C_k are given by

$$\bar{C}_k(z) = \bar{v}_{fi} + V_{gj} + \Omega_0 \left[F(z) - F(\bar{\lambda}) \right] \tag{V2.16}$$

and

$$\delta C_k(t) = \delta v(t) - \Omega_0 f(\bar{\lambda})\, \delta\lambda(t) \tag{V2.17}$$

or from Eq. (V1.7) and (V1.44)

$$\delta C_k(t) = \varepsilon e^{st} \Lambda_3(s) \tag{V2.18}$$

here
$$\Lambda_3(s) \equiv 1 - \Omega_c f(\bar{\lambda}) \Lambda_2(s) \tag{V2.19}$$

C. Perturbation Method on the Density Propagation Equation

In the previous analysis, the volumetric flux equation was solved and the kinematic wave velocity was expressed by the independent variables z and t. Therefore, by substituting Eq. (V2.15) into Eq. (V2.8), we can solve for the density of the mixture. By introducing a new variable ϕ which is defined by

$$\phi(z,t) \equiv \ln \left[\frac{\rho_m(z,t)}{\rho_f} \right] \tag{V2.20}$$

the density propagation equation (V2.8) can be written as

$$\frac{\partial \phi}{\partial t} + C_k(z,t) \frac{\partial \phi}{\partial z} = -\Omega_0 f(z) \tag{V2.21}$$

In order to apply the perturbation method, we define

$$\phi(z,t) = \bar{\phi}(z) + \delta\phi(z,t) \tag{V2.22}$$

and

$$\rho_m(z,t) = \bar{\rho}_m(z) + \delta\rho_m(z,t) \tag{V2.23}$$

The relations between $\bar{\phi}$ and $\bar{\rho}_m$ and $\delta\phi$ and $\delta\rho_m$ can be obtained from Eq. (V2.20) using the order of magnitude analysis. Hence

$$\bar{\phi}(z) = \ln\left[\frac{\bar{\rho}_m(z)}{\rho_f}\right] \tag{V2.24}$$

and

$$\delta\phi(z,t) = \frac{\delta\rho_m}{\bar{\rho}_m} \tag{V2.25}$$

Substituting Eq. (V2.15) and (V2.22) into Eq. (V2.21) and using the perturbation method, we get

for the zero$^{\text{th}}$ order

$$\bar{C}_k(z)\frac{d\bar{\phi}}{dz} = -\Omega_0 f(z) \tag{V2.26}$$

and

for the first order

$$\frac{\partial\delta\phi}{\partial t} + \bar{C}_k(z)\frac{\partial\delta\phi}{\partial z} = \delta C_k(t)\,\frac{\Omega_0 f(z)}{\bar{C}_k(z)} \tag{V2.27}$$

D. Steady State (Zero$^{\text{th}}$ Order) Solution for the Density

We integrate Eq. (V2.25) from $\bar{\lambda}$ to z, then

$$\bar{\Phi}(z) - \bar{\Phi}(\bar{\lambda}) = - \int_{\bar{\lambda}}^{z} \frac{\Omega_0 f(z)}{\bar{C}_k(z)} dz \tag{V2.28}$$

But, from Eq. (V2.16) and (V1.11)

$$\Omega_0 f(z) = \frac{d}{dz} \bar{C}_k(z) \tag{V2.29}$$

Hence

$$\bar{\Phi}(z) - \bar{\Phi}(\bar{\lambda}) = - \int_{\bar{\lambda}}^{z} \frac{d\bar{C}_k(z)}{\bar{C}_k(z)} = - \ln\left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right] \tag{V2.30}$$

From Eq. (V2.30) and (V2.24), we obtain

$$\frac{\rho_m(z)}{\rho_f} = \frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)} \tag{V2.31}$$

or

$$\frac{\rho_m(z)}{\rho_f} = \frac{\bar{v}_{fi} + V_{gj}}{\bar{v}_{fi} + V_{gj} + \Omega_0 \left[\bar{F}(z) - \bar{F}(\bar{\lambda})\right]} \tag{V2.32*}$$

E. Perturbed Part (First Order) Solution for the Density

The differential equation for $\delta\Phi$, i.e., Eq. (V2.27), can be transformed to Lagrangian form (59), thus

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[*] This equation has been obtained assuming V_{gj} = const.; however, by solving directly the mixture continuity equation, it can be shown that it also holds for the case when V_{gj} varies along the channel.

$$dt = \frac{dz}{\bar{C}_k(z)} = \frac{C_k(z)\, d(\delta\Phi)}{\delta C_k(t)\,\Omega_c\, f(z)} \qquad \text{(V2.33)}$$

Let us solve this equation by the method of characteristics, considering the particle entering the heated mixture region at t = τ_2. Taking the first equality, we can integrate directly

$$t - \tau_2 = \int_{\lambda(\tau_2)}^{z} \frac{dz}{\bar{C}_k(z)} \qquad \text{(V2.34)}$$

Let us define a new function E(z) by

$$E(z) \equiv \int \frac{1}{\bar{C}_k(z)}\, dz \qquad \text{(V2.35)}$$

Then Eq. (V2.35) becomes

$$t - \tau_2 = E(z) - E(\bar{\lambda} + \delta\lambda(\tau_2)) \qquad \text{(V2.36)}$$

The expansion of the second term into a Taylor series and retaining up to the first order terms of ϵ, we get

$$t = \tau_2 + [E(z) - E(\bar{\lambda})] - \frac{1}{\bar{C}_k(\bar{\lambda})}\,\delta\lambda(\tau_2) \qquad \text{(V2.37)}$$

On the other hand, by considering the first and third terms of Eq. (V2.33) we obtain

$$d(\delta\Phi) = \frac{1}{[\bar{C}_k(z)]^2}\,\frac{d\bar{C}_k(z)}{dz}\,\delta C_k(t)\, dz \qquad \text{(V2.38)}$$

The expressions for δC_k (t) and the particle path equation between t and z, i.e., Eq. (V2.18) and (V2.37), respectively, have already been obtained. Thus, by substituting these equations into (V2.38) and neglecting the higher order of ε, we get

$$d(\delta\Phi) = \varepsilon e^{s\tau_2} \frac{1}{[\bar{C}_k(z)]^2} e^{s\{E(z)-E(\bar{\lambda})\}} \Lambda_3(s) \frac{d\bar{C}_k(z)}{dz}\, dz \tag{V2.39}$$

By defining

$$H(z,s) = \int \frac{1}{[\bar{C}_k(z)]^2} e^{s\{E(z)-E(\bar{\lambda})\}} \Lambda_3(s) \frac{d\bar{C}_k}{dz}\, dz \tag{V2.40}$$

and integrating Eq. (V2.39) from $\lambda(\tau_2)$ to z and retaining only the first order of ε, we obtain

$$\delta\Phi(z,\tau_2) - \delta\Phi(\lambda(\tau_2)) = \varepsilon e^{s\tau_2} \Lambda_3(s)\left[H(z,s) - H(\bar{\lambda},s)\right] \tag{V2.41}$$

In order to determine the boundary value, we combine the zeroth and first order solutions, i.e., Eq. (V2.30) and (V2.41), respectively. Thus

$$\begin{aligned} \Phi(z,\tau_2) &= \bar{\Phi}(z) + \delta\Phi(z,\tau_2) \\ &= \ln\left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)}\right] + \left\{\delta\Phi(\lambda(\tau_2)) + \varepsilon e^{s\tau_2}\Lambda_3(s)\left[H(z,s) - H(\bar{\lambda},s)\right]\right\} \end{aligned} \tag{V2.42}$$

and the boundary condition for Φ is given by

$$\phi(\lambda(\tau_2),\tau_2)=0 \tag{V2.43}$$

Applying the above condition to Eq. (V2.42), we get the solution in terms of z and τ_2

$$\delta\phi(z,\tau_2)=\varepsilon e^{s\tau_2}\left\{\frac{\Omega_0 f(\bar{\lambda})\Lambda_2(s)}{\bar{C}_k(\bar{\lambda})}+\Lambda_2(s)\left[H(z,s)-H(\bar{\lambda},s)\right]\right\} \tag{V2.44}$$

Recalling Eq. (V2.37), we have

$$\varepsilon e^{s\tau_2} \doteq \varepsilon e^{st} e^{-s\left\{E(z)-E(\bar{\lambda})\right\}} \tag{V2.45}$$

We substitute Eq. (V2.45) into Eq. (V2.44), then the solution for $\delta\phi$ can be expressed by the independent variables t and z. Thus

$$\begin{aligned}\delta\phi(z,t)= {} & \varepsilon e^{st} e^{-s\left\{E(z)-E(\bar{\lambda})\right\}}\left[\frac{\Omega_0 f(\bar{\lambda})\Lambda_2(s)}{\bar{C}_k(\bar{\lambda})}+\right. \\ & \left. +\Lambda_3(s)\left[H(z,s)-H(\bar{\lambda},s)\right]\right]\end{aligned} \tag{V2.46}$$

Finally, we obtain the response of the mixture density ρ_m from Eq. (V2.46) and (V2.31), hence

$$\frac{\rho_m(z,t)}{\rho_f}=\frac{\bar{\rho}_m(z)}{\rho_f}+\frac{\delta\rho_m(z,t)}{\rho_f} \tag{V2.47}$$

For the steady state part we have

$$\frac{\rho_m(z)}{\rho_f} = \frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)} \qquad \text{(V2.48)}$$

whereas for the perturbed part

$$\frac{\delta\rho_m}{\rho_f} = \varepsilon e^{st} \Lambda_4(z,s) \qquad \text{(V2.49)}$$

with

$$\Lambda_4(z,s) \equiv \left[\frac{\bar{C}_k(\bar{\lambda})}{C_k(z)}\right] e^{-s[E(z)-E(\bar{\lambda})]} \left\{ \frac{\Omega_c f(\bar{\lambda})}{\bar{C}_k(\bar{\lambda})} \Lambda_2(s) + \right. \qquad \text{(V2.50)}$$

$$\left. + \Lambda_3(s)\left[H(z,s) - H(\bar{\lambda},s)\right]\right\}$$

F. Center of Mass Velocity v_m

In previous analyses, the solutions for the volumetric flux j and for the mixture density ρ_m have been obtained. Whereas the solution for the center of mass velocity v_m can be expressed by j and ρ_m as

$$v_m = j - \left[\frac{\rho_f}{\rho_m} - 1\right] V_{gj} \qquad \text{(V2.51)}$$

Recalling the definition of C_k, i.e., Eq. (V2.9), and using the series expansion around the steady state solution, we obtain

$$v_m(z,t) \doteq \left\{\bar{C}_k(z) - \frac{\rho_f}{\bar{\rho}_m} V_{gj}\right\} + \left\{\delta C_k(t) + \frac{\rho_f}{\bar{\rho}_m}\frac{\delta\rho_m}{\bar{\rho}_m} V_{gj}\right\} \qquad \text{(V2.52)}$$

By using Eq. (V2.48) we get

$$\frac{v_m(z,t)}{\bar{v}_{fi}} = \left\{\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right\} + \left\{\frac{\delta C_k(t)}{\bar{v}_{fi}} + \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 \frac{V_{gj}}{\bar{v}_{fi}}\, \frac{\delta \rho_m}{\rho_f}\right\} \qquad \text{(V2.53)}$$

Substituting Eq. (V2.19) and (V2.49), we obtain for v_m the following expression

$$\frac{v_m(z,t)}{\bar{v}_{fi}} = \left\{\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right\} + \frac{\varepsilon e^{st}}{\bar{v}_{fi}} \left\{\Lambda_3(s) + \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 V_{gj} \Lambda_4(z,s)\right\} \qquad \text{(V2.54)}$$

Finally we obtain the solution for v_m in a perturbed form

$$\frac{v_m(z,t)}{\bar{v}_{fi}} = \frac{\bar{v}_m(z)}{\bar{v}_{fi}} + \frac{\delta v_m(z,t)}{\bar{v}_{fi}} \qquad \text{(V2.55)}$$

For the steady state part we have

$$\frac{\bar{v}_m(z)}{\bar{v}_{fi}} = \frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})} \qquad \text{(V2.56)}$$

whereas for the perturbed part we obtain

$$\frac{\delta v_m(z,t)}{\bar{v}_{fi}} = \frac{\varepsilon e^{st}}{\bar{v}_{fi}}\, \Lambda_5(z,s) \qquad \text{(V2.57)}$$

Here

$$\Lambda_5(s) \equiv \Lambda_3(s) + \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 V_{gj} \Lambda_4(z,s) \qquad \text{(V2.58)}$$

G. Enthalpy i_m of the Mixture

Knowing the response of the density, i.e., Eq. (V2.47,48,49, and 50) the solution for the enthalpy can be calculated directly from Eq. (V2.1). Thus

$$\frac{i_m(z,t)}{i_{fs}} = 1 + \frac{\rho_g \Delta i_{fg}}{\Delta\rho\, i_{fs}} \frac{\Omega_c\left[F(z) - \bar{F}(\bar{\lambda})\right]}{C_k(\bar{\lambda})} - \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 \frac{\rho_g \Delta i_{fg}}{\Delta\rho\, i_{fs}} \frac{\delta\rho_m}{\rho_f} \qquad \text{(V2.59)}$$

H. The Residence Time

It is of interest to evaluate now the steady state residence time, i.e., the transit time of a particle in the heated duct. Recalling that the particle entering the mixture region at τ_2 reaches the end of the heated duct at τ_3, we can calculate the residence time τ_{23} from Eq. (V2.30). Hence

$$\tau_{23} = E(\ell) - E(\bar{\lambda}) - \frac{1}{C_k(\bar{\lambda})}\, \delta\lambda(\tau_2) \qquad \text{(V2.60)}$$

From Eq. (V2.60), we can decide the steady state $\bar{\tau}_{23}$ and its perturbation $\delta\tau_{23}$. Hence

$$\bar{\tau}_{23} = E(\ell) - E(\bar{\lambda}) \qquad \text{(V2.61)}$$

and

$$\delta\tau_{23}(t) = -\frac{\varepsilon e^{st}}{C_k(\bar{\lambda})}\, e^{-s[E(\ell) - E(\bar{\lambda})]}\, \Lambda_2(s) \qquad \text{(V2.62)}$$

V-3. Kinematics of the Downstream Un-heated Region (D)

The set of governing equations for the downstream un-heated region has been derived in Section IV-5. Under the isochoric conditions, the continuity equation takes the form of Eq. (IV5.2). Hence, the velocity of the mixture is independent of the position and is given by Eq. (V2.55, 56,57, and 58). Thus

$$\frac{v_{me}}{\bar{v}_{fi}} = \frac{\bar{v}_{me}}{\bar{v}_{fi}} + \frac{\delta v_{me}(t)}{\bar{v}_{fi}} = \left[\frac{A_c}{A_e}\right]\left\{\frac{\bar{C}_k(\ell)}{\bar{C}_k(\bar{\lambda})} + \frac{\varepsilon e^{st}}{\bar{v}_{fi}}\Lambda_5(\ell,s)\right\} \tag{V3.1}$$

On the other hand, the equation of state, i.e., Eq. (IV5.1), is given by

$$\frac{\partial \rho_{me}}{\partial t} + v_{me}(t)\frac{\partial \rho_{me}}{\partial z} = 0 \tag{V3.2}$$

whose characteristic equation is

$$dt = \frac{dz}{v_{me}(t)} \tag{V3.3}$$

The integration of Eq. (V3.3) yields

$$z - \ell = \bar{v}_{fi}\left[\frac{A_c}{A_e}\right]\left\{\frac{\bar{C}_k(\ell)}{\bar{C}_k(\bar{\lambda})}(t - \tau_3) + \frac{\Lambda_5(\ell,s)}{\bar{v}_{fi}}\varepsilon\left[\frac{e^{st} - e^{s\tau_3}}{s}\right]\right\} \tag{V3.4}$$

However, the initial condition for the density ρ_{me} is given by Eq. (V2.47,48,49, and 50) as

$$\frac{\rho_{me}(z,\tau_3)}{\rho_f} = \frac{\rho_m(\ell,\tau_3)}{\rho_f} = \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(\ell)}\right] + \varepsilon e^{s\tau_3}\Lambda_4(\ell,s) \tag{V3.5}$$

Now we cancel the parameter τ_3 between Eq. (V3.4) and (V3.5), retaining the first order of ε only. Thus

$$\frac{\rho_{me}(z,t)}{\rho_f} = \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(\ell)}\right] + \varepsilon e^{st}\left\{e^{-s\left[\frac{z-\ell}{\bar{v}_{fi}}\right]\left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)}\right]\left[\frac{A_e}{A_c}\right]}\Lambda_4(\ell,s)\right\} \tag{V3.6}$$

By defining Λ_6 as

$$\Lambda_6(z,s) = e^{-s\left[\frac{z-\ell}{\bar{v}_{fi}}\right]\left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(\ell)}\right]\left[\frac{A_e}{A_c}\right]}\Lambda_4(\ell,s) \tag{V3.7}$$

we get

$$\frac{\rho_{me}(t,z)}{\rho_f} = \frac{\bar{\rho}_{me}}{\rho_f} + \frac{\delta\rho_{me}}{\rho_f} = \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(\ell)}\right] + \varepsilon e^{st}\Lambda_6(z,s) \tag{V3.8}$$

The characteristic residence time τ_{34} can be obtained by setting $z - \ell = \ell e$ in Eq. (V3.4) and retaining only the first order of ε. Thus

$$\tau_{34} = \bar{\tau}_{34} + \delta\tau_{34}(t) \tag{V3.9}$$

$$= \frac{\ell_e}{\bar{v}_{fi}}\left(\frac{A_e}{A_c}\right)\left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(\ell)}\right] - \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(\ell)}\right]\frac{\Lambda_5(\ell,s)}{\bar{v}_{fi}}\,\varepsilon e^{st}\frac{\left[1-e^{-s\bar{\tau}_{34}}\right]}{s}$$

CHAPTER VI

DYNAMICS OF THE SYSTEM

VI-1. Pressure Drop in the Upstream Un-heated Region (A)

In the previous chapters, the kinematics of the fluid has been solved. Knowing the veolcity field and the density variation, the pressure drop can be calculated by integrating the momentum equation. For the upstream un-heated region (A) the velocity and density are given by Eq. (V1.1) and (V1.4), respectively. Hence, the momentum equation (IV2.3) can be integrated directly. In single phase turbulent flows, the perturbation of the friction factor due to the fluctuation in the velocity may be neglected for two reasons. First, the friction factor is inversely proportional to only one-fourth power of the Reynolds number, second the existing large turbulent eddies may not be sensitive to the mathematically introduced small perturbation of the local velocity. Thus, in the following analysis, we assume that the friction factor takes the value of the steady state. Integrating Eq. (IV2.3) from $-\ell_o$ to 0 with an inlet restriction, Eq. (IV2.4), we obtain

$$\Delta P_{e1} = \int_{-\ell_o}^{0} \rho_f \left\{ \frac{\partial v_{fo}}{\partial t} + v_{fo}\frac{\partial v_{fo}}{\partial z} + g_c + \frac{f_o}{2D_o} v_{fo}^2 \right\} dz + k_i \rho_f v_{fi}^2 \qquad \text{(VI1.1)}$$

In view of Eq. (V1.1) and (V1.7), the integrated momentum equation yields,

$$\Delta P_{c1} = \rho_f \left\{ \frac{\partial v_{fc}}{\partial t} + g_c + \frac{f_c}{2D_c} (\bar{v}_{fc} + \delta v_{fc})^2 \right\} \ell + k_i \rho_f (\bar{v}_{fi} + \delta v)^2 \quad \text{(VI1.2)}$$

Linearizing Eq. (VI1.2) and retaining only the terms with the first power of ϵ, it can be shown that

$$\Delta P_{c1} = \rho_f \left\{ g_c \ell_c + \frac{f_c \ell_c}{2D_c} \left(\frac{A_c}{A_i}\right)^2 \bar{v}_{fi} + k_i \bar{v}_{fi}^2 \right\} + \quad \text{(VI1.3)}$$

$$+ \rho_f \left\{ \left(\frac{A_c}{A_i}\right) s \ell_c + \frac{f_c \ell_c}{2D_c} 2 \left(\frac{A_c}{A_i}\right)^2 \bar{v}_{fi} + 2 k_i \bar{v}_{fi} \right\} \delta v$$

We now obtain the steady state pressure drop and the pressure drop response to the disturbance as

$$\overline{\Delta P_{c1}} = \rho_f \left\{ g_c \ell_c + \frac{f_c \ell_c}{2D_c} \left(\frac{A_c}{A_i}\right)^2 \bar{v}_{fi}^2 + k_i \bar{v}_{fi}^2 \right\} \quad \text{(VI1.4)}$$

$$\delta \Delta P_{c1} = \rho_f \left\{ \left(\frac{A_c}{A_i}\right) s \ell_c + \frac{f_c \ell_c}{2D_c} 2 \left(\frac{A_c}{A_i}\right)^2 \bar{v}_{fi} + 2 k_i \bar{v}_{fi} \right\} \varepsilon e^{st} \quad \text{(VI1.5)}$$

Hence

$$\Delta P_{c1} = \overline{\Delta P_{c1}} + \delta \Delta P_{c1} \quad \text{(VI1.6)}$$

It should be noted that three terms in the right hand side of Eq. (VI1.5) represent the inertia, frictional, and inlet orifice effects, respectively.

VI-2. Pressure Drop in the Heated Liquid Region (B)

In this region, the pressure drop response can be obtained from the momentum equation (IV3.3) by using the expression for the velocity, i.e., Eq. (V1.7). Recalling the boundary of the region (B), i.e., Eq. (V1.22), (V1.42), (V1.43), and (V1.44), the integrated momentum equation becomes

$$\Delta P_{12} = \rho_f \int_c^{\bar{\lambda}+\delta\lambda} \left\{ \frac{\partial v_f}{\partial t} + v_f \frac{\partial v_f}{\partial z} + g + \frac{f_s}{2D} v_f^2 \right\} dz \qquad \text{(VI2.1)}$$

where we have taken into account the assumption that the density is constant in region (B). Upon substitution of Eq. (V1.7) into Eq. (VI2.1) and taking into account only the first order terms of ϵ, we obtain

$$\Delta P_{12} = \rho_f \left\{ g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right\} \bar{\lambda} + \qquad \text{(VI2.2)}$$

$$+ \rho_f \left\{ S + \frac{f_s}{2D} 2 \bar{v}_{fi} \right\} \bar{\lambda}\, \delta v + \rho_f \left[g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right] \delta\lambda$$

Hence the steady state pressure drop can be expressed as

$$\Delta P_{12} = \rho_f \left[g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right] \bar{\lambda} \qquad \text{(VI2.3)}$$

And, in view of Eq. (V1.7) and Eq. (V1.44), the perturbed pressure drop response becomes

$$\delta \Delta P_{12} = \rho_f \left\{ S\bar{\lambda} + \frac{f_s}{2D} 2\bar{v}_{fi} \bar{\lambda} + \left(g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right) \Lambda_2(S) \right\} \varepsilon e^{St} \qquad \text{(VI2.4)}$$

Hence

$$\Delta P_{12} = \overline{\Delta P_{12}} + \delta \Delta P_{12} \tag{VI2.5}$$

The right hand side of Eq. (VI2.4) shows various effects on the response of the pressure. The first term is the effect of the inertia, the second term is the effect of the velocity perturbation on the frictional pressure drop, whereas the last group shows the effect of the fluctuating boiling boundary.

VI-3. Pressure Drop in the Heated Mixture Region (C)

A. Integration of Mixture Momentum Equation

The mixture momentum equation will be integrated now in a manner similar to that used in the single phase region. However, in the heated mixture region, the kinematic variables, i.e., the density and velocity of the mixture, are not only functions of time, but also of the axial coordinate z. Hence, the integration of Eq. (IV5.8) is more complicated than the previous ones in the liquid region. Furthermore, the dynamic characteristic of the two-phase friction factor f_m is quite unknown in the present state of art, consequently, certain dynamic modeling for f_m is required. In this respect, we employ two distinct approaches: 1) assuming that the dynamic characteristic of the frictional pressure drop is similar to that of the inertia force $\rho_m v_m^2$, it reduces to the time independent f_m; 2) the constitutive functional forms of f_m for the dynamic and steady state cases are the same, and it can be expressed by the kinematic variables. In either case, without going to the detailed form, we can express the perturbed two-phase friction factor as follows

$$f_m = \overline{f}_m(z) + \delta f_m(z,t) \qquad \text{(VI3.1)}$$

with

$$\delta f_m = 0 \qquad \text{for case (1)} \qquad \text{(VI3.2)}$$

$$\delta f_m = \varepsilon e^{st} \Lambda_7(s,z) \qquad \text{for case (2)} \qquad \text{(VI3.3)}$$

Once the friction factor model is specified, the functions $\overline{f}_m$, δf_m, or Λ_7, can be obtained by the perturbation method.

In view of Eq. (V2.47-50), (V2.55-58), and (VI3.1-3), the momentum equation for the mixture, Eq. (IV4.4), can be integrated from $\lambda(t)$ to ℓ, thus

$$\Delta P_{34} = \int_{\lambda(t)}^{\ell} \Big\{ \rho_m \Big[\frac{\partial v_m}{\partial t} + v_m \frac{\partial v_m}{\partial z} \Big] + g\rho_m + \frac{f_m}{2D} \rho_m v_m^2 + \frac{\partial}{\partial z} \Big[\frac{\rho_f - \rho_m}{\rho_m - \rho_g} \, \frac{\rho_g \rho_f}{\rho_m} V_{gj}^2 \Big] \Big\} dz \qquad \text{(VI3.4)}$$

Now, we shall consider each term of Eq. (VI3.4) separately.

B. The Inertia Term

The inertia term, i.e., the acceleration pressure drop in the momentum equation, is given by

$$\Delta P_{23a} = \int_{\bar{\lambda}+\delta\lambda}^{\ell} \rho_m \frac{\partial v_m}{\partial t}\, dz \tag{VI3.5}$$

Substituting Eq. (V2.47-50) and (V2.55-58) into Eq. (VI3.5) and retaining only up to the first power of ϵ, one can show that

$$\Delta P_{23a} = \varepsilon e^{st} s \rho_f \int_{\bar{\lambda}}^{\ell} \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)}\right] \Lambda_5(z,s)\, dz \tag{VI3.6}$$

It should be noted that here we used the mean value theorem between λ and $\bar{\lambda}$ to change the integral limit. By defining a transfer function Λ_8 as

$$\Lambda_8 \equiv \rho_f s \int_{\bar{\lambda}}^{\ell} \left[\frac{\bar{C}_k(\bar{\lambda})}{C_k(z)}\right] \Lambda_5(z,s)\, dz \tag{VI3.7}$$

the steady state and perturbed acceleration pressure drops become

$$\Delta P_{23a} = \overline{\Delta P}_{23a} + \delta P_{23a} \tag{VI3.8}$$

with

$$\overline{\Delta P}_{23a} = 0 \tag{VI3.9}$$

and

$$\delta P_{23a} = \varepsilon e^{st} \Lambda_8(s) \tag{VI3.10}$$

C. The Convective Acceleration Term

The convective acceleration term in Eq. (VI 3.4) is given by

$$\Delta P_{23c} = \int_{\bar{\lambda}+\delta\lambda}^{\ell} \rho_m v_m \frac{\partial v_m}{\partial z}\, dz$$

Using the mixture continuity equation (IV4.1), the above integral can be transformed to

$$\Delta P_{23c} = \rho_m v_m^2 \Big|_{\bar{\lambda}+\delta\lambda}^{\ell} + \int_{\bar{\lambda}+\delta\lambda}^{\ell} v_m \frac{\partial \rho_m}{\partial t}\, dz \qquad \text{(VI3.11)}$$

Substituting Eq. (V2.47-50) and (V2.55-58) into Eq. (VI3.11) and neglecting the second and higher order of δ, we obtain

$$\Delta P_{23c} = \overline{\Delta P_{23c}} + \delta \Delta P_{23c} \qquad \text{(VI3.12)}$$

with

$$\Delta P_{23c} = \rho_f \bar{v}_{fi}^2 \left(C_r^* - 1 \right) \qquad \text{(VI3.13)}$$

and

$$\delta \Delta P_{23c} = \varepsilon e^{st} \Lambda_9(s) \qquad \text{(VI3.14)}$$

Here C_r^* and Λ_9 are defined by

$$C_r^{*} = \frac{\bar{C}_k(\ell)}{\bar{C}_k(\bar{\lambda})} = \frac{\rho_f}{\bar{\rho}_m(\ell)} \qquad \text{(VI3.15)}$$

and

$$\Lambda_7(S) = \rho_f \bar{v}_{fi}^2 \Big\{ \big[C_r^{*2} \Lambda_4(\ell, S) - \Lambda_4(\bar{\lambda}, S) \big] + \qquad \text{(VI3.16)}$$

$$+ \frac{2}{\bar{v}_{fi}} \big[\Lambda_5(\ell, S) - \Lambda_5(\bar{\lambda}, S) \big] - \frac{\Omega_o}{\bar{C}_k(\bar{\lambda})} f(\bar{\lambda}) \Lambda_2(S)$$

$$+ \frac{1}{\bar{v}_{fi}} S \int_{\bar{\lambda}}^{\ell} \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})} \right] \Lambda_4(z, S)\, dz$$

D. The Gravitational Term

The gravitational term in Eq. (VI3.4) is given by

$$\Delta P_{23g} = \int_{\bar{\lambda}+\delta\lambda}^{\ell} g \rho_m \, dz \qquad \text{(VI3.17)}$$

Applying the mean value theorem between λ and $\bar{\lambda}$ and substituting Eq. (V2.47) into Eq. (VI3.17), we obtain

$$\Delta P_{23g} = g \int_{\bar{\lambda}}^{\ell} \bar{\rho}_m \, dz + g \int_{\bar{\lambda}}^{\ell} \delta \rho_m \, dz - g \rho_f \delta\lambda \qquad \text{(VI3.18)}$$

In view of Eq. (V2.48) and (V2.49), it reduces to

$$\Delta P_{23g} = g \rho_f \left\{ \int_{\bar{\lambda}}^{\ell} \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})} \right] dz + \varepsilon e^{st} \left[\int_{\bar{\lambda}}^{\ell} \Lambda_4(z, S)\, dz - \Lambda_2(S) \right] \right\} \qquad \text{(VI3.19)}$$

Hence the gravitational pressure drop can be expressed as

$$\Delta P_{23g} = \overline{\Delta P_{23g}} + \delta \Delta P_{23g} \qquad \text{(VI3.20)}$$

with the steady state part given by

$$\overline{\Delta P_{23g}} = g \rho_f \int_{\bar{\lambda}}^{\ell} \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)} \right] dz \qquad \text{(VI3.21)}$$

and the perturbed part by

$$\delta \Delta P_{23g} = \varepsilon e^{st} \left[\Lambda_{10}(s) - g \rho_f \Lambda_2(s) \right] \qquad \text{(VI3.22)}$$

Here the transfer function Λ_{10} is defined by

$$\Lambda_{10}(s) = g \rho_f \int_{\bar{\lambda}}^{\ell} \Lambda_f(z, s)\, dz \qquad \text{(VI3.23)}$$

E. The Frictional Term

The frictional pressure drop term in Eq. (VI3.4) is given by

$$\Delta P_{23f} = \int_{\bar{\lambda} + \delta\lambda}^{\ell} \frac{f_m}{2D} \rho_m v_m^2\, dz \qquad \text{(VI3.24)}$$

Using the mean value theorem, the above equation reduces to

$$\Delta P_{23f} = \int_{\bar{\lambda}}^{\ell} \frac{f_m}{2D} \rho_m v_m^2\, dz - \frac{f_s}{2D} \rho_f v_f^2\, \delta\lambda \qquad \text{(VI3.25)}$$

In view of Eq. (VI3.1-3), (V2.47-50), and (V2.55-58) and retaining only the first power of ϵ, it can be shown that the steady state part becomes,

$$\overline{\Delta P}_{23f} = \int_{\bar{\lambda}}^{l} \frac{f_m}{2D}\rho_f \bar{v}_{fi}^2 \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right] dz \tag{VI3.26}$$

and the perturbed part is given by

$$\delta\Delta P_{23f} = \varepsilon e^{st}\left\{\Lambda_{11}(s) - \frac{f_s}{2D}\rho_f \bar{v}_{fi}^2 \Lambda_2(s)\right\} \tag{VI3.27}$$

where the transfer function Λ_{11} is defined by

$$\Lambda_{11}(s) = \int_{\bar{\lambda}}^{l} \frac{1}{2D}\left\{\bar{f}_m \rho_f \bar{v}_{fi}^2 \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 \Lambda_4(z,s) + \right. \tag{VI3.28}$$

$$\left. + 2\bar{f}_m \rho_f \bar{v}_{fi}\Lambda_5(z,s) + \rho_f \bar{v}_{fi}^2\left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]\Lambda_4(z,s)\right\} dz$$

Hence, the frictional pressure drop can be expressed by the above three equations as

$$\begin{aligned} \Delta P_{23f} &= \overline{\Delta P}_{23f} + \delta\Delta P_{23f} \\ &= \overline{\Delta P}_{23f} + \varepsilon e^{st}\left[\Lambda_{11}(s) - \frac{f_s}{2D}\rho_f \bar{v}_{fi}^2 \Lambda_2(s)\right] \end{aligned} \tag{VI3.29}$$

F. The Drift Stress Term

The drift stress term in the momentum equation is given by

$$\Delta P_{23d} = \int_{\bar{\lambda}+\delta\lambda}^{l} \frac{\partial}{\partial z}\left\{\frac{\rho_f - \rho_m}{\rho_m - \rho_g}\,\frac{\rho_g \rho_f}{\rho_m} V_{gj}^2\right\} dz \tag{VI3.30}$$

After integration and with proper boundary conditions for ρ_m, Eq. (VI3.30) yields,

$$\Delta P_{23d} = \frac{\rho_f - \rho_m(\ell,t)}{\rho_m(\ell,t) - \rho_g} \frac{\rho_g \rho_f}{\rho_m(\ell,t)} V_{gj}^2(\ell) \qquad \text{(VI3.31)}$$

We recall that, in the present analysis, the vapor drift velocity V_{gj} has been treated as a constant. However, this same assumption leads us to a singularity problem on the drift pressure drop term as the mixture density approaches to the vapor density, which occurs either at the critical pressure or at the complete vaporization point. Because this singularity is not important at all from the physical point of view, since the drift pressure drop term approaches zero,* we shall eliminate it by further assuming that, for the case $x \nrightarrow 0$

$$\frac{V_{gj}^2(\ell)}{\rho_m(\ell) - \rho_g} \approx \frac{V_{gj}^2}{\rho_m(\ell)} \qquad \text{(VI3.32)}$$

In view of Eq. (V2.47-50) and retaining only the first power of ϵ, we obtain

$$\Delta P_{23d} = (C_r^* - 1) C_r^* \rho_g V_{gj}^2 + (1 - 2C_r^*) C_r^{*2} \rho_g \Lambda_4(\ell, s) V_{gj}^2 \varepsilon e^{st} \qquad \text{(VI3.33)}$$

Here, C_r^* is the ratio of ρ_f to $\bar{\rho}_m(\ell)$ defined by Eq. (VI3.15). By setting the steady state and perturbed parts of the drift pressure drop as

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*In both cases the fluid becomes homogeneous, thus the vapor drift velocity is $V_{gj} = 0$. Thus, in order to retain the exact form of the drift pressure drop, it is recommended that the exit value of V_{gj} be used.

$$\overline{\Delta P}_{23d} = (C_r^* - 1) C_r^* \rho_g V_{gj}^2 \tag{VI3.34}$$

and

$$\delta \Delta P_{23d} = \varepsilon e^{st} \Lambda_{12}(s) \tag{VI3.35}$$

with

$$\Lambda_{12}(s) = (1 - 2C_r^*) C_r^{*2} \rho_g \Lambda_4(\ell, s) V_{gj}^2 \tag{VI3.36}$$

we get finally,

$$\Delta P_{23d} = \overline{\Delta P}_{23d} + \delta \Delta P_{23d} \tag{VI3.37}$$

G. Total Pressure Drop in the Heated Mixture Region

In the previous section, we integrated the right hand side of Eq. (VI3.4) separately. By adding the inertia, convective, gravitational, frictional, and drift terms, i.e., Eq. (VI3.8,12,20,29, and 37), the total pressure drop in the heated mixture region becomes

$$\Delta P_{23} = \overline{\Delta P_{23c}} + \overline{\Delta P_{23g}} + \overline{\Delta P_{23f}} + \overline{\Delta P_{23d}} + \tag{VI3.38}$$

$$+ \delta \Delta P_{23a} + \delta \Delta P_{23c} + \delta \Delta P_{23g} + \delta \Delta P_{23f} + \delta \Delta P_{23d}$$

Hence the steady state pressure drop $\overline{\Delta P}_{23}$ is given by

$$\overline{\Delta P}_{23} = \overline{\Delta P}_{23c} + \overline{\Delta P}_{23g} + \overline{\Delta P}_{23f} + \overline{\Delta P}_{23d} \qquad \text{(VI3.39)}$$

In Eq. (VI3.39), each term on the right hand side represents various pressure drops due to different mechanisms which are given by Eq. (VI3.13), (VI3.21), (VI3.26), and (VI3.34), respectively.

On the other hand, the perturbed pressure drop $\delta\Delta P_{23}$ is given by

$$\delta\Delta P_{23} = \delta\Delta P_{23a} + \delta\Delta P_{23c} + \delta\Delta P_{23g} + \delta\Delta P_{23f} + \delta\Delta P_{23d} \qquad \text{(VI3.40)}$$

or, in terms of the transfer functions,

$$\delta\Delta P_{23} = \varepsilon e^{st}\left\{\Lambda_8(s) + \Lambda_9(s) + \Lambda_{10}(s) + \Lambda_{11}(s) + \Lambda_{12}(s)\right\} - \qquad \text{(VI3.41)}$$

$$- \varepsilon e^{st}\left\{g\rho_f + \frac{f_s}{2D}\rho_f\overline{v}_{fi}^{2}\right\}\Lambda_2(s)$$

where Λ_2 and Λ_8 to Λ_{12} are given by Eq. (V1.43), (VI3.7), (VI3.16), (VI3.23), (VI3.28), and (VI3.36), respectively. Hence the total pressure drop in (C) becomes

$$\Delta P_{23} = \overline{\Delta P}_{23} + \delta\Delta P_{23} \qquad \text{(VI3.42)}$$

VI-4. Pressure Drop in the Downstream Un-heated Region (D)

In order to obtain the pressure drop response in this un-heated mixture region (D), we shall integrate the momentum Eq. (IV5.3) with Eq. (IV5.4). Thus we have,

$$\Delta P_{34} = k_e \rho_{in}(\ell)\, v_m^2(\ell) + \int_{\ell}^{\ell+\ell_e} \Big\{ \rho_{me} \Big[\frac{\partial v_{me}}{\partial t} + v_{me} \frac{\partial v_{me}}{\partial z} \Big] + \tag{VI4.1}$$

$$+ g_e \rho_{me} + \frac{f_{me}}{2D_e} \rho_{me} v_{me}^2 + \frac{\partial}{\partial z} \Big[\frac{\rho_f - \rho_{me}}{\rho_{me} - \rho_g} \frac{\rho_f \rho_g}{\rho_{me}} V_{gj}^2 \Big] \Big\} dz$$

The terms on the right hand side of the above equation represent the exit orifice, inertia, convective, gravitational, frictional, and drift pressure drops, respectively. Each term can be obtained separately in a manner similar to that demonstrated in the previous section. The friction factor f_{me} will be given by

$$f_{me} = \bar{f}_{me} + \delta f_{me}(z,t) \tag{VI4.2}$$

with

$$\delta f_{me} = 0 \qquad \text{for case (1)} \tag{VI4.3}$$

$$\delta f_{me} = \varepsilon e^{st} \Lambda_\eta'(s,z) \qquad \text{for case (2)} \tag{VI4.4}$$

Here we followed the assumption made in Section VI-3.

Recalling that the velocity v_{me} is a function only of t, and not of both t and z, one can show the following results.

<u>For the exit pressure drop</u>

From Eq. (V2.47-50) and Eq. (V2.55-58), we have

$$\Delta P_{34e} = k_e \rho_m U_m^2 = \overline{\Delta P_{34e}} + \delta \Delta P_{34e} \qquad \text{(VI4.5)}$$

with

$$\overline{\Delta P_{34e}} = k_e \rho_f \bar{v}_{fi}^2 C_r^* \qquad \text{(VI4.6)}$$

and

$$\delta \Delta P_{34e} = \varepsilon e^{st} \Lambda_{13}(s) \qquad \text{(VI4.7)}$$

Here the transfer function $\Lambda_{13}(S)$ is defined by

$$\Lambda_{13}(S) = 2 k_e \rho_f \bar{v}_{fi} \Lambda_5(\ell, S) + k_e \rho_f \bar{v}_{fi}^2 C_r^* \Lambda_4(\ell, S) \qquad \text{(VI4.8)}$$

<u>For the acceleration term</u>

From Eq. (V3.1), (V3.7), and (V3.8), we obtain

$$\Delta P_{34a} = \int_{\ell}^{\ell+\ell_e} \rho_{me} \frac{\partial v_{me}}{\partial t} = \overline{\Delta P_{34a}} + \delta \Delta P_{34a} \qquad \text{(VI4.9)}$$

with

$$\overline{\Delta P}_{34a} = 0 \qquad \text{(VI4.10)}$$

and

$$\delta \Delta P_{34a} = \varepsilon e^{st} \Lambda_{14}(S) \qquad \text{(VI4.11)}$$

where $\Lambda_{14}(S)$ is given by

$$\Lambda_{14}(S) = S \rho_f \frac{1}{C_r^*} l_e \left(\frac{A_c}{A_e} \right) \Lambda_5(l, S) \qquad \text{(VI4.12)}$$

The convective acceleration pressure drop is zero, since we assumed the isochoric process in the region, and hence the divergence of the velocity is zero.

For the gravitational pressure drop

From Eq. (V3.7), (V3.8), and (V3.9), we get

$$\Delta P_{34g} = \int_{l}^{l+l_e} g_e \rho_{me} \, dz = \overline{\Delta P_{34g}} + \delta \Delta P_{34g} \qquad \text{(VI4.13)}$$

with

$$\overline{\Delta P_{34g}} = g_e \frac{\rho_f}{C_r^*} l_e \qquad \text{(VI4.14)}$$

and

$$\delta \Delta P_{34g} = \varepsilon e^{st} \Lambda_{15}(S) \qquad \text{(VI4.15)}$$

Here the transfer function $\Lambda_{15}(S)$ is defined by

$$\Lambda_{15}(S) = g_e \rho_f \bar{v}_{fi} C_r^* \left(\frac{A_c}{A_e}\right) \frac{1 - e^{-S\bar{\tau}_{34}}}{S} \Lambda_4(\ell, S) \tag{VI4.16}$$

<u>For the frictional pressure drop</u>

From Eq. (VI4.2-4), (V3.1), (V3.7), and (V3.8), we obtain

$$\Delta P_{34f} = \int_{\ell}^{\ell+\ell_e} \frac{f_{me}}{2D_e} \rho_{me} v_{me}^2 \, dz = \Delta\bar{P}_{34f} + \delta\Delta P_{34f} \tag{VI4.17}$$

with

$$\Delta\bar{P}_{34f} = \frac{\bar{f}_{me}}{2\,D_e} \rho_f \bar{v}_{fi}^2 C_r^* \left(\frac{A_c}{A_e}\right)^2 \ell_e \tag{VI4.18}$$

and

$$\delta\Delta P_{34f} = \varepsilon e^{st} \Lambda_{16}(S) \tag{VI4.19}$$

where $\Lambda_{16}(S)$ is defined by

$$\Lambda_{16}(S) = \frac{1}{2D_e} \Big\{ 2\,\bar{f}_{me} \rho_f \bar{v}_{fi} \left(\frac{A_c}{A_e}\right)^2 \ell_e \Lambda_5(\ell, S) + \tag{VI4.20}$$

$$+ \bar{f}_{me} \rho_f \bar{v}_{fi}^3 \left(\frac{A_c}{A_e}\right)^3 C_r^{*3} \left(\frac{1 - e^{-S\bar{\tau}_{34}}}{S}\right) \Lambda_4(\ell, S)$$

$$+ \rho_f \bar{v}_{fi}^2 C_r^* \left(\frac{A_c}{A_e}\right)^2 \int_{\ell}^{\ell+\ell_e} \Lambda_7(S, z) \, dz \Big\}$$

For the drift pressure drop

From Eq. (V3.7) and (V3.8), we have

$$\Delta P_{34d} = \int_{\ell}^{\ell+\ell_e} \frac{\partial}{\partial z} \left\{ \frac{\rho_f - \rho_{me}}{\rho_{me} - \rho_f} \frac{\rho_g \rho_f}{\rho_{me}} V_{gj}^{2} \right\} dz = \overline{\Delta P}_{34d} + \delta \Delta P_{34d} \quad \text{(VI4.21)}$$

which is approximated by

$$\Delta P_{34d} \approx \frac{\rho_f - \rho_{me}}{\rho_{me}} \frac{\rho_g \rho_f}{\rho_{me}} V_{gj}^{2} \Big|_{\ell}^{\ell+\ell_e} \quad \text{(VI4.22)}$$

Hence

$$\overline{\Delta P_{34d}} = 0 \quad \text{(VI4.23)}$$

and

$$\delta \Delta P_{34d} = \varepsilon e^{st} \Lambda_{12}(s) \left(e^{-s\bar{\tau}_{34}} - 1 \right)$$

where the residence time $\bar{\tau}_{34}$ is defined by Eq. (V3.9).

By adding Eq. (VI4.5), (VI4.9), (VI4.13), (VI4.17), and (VI4.21), we obtain the total pressure drop in the downstream un-heated region (D). Thus

$$\Delta P_{34} = \overline{\Delta P_{34e}} + \overline{\Delta P_{34g}} + \overline{\Delta P_{34f}} + \quad \text{(VI4.24)}$$

$$+ \delta \Delta P_{34e} + \delta \Delta P_{34a} + \delta \Delta P_{34g} + \delta \Delta P_{34f} + \delta \Delta P_{34d}$$

Here each term on the right hand side of the equation is given by Eq. (VI4.6), (VI4.14), and (VI4.18).

On the other hand, the perturbed pressure drop $\delta\Delta P_{34}$ is given by

$$\delta\Delta P_{34} = \delta\Delta P_{34e} + \delta\Delta P_{34a} + \delta\Delta P_{34q} + \delta\Delta P_{34s} + \delta\Delta P_{34d} \tag{VI4.24}$$

or in terms of the various transfer functions.

$$\delta\Delta P_{34} = \varepsilon e^{st} \Big\{ \Lambda_{13}(s) + \Lambda_{14}(s) + \Lambda_{15}(s) + \tag{VI4.25}$$
$$+ \Lambda_{16}(s) + \Lambda_{12}(s)\left(e^{-s\bar{\tau}_{34}} - 1\right) \Big\}$$

where Λ_{12} to Λ_{16} are given by Eq. (V3.36), (VI4.8), (VI4.12), (VI4.16), and (VI4.20), respectively.

Hence the total pressure drop in (D) becomes

$$\Delta P_{34} = \overline{\Delta P_{34}} + \delta\Delta P_{34} \tag{VI4.26}$$

VI-5. Pressure Response of the System

In previous sections, the steady state as well as the perturbed pressure drops of each region have been derived separately. Here, we consider the total pressure drop ΔP_{ex} imposed at the boundary of the system.

In a perturbed form, the external pressure drop ΔP_{ex} can be given

$$\Delta P_{ex} = \overline{\Delta P}_{ex} + \delta \Delta P_{ex} \tag{VI5.1}$$

which should be the same as the internal pressure drop obtained by adding the terms expressed by Eq. (VI1.6), (VI2.5), (VI3.42), and (VI4.26). Thus

$$\Delta P_{ex} = \left(\overline{\Delta P}_{01} + \overline{\Delta P}_{12} + \overline{\Delta P}_{23} + \overline{\Delta P}_{34} \right) + \tag{VI5.2}$$

$$+ \left(\delta \Delta P_{01} + \delta \Delta P_{12} + \delta \Delta P_{23} + \delta \Delta P_{34} \right)$$

If the system operates under the steady state condition, all perturbations go to zero. Hence we have

$$\overline{\Delta P}_{ex} = \overline{\Delta P}_{01} + \overline{\Delta P}_{12} + \overline{\Delta P}_{23} + \overline{\Delta P}_{34} \tag{VI5.3}$$

Equation (VI5.3) can also be expressed in terms of the system parameters by substituting Eq. (VI1.4), (VI2.3), and (VI3.39) with (VI3.12,21,26, and 34) and (VI4.25) with (VI4.6,14, and 18).

Hence

$$\overline{\Delta P}_{ex} = \rho_f \left[g_c \ell_c + \frac{f_c \ell_c}{2D_o} \left(\frac{A_c}{A_o} \right)^2 \bar{U}_{fi}^{\,2} + k_i \bar{U}_{fi}^{\,2} \right] + \rho_f \left[g \bar{\lambda} + \frac{f_s}{2D} \bar{U}_{fi}^{\,2} \bar{\lambda} \right] + \tag{VI5.4}$$

$$+ \left\{ \rho_f \bar{U}_{fi}^{\,2} \left(C_r^* - 1 \right) + g \rho_f \int_{\bar{\lambda}}^{\ell} \frac{C_k(\bar{\lambda})}{C_k(z)} dz + \right.$$

$$\left. + \frac{\rho_f \bar{U}_{fi}^{\,2}}{2D} \int_{\bar{\lambda}}^{\ell} \bar{f}_m \left[\frac{C_k(z)}{C_k(\bar{\lambda})} \right] dz + \rho_g \left(C_r^* - 1 \right) V_{gj}^{\,2} \right\} + \qquad \text{(Continued)}$$

$$+ \left\{ k_e \rho_f \bar{v}_{fi}^2 C_r^* + g_e \rho_f \frac{1}{C_r^*} l_e + \right.$$

$$\left. + \frac{\bar{f}_{me}}{2D_e} \rho_f \bar{v}_{fi}^2 C_r^* \left(\frac{A_c}{A_e}\right)^2 l_e \right\}$$

which can be rearranged in the following form

$$\Delta P_{ex} = k_i \rho_f \bar{v}_{fi}^2 + \qquad \text{inlet orifice} \qquad (VI5.5)$$

$$+ \rho_f \bar{v}_{fi}^2 (C_r^* - 1) + \qquad \text{convective}$$

$$+ \rho_f \left\{ g_c l_c + g\left[\bar{\lambda} + \int_{\bar{\lambda}}^{\ell} \frac{C_k(\bar{\lambda})}{C_k(z)} dz\right] + g_e \frac{l_e}{C_r^*} \right\} + \qquad \text{gravitational}$$

$$+ \rho_f \bar{v}_{fi}^2 \left\{ \frac{f_c l_c}{2D_c}\left(\frac{A_c}{A_c}\right)^2 + \frac{f_s \bar{\lambda}}{2D} + \frac{1}{2D}\int_{\bar{\lambda}}^{1} \bar{f}_m \frac{C_k(z)}{C_k(\bar{\lambda})} dz + \frac{\bar{f}_{me}}{2D_e} C_r^* \left(\frac{A_c}{A_e}\right)^2 l_e \right\} + \qquad \text{frictional}$$

$$+ \rho_g C_r^* (C_r^* - 1) V_{gj}^2 \qquad \text{drift}$$

$$+ k_e \rho_f \bar{v}_{fi}^2 C_r^* \qquad \text{exit orifice}$$

On the other hand, the perturbed system pressure drop $\delta\Delta P_{ex}$ can be obtained by subtracting Eq. (VI5.3) from Eq. (VI5.2). Thus

$$\delta\Delta P_{ex} = \delta\Delta P_{01} + \delta\Delta P_{12} + \delta\Delta P_{23} + \delta\Delta P_{34} \qquad (VI5.6)$$

where each term on the right hand side of the equation is given by Eq. (VI1.5), (VI2.4), (VI3.41), and (VI4.25), respectively. Upon substituting these expressions into Eq. (VI5.6), we can obtain the relation between $\delta \Delta P_{ex}$ and δv, i.e., the perturbed system pressure drop and velocity. Thus

$$\frac{\delta \Delta P_{ex}}{\delta v} = \rho_f \left\{ \left(\frac{A_c}{A_o}\right) S \ell_c + \frac{f_c \ell_c}{2D} 2 \left(\frac{A_c}{A_o}\right)^2 \bar{v}_{fi} + 2 k_i \bar{v}_{fi} \right\} + \tag{VI5.7}$$

$$+ \rho_f \left\{ S \bar{\lambda} + \frac{f_s}{2D} 2 \bar{v}_{fi} \bar{\lambda} + \left(g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right) \Lambda_2(S) \right\} +$$

$$+ \left\{ \Lambda_8(S) + \Lambda_9(S) + \Lambda_{10}(S) + \Lambda_{11}(S) + \Lambda_{12}(S) - \rho_f \left(g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right) \Lambda_2(S) \right\} +$$

$$+ \left\{ \Lambda_{13}(S) + \Lambda_{14}(S) + \Lambda_{15}(S) + \Lambda_{16}(S) + \Lambda_{12}(S) \left(e^{-S \bar{\tau}_{34}} - 1 \right) \right\}$$

Here each group on the right hand side of the equation gives the transfer functions between the regional pressure drop responses and the velocity perturbation.

For simplicity, let us define the new transfer functions Λ_A, Λ_B, Λ_C, and Λ_D by

$$\Lambda_A(S) \equiv \rho_f \left\{ \left(\frac{A_c}{A_o}\right) S \ell_c + \frac{f_o \ell_o}{2D_o} 2 \left(\frac{A_c}{A_c}\right)^2 \bar{v}_{fi} + 2 k_i \bar{v}_{fi} \right\} \tag{VI5.8}$$

$$\Lambda_B(S) \equiv \rho_f \left[S \bar{\lambda} + \frac{f_s}{2D} 2 \bar{v}_{fi} \bar{\lambda} \right] + \rho_f \left(g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right) \Lambda_2(S) \tag{VI5.9}$$

$$\Lambda_C(S) \equiv \Lambda_8(S) + \Lambda_9(S) + \Lambda_{10}(S) + \Lambda_{11}(S) + \Lambda_{12}(S) - \rho_f \left(g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right) \Lambda_2(S) \tag{VI5.10}$$

and

$$\Lambda_D(S) \equiv \Lambda_{13}(S) + \Lambda_{14}(S) + \Lambda_{15}(S) + \Lambda_{16}(S) + \Lambda_{12}(S) \left(e^{-S \bar{\tau}_{34}} - 1 \right) \tag{VI5.11}$$

where subscripts A, B, C, and D refer to the regions (A), (B), (C), and (D), respectively. Furthermore, the functions Λ_8 to Λ_{16} are defined by Eq. (VI3.7,16,23,28, and 36) and Eq. (VI4.8,12,16, and 20).

By substituting Eq. (VI5.8,9,10, and 11) into Eq. (VI5.7), we obtain

$$\delta \Delta P_{ex} = \varepsilon e^{st}\left[\Lambda_A(s)+\Lambda_B(s)+\Lambda_C(s)+\Lambda_D(s)\right] \qquad \text{(VI5.12)}$$

Equation (VI5.12) is a solution for the dynamics of the system, since this expression gives the response of the system to the initial flow perturbation or vice versa in terms of the transfer functions.

VI-6. General Characteristic Equations

In the preceding sections, the perturbation method has been applied to the analysis of the dynamical problem of the two-phase flow systems. The solution has been obtained in a form of a functional relation between the perturbation of the system pressure drop ΔP_{ex} and of the incoming fluid velocity δv, i.e., Eq. (VI5.12), which is expressed in terms of the transfer functions defined in Section VI. Thus we have

$$\delta\Delta P_{ex}(s,t) = \delta v(s,t)\left[\Lambda_A(s)+\Lambda_B(s)+\Lambda_C(s)+\Lambda_D(s)\right] \qquad \text{(VI6.1)}$$

In order to examine the stability of the system, it is necessary to specify the generalized input force and the output displacement. Although in our analysis we assumed the initial flow perturbation δv and obtained the response of ΔP_{ex} to it, the physical process occurring in the system is exactly the opposite. In other words, the input force imposed on the system is the pressure drop perturbation $\delta\Delta P_{ex}$, and it induces the change in the flow field. Hence δv is the generalized displacement. This relationship between input and output can be conveninetly displayed by the block diagram shown in Fig. 7. Here $1/Q(s)$ is called the system transfer function, and the dynamic response can be represented by the transformation

$$\delta v = \left[\frac{1}{Q(s)}\right] \delta \Delta P_{ex} \tag{VI6.2}$$

where the characteristic function $Q(s)$ can be obtained from Eq. (VI6.1) and (VI6.2). Thus

$$Q(s) = \Lambda_A(s) + \Lambda_B(s) + \Lambda_C(s) + \Lambda_D(s) \tag{VI6.3}$$

According to control theory, the asymptotic stability of the system can be determined by the nature of the roots of the characteristic equation given by

$$Q(s) = 0 \tag{VI6.4}$$

With Eq. (VI6.4), the formulation is now essentially complete, since the initial problem of determining the dynamic stability of the physical system is reduced to the mathematical problem of the complex functional analysis. More specifically, now our problem becomes: to examine the nature of the roots in the complex plane for the character-

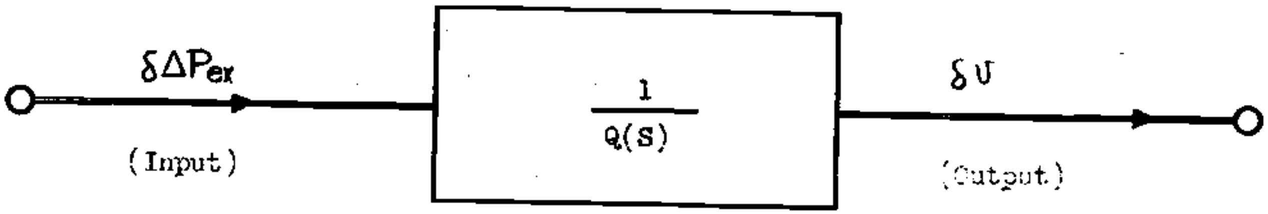

Figure 7. Block Diagram of the System

istic equation given by

$$Q(S) \equiv \Lambda_A(S) + \Lambda_B(S) + \Lambda_C(S) + \Lambda_D(S) = 0 \qquad \text{(VI6.5)}$$

Here it can be mentioned briefly that, if the characteristic equation (VI6.5) has all its roots in the left half of the S-plane, every component of disturbance tends to zero as t → ∞. Thus, this is the necessary condition for the <u>asymptotic stability</u>. Furthermore, if the characteristic equation has a root with a positive real part, the disturbance grows with time and hence it is unstable.

On the other hand, when the <u>excursive instability</u> is considered, it is useful to transform Eq. (VI5.5) in the following form

$$\frac{\delta \Delta P_{ex}}{\delta U} = \frac{\delta \Delta P_{c1}}{\delta U} + \frac{\delta \Delta P_{12}}{\delta U} + \frac{\delta \Delta P_{23}}{\delta U} + \frac{\delta \Delta P_{34}}{\delta U} = Q(S) \qquad \text{(VI6.6)}$$

By taking the limit S → 0, the perturbation becomes constant. Hence the excursive stability condition can be given by

$$\lim_{S \to 0} \frac{\delta \Delta P_{ex}}{\delta U} > 0 \qquad \text{(VI6.6)}$$

Recalling that ϵ, the magnitude of the velocity perturbation, is an arbitrarily small constant, the inequality (VI6.7) reduces to the <u>criteria obtained by Ledinegg</u> (2). Namely,

$$\frac{\partial \Delta P_{ex}}{\partial U} > 0 \qquad \text{(VI6.8)}$$

From Eq. (VI6.6) and (VI6.7), it follows that

$$\lim_{S \to 0} Q(S) > 0 \tag{VI6.9}$$

is the condition for the <u>excursive stability</u>. In view of Eq. (VI6.5), this corresponds to a singularity in the dynamic stability analysis. Therefore, the examination of the characteristic equation (VI6.5) and of the nature of its roots in the complex S-plane are sufficient for <u>both the dynamic stability and excursive stability analyses</u>.

<u>VI-7. Summary on the Transfer Function</u>

In the following, we shall summarize the various transfer functions as well as the characteristic functions which have been obtained from the theoretical analysis developed in the preceding chapters.

$$\frac{\delta \tau_{12}}{\delta v} \equiv \Lambda_1(S) = -\frac{1}{f(\bar{\lambda})\bar{v}_{fi}} \left\{ g(\bar{\lambda}, S) - \left[f(\bar{\lambda}) - f(0)\right]\right\} \frac{e^{-S\bar{\tau}_{12}}}{S} \tag{VI7.1}$$

$$\frac{\delta \lambda}{\delta v} \equiv \Lambda_2(S) = \frac{1 - e^{-S\bar{\tau}_{12}}}{S} + \bar{v}_{fi}\, \Lambda_1(S) \tag{VI7.2}$$

$$\frac{\delta C_k}{\delta v} \equiv \Lambda_3(S) = 1 - \Omega_0 f(\bar{\lambda})\, \Lambda_2(S) \tag{VI7.3}$$

$$\frac{1}{\rho_f}\frac{\delta P_m}{\delta v} \equiv \Lambda_4(z, S) = \left[\frac{C_k(\bar{\lambda})}{C_k(z)}\right] e^{-S\left[E(z) - E(\bar{\lambda})\right]} \left\{\frac{\Omega_0 f(\bar{\lambda})}{C_k(\bar{\lambda})}\Lambda_2(S) + \right. \tag{VI7.4}$$

$$\left. + \Lambda_3(S)\left[H(z, S) - H(\bar{\lambda}, S)\right]\right\}$$

$$\frac{\delta v_m}{\delta v} = \Lambda_5(z, S) = \Lambda_3(S) + \left[\frac{C_k(z)}{C_k(\bar{\lambda})}\right]^2 V_{gj}\, \Lambda_4(z, S) \tag{VI7.5}$$

$$\frac{1}{\rho_f}\frac{\delta f_{me}}{\delta v} \equiv \Lambda_6(z,s) = e^{-s\left[\frac{z-\ell}{\bar{v}_{fi}C_r^*}\right]\left(\frac{A_e}{A_c}\right)}\Lambda_4(\ell,s) \tag{VI7.6}$$

$$\frac{\delta f_{mi}}{\delta v} \equiv \Lambda_7(z,s) \tag{VI7.7}$$

$$\frac{\delta f_{me}}{\delta v} \equiv \Lambda_7'(z,s) \tag{VI7.8}$$

$$\frac{\delta \Delta P_{23a}}{\delta v} \equiv \Lambda_8(s) = s\,\rho_f\int_{\bar{\lambda}}^{\ell}\left[\frac{C_k(\bar{\lambda})}{C_k(z)}\right]\Lambda_5(z,s)\,dz \tag{VI7.9}$$

$$\frac{\delta \Delta P_{23c}}{\delta v} \equiv \Lambda_9(s) = \rho_f\bar{v}_{fi}^{\,2}\Big\{\left[C_r^*\Lambda_4(\ell,s) - \Lambda_4(\bar{\lambda},s)\right] + \tag{VI7.10}$$

$$+\frac{2}{\bar{v}_{fi}}\left[\Lambda_5(\ell,s)-\Lambda_5(\bar{\lambda},s)\right] - \frac{\Omega_0 f(\bar{\lambda})}{C_k(\bar{\lambda})}\Lambda_2(s) + \frac{1}{\bar{v}_{fi}}s\int_{\bar{\lambda}}^{\ell}\left[\frac{C_k(z)}{C_k(\bar{\lambda})}\right]\Lambda_4(z,s)\,dz$$

$$\frac{\delta \Delta P_{23g}}{\delta v} \equiv \Lambda_{10}(s) - g\,\rho_f\,\Lambda_2(s) \tag{VI7.11}$$

$$\Lambda_{10}(s) \equiv g\,\rho_f\int_{\bar{\lambda}}^{\ell}\Lambda_4(z,s)\,dz \tag{VI7.12}$$

$$\frac{\delta \Delta P_{23f}}{\delta v} \equiv \Lambda_{11}(s) - \frac{f_s}{2D}\,\rho_f\,\bar{v}_{fi}^{\,2}\Lambda_2(s) \tag{VI7.13}$$

$$\Lambda_{11}(s) \equiv \int_{\bar{\lambda}}^{\ell}\frac{1}{2D}\Big\{\bar{f}_m\,\rho_f\,\bar{v}_{fi}^{\,2}\left[\frac{C_k(z)}{C_k(\bar{\lambda})}\right]^2\Lambda_4(z,s) + \tag{VI7.14}$$

$$+\,2\,\bar{f}_m\,\rho_f\,\bar{v}_{fi}\,\Lambda_5(z,s) + \rho_f\,\bar{v}_{fi}^{\,2}\left[\frac{C_k(z)}{C_k(\bar{\lambda})}\right]\Lambda_7(z,s)\Big\}\,dz$$

$$\frac{\delta \Delta P_{23d}}{\delta v} \equiv \Lambda_{12}(s) = (1-2C_r^*)\,C_r^{*2}\,\rho_g\,\Lambda_4(\ell,s)\,V_{gj}^{\,2} \tag{VI7.15}$$

$$\frac{\delta \Delta P_{34e}}{\delta v} \equiv \Lambda_{13}(s) = \rho_f \bar{v}_{fi} k_e \left[2 \Lambda_5(\ell, s) + \bar{v}_{fi} C_r^* \Lambda_4(\ell, s) \right] \quad \text{(VI7.16)}$$

$$\frac{\delta \Delta P_{34d}}{\delta v} \equiv \Lambda_{14}(s) = s \rho_f \frac{1}{C_r^*} \ell_e \left(\frac{A_c}{A_e} \right) \Lambda_5(\ell, s) \quad \text{(VI7.17)}$$

$$\frac{\delta \Delta P_{34g}}{\delta v} \equiv \Lambda_{15}(s) = g_e \rho_f \bar{v}_{fi} C_r^* \left(\frac{A_c}{A_e} \right) \frac{1 - e^{-s\bar{\tau}_{34}}}{s} \Lambda_4(\ell, s) \quad \text{(VI7.18)}$$

$$\frac{\delta \Delta P_{34f}}{\delta v} \equiv \Lambda_{16}(s) = \frac{1}{2D_e} \rho_f \bar{v}_{fi} \left(\frac{A_c}{A_e} \right)^2 \Big\{ 2 \bar{f}_{me} \ell_e \Lambda_5(\ell, s) + \quad \text{(VI7.19)}$$

$$+ \bar{f}_{me} \bar{v}_{fi}^2 C_r^* \left(\frac{1 - e^{-s\bar{\tau}_{34}}}{s} \right) \Lambda_4(\ell, s) + \bar{v}_{fi} C_r^* \int_{\ell}^{\ell + \ell_e} \Lambda_7'(s, z)\, dz$$

$$\frac{\delta \Delta P_{34d}}{\delta v} \equiv \Lambda_{12}(s) \left(e^{-s\bar{\tau}_{34}} - 1 \right) \quad \text{(VI7.20)}$$

$$\frac{\delta \Delta P_{01}}{\delta v} \equiv \Lambda_A(s) = \rho_f \left\{ \left(\frac{A_c}{A_c} \right) s \ell_c + \frac{f_c}{2D_c} 2 \left(\frac{A_c}{A_o} \right)^2 \bar{v}_{fi} + 2 k_i \bar{v}_{fi} \right\} \quad \text{(VI7.21)}$$

$$\frac{\delta \Delta P_{12}}{\delta v} \equiv \Lambda_B(s) = \rho_f \left\{ s \bar{\lambda} + \frac{f_s}{2D} 2 \bar{v}_{fi} \bar{\lambda} \right\} + \rho_f \left(g + \frac{f_s \bar{v}_{fi}^2}{2D} \right) \Lambda_2(s) \quad \text{(VI7.22)}$$

$$\frac{\delta \Delta P_{23}}{\delta v} \equiv \Lambda_C(s) = \Lambda_8(s) + \Lambda_9(s) + \Lambda_{10}(s) + \Lambda_{11}(s) + \quad \text{(VI7.23)}$$

$$+ \Lambda_{12}(s) - \rho_f \left(g + \frac{f_s}{2D} \bar{v}_{fi}^2 \right) \Lambda_2(s)$$

$$\frac{\delta \Delta P_{34}}{\delta v} \equiv \Lambda_D(s) \equiv \Lambda_{13}(s) + \Lambda_{14}(s) + \Lambda_{15}(s) + \Lambda_{16}(s) + \Lambda_{12}(s) \left(e^{-s\bar{\tau}_{34}} - 1 \right) \quad \text{(VI7.24)}$$

The characteristic function Q(s)

$$\frac{\delta \Delta P_{ex}}{\delta v} \equiv Q(s) = \Lambda_A(s) + \Lambda_B(s) + \Lambda_C(s) + \Lambda_D(s) \quad \text{(VI7.25)}$$

The various functions appearing in the above formulations are defined as follows.

$$f(z) = \frac{q_w''(z)}{\frac{1}{\ell}\int_0^\ell q_w''(z)\,dz} \tag{VI7.26}$$

$$F(z) = \int_0^z f(z)\,dz \tag{VI7.27}$$

$$g(\eta,s) = \int_0^\eta f'(\eta)\, e^{s\eta/\bar{v}_{fi}}\, d\eta \tag{VI7.28}$$

$$\bar{C}_k(z) = \bar{v}_{fi} + V_{gj} + \Omega_0\left[F(z) - F(\bar{\lambda})\right] \tag{VI7.29}$$

$$E(z) = \int \frac{1}{C_k(z)}\,dz \tag{VI7.30}$$

$$H(z,s) = -\int e^{s\left[E(z) - E(\bar{\lambda})\right]}\frac{d^2E}{dz^2}\,dz \tag{VI7.31}$$

where the boiling length $\bar{\lambda}$ is given by

$$\bar{\lambda} = F^{-1}\left(\frac{\Delta i_{12} A_c \rho_f \bar{v}_{fi}}{q_0'' \xi}\right) \tag{VI7.32}$$

and the time lags $\bar{\tau}_{12}$, $\bar{\tau}_{23}$, and $\bar{\tau}_{34}$ by

$$\bar{\tau}_{12} = \frac{\bar{\lambda}}{\bar{v}_{fi}} \tag{VI7.33}$$

$$\bar{\tau}_{23} = E(\ell) - E(\bar{\lambda}) \tag{VI7.34}$$

$$\bar{\tilde{\tau}}_{34} = \frac{l_e}{\bar{v}_{f1}\, C_r^*}\left(\frac{A_e}{A_c}\right) \qquad \text{(VI7.35)}$$

VI-8. Discussion on the Transfer Functions

Before we proceed with the application of our analysis, we shall discuss some of the important aspects of the present results in comparison with those reported previously in (23, 25, and 1). It can be said that mathematical methods applied to the previous analysis and to the present study are basically the same. Thus, all of them used the perturbation method, the linearization technique, and the assumption $\rho_m = \rho_m(i_m)$, which decouples the momentum equation from the energy and continuity equations. Furthermore, the unsteady state was generated by the inlet flow perturbation δv, which is in contrast to the analysis made in (21) where the heat flux was perturbed.

In the present analysis, nonuniform heat flux and the relative velocity between the phases has been taken into account, whereas in (23, 25, 1, and 21) the heat flux was assumed to be uniform in the axial direction. As it was mentioned in Section II-2, the analysis of (23, 25) is limited to homogeneous flow; however, this shortcoming has been removed in the analysis of Zuber (1), who used the vapor drift velocity in the formulation in order to include the diffusion effects and transformed the continuity equation into the density propagation equation (30, 32) based on the kinematic wave velocity (14). Exactly the same approach has been introduced in our analysis. Thus, the effects of the relative velocity on the characteristic equation should be identical to those reported in (1, 41).

<u>The nonuniform heat flux effects</u> on the characteristic equation are clearly demonstrated in the present analysis. Among them, the most significant and direct result is the existence of <u>the perturbation on the time lag</u> τ_{12}, i.e., the particle residence time in the heated liquid region. The transfer function between $\delta\tau_{12}$ and δV was denoted by $\Lambda_1(S)$, thus

$$\frac{\delta\tau_{12}}{\delta v} = \Lambda_1(S) = -\frac{1}{f(\bar{\lambda})\bar{v}_{fi}}\left\{\int_0^{\bar{\lambda}} f'(z)\, e^{\frac{Sz}{\bar{v}_{fi}}}\, dz - [f(\bar{\lambda}) - f(0)]\right\}\frac{e^{-S\bar{\tau}_{12}}}{S} \qquad \text{(VI8.1)}$$

In view of Section VI-7, this Λ_1 appears also in the perturbation of the boiling boundary $\delta\lambda$ in a form

$$\frac{\delta\lambda}{\delta v} = \frac{1 - e^{-S\bar{\tau}_{12}}}{S} + \bar{v}_{fi}\,\Lambda_1(S) \qquad \text{(VI8.2)}$$

where the first term on the right hand side is the familiar transfer function for the case of the uniform heat flux, (1, 22, and 25), although the steady state residence time $\bar{\tau}_{12}$ may be different from that of the q_w'' constant. Hence the change in the time lag represented by $\Lambda_1(S)$ influences $\delta\lambda$ can be shown in a slightly different way by integrating Eq. (VI8.1) by part and substituting into Eq. (VI8.2), which yields

$$\frac{\delta\lambda}{\delta v} = \int_0^{\bar{\tau}_{12}} e^{-S(\bar{\tau}_{12} - t)}\left[\frac{f(t\,\bar{v}_{fi})}{f(\bar{\tau}_{12}\bar{v}_{fi})}\right] dt \qquad \text{(VI8.3)}$$

Therefore, the time lag effect on $\delta\lambda$, i.e., the integration of the exponential term, is weighted by the ratio of the local heat flux to that of the boiling boundary. This result is rather significant, since it shows that as heat flux in the neighborhood of the boiling boundary increases, i.e., the weight function becomes less than unity, the effect of the fluctuation of $\delta\lambda$ decreases, which can be considered as a stabilizing effect. This is explained by taking the limiting case of impulse heat flux which can remove the subcooling of the liquid at once. In this case, as we can see from Eq. (VI8.3), the motion of the boiling boundary becomes zero and hence it has no influences on the pressure drop response. Consequently, there is no phase angle shift due to the residence time in the liquid region, i.e., no time lag effect from the subcooling. However, the waves propagate with finite velocities in the mixture region, and thus there is a time lag effect in the two-phase flow region. Recalling the mechanism of instability due to time lag, i.e., the phase angle shifts, it can be said that the disappearance of one of the delay effects stabilizes* the system.

The transfer functions, obtained for a general nonuniform heat flux distribution and listed in VI-7, show that they may not be expressible in terms of elementary functions depending on the functional form of the kinematic wave velocity $\overline{C}_k(z)$. In such a case, it is difficult to pursue a closed form stability analysis because of the fact that most functions to be integrated are complex variable functions. However, in

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*Relative stabilization, i.e., it does not mean that the unstable system becomes stable. In comparison with the case of the uniform heat flux, we should consider also the change in τ_{12} which has a strong influence on the stability. See Chapter XI for a more detailed discussion.

many practical applications, the kinematic wave velocity $\bar{C}_k(z)$ can be approximated by a linear function of z, though heat flux itself is not uniform. For example, if the system has a chopped sine heat flux distribution with a sufficiently high base heat flux, the errors introduced in $\bar{C}_k(z)$ by taking an average heat flux in the mixture region are sufficiently small in view of Eq. (V2.9) and (V2.12). This is illustrated in Fig. 8. This approximation seems to be quite reasonable, since most integrands appearing in the various transfer functions can be treated as functions only of $\bar{C}_k(z)$. Furthermore, they are integrated over the range $z = \bar{\lambda}$ to ℓ. Thus, what we assumed, is similar to that of the integral analysis in the boundary layer theory.* Yet, by doing so, we still retain some of the most important aspects of the nonuniformly heated system. Indeed, the perturbation of the residence time $\delta\tau_{12}$ can be taken into account. Thus, the transfer functions for the boiling boundary and for the kinematic wave velocity are exact, and they carry the effect of the nonuniform heat flux into the density, velocity, and pressure responses.

Following the above discussion, we recommend that the steady state kinematic wave velocity $\bar{C}_k(z)$, i.e., Eq. (29) of VI-7, be replaced by an expression

$$\bar{C}_k(z) \approx \bar{v}_{fi} + V_{gj} + \Omega_\infty (z - \bar{\lambda}) \qquad \text{(VI8.4)}$$

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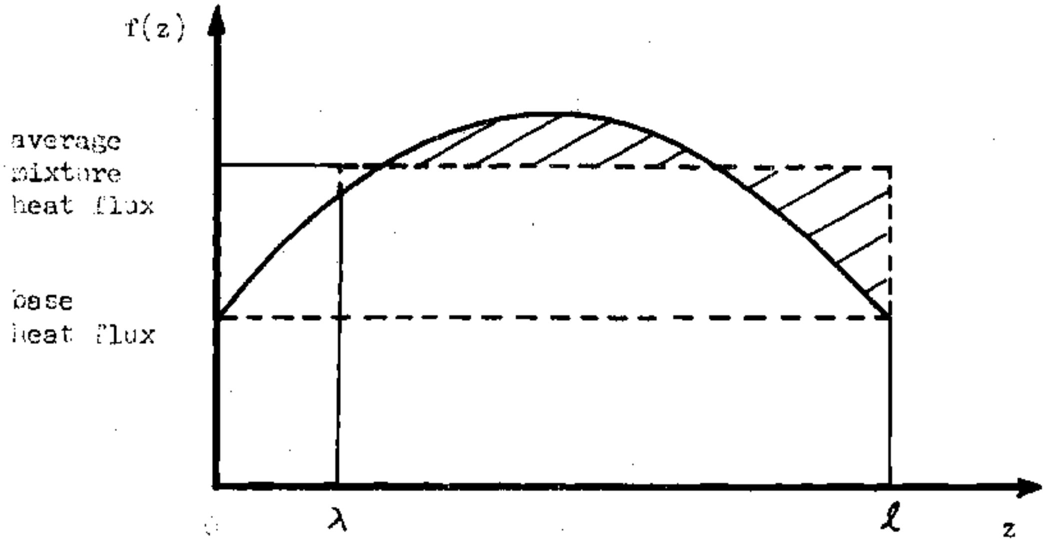
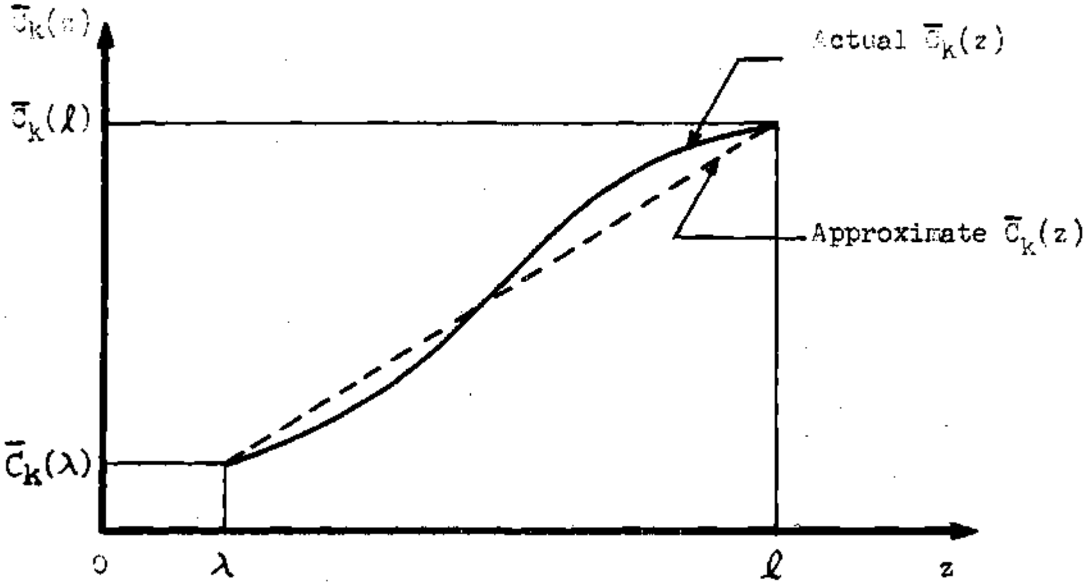
*What we are interested in is the result of the integrated variables.



(A) Dimensionless Heat Flux



(B) Kinematic Wave Velocity

Figure 8. Effect of Convex Heat Flux Profile

where

$$\Omega_{\alpha} \equiv \frac{1}{\ell-\bar{\lambda}} \int_{\bar{\lambda}}^{\ell} \Omega_{c} \left[F(z) - F(\bar{\lambda}) \right] dz \qquad \text{(VI8.5)}$$

Hence, the transfer functions in the mixture region becomes similar to those for the uniform heat flux case, except that the expressions for $\Lambda_2(S)$ and $\Lambda_3(S)$ are different.

Besides the effects of the relative velocity and of the nonuniform heat flux, one more important aspect, i.e., the time dependent or dynamic two-phase flow friction factor, has been considered on the present analysis. Thus we obtain the terms represented by the transfer functions $\Lambda_7(S)$ and $\Lambda_7'(S)$ which were neglected in (1, 19, 22, and 25). Since the functional forms of Λ_7 and Λ_7' depend on the two-phase frictional pressure drop models in use, a more detailed discussion in this connection will be given in Part 2 where we use the characteristic equation to determine the stability of the system. Finally, it should be noted that the transfer functions and the characteristic equation are for a distributed parameter system, since four field equations governing the system have been formally integrated.

CHAPTER VII

VARIOUS CHARACTERISTIC EQUATIONS

VII-1. Nonuniformly Heated System with Constant Two-Phase Friction Factor

In this section, we shall express the general characteristic equation in Chapter VI in terms of a linear kinematic wave velocity $\bar{C}_k(z)$, and the constant two-phase friction factor. As discussed in Section VI-8, the kinematic wave velocity $\bar{C}_k(z)$ for many practical cases can be approximated by

$$\bar{C}_k(z) \simeq \bar{v}_{fi} + V_{gj} + \Omega_\infty (z - \bar{\lambda}) \qquad \text{(VII1.1)}$$

where the average mixture reaction frequency Ω_∞ is given by

$$\Omega_\infty = \frac{1}{\ell - \bar{\lambda}} \int_{\bar{\lambda}}^{\ell} \Omega_o \left[F(z) - F(\bar{\lambda}) \right] dz \qquad \text{(VII1.2)}$$

We will express the two-phase friction factor f_m by the following relation

$$f_m = \bar{f}_m = \text{constant} \qquad \text{(VII1.3)}$$

Thus, we neglect the axial change of the friction factor as well as the dynamic response due to the perturbations.

Substituting Eq. (VII1.1) into Eq. (VI7.30), we get

$$E(z) - E(\bar{\lambda}) = \frac{1}{\Omega_\infty} \ln\left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right] \qquad \text{(VII1.4)}$$

which corresponds to the steady state residence time in the mixture region at z = z. Hence, the function H(z,S) now can be obtained in view of Eq. (VII1.1), (VII1.4), and (VI7.31). Thus

$$H(z,S) = \frac{\Omega_\infty}{S-\Omega_\infty}\left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^{\frac{S}{\Omega_\infty}-1} \frac{1}{\bar{C}_k(\bar{\lambda})} \qquad \text{(VII1.5)}$$

By substituting Eq. (VII1.4) and (VII1.5) into the mixture density transfer function, i.e., Eq. (VI7.4), we get

$$\frac{1}{\rho_f}\frac{\delta\rho_m}{\delta v} = \Lambda_4(z,S) = \qquad \text{(VII1.6)}$$

$$= \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 \frac{1}{\bar{C}_k(\bar{\lambda})}\frac{\Omega_\infty}{S-\Omega_\infty}\Lambda_3(S) + \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^{\frac{S}{\Omega_\infty}+1} \frac{1}{C_k(\bar{\lambda})}\left(1-\frac{S}{S-\Omega_\infty}\Lambda_3\right)$$

where Λ_2 has been eliminated by means of Eq. (VI7.3).

Accordingly, the mixture velocity transfer function Λ_5 can be obtained from Eq. (VII1.6) and (VI7.5). Thus

$$\frac{\delta v_m}{\delta v} = \Lambda_5(z,S) = \left(1+\frac{V_{gj}}{\bar{C}_k(\bar{\lambda})}\frac{\Omega_\infty}{S-\Omega_\infty}\right)\Lambda_3 + \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)}\right]^{\frac{S}{\Omega_\infty}-1}\frac{V_{gj}}{C_k(\bar{\lambda})}\left[1-\frac{S}{S-\Omega_\infty}\Lambda_3\right] \qquad \text{(VII1.7)}$$

Recalling that the two-phase friction factor f_m is constant, we have

$$\Lambda_7 = \Lambda_7' \qquad \text{(VII1.8)}$$

In view of Eq. (VII1.1), (VII1.6), and (VII1.7), all integrations appearing in the pressure drop transfer functions reduce to that of the binomial, hence they are trivial. Also we note that, from Eq. (VII1.4) and (VI7.34), we have

$$\bar{\tau}_{23} = \frac{1}{\Omega_\infty} \ln C_r^* \tag{VII1.9}$$

and hence

$$[C_r^*]^{-\frac{S}{\Omega_\infty}} = e^{-S\bar{\tau}_{23}} \tag{VII1.10}$$

which gives the time lag in the mixture region. Furthermore, from Eq. (VI8.3) we have

$$\Lambda_2 = e^{-S\bar{\tau}_{12}} \int_0^{\bar{\tau}_{12}} e^{St} \left\{ \frac{f(t\,\bar{v}_{1i})}{f(\bar{\tau}_{12}\,\bar{v}_{3i})} \right\} dt \tag{VII1.11}$$

and

$$\Lambda_3 = 1 - \Omega_0 f(\bar{\lambda})\, \Lambda_2(S) \tag{VII1.12}$$

Following the above development, it can be shown that the characteristic equation becomes

$$Q\left(S, \frac{1}{S}, \frac{1}{S-\Omega_\infty}, \frac{1}{S-2\Omega_\infty}, \Lambda_3(S), e^{-S\bar{\tau}_{13}}, e^{-S\bar{\tau}_{3*}}\right) = 0 \tag{VII1.13}$$

where the function is algebraic in terms of the arguments appearing in Eq. (VII1.13), and the detailed form of the characteristic equation for this case, expressed through operational and geometric parameters, is given in Appendix D.

It is important to note here that there are no poles for function Q(S), which can be justified by the theorem of interchanging an integral with a limit operation.*

Equation (VIII.13) clearly shows the time delay effects of the subcooling, boiling, and downstream un-heated regions represented by Λ_3, $e^{-s\bar{\tau}_{21}}$, and $e^{-s\bar{\tau}_{34}}$, respectively.

The form of the function Λ_3 depends on the dimensionless heat flux f(z); however, as we can see from Eq. (VIII.12), it has a characteristic of exponential $e^{-s\bar{\tau}_{12}}$. As a matter of fact, if f(z) has the form of a trigonometric fundtion and/or polynomials of z, $\Lambda_3(S)$ can be expressed by exponential and rational functions of S. In particular, if f(z) is a polynomial of z only, $\Lambda_3(S)$ reduces to

$$\Lambda_3(S) = 1 - \Omega_0\left[R_1(S) - R_2(S)e^{-s\bar{\tau}_{12}}\right] \qquad \text{(VIII.14)}$$

where $R_1(S)$ and $R_2(S)$ are rational functions of .

In either case, the characteristic equation can be transformed to exponential polynomials, which shows that the system is essentially governed by the difference differential equation. The order of polynomials depends on the heat flux profile f(z), and it takes the minimum of the fifth order when the system is uniformly heated.

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*The theorem is applicable, if the integrand is continuous at that limit. Indeed, this condition holds for our case (see Eq. (VIII.11) and various transfer functions appearing in Section VI-7).

VII-2. Uniformly Heated System with Constant Two-Phase Friction Factor

By treating the nonuniformly heated system, we cannot avoid arbitrariness in the characteristic equation due to the fact that f(z), the heat flux profile, is an arbitrary function. In order to eliminate this indefinite nature in our analysis, we now study in great detail both qualitatively and quantitatively the system, which is uniformly heated along the channel. There are two reasons for doing this, i.e.,:

1. For the uniformly heated system, the characteristic equation becomes simple, since the function f(z) is known and thus the parametric study is possible.

2. The characteristic equation for the uniformly heated system is a corollary of the one with nonuniform heat flux derived in VII-1. Thus the examination of the validity in the first case is quite sufficient for the proof of the applicability of both models. (Remember that there are no singularities or poles in either case.)

In the following, we take the case when the two-phase friction factor can be assumed as a constant.

By taking the heat flux as uniform, we see from Eq. (V1.9) that the average heat flux q_o'' is

$$q_o'' = q_w'' \tag{VII2.1}$$

It follows that the dimensionless heat flux f(z) is

$$f(z) = 1 \tag{VII2.2}$$

Hence, from Eq. (V1.11) we have

$$F(z) = z \tag{VII2.3}$$

In view of Eq. (V1.27) and (VII2.3), the boiling length $\bar{\lambda}$ becomes

$$\bar{\lambda} = \frac{\Delta i_{12} A_c \rho_f \bar{v}_{fi}}{q_w'' \xi} \tag{VII2.4}$$

and the residence time in the heated liquid region is

$$\bar{\tau}_{12} = \frac{\bar{\lambda}}{\bar{v}_{fi}} \tag{VII2.5}$$

The reaction frequencies are defined by Eq. (V2.4), (V2.6), and (VII1.2) and thus become identical.

$$\Omega = \Omega_c = \Omega_{co} = \frac{q_w'' \xi}{A_c \Delta i_{fg}} \frac{\Delta \rho}{\rho_g \rho_f} \tag{VII2.6}$$

By substituting Eq. (VII2.6) into Eq. (V2.16), the kinematic wave velocity has the form

$$\bar{C}_k(z) = \bar{v}_{fi} + V_{gj} + \Omega(z - \bar{\lambda}) \tag{VII2.7}$$

Furthermore, the uniform heat flux condition corresponds to no perturbation on the residence time. Thus from Eq. (VI7.1)

$$\frac{\delta \tau_{23}}{\delta v} = \Lambda_1(s) = 0 \tag{VII2.8}$$

Hence the fluctuation of the boiling boundary expressed by Eq. (VI7.2) reduces to

$$\frac{\delta\lambda}{\delta v} = \Lambda_2(s) = \frac{1 - e^{-s\bar{\tau}_{12}}}{s} \qquad \text{(VII2.9)}$$

By substituting Eq. (VII2.9) and (VII2.6) into (VI7.3), we have

$$\frac{\delta c_k}{\delta v} = \Lambda_3(s) = 1 - \frac{\Omega\,(1 - e^{-s\bar{\tau}_{12}})}{s} \qquad \text{(VII2.10)}$$

In view of Eq. (VII2.2) to (VII2.10), the characteristic equation for the uniformly heated system can be readily obtained from Appendix D. Thus we have, for the characteristic equation

$$Q(s) = Q\left(s, \frac{1}{s}, \frac{1}{s-\Omega}, \frac{1}{s-2\Omega}, e^{-s\bar{\tau}_{12}}, e^{-s\bar{\tau}_{23}}, e^{-s\bar{\tau}_{34}}\right) = 0 \qquad \text{(VII2.11)}$$

Here the function Q is algebraic in terms of the arguments appearing in Eq. (VII2.11). By recalling that $S = 0$, $s - \Omega = 0$, and $s - 2\Omega = 0$ are not poles of the characteristic equation, it can be rewritten as

$$s\,(s - \Omega)^2\,(s - 2\,\Omega)\,Q(s) = 0 \qquad \text{(VII2.12)}$$

which in expanded form reduces to the fifth order exponential polynomials with three time delays $e^{-s\bar{\tau}_{12}}$, $e^{-s\bar{\tau}_{23}}$, $e^{-s\bar{\tau}_{34}}$. This equation is exactly the same as that obtained in (1), except in our analysis we have the effects of the upstream and downstream un-heated regions. Thus, one additional time delay $e^{-s\bar{\tau}_{34}}$, which takes into account the effect of the wave propagation time in the adiabatic two-phase region, enters into the characteristic equation.

VII-3. Effects of Various Fraction Factor Models on the Characteristic Equation

The frictional pressure drop response in the two-phase region largely depends on the friction factor model used. It is best characterized by the transfer function Λ_{11} , i.e., Eq. (VI7.14), although the downstream un-heated region frictional pressure drop response Λ_{1L} , i.e., Eq. (VI7.19), is also affected.

To simplify the analysis, let us consider now the effects of various friction factor models only in the boiling region.

Recalling Eq. (VI3.1), we have

$$f_m = \bar{f}_m + \delta f_m(z,t) \tag{VII3.1}$$

where the first term on the right hand side represents the steady state value for f_m and the second term gives its perturbed part. The equation for δf_m is given by Eq. (VI7.7), hence

$$\frac{\delta f_m}{\delta v} = \Lambda_7 (z,s) \tag{VII3.2}$$

In view of Eq. (VII3.1) and (VII3.2), the general frictional pressure drop response can be given by Eq. (VI7.14). Thus

$$\Lambda_{11}(s) = \int_{\bar{\lambda}}^{\ell} \frac{1}{2D} \left\{ \bar{f}_m \rho_f \bar{v}_{fi}^2 \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 \Lambda_4(z,s) + \right. \tag{VII3.3}$$

$$\left. + 2 \bar{f}_m \rho_f \bar{v}_{fi} \Lambda_5(z,s) + \rho_f \bar{v}_{fi}^2 \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right] \Lambda_7(z,s) \right\} dz$$

For a given functional form of $\bar{f}_m(z)$ and $\Lambda_7(z,s)$, the right hand sides of Eq. (VII3.3) can be integrated. In order to carry out the integration in simple form as well as to decouple the effects of nonuniform heat flux profiles from those of the different friction factor models, we shall consider only the case of a uniformly heated system in the following analysis. The problem will be considered from both the static aspect, i.e., the effects of magnitude of friction factor, and the dynamic aspect, i.e., the effects of perturbation in f_m.

A. Static Aspect of the Friction Factor f_m

A-1. General Form of the Transfer Function. By assuming that the two-phase friction factor depends only on the steady state effects, we have

$$f_m = \bar{f}_m(z) \tag{VII3.4}$$

and

$$\delta f_m = 0 \tag{VII3.5}$$

This approximation leads to the conclusion that the dynamic response of the frictional pressure drop is similar to the inertia wave, i.e., the response of $\rho_m v_m^2$. Hence the transfer function Λ_{11} reduces to

$$\Lambda_{11}(s) = \int_{\bar{\lambda}}^{\ell} \frac{1}{2D} \left\{ \bar{f}_m \rho_f \bar{v}_{fi}^2 \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 \Lambda_4(z,s) + 2 \bar{f}_m \rho_f \bar{v}_{fi} \Lambda_5(z,s) \right\} dz \tag{VII3.6}$$

In case of a uniformly heated system, $\bar\lambda$, $\bar\tau_{12}$, Ω, $\bar C_k(z)$, $\Lambda_1(S)$, $\Lambda_2(S)$, Λ_3 (S) are given by Eq. (VII2.4,5,6,7,8,9, and 10), respectively. Thus, from Eq. (VI7.4) and (VI7.5), we obtain

$$\frac{1}{\rho_f}\frac{\delta\rho_m}{\delta v} = \Lambda_4(z,s) = \frac{1}{\bar C_k(\bar\lambda)}\,\frac{s}{s-\Omega}\left\{\left[\frac{\bar C_k(\bar\lambda)}{\bar C_k(z)}\right]^2\Lambda_3 - \left[\frac{\bar C_k(\bar\lambda)}{\bar C_k(z)}\right]^{\frac{s}{\Omega}+1} e^{-s\bar\tau_{12}}\right\} \qquad \text{(VII3.7)}$$

and

$$\frac{\delta v_m}{\delta v} = \Lambda_5(z,s) = \Lambda_3(s) + \frac{\Omega}{s-\Omega}\,\frac{V_{gj}}{\bar C_k(\bar\lambda)}\left\{\Lambda_3 - \left[\frac{\bar C_k(\bar\lambda)}{\bar C_k(z)}\right]^{\frac{s}{\Omega}-1} e^{-s\bar\tau_{12}}\right\} \qquad \text{(VII3.8)}$$

A-2. Lumped Friction Factor Model $f_m = f_s\bar C_m$. The simplest form of the friction factor model can be obtained by the lumped parameter method. Thus f_m is given by the following equation

$$\bar f_m = f_s\bar C_m \qquad \text{(VII3.9)}$$

where f_s is the liquid friction factor at the boiling boundary, and $\bar C_m$ = constant.

By substituting Eq. (VII3.7), (VII3.8), and (VII3.9) into Eq. (VII3.6), it can be shown that

$$\Lambda_{11}(s) = \frac{f_s\bar C_m}{2D}\,\rho_f\,\bar v_{12}\,(\ell-\bar\lambda)\left\{\left[2\left(\frac{s-\Omega}{\Omega}\right) + \frac{\bar C_k(\bar\lambda)+V_{gj}}{\bar C_k(\bar\lambda)}\right]\frac{\Omega}{s-\Omega}\Lambda_3(s) - \right. \qquad \text{(VII3.11)}$$

$$\left. -\left[\frac{\bar C_k(\bar\lambda)+V_{gj}}{\bar C_k(\bar\lambda)}\right]\left(\frac{\Omega}{s-\Omega}\right)\left(\frac{\Omega}{s-2\Omega}\right)\left[\frac{e^{-s\bar\tau_{12}} - C_r^{*2}e^{-s\bar\tau_{13}}}{C_r^{*}-1}\right]\right\}$$

<u>A-3. Friction Factor Based on the Void Fraction</u>. In order to take into account the effect of the void fraction on the friction factor, we consider a simple model given by

$$\bar{f}_m \approx \frac{f_s}{(1-\alpha)^{n'}} \approx f_s \left(\frac{\rho_f}{\rho_m}\right)^{n'} \qquad \text{(VII3.12)}$$

Since the steady state density ratio can be expressed in terms of the kinematic wave velocity, i.e., Eq. (V2.48), we have

$$\bar{f}_m = f_s \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^{n'} \qquad \text{(VII3.13)}$$

where

$$0 < n' < 1 \qquad \text{(VII3.14)}$$

The magnitude of n' depends on the inlet Reynolds number N_{Res} as well as on the system pressure.

In view of Eq. (VII3.7), (VII3.8), and (VII3.13), the integration in Eq. (VII3.6) reduces to that of the binomials, hence it can be shown that

$$\Lambda_{11}(S) = \frac{f_s}{2D} \rho_f \bar{v}_{fi} (\ell - \bar{\lambda}) \left\{ \left[2\left(\frac{S-\Omega}{\Omega}\right) + \frac{C_k(\bar{\lambda}) + V_{gj}}{\bar{C}_k(\bar{\lambda})} \right] \frac{1}{n'+1} \left(\frac{C_r^{*n'}+1}{C_r^{*}-1}\right) \frac{\Omega}{S-\Omega} \Lambda_3(S) - \right. \qquad \text{(VII3.15)}$$

$$\left. - \left[\frac{\bar{C}_k(\bar{\lambda}) + V_{gj}}{\bar{C}_k(\bar{\lambda})}\right] \left(\frac{\Omega}{S-\Omega}\right) \left(\frac{\Omega}{S-[2+n']\Omega}\right) \left(\frac{e^{-S\bar{\tau}_{12}} - C_r^{*(2+n')} e^{-S\bar{\tau}_{13}}}{C_r^{*}-1}\right) \right\}$$

By comparing Eq. (VII3.11) and (VII3.15), it can be seen that the forms of the transfer function Λ_{11} for the case of the constant friction factor and of the distributed $\bar{f}_m$ are quite similar. The first terms of each transfer function represent the effects of the kinematic wave velocity. They differ only by a constant. The second term corresponds to the time delay effects in the mixture region; there is, however, a slight difference in the functional form between the models. However, $S=(2+n')\Omega$ and $S=2\Omega$ are not poles of Λ_{11}, consequently, the latter difference is not essential in a dynamic analysis.

By setting the first terms of each transfer function, i.e., Eq. (VII3.11) and (VII3.15), to be equal, we obtain

$$\bar{C}_m = \frac{1}{1+n'}\left(\frac{C_r^{*\,1+n'}-1}{C_r^* - 1}\right) \qquad \text{(VII3.16)}$$

On the other hand, the steady state solution requires

$$\bar{C}_m = \frac{2}{2+n'}\left(\frac{C_r^{*\,2+n'}-1}{C_r^* - 1}\right) \qquad \text{(VII3.17)}$$

For high pressure systems with reasonably high exit qualities, the value for $\bar{C}_m$ ranges from 1.5 to 2.5, thus we may approximate $\bar{C}_m$ by

$$\bar{C}_m \approx 2.0 \qquad \text{(VII3.18)}$$

<u>A-4. Other Frictional Pressure Drop Models</u>. Although most of the two-phase frictional pressure drop models are not written in the form of Eq. (VI3.24), the equivalent two-phase friction factor corresponding to f_m can be obtained easily. The resulting friction factor is usually a function of the void fraction α, the quality x, and the density ratio ρ_g^*.

For example, the well known correlation of Martinelli and Nelson (43) can be closely approximated by

$$f_m \doteq f_s \left\{ \frac{(1-x)^{1.75}}{(1-\alpha)^2} \frac{\rho_m}{\rho_f} \right\} \tag{VII3.19}$$

On the other hand, the model obtained by Wallis (44) gives

$$f_m \doteq 0.02 \left\{ \frac{(1-x)^2}{(1-\alpha)^2} \frac{\rho_m}{\rho_f} \right\} \tag{VII3.20}$$

Following the assumption given by Eq. (VII3.4), the variables x, α, and ρ_m take values at corresponding steady state conditions. Thus, the liquid quality is given by

$$1 - \bar{x} = 1 - \frac{\rho_g}{\Delta\rho} \frac{\bar{U}_{fi} + V_{gj}}{\bar{U}_{fi}} \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})} - 1 \right] \tag{VII3.21}$$

and the liquid void fraction by

$$1 - \bar{\alpha} = \frac{\rho_f}{\Delta\rho} \left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)} \right] - \frac{\rho_g}{\Delta\rho} \tag{VII3.22}$$

Recalling Eq. (V2.48), the mixture density can be expressed by

$$\frac{\bar{\rho}_m}{\rho_f} = \frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)} \tag{VII3.23}$$

In view of Eq. (VII3.7,8,21,22, and 23), the variable of integration in Eq. (VII3.6) can be transformed from z to $C_k(z)$. However, due to the existence of the $(1-\alpha)^2$ term in the denominator of the two-phase friction factor models, i.e., Eq. (VII3.19) or (VII3.20), the analytical expression for Λ_{11} may not yield a simple form. This difficulty can be eliminated by taking only the first term of the right hand side of Eq. (VII3.22) under the condition that $\rho_f \gg \rho_g$ and the quality x is sufficiently less than 1. In any case, the dynamic characteristic of the transfer function Λ_{11} corresponding to the Martinelli and Nelson correlation or to the Wallis model is expected to be quite similar to those obtained in Sections A-2 and A-3. This can be justified by comparing Eq. (VII3.19) or (VII3.20) to Eq. (VII3.31). By taking a proper value for in Eq. (VII3.13), the former two equations can be closely approximated by the latter for a wide range of the void fraction α. Furthermore, it should be remembered that the transfer function Λ_{11} does not have any poles. Thus, the continuity of the solution for Λ_{11} is expected.

B. Dynamic Aspect of the Friction Factor f_m

The dynamic effects of the friction factor f_m can be studied by considering the perturbation δf_m, i.e., Eq. (VII3.1), then the general form of the transfer function $\Lambda_{11}(S)$ is given by Eq. (VII3.3). The first two terms of the right hand side of Eq. (VII3.3) represent static effects of the two-phase friction factor, whereas the last term takes into account the dynamic response of f_m due to the velocity perturbation δU, i.e., $\Lambda_7(z,S)$. In order to carry out the integration in Eq. (VII3.3), it is necessary to know the equation for f_m under the steady state as well as the dynamic condition. As it has been explained in Chapter VI, we assume that the steady state equation for f_m can be written in terms of kinematic variables and, furthermore, this relation can be extended to the unsteady conditions with perturbed kinematic variables.

Thus, if the two-phase friction factor can be written as

$$f_m = f_m \left(\rho_m , v_m , X , \alpha \right) \tag{VII3.24}$$

then the steady-state part is given by

$$\bar{f}_m = \bar{f}_m \left(\bar{\rho}_m , \bar{v}_m , \bar{X} , \bar{\alpha} \right) = \bar{f}_m \left(\bar{C}_k(z) \right) \tag{VII3.25}$$

and the perturbed part by

$$\frac{\partial f_m}{\delta \upsilon} = \Lambda_7(z,S) = \frac{\partial \bar{f}_m}{\partial \bar{\rho}_m}\left(\frac{\delta \rho_m}{\delta \upsilon}\right) + \frac{\partial \bar{f}_m}{\partial \bar{v}_m}\left(\frac{\delta v_m}{\delta \upsilon}\right) + \frac{\partial \bar{f}_m}{\partial \bar{X}}\left(\frac{\delta \bar{X}}{\delta \upsilon}\right) + \frac{\partial \bar{f}_m}{\partial \bar{\alpha}}\left(\frac{\delta \alpha}{\delta \upsilon}\right) \tag{VII3.26}$$

The void transfer function can be obtained from the definition of the mixture density, thus

$$\frac{\delta \alpha}{\delta \upsilon} = - \frac{1}{\Delta \rho} \frac{\delta \rho_m}{\delta \upsilon} = - \frac{\rho_f}{\Delta \rho} \Lambda_4(z,S) \tag{VII3.27}$$

And the vapor quality transfer function is given by

$$\frac{\delta X}{\delta \upsilon} = \frac{\rho_g}{\Delta \rho}\left[1 - \frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)}\right]\frac{1}{\bar{v}_{fi}}\left\{\Lambda_3(S) - \frac{\bar{C}_k(\bar{\lambda})}{\bar{v}_{fi}}\Lambda_5(z,S)\right\} - \tag{VII3.28}$$

$$- \frac{\rho_g}{\Delta \rho} \frac{C_k(\bar{\lambda})}{\bar{v}_{fi}} \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right]^2 \Lambda_4(z,S)$$

where Λ_3, Λ_4, and Λ_5, i.e., the transfer functions for the kinematic wave velocity, for the mixture density, and for the mixture velocity, are defined in Section VI-7.

Hence, by knowing the steady-state friction factor model, i.e., Eq. (VII3.25), the dynamic response of the friction factor $\Lambda_7(z,S)$ can be obtained. Substituting the expression for $\bar{f}_m$ and for Λ_7, i.e., Eq. (VII3.25) and (VII3.26), respectively, into Eq. (VII3.3), we get the two-phase frictional pressure drop response $\Lambda_{11}(S)$. Since the quality $\bar{x}$ and the void fraction $\bar{\alpha}$ as well as $\bar{\rho}_m$ and $\bar{v}_m$ can be written in terms of $C_k(z)$, the variable of integration in Eq. (VII3.3) may be transformed from z to $C_k(z)$.

In view of Eq. (VII3.7,8,26,27, and 28), we can see that the integrand of Eq. (VII3.3) increases in complexity as the degree of sophistication in the two-phase friction factor model f_m, i.e., Eq. (VII3.24), increases. From the numerical point of view, it is desirable to have as few terms as possible in the expression for Λ_7, the friction factor response function. By considering the entire functional dependence of ρ_m, v_m, $\bar{x}$, and $\bar{\alpha}$ in f_m, it can be said that the two-phase frictional pressure drop response δf_m generates a large number of terms in the characteristic equation. Thus, a higher degree of accuracy is required to determine the dynamic response of f_m. On the other hand, our analysis is based on the assumption that the steady state relation for f_m can be extended to the unsteady operation. The reliability of Eq. (VII3.26) is limited. As a conclusion, a simple friction factor model such as the one based on the void fraction is better suited for our analysis than the correlations given by Martinelli and Nelson (43) or by Wallis (44).*

Thus we recommend the use of

$$f_m = \frac{f_s}{(1-\alpha)^{n'}} \qquad \text{(VII3.29)}$$

for the <u>analysis of dynamic response of the friction factor</u>. However, the relation given by Eq. (VII3.29) has a singularity at $\alpha=1$; the modification similar to the one made in A-3 is desirable.

Hence, we approximate

$$f_m \approx f_s \left(\frac{\rho_f}{\rho_m}\right)^{n'} \qquad \text{(VII3.30)}$$

which we call the <u>density type friction factor</u>. On the other hand, Eq. (VII3.29) can also be approximated by the velocity ratio instead of the density ratio. Thus, we have the <u>velocity type friction factor</u>

$$f_m \doteq f_s \left(\frac{v_m}{\bar{v}_{fi}}\right)^{n'} \qquad \text{(VII3.31)}$$

In view of Eq. (IV4.4) and (VII3.4,30, and 31), it can be seen that the dynamic response of the two-phase frictional pressure drop takes the form for <u>the static type model</u>, i.e., Eq. (VII3.4)

$$\delta\left(\frac{\partial P}{\partial z}\right)_{2Pf} \sim \delta\left(\rho_m v_m^2\right) \qquad \text{(VII3.32)}$$

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*Note that the form is as justified as that of Wallis or Martinelli. Also that it gives good agreement with experimental data for the steady state.

the density type model, i.e., Eq. (VII3.30)

$$\delta\left(\frac{\partial P}{\partial z}\right)_{2Pf} \sim \delta\left(\rho_m^{1-n'} v_m^2\right) \qquad \text{(VII3.33)}$$

and the velocity type model, i.e., Eq. (VII3.31)

$$\delta\left(\frac{\partial P}{\partial z}\right)_{2Pf} \sim \delta\left(\rho_m v_m^{2+n'}\right) \qquad \text{(VII3.34)}$$

The three relations above clearly show the responses characteristic of $\left.\frac{\partial P}{\partial z}\right)_{2Pf}$ with respect to the inertia of the mixture $\rho_m v_m^2$ for the particular friction factor models, whereas the steady state friction factor f_m is given by the common expression for these three cases. Thus

$$f_m = f_S \left\{\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right\}^{n'} \qquad \text{(VII3.35)}$$

B-1. Density Type Friction Factor. First let us consider the case of the density type friction factor model, i.e., Eq. (VII3.26). Recalling Eq. (V2.48), (V2.49), and (VII3.2), the friction factor transfer function Λ_7 becomes

$$\Lambda_7(z,s) = -n' \bar{f}_m \left[\frac{\bar{C}_k(z)}{\bar{C}_k(\bar{\lambda})}\right] \Lambda_4(z,s) \qquad \text{(VII3.36)}$$

which shows the dependence of the friction factor on the mixture density, i.e., Eq. (VII3.30).

Substituting Eq. (VII3.35) and (VII3.36) into Eq. (VII3.3), the right hand side integration can be calculated with Eq. (VII2.7). Thus we obtain the two-phase frictional pressure drop transfer function Λ_{11} ,

$$\Lambda_{11}(S) = \frac{f_s}{2D}\rho_f\bar{v}_{fi}(\ell-\bar{\lambda})\left\{\left[2\left(\frac{S-\Omega}{\Omega}\right)+A\right]\left(\frac{1}{n'+1}\right)\left(\frac{C_r^{*\,n'+1}+1}{C_r^{*}-1}\right)\left(\frac{\Omega}{S-\Omega}\right)\Lambda_3 - \right. \quad \text{(VII3.37)}$$

$$\left. -A\left(\frac{\Omega}{S-\Omega}\right)\left(\frac{\Omega}{S-[n'+2]\Omega}\right)\left[\frac{e^{-S\bar{\tau}_{12}}-C_r^{*(2+n')}e^{-S\bar{\tau}_{13}}}{C_r^{*}-1}\right]\right\}$$

where A is given by

$$A = \frac{(1-n')C_k(\bar{\lambda})+(1+n')V_{fi}}{C_k(\bar{\lambda})} \qquad \text{(VII3.38)}$$

By comparing this result to the one with the static friction factor model, i.e., Eq. (VII3.19), it can be seen that the dynamic friction factor influence appears only in the terms represented by A. Furthermore, these effects are limited to the shifting of the coefficients and thus the functional form of Λ_{11} is preserved.

<u>B-2. Velocity Type Friction Factor</u>. For the case of the velocity type friction factor model, i.e., Eq. (VII3.31), the transfer function Λ_7 , can be obtained from Eq. (V2.56), (V2.57), and (VII3.2). Thus we have

$$\Lambda_7 = n'\bar{f}_m\left[\frac{\bar{C}_k(\bar{\lambda})}{\bar{C}_k(z)}\right]\frac{1}{\bar{v}_{fi}}\Lambda_5 \qquad \text{(VII3.39)}$$

which shows the dependence of f_m on the mixture velocity v_m. Substituting Eq. (VII3.35) and (VII3.39) into Eq. (VII3.3) and integrating the right hand side, we get

$$\Lambda_{11} = \frac{f_s}{2D} \rho_f \bar{V}_{gi} (\ell - \bar{\lambda}) \left\{ \left[2\left(1+\frac{n'}{2}\right)\left(\frac{S-\Omega}{\Omega}\right) + B \right] \left(\frac{C_r^{*\,n'+1}}{C_r^* - 1} \right) \left(\frac{\Omega}{S-\Omega} \right) \Lambda_3(S) - \right. \tag{VII3.40}$$

$$\left. - B \left(\frac{\Omega}{S-\Omega} \right) \left(\frac{\Omega}{S-(n'+2)\Omega} \right) \left[\frac{e^{-S\bar{\tau}_{12}} - C_r^{*(2+n')} e^{-S\bar{\tau}_{13}}}{C_r^* - 1} \right] \right\}$$

where B is given by

$$B = \frac{C_k(\bar{\lambda}) + (1+n') V_{gj}}{C_k(\bar{\lambda})} \tag{VII3.41}$$

Comparing these results to the case of the static friction factor or of the density type friction factor, i.e., Eq. (VII3.19) and Eq. (VII 3.37), respectively, it can be seen that the differences are limited to the magnitude of the coefficients, whereas the functional forms of Λ_{11} for each case are the same.

As a conclusion, the effect of the two-phase frictional pressure drop on the dynamics of the system can be expressed by a similar transfer function Λ_{11} <u>either for the static type or dynamic type friction factor model</u>.

Part 2

APPLICATION OF THE ANALYSIS

In what follows, we shall demonstrate the applicability of the analysis developed in Part 1. By considering the system with an arbitrary heat flux profile, the characteristic equation includes the unknown function f(z), which precludes a parametric study of the stability of a system. For example, in order to have a similarity between two systems, it is obvious from the discussion given in Section III-2 that we should have the same dimensionless heat flux f(z). Since the system with a uniform heat flux profile has the simplest characteristic equation and the parametric study on this model can give some general trend of the effects of various parameters on the stability of general systems, we shall choose this as a basic model. The analysis of any other nonuniformly heated system can be developed in exactly the same way provided the function f(z) is known. However, in such a case it is expected that the order of the exponential polynomial will increase.

By taking the system with the uniform heat flux profile, we shall express first the characteristic equation in a dimensionless form in a manner quite similar to that in Section III-2. From this we can obtain the similarity parameters which govern the stability of the system. After a brief introduction on the stability theorems (such as D-Partition Method and Nyquist Criterion), which are important for the analysis of the system governed by the exponential polynomials in terms of the auxiliary parameter S, we shall solve the characteristic equation for the stability boundary.

CHAPTER VIII

SIMILARITY GROUPS GOVERNING THE STABILITY OF THE SYSTEM

VIII-1. Dimensionless Characteristic Equation

Dimensionless Parameters

In the following analysis, we shall non-dimensionalize the characteristic equation for the uniformly heated system in a way very similar to that employed in Appendix C. We choose the length scale as ℓ, the heated channel length, and the time scale as $1/\Omega$, the reaction time in the mixture given by Eq. (VII2.6).

Based on the above fundamental scales for the time and length, we define the following dimensionless parameters.

The geometric parameters:

$$z^* = \frac{z}{\ell}, \qquad \ell_o^* = \frac{\ell_o}{\ell}, \quad \ell_e^* = \frac{\ell_e}{\ell}, \quad D_o^* = \frac{D_o}{\ell}, \qquad \text{(VIII1.1)}$$

$$D^* = \frac{D}{\ell}, \qquad D_e^* = \frac{D_e}{\ell}, \quad A_e^* = \frac{A_e}{A_c}, \quad A_o^* = \frac{A_o}{A_c}$$

The boiling length λ^* from Eq. (VII2.4)

$$\lambda^* = \frac{\bar{\lambda}}{\ell} = \frac{\Delta i_{12} A_c \rho_f \bar{v}_{fi}}{q_w' \xi \ell} \qquad \text{(VIII1.2)}$$

The velocity field:

For the inlet velocity

$$v_{fi}^{*} = \frac{\bar{v}_{fi}}{\Omega \ell} = \frac{\bar{v}_{fi} A_c \Delta i_{fg}}{q_w'' \xi \ell} \frac{\rho_g \rho_f}{\Delta \rho} \tag{VIII1.3}$$

where Ω is given by

$$\Omega = \Omega_o = \Omega_\infty = \frac{q_w'' \xi}{A_c \Delta i_{fg}} \frac{\Delta \rho}{\rho_g \rho_f}$$

For the drift velocity

$$V_{gj}^{*} = \frac{V_{gj}}{\bar{v}_{fi}} = \frac{V_{gj}}{\Omega \ell \ v_{fi}^{*}} \tag{VIII1.4}$$

For the kinematic wave velocity

$$C_k^{*}(z) = \frac{\bar{C}_k(z)}{\Omega \ell} \tag{VIII1.5}$$

Hence, from Eq. (VII2.7), we have

$$C_k^{*}(\ell) = v_{fi}^{*}(1 + V_{gj}^{*}) + (1 - \lambda^{*}) \tag{VIII1.6}$$

and

$$C_k^{*}(\bar{\lambda}) = v_{fi}^{*}(1 + V_{gj}^{*}) \tag{VIII1.7}$$

From Eq. (VIII1.6), (VIII1.7), and (VI3.15), we have

$$C_r^{*} = \frac{\bar{C}_k(\ell)}{\bar{C}_k(\bar{\lambda})} = 1 + \frac{1 - \lambda^{*}}{v_{fi}^{*}(1 + V_{gj}^{*})} \tag{VIII1.8}$$

The residence time:

We define the dimensionless time by

$$t^* = t\Omega \qquad \text{(VIII1.9)}$$

Hence, for the residence time in the heated liquid reqion from Eq. (VII2.4) and (VII2.5), we have

$$\tau_{12}^* = \bar{\tau}_{12}\Omega = \frac{\lambda^*}{\bar{v}_{fi}^*} = \frac{\Delta i_{12}}{\Delta i_{fg}} \frac{\Delta\rho}{\rho_g} \qquad \text{(VIII1.10)}$$

For the residence time in the heated mixture region from Eq. (VII1.9), we have

$$\tau_{13}^* = \bar{\tau}_{23}\Omega = \ln C_r^* \qquad \text{(VIII1.11)}$$

and for the residence time in the downstream un-heated region from Eq. (VI7.35), we have

$$\tau_{34}^* = \bar{\tau}_{34}\Omega = \frac{l_e^* A_e^*}{\bar{v}_{fi}^* C_r^*} \qquad \text{(VIII1.12)}$$

The independent variable

$$S^* = \frac{S}{\Omega} = \frac{a}{\Omega} + j\frac{\omega}{\Omega} = a^* + j\omega^* \qquad \text{(VIII1.13)}$$

The density ratio

$$\rho_g^* = \frac{\rho_g}{\rho_f} \qquad \text{(VIII1.14)}$$

The pressure drop

$$\Delta P^* = \frac{\Delta P}{(\Omega \ell)^2 \rho_f} \qquad \text{(VIII1.15)}$$

The gravitational force

$$g^* = \frac{g}{\Omega^2 \ell} \quad , \quad g_e^* = \frac{g_e}{\Omega \ell} \qquad \text{(VIII1.16)}$$

or

$$N_{Fr} = \frac{\bar{v}_{fi}^{\,2}}{g \ell} = \frac{v_{fi}^{*2}}{g^*} \,, \quad N_{Fre} = \frac{v_{fi}^{*2}}{g_e^*} \qquad \text{(VIII1.17)}$$

Dimensionless Characteristic Equation (Uniform Heat Flux)

Using the dimensionless parameters defined in A, the characteristic equation corresponding to Eq. (VII2.11) reduces to

$$\frac{Q(S)}{\rho_f (\Omega \ell) v_{fi}^*} \equiv Q^* = \frac{\Lambda_A^*}{v_{fi}^*} + \frac{\Lambda_B^*}{v_{fi}^*} + \frac{\Lambda_C^*}{v_{fi}^*} + \frac{\Lambda_D^*}{v_{fi}^*} \qquad \text{(VIII1.18)}$$

where each term in the expression for Q*(S*) represents the dimensionless pressure drop responses of the upstream un-heated, heated liquid, heated mixture, and downstream un-heated regions, respectively. The parametric expressions of the above terms are given by the following expressions.

for the upstream un-heated region

$$\frac{\delta \Delta P_{01}^*}{\delta v^* v_{fi}^*} = \frac{\Lambda_A^*}{v_{fi}^*} = 2k_i + \left(\frac{\ell_0^*}{A_0^*}\right)\frac{1}{v_{fi}^*} S^* + \frac{f_0}{2D_0^*}\, 2\,\ell_0^* \left(\frac{1}{A_0^*}\right)^2 \qquad \text{(VIII1.19)}$$

for the liquid heated region

$$\frac{\delta \Delta P_{12}^*}{\delta \upsilon^* v_{fi}^*} = \frac{\Lambda_B^*}{v_{fi}^*} = \tau_{12}^* S^* + \frac{f_s}{2D^*} 2\lambda^* + \underline{\left[\frac{v_{fi}^*}{N_{Fr}} + \frac{f_s}{2D^*} v_{fi}^*\right](1-\Lambda_3^*)} \qquad \text{(VIII1.20)}$$

for the heated mixture region

$$\frac{\delta \Delta P_{23}^*}{\delta \upsilon^* v_{fi}^*} = \frac{\Lambda_C^*}{v_{fi}^*} = \left\{ \ln C_r^* \left[(1+V_{gj}^*) + V_{gj}^* \frac{1}{S^*-1} \right] S^* \Lambda_3^* - \right. \qquad \text{(VIII1.21)}$$

$$\left. - V_{gj}^* \left(\frac{1}{S^*-1}\right)^2 S^* C_2^* \right\} + \left\{ - \frac{1}{1+V_{gj}^*}(1-\Lambda_3^*) + \ln C_r^* \left(\frac{S^*}{S^*-1}\right)\Lambda_3^* - \left[\left(\frac{1}{S^*-1}\right)^2 - \frac{V_{gj}^*}{1+V_{gj}^*}\frac{1}{S^*-1} \right] C_2^* \right\}$$

$$+ \left(\frac{v_{fi}^*}{N_{Fr}}\right) \left\{ \left(1 - \frac{1}{C_r^*}\right) \frac{\Lambda_3^*}{S^*-1} - \frac{1}{S^*} \frac{1}{S^*-1} C_1^* - \underline{(1-\Lambda_3^*)} \right\} + \frac{f_m}{2D^*} \left\{ (1-\chi^*)\left(2S^* - \frac{1}{1+V_{gj}^*}\right) \frac{\Lambda_3^*}{S^*-1} - \right.$$

$$\left. - v_{fi}^* (1+2V_{gj}^*) \left(\frac{1}{S^*-1}\right) \left(\frac{1}{S^*-2}\right) C_3^* \right\} - \underline{\frac{f_s}{2D^*} v_{fi}^* (1-\Lambda_3^*)} - \rho_g^* \frac{V_{gj}^{*2}}{1+V_{gj}^*} \left[2C_r^* - 1\right] \frac{C_4^*}{S^*-1}$$

for the downstream un-heated region

$$\frac{\delta \Delta P_{34}^*}{\delta \upsilon^* v_{fi}^*} = \frac{\Lambda_D^*}{v_{fi}^*} = k_e \left[2\Lambda_3^* + \frac{(1+2V_{gj}^*)}{(1+V_{gj}^*)} \frac{1}{S^*-1} C_4^* \right] + \qquad \text{(VIII1.22)}$$

$$+ \frac{1}{v_{fi}^*} \left(\frac{l_e^*}{A_e^*}\right) \frac{1}{C_r^*} \left[S^* \Lambda_3^* + \frac{V_{gj}^*}{1+V_{gj}^*} \frac{S^*}{S^*-1} C_4^* \right] +$$

$$+ \left(\frac{v_{fi}^*}{N_{Fre}}\right) \frac{1}{1+V_{gj}^*} \frac{1}{C_r^*} \frac{1}{S^*-1} \left(\frac{1-e^{-S^*\tau_{34}^*}}{S^*} \right) C_4^* + \frac{f_{me}}{2D_e^*} \left(\frac{1}{A_e^*}\right)^2 \left\{ 2 l_e^* \left[\Lambda_3^* + \frac{V_{gj}^*}{1+V_{gj}^*} \frac{1}{S^*-1} C_4^* \right] + \right.$$

$$\left. + \frac{v_{fi}^*}{1+V_{gj}^*} \frac{(1-e^{-S^*\tau_{34}^*})}{S^*(S^*-1)} C_4^* \right\} - \rho_g^* \frac{V_{gj}^{*2}}{(1+V_{gj}^*)} \left[2C_r^* - 1\right] \frac{1}{S^*-1} C_4^* \left(e^{-S^*\tau_{34}^*} - 1\right)$$

where Λ_3^*, C_1^*, C_2^*, C_3^*, and C_4^* are given by

$$\Lambda_3^* = \frac{\delta C_k^*}{\delta U^*} = 1 - \frac{(1 - e^{-S^*\tau_{12}^*})}{S^*} \qquad \text{(VIII1.23)}$$

$$C_1^* = e^{-S^*\tau_{12}^*} - e^{-S^*\tau_{13}^*} \qquad \text{(VIII1.24)}$$

$$C_2^* = e^{-S^*\tau_{12}^*} - C_r^* e^{-S^*\tau_{13}^*} \qquad \text{(VIII1.25)}$$

$$C_3^* = e^{-S^*\tau_{12}^*} - C_r^{*2} e^{-S^*\tau_{13}^*} \qquad \text{(VIII1.26)}$$

$$C_4^* = \Lambda_3^* - C_r^* e^{-S^*\tau_{13}^*} \qquad \text{(VIII1.27)}$$

and τ_{13}^* is the residence time in the heated region and thus given by

$$\tau_{13}^* = \tau_{12}^* + \tau_{23}^* = \tau_{12}^* + \ln C_r^* \qquad \text{(VIII1.28)}$$

We note here that the terms underlined in the expressions of the pressure drop response cancel each other in the characteristic equation (VIII1.18). Hence, they have <u>no effect</u> on the stability of the systems.

Furthermore, in this characteristic equation, the two-phase friction factor was taken as a constant for simplicity in the analysis. However, more sophisticated, though far more complicated, friction factor models could be used as we demonstrated in Section VII-3.

VIII-2. Similarity Groups from the Characteristic Equation

A. Similarity Groups from the Heated Regions

In order to satisfy the similarity condition in the heated region, it is necessary that the dimensionless total pressure drop response of the heated region should be the same between the two systems.

Recalling that S*, the perturbation parameter, is an unknown in the characteristic equation given by Eq. (VIII1.18), the above requirement is satisfied if and only if all the coefficients of the pressure drop responses are the same for the two different systems. Hence, in view of Eq. (VIII1.20,21,23,24,25, and 26), we obtain 10 dimensionless groups, i.e., τ_{12}^*, $f_s/2D^*$, λ^*, v_{fi}^*, N_{Fr}, C_r^*, V_{gj}^*, τ_{13}^*, $\bar{f}_m/2D^*$ and ρ_g^*.

However, not all of the above groups are independent, since they are interrelated by the parametric equations given by Eq. (VIII1.8), (VIII1.10), and (VIII1.28). As a consequence, we have only seven basic similarity parameters which are

$$\tau_{12}^* , \quad v_{fi}^* , \quad V_{gj}^* , \quad \rho_g^* , \quad N_{Fr} , \quad \frac{f_s}{2D^*} \quad \text{and} \quad \frac{\bar{f}_m}{2D^*} .$$

B. Similarity Groups from the Un-heated Regions

The similarity groups for the un-heated regions can be obtained in view of Eq. (VIII1.19) and (VIII1.22). Thus we have three similarity parameters for the upstream un-heated region and five for the downstream un-heated region in addition to the dimensionless groups obtained from the heated regions in A. They are

$$k_i , \quad l_i^*/A_i^* , \quad \frac{f_o}{2D_o^*} l_o^* \left(\frac{1}{A_o^*}\right)^2 , \quad k_e , \quad l_e^* , \quad A_e^* , \quad N_{Fre}, \text{ and } \quad \frac{\bar{f}_{me}}{2D^*} .$$

C. Similarity Groups for the System

The foregoing analysis shows that a complete geometrical similarity is not necessary between two systems in order to have the same dynamic characteristics, though they are restricted in some aspects. However, in order to perform a parametric study by changing the kinematic and thermal operational conditions, it is important to have a complete geometrical similarity, owing to the fact that once the geometry of the model system is chosen the friction factors and Froude numbers cannot be independent of the flow fields. Thus, in the following, we assume that the geometrical similarities are established. Consequently, we have nine parameters, i.e., k_i, D_o^*, ℓ_o^*, A_o^*, D^*, k_e, ℓ_e^*, A_e^*, and D_e^*, which are basically the hydrodynamic conditions. We note here that the thermal condition, $A_c/\xi\ell$ need not enter explicitly into the geometrical restrictions.

The remaining parameters to be examined are τ_{12}^*, v_{fi}^*, V_{gj}^*, ρ_g^*, N_{Fr}, N_{Fre}, f_s, $\overline{f}_m$, and $\overline{f}_{me}$. In view of Eq. (VIII1.17), the two Froude numbers are related by the geometrical orientations of the heated duct and the downstream un-heated region. By taking the angle between the two portions of the system as θ, we have two similarity parameters, N_{Fr} (or N_{Fre}) and $\cos\theta$, where the latter is a pure geometrical parameter.

Now let us examine the relations between friction factors f_s, $\overline{f}_m$, and $\overline{f}_{me}$. The liquid friction factor f_s can be considered as a function only of the Reynolds number N_{Res} for the smooth pipe. Thus, we have

$$f_s = f_s(N_{Res}) \doteq \frac{0.184}{N_{Res}^{0.2}} \qquad \text{(VIII2.1)}$$

On the other hand, the average two-phase friction factor $\overline{f}_m$ and $\overline{f}_{me}$ can be assumed to be a function of N_{Res}, the kinematic conditions at the end of the heated duct and the geometry of the system. Hence we have

$$f_m = C_m f_s \tag{VIII2.2}$$

and

$$\overline{f}_{me} = C_{me} f_s \tag{VIII2.3}$$

where

$$C_m = C_m\ (\tau_{12}{}^*,\ v_{fi}{}^*,\ V_{gj}{}^*,\ \rho_g{}^*) \tag{VIII2.4}$$

and

$$C_{me} = C_{me}\ (\tau_{12}{}^*,\ v_{fi}{}^*,\ V_{gj}{}^*,\ \rho_g{}^*,\ D_e{}^*) \tag{VIII2.5}$$

As a consequence, we can replace the three friction factors by N_{Res}, C_m, and C_{me}. The remaining parameters $\tau_{12}{}^*$, $v_{fi}{}^*$, $V_{gj}{}^*$, and $\rho_g{}^*$ are independent and important for the dynamic characteristic of the system.

D. Summary and Discussion on the Similarity Parameters

As a summary, we obtain the following dimensionless groups from the characteristic equation (VIII1.18)

1) The Geometrical Similarity Groups

k_i, k_e, $D_o{}^*$, D^*, $D_e{}^*$, $A_o{}^*$, $A_e{}^*$, $\ell_o{}^*$, $l_e{}^*$, $\cos\theta$

From now on we shall represent the geometrical similarity groups, except k_i and k_e, by N_G. Thus we have k_i, k_e, and N_G.

2) The Froude Number N_{Fr} from Eq. (VIII1.17)

$$N_{Fr} = \frac{v_{fi}{}^2}{g\ell} \tag{VIII2.6}$$

3) The Reynolds Number N_{Res} from Eq. (VIII2.1)

$$N_{Res} = \frac{\rho_f v_{fi} D}{\mu_f} \tag{VIII2.7}$$

4). The Subcooling Number N_{sub} from Eq. (VIII1.10)

$$N_{sub} = \zeta_{12}^* = \frac{\Delta i_{12}}{\Delta i_{fg}} \frac{\Delta \rho}{\rho_g} \qquad \text{(VIII2.8)}$$

5) The Phase Change Number N_{pch} from Eq. (VIII1.3)

for a mixture in thermodynamic equilibrium

$$N_{pch} = \frac{1}{v_{fi}^*} = \frac{q_w'' \xi \ell}{A_c \bar{v}_{fi} \Delta i_{fg}} \frac{\Delta \rho}{\rho_g \rho_f} \qquad \text{(VIII2.9)}$$

or more generally for a mixture in thermodynamic nonequilibrium

$$N_{pch} = \frac{\Omega \ell}{\bar{v}_{fi}} = \frac{\Gamma_g \ell}{\bar{v}_{fi}} \frac{\Delta \rho}{\rho_g \rho_f} \qquad \text{(VIII2.9a)}$$

6) The Drift Number N_d from Eq. (VIII1.4)

$$N_d = V_{gj}^* = \frac{V_{gj}}{\bar{v}_{fi}} \qquad \text{(VIII2.10)}$$

7) The Density Number N_ρ from Eq. (VIII1.14)

$$N_\rho = \rho_g^* = \frac{\rho_g}{\rho_f} \qquad \text{(VIII2.11)}$$

The above groups are independent of each other. Thus they are the basic parameters governing the dynamics of the system. In addition to them, we should supply parametric equations given by Eq. (VIII1.8), (VIII1.10), (VIII1.28), and (VIII2.1,2,3,4, and 5) with the relation between N_{Fr} and N_{Fre} through θ. The geometrical parameters, the Froude number N_{Fr}, and the Reynolds number N_{Res} have the standard significances. On the other hand, the subcooling, phase change, drift, and density numbers given by N_{sub}, N_{pch}, N_d, and N_ρ are associated with the two-phase flow systems.

The subcooling number N_{sub} takes into account the time lag effects in the liquid region due to the subcooling of the fluid entering the heated duct. Thus it is one of the important parameters for the stability analysis.

The phase change number N_{pch} corresponds to Damkoeler's Group I (45) in chemical kinetics, and it scales the change of phase due to the heat transfer to the system. In view of Eq. (VIII1.8), (VIII1.10), and (VIII1.11), it can be seen that N_{pch} is one of the decisive parameters for the time lag in the mixture region. Both N_{sub} and N_{pch} are significant not only for stability analyses, but also for the description of the steady state operational conditions.

Since we assumed the thermal equilibrium in the mixture, the expression for N_{pch} takes the form of Eq. (VIII2.9), which is inversely related to the ones proposed by Bouré (24). However, in general, it is given by (VIII2.9a) which is based on the vapor generation Γ_g and proposed by Zuber et al. (33). The latter expression takes into account the effect of the thermal nonequilibrium and the equality of the phase change number N_{pch} in two different systems and ensures that the phase change has progressed equally in both.

The drift number N_d takes account of the diffusion effects due to the relative motion of the fluids and thus plays a role in two-phase flow similar to that of Damkoeler's Group II (45) in chemical reaction kinetics. Since the vapor drift velocity V_{gj} depends on the flow regime (30,32), this group characterizes the flow pattern. We note here that two-phase diffusion, i.e., drift, processes are not due to the molecular random motions but to the macroscopic geometrical orientation of each phase.

The drift number is important in the kinematics of mixture and if $N_d \gg N_{pch}$, then the change of the density and velocity are controlled by the drift, i.e., redistribution of phases.

The density number N_ρ explicitly appears only in the drift stress term of the characteristic Eq. (VIII1.18). Depending on the constitutive equation for C_m and C_{me}, i.e., Eq. (VIII2.4) and (VIII2.5), this group N_ρ may also appear in the two-phase frictional pressure drop terms. The density number N_ρ actually scales the system pressure Ps, and this is quite important in connection with the above statements on the drift and frictional pressure drop terms.

For example, if the condition given by Eq. (III2.17) is satisfied, then

$$N_\rho N_d^2 \ll 1 \qquad \text{(VIII2.12)}$$

And also, if we can assume that, for a wide range of system pressures

$$\frac{\partial C_m}{\partial N_\rho} \approx \frac{\partial C_{me}}{\partial N_\rho} \approx 0 \qquad \text{(VIII2.13)}$$

then the system pressure influence can be effectively taken into account by the groups N_{sub} and N_{pch}, given by Eq. (VIII2.8) and (VIII2.9). In this case, N_ρ can be eliminated.

Following the above similarity analysis, the characteristic equation given by Eq. (VIII1.18) can be rewritten as

$$0 = Q^* = Q^* \left(k_i, k_e N_G, N_{Fr}, N_{Res}, N_{sub}, N_{pch}, N_d, N_\rho, S^*\right) \quad \text{(VIII2.14)}$$

Equation (VIII2.14) shows complete parametric dependence of the characteristic equation. In usual circumstances, N_G cannot be considered as an operational parameter, but rather as a design parameter. Thus, once the system is given, the dynamic behavior can be represented in eight dimensional space with S^* as a parameter. It should be noted here that Bouré (24) obtained three groups, N_{sub}, $1/N_{pch}$, and $1/N_{Fr}\, N_{pch}^2$, in addition to the geometrical groups.

CHAPTER IX

STABILITY ANALYSIS

IX-1. Stability Theorem

The systems which we have analyzed in the previous chapters are governed by the differential difference equations. Therefore, the solution at any instance is coupled with the past history of the process. Such a system is characterized by time delay effects induced by the finite time that is required for the propagation of the signals. As it has been discussed in Chapter VII, if the kinematic wave velocity is a linear function of the axial coordinate z, the characteristic equation can be expressed by the combination of the <u>rational functions and the exponential functions</u>, i.e., Eq. (VII2.11). Thus, in nondimensional form, we obtain the characteristic equation

$$Q^*\left(S^*, \frac{1}{S^*}, \frac{1}{S^*-1}, \frac{1}{S^*-2}, e^{-S^*\tau_{12}^*}, e^{-S^*\tau_{23}^*}, e^{-S^*\tau_{34}^*}\right) = 0 \qquad \text{(IX1.1)}$$

which, in terms of the operational parameters, is given by Eq. (VIII1.18) or Eq. (VIII2.14). It should be noted here that Eq. (IX1.1) does not have a form of exponential polynomials. However, by recalling that S* = 0, 1 and 2 are not the poles of Eq. (IX1.1), we can define a <u>shifted characteristic equation</u> as

$$L^* \equiv S^*(S^*-1)^2(S^*-2)\,Q^* = 0 \qquad \text{(IX1.2)}$$

The function L* takes a form of exponential characteristic polynomials with a single root at S* = 0 or 2, and a double root at S* = 1. Therefore, the stability analysis using Eq. (IX1.1) and (IX1.2) is slightly different. For the system with Eq. (IX1.1), no roots in the right hand half plane of S* are necessary for the stability of the system, whereas by using Eq. (IX1.2) we should take into account the roots which we have created artificially. In particular, the shifted characteristic function L* has one of its zeros on the imaginary axis, which causes some difficulties in the application of the existing stability criteria. We recommend the form

$$Z^* \equiv (S^* - 1)^2 (S^* - 2)\, Q^* = 0 \tag{IX1.3}$$

The stability theorem applicable to our characteristic equation can be classified in two groups, the stability test criterion and the stability boundary criterion. In the following, we shall discuss the above groups separately.

A. Graphical Method Based on Encirclement Theorem (Mikhailov Criterion)

The stability criterion from the encirclement theorem was first developed for the linear system with no time delays by Mikhailov (46), Leonhard (47), and slightly modified by Nyquist (48) for the closed loop control system. The theorem is well known and not restricted to the rational algebraic functions. The extension of the criterion to the time-delay system was studied by A. Sokolov, Miasnikov, and Satche among others. The detailed review of these workers can be found in fine books by E. P. Popov (49) or by B. Porter (50). Briefly, the criterion was obtained from the application of conformal mapping and the residue theorem.

Let us take a complex variable function F, which can be considered as the combination of the rational and exponential functions. Thus

$$F = F(S^*, e^{S^*}) \tag{IX1.4}$$

Then the encirclement theorem (or the principle of argument) gives that

$$\frac{1}{2\pi} \Delta C_+ \arg F = P - Z \tag{IX1.5}$$

where Z and P are the number of zeros and poles inside the contour C, ΔC_+ arg F is the total change in the argument of F or the anti-clockwise encirclements of the origin in the F plane around C_+. The theorem is derived under the condition that F is analytic except for a finite number of poles inside C, and that F has no zeros on C.

When the characteristic equation does not have roots on the imaginary axis and at the infinity of the right half plane, the stability of the system can be studied by direct application of the above theorem. By taking the contour C shown in Fig. 9, the right hand side plane is entirely covered. Now we use the mapping of S* on C into the F plane, thus the right hand side of Eq. (IX1.5) can be found. Hence, by knowing the number of poles in the right hand half plane, <u>the number of zeros Z</u> can be found.

In our application, the characteristic equation has

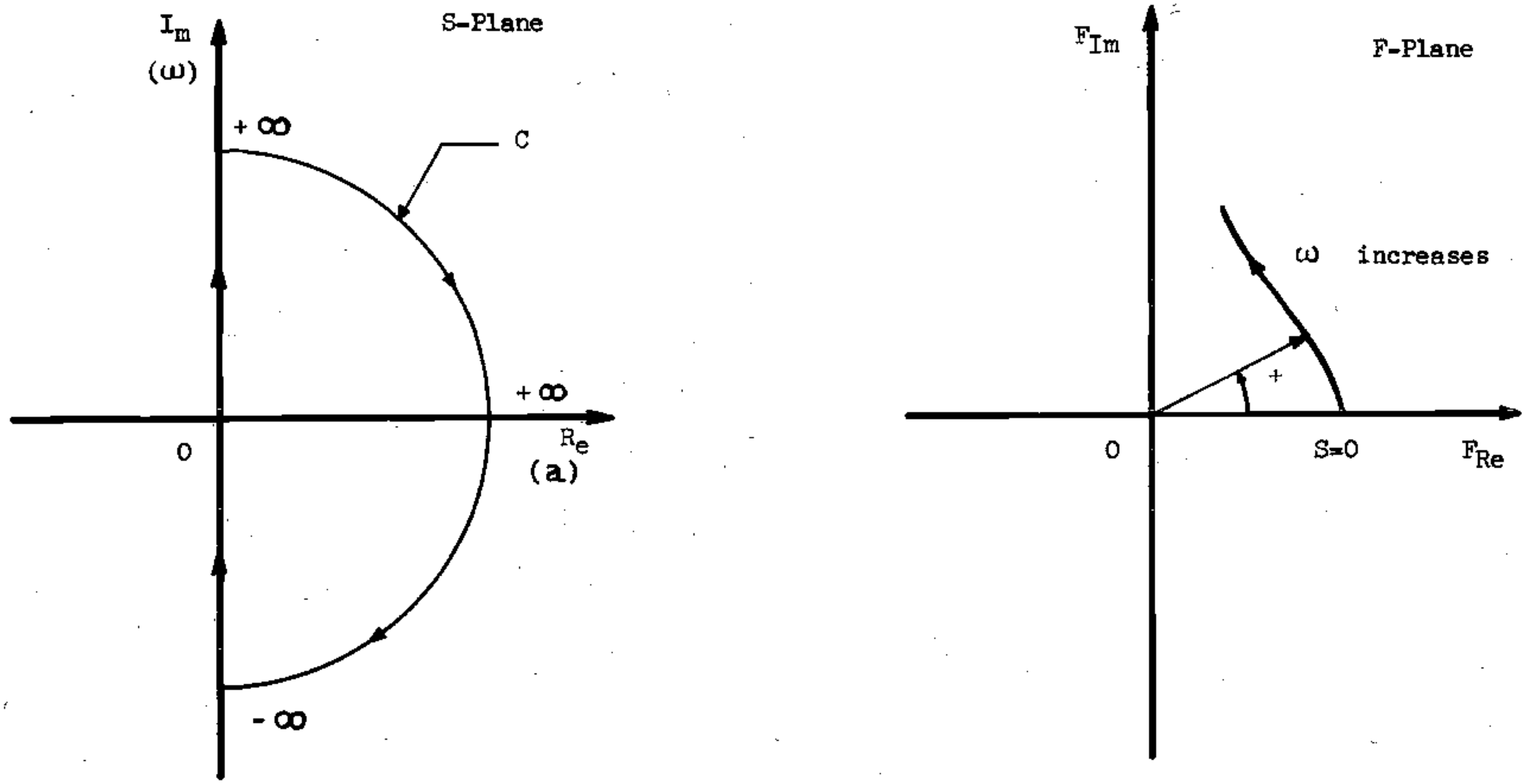

Figure 9. Conformal Maping

1. the principal term S^{*n} (the highest polynomial with positive coefficient)

2. no poles

3. no zeros on the contour C and k zeros in C.

Then the theorem (IX1.5) can be modified to

$$k = \frac{1}{2}\left\{ n - \frac{1}{\pi}\left[\Delta F\right]_{-\infty}^{+\infty} \right\} \tag{IX1.6}$$

where $\left[\Delta F\right]_{-\infty}^{+\infty}$ corresponds to the change of argument of F when S* travels on the imaginary part of the contour C.

If the original characteristic equation (IX1.1) is used in place of F, the number of roots k should be zero for the stability of the system, whereas if the shifted characteristic equation (IX1.3) is in use, k should be 3 for the stability.

The above criterion is extensively based on the graphical means, and the plotting of the Nyquist or Mikhailov diagram is required. The algebraic criteria of stability for the linear time delay system were developed by L. Pontriagin (51); however, the application of the criteria to the practical problem usually becomes quite complicated. Therefore, it is not recommended here.

B. Stability Boundary Criterion (D-Partition Method)

The criterion discussed in the foregoing section is best suited for the examination of the stability when all operational parameters are known constant. On the other hand, if a parametric study for the system is required, the D-Partition Method is preferable. It was first studied by Y. Neimark (52) and the detailed discussion on the application for the time-delay system could be found in (49, 51). By taking the characteristic function F, we have

$$F(S^*, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_m) = 0 \tag{IX1.7}$$

where α_1 to α_m represents the parameters which can be changed independently. Now we consider the harmonic oscillations. Thus we take

$$S^* = j\omega^* \tag{IX1.8}$$

By substituting Eq. (IX1.8) into (IX1.7), we obtain

$$F(j\omega^*, \alpha_1, \cdots, \alpha_m) = F_{Re}(\omega^*, \alpha_1, \cdots, \alpha_m) + jF_{Im}(\omega^*, \alpha_1, \cdots, \alpha_m) \tag{IX1.9}$$

which reduces to

$$F_{Re}(\omega^*, \alpha_1, \cdots, \alpha_m) = 0 \tag{IX1.10}$$

$$F_{Im}(\omega^*, \alpha_1, \cdots, \alpha_m) = 0 \tag{IX1.11}$$

Equations (IX1.10) and (IX1.11) give the harmonic frequency surfaces in m-dimensional space, i.e., $\alpha_1, \ldots, \alpha_m$ as coordinates, with ω^* as an auxiliary parameter.

Since the complex roots are always conjugate for the function with real coefficients, the domain of ω^* becomes

$$0 < \omega^* < \infty \qquad \text{(IX1.12)}$$

with two singular surfaces

$$\lim_{\omega^* \to 0} F = 0 \qquad \text{(IX1.13)}$$

and

$$\lim_{\omega^* \to \infty} F = 0 \qquad \text{(IX1.14)}$$

Now the m-dimension space is divided into regions bounded by the harmonic frequency surfaces, i.e., Eq. (IX1.10) and (IX1.11) with Eq. (IX1.12) and two singular surfaces given by Eq. (IX1.13) and (IX1.14).

The theorem states that the number of roots lying in the right half S* plane for each region divided by the surfaces do not change within a subdivision. Therefore, the stability of each region can be decided by testing the stability for any point in that region. For this purpose, the criterion derived in the previous section is useful. We note here that the D-Partition parameter $D(\eta)$ denotes the existence of η roots in the right half S* plane for that particular region.

IX-2. Stability Plane

The parametric study of the stability of the system can proceed by using the D-Partition method discussed in Section IX-1. The governing parameters obtained from dimensional analysis in Chapter VIII are k_i, k_e, N_G, N_{Fr}, N_{Res}, N_{sub}, N_{pch}, N_d, and N_ρ, whereas the harmonic frequency ω^* is an auxiliary variable at the stability boundaries. As has been shown in a previous section, the neutral stability surfaces in a multi-dimensional space are given by setting $S^* = j\omega^*$ in the characteristic equation. However, in order to present these stability boundaries in a two-dimensional plane, it is necessary to select two representative parameters for the coordinates of such a <u>stability plane</u>. Since, for constant system pressure and inlet velocity with fixed geometry, the parameters k_i, k_e, N_G, N_{Fr}, N_{Res}, N_d, and N_ρ are fixed, the Subcooling and Phase Change[*] numbers are best suited for the coordinate of such a plane. For example, it was found that the <u>operational domain</u> in the stability plane was <u>bounded</u> by the physical restrictions on the subcooling and the heat flux. From the condition that the subcooling is a positive entry and is upper bounded by Δi_s, corresponding to the freezing point, we have

$$0 \leq N_{sub} \leq \frac{\Delta i_s}{\Delta i_{fg}} \frac{\Delta \rho}{\rho_g} \qquad \text{(IX2.1)}$$

On the other hand, from the condition that indeed boiling takes place in the channel and the super-heat of the vapor would not occur, we obtain

$$N_{pch} - \frac{\Delta \rho}{\rho_g} < N_{sub} < N_{pch} \qquad \text{(IX2.2)}$$

In addition to the stability boundaries, some important operational characteristics can be represented in simple form on the stability plane. For example, the constant exit quality line is given by

---

[*] The expression used for N_{pch}, i.e., Eq. (VIII2.9), is for the case of thermal equilibrium.

$$N_{sub} = N_{pch} - \chi_e \left(\frac{\Delta\rho}{\rho_g}\right) \qquad \text{(IX2.3)}$$

whereas the length of the non boiling region is given by

$$\lambda^* = N_{sub}/N_{pch} \qquad \text{(IX2.4)}$$

The basic characteristics of this stability plane are illustrated in Fig. 10.

IX-3. Parametric Study of the System

(Analytical Conclusions and the Scheme for the Numerical Study)

In Section IX-1 we have studied the *stability theorem* applicable to our case, and then in Section IX-2 the *stability plane* suitable to the parametric study has been chosen. Now we shall build up the scheme for the parametric analysis on the stability of the system.

A. Influence of N_{sub} and N_{pch} (Subcooling and Heat Flux)

Following the discussion in Section IX-2, first we fix the parameters, k_i, k_e, N_G, N_{Fr}, N_{Res}, N_d, and N_ρ. Thus, the system pressure, inlet velocity, flow regime, and geometry are temporarily frozen. The remaining free parameters are the *subcooling* and *phase change* numbers which correspond to the subcooling and heat input to the system. Hence Eq. (IX1.7), in terms of the characteristic function Q*, can be written as

$$Q^*(s^*, N_{sub}, N_{pch}) = 0 \qquad \text{(IX3.1)}$$

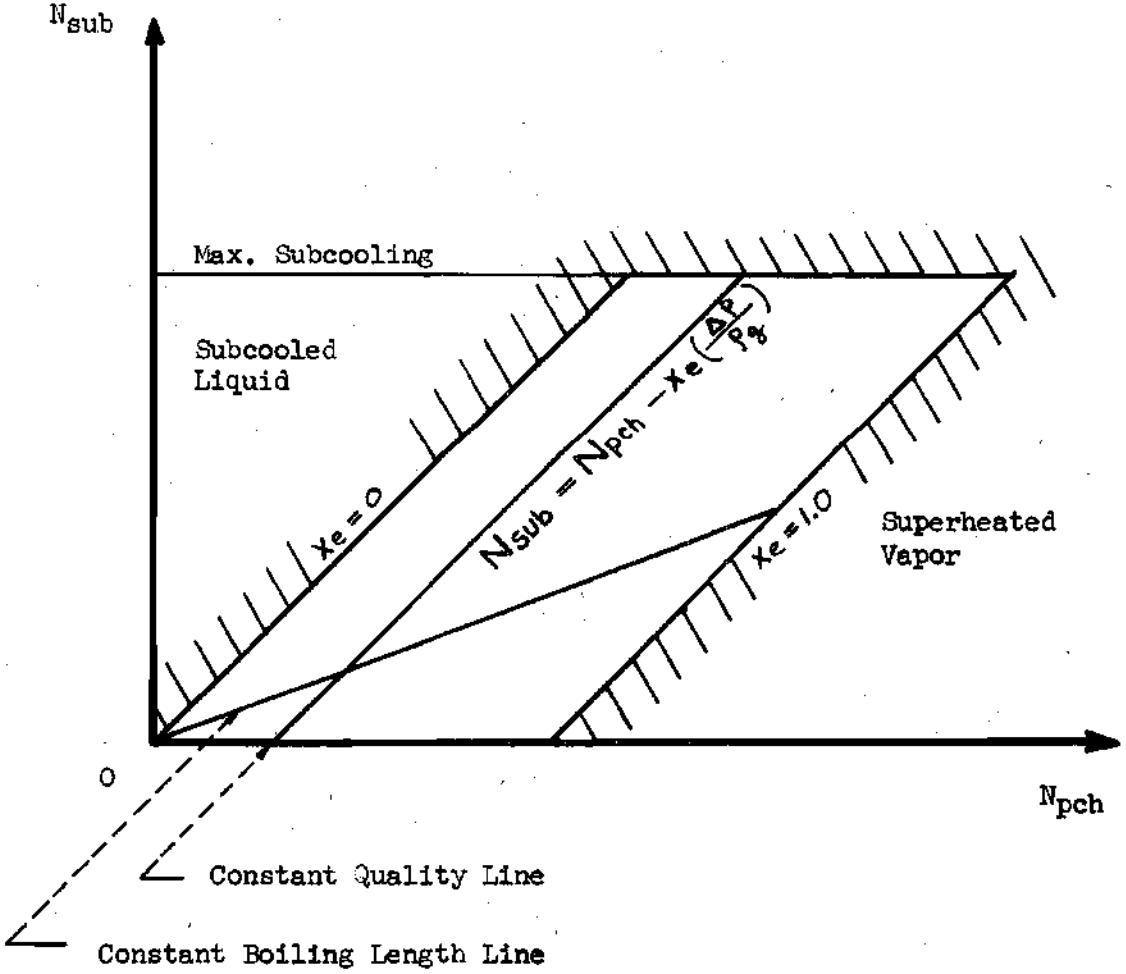

Figure 10. Stability Plane

or in terms of the shifted characteristic function Z*

$$Z^* (X^*, N_{sub}, N_{pch}) = (S^*-1)^2 (S^*-2) Q^* = 0 \qquad \text{(IX3.2)}$$

By setting S* = jω* and separating the imaginary and real parts, we obtain

$$Z^*_{Re} (\omega^*, N_{sub}, N_{pch}) = 0 \qquad \text{(IX3.3)}$$

and

$$Z^*_{Im} (\omega^*, N_{sub}, N_{pch}) = 0 \qquad \text{(IX3.4)}$$

with

$$0 < \omega^* < \infty \qquad \text{(IX3.5)}$$

Since Eq. (IX3.2) already has three roots in the right half S* plane which are artificially produced by multiplying $(S^*-1)^2 (S^*-2)$ on Q*, the system is stable in the region of D(3). The other regions with D(4), D(5), etc. should be examined by the criterion based on the encirclement theorem. The singularity corresponds to ω* → ∞ which has no importance in our case because Z* → ∞.* On the other hand, the second singularity given by Eq. (IX1.14) has a significant physical interpretation. Recalling the inequality (VI6.9), the excursive stability condition is given by

$$\lim_{\omega^* \to 0} Z^*(\omega^*, N_{sub}, N_{pch}) < 0 \qquad \text{(IX3.6)}$$

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*Actually the condition gives $C_k^* (\bar{\lambda}) = v_{fi}^* (1 + V_{gj}^*) = 0$. Thus it corresponds to the starting point of flooding. However, this usually happens for the downward flow and is not important for the system operating in standard circumstances. It may be important, however, for space application, where the acceleration force may act in the direction of the inlet liquid flow.

By taking the limit $\omega^* \rightarrow 0$, it can be shown that

$$\lim_{\omega^* \to 0} Z^* = \lim_{\omega^* \to 0} Z^*_{Re} (\omega^*, N_{sub}, N_{pch}) \tag{IX3.7}$$

Hence the singularity expressed by Eq. (IX1.13) corresponds to the excursive stability boundary. In view of Eq. (IX3.7) and the inequality Eq. (IX3.6), we obtain the condition for the excursive stability. Thus

$$\lim_{\omega^* \to 0} Z^*_{Re} (\omega^*, N_{sub}, N_{pch}) < 0 \tag{IX3.8}$$

Now we can plot the neutral stability curves expressed by the relations (IX3.3,4, and 5), where ω^* is an auxiliary parameter, and the excursive stability condition expressed by the inequality (IX3.8) on the stability plane, Fig. 10. As a result, we shall obtain various regions bounded by the above curves and the lines specifying the operational domain, i.e., the relations (IX2.1) and (IX2.2). The stability of each region should be examined by taking any point inside the subdivision and applying the Mikhailov criterion at that point. By taking i^{th} region and denoting any sample point inside of it by (N_{sub_i}, N_{pch_i}), the stability criterion can be obtained from Eq. (IX1.6). (Recall that $k > 3$ corresponds to an unstable condition.) Thus we obtain the condition for the dynamical stability

$$\frac{1}{2} \left\{ 4 - \frac{1}{\pi} \left[\Delta Z^* \right]_{-\infty}^{+\infty} \right\} \leq 3 \tag{IX3.9}$$

where we have substituted $n = 4$, since the highest order of the polynomials in Z* is four.

The number of roots for the characteristic equation Q*, lying in the right half S* plane, is given by

$$R = -1 - \frac{1}{\pi}\left[\Delta Z^*\right]_0^{+\infty} \qquad \text{(IX3.10)}$$

where $\left[\Delta Z^*\right]_0^{+\infty}$ denote the change of argument of Z* (jω*) in the anticlockwise diversion when ω* travels from 0 to +∞.

Due to the complexities of the characteristic equation and of the coefficients in terms of N_{sub} and N_{pch}, the above steps cannot be carried out analytically, and the use of a computer is required. However, the computer programming is quite simple, since it basically reduces to the problem of calculating the known function Z_{Re}* and Z_{Im}* for various values of the parameters ω*, N_{sub} and N_{pch}. Furthermore, we note that these variables are bounded. The domain of (N_{sub} and N_{pch}) is expressed by the relations (IX2.1) and (IX2.2). For ω* we should recall the characteristic Eq. (VIII1.18) with Eq. (VIII1.19,20,21, and 22) and Eq. (IX1.3). From the above equations it can be seen that the highest order of ω* in Z_{Re}* is four and appears as <u>a pure polynomial</u>; hence, it is easy to show that there exists a number M such that for ω* > M, Eq. (IX3.3) has no roots. As a consequence, we only need to examine for ω* < M. Actually M is a function of N_{sub} and N_{pch}; however, the crude upper boundary for ω* is easily found from the coefficients of Z_{Re}*. (It is <u>not necessary</u> to obtain the <u>least</u> upper boundary for ω*.)

Typical plots of the Mikhailov diagram for Z* are shown in Figs. 11-14.

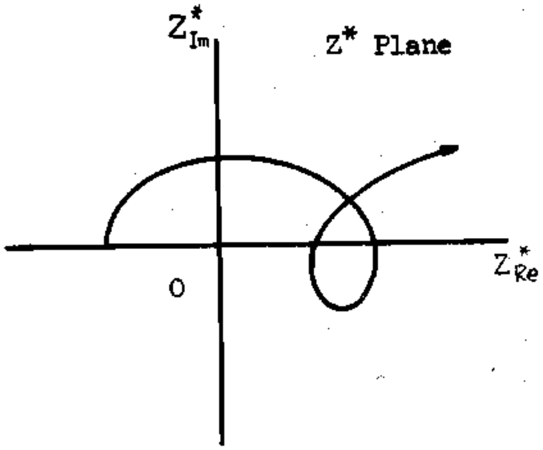


Figure 11. Case of D(3)

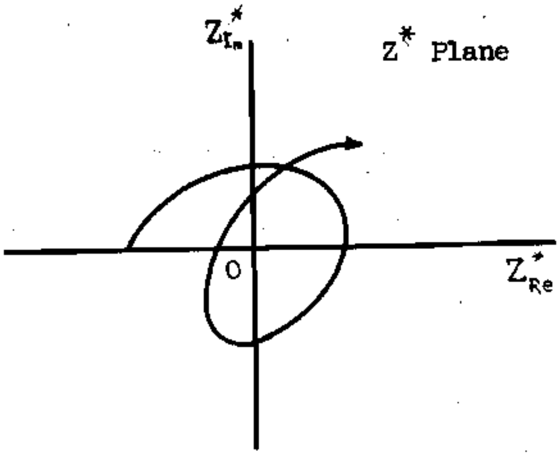


Figure 12. Case of D(5)

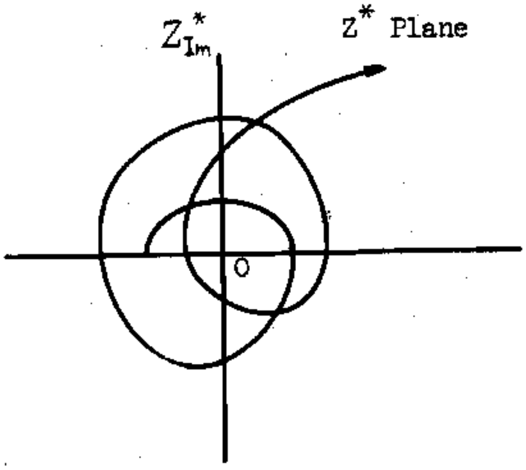


Figure 13. Case of D(7)

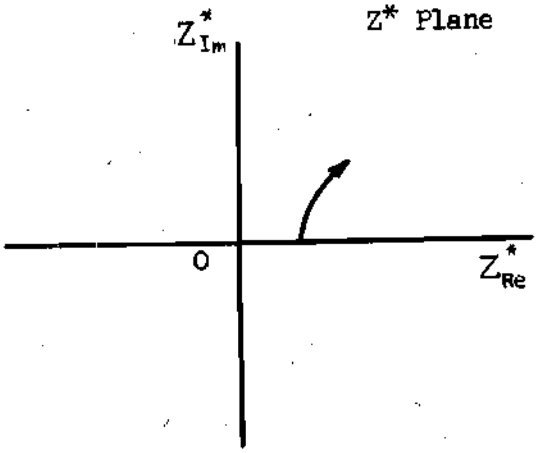


Figure 14. Case of $Z_{Re}(0) > 0$

From the four diagrams, only the one shown in Fig. 11 expresses the stable system. The number of roots in the right half S* plane for the ones shown in Figs. 12 and 13 is two and four, respectively, and thus they are dynamically unstable. The diagram shown in Fig. 14 represents the excursive unstable system.

B. Influence of the Inlet Velocity

Following A, we shall now examine the influence of the inlet velocity on the stability boundaries. Assuming that the system pressure and the geometrical parameters are fixed, the change of the inlet velocity $\overline{v}_{fi}$ is characterized by the shift in the Reynolds number N_{Res}; however, the two other parameters, N_{Fr} and N_d, are also affected through Eq. (VIII2.6), (VIII2.7), and (VIII2.10). From the characteristic equation (VIII1.18) with Eq. (VIII1.19,20), it can be seen that the increase of $\overline{v}_{fi}$ reduces the effects of both the gravitational force and the relative velocity. By keeping in mind the above consideration, we can obtain another set of the stability boundary curves in the stability plane by the method described in A. This new set of boundaries will be compared with previous curves having a different Reynolds number.

It should be noted here that the kinematic effects of the inlet velocity are largely taken into account by the coordinate itself, i.e., Eq. (VIII2.9). Thus, by recalling that the turbulent f_s is proportional to only one fourth power of $1/N_{Res}$, and assuming that the dynamics of flow are dominated by the inertial, frictional, and orifice pressure drop, it is easy to conclude that the most significant influence of the inlet velocity is included in the phase change number N_{pch}.

C. Influence of the Relative Velocity

The influence of the relative velocity on the stability of the system can be examined by changing the drift number N_d, i.e., Eq. (VIII-2.10). For example, the result of the homogeneous flow assumption can be obtained by merely setting $N_d = 0$. It is expected that the effect of the relative velocity is to stabilize the flow, since an increase of N_d decreases the void fraction α and thereby increases the mixture density which has a stabilizing effect.

D. Influence of the System Pressure

The influence of the system pressure is characterized by the density number N_ρ which appears in the drift pressure drop term and in the friction factor coefficients C_m and $C_{\bar{m}e}$, i.e., Eq. (VIII2.2,3,4, and 5). From the discussion in Section VIII-2, the effect of the system pressure on the stability boundaries in the N_{sub}-N_{pch} plane is quite limited. Thus in view of the expressions for N_{sub} and N_{pch}, i.e., Eq. (VIII2.8) and (VIII2.9), it can be concluded that the system pressure effects are largely taken into account by the coordinates of the stability plane. Therefore, in terms of the exit quality, the increase of pressure stabilizes the flow.

E. Influence of k_i and k_e

The effects of throttling at the inlet and the exit can be studied by comparing the stability maps for different values of k_i and k_e. By taking only the inlet and exit orifice terms in the characteristic equation (VIII1.8), it is straightforward to conclude that the increase of k_i is a stabilizing effect, whereas the increase of k_e destabilizes the system.

IX-4. Simple Stability Criteria (Dynamic)

The stability analysis described in the previous sections is rather complicated in view of its algebraic calculations, though once the computer program is written, the stability test can be carried out in a matter of seconds. Here we shall develop a simple stability criterion which can be used for the first hand design purpose without going into the detailed calculations. It can be said that the sophistication of the model is coupled with the complexity of the stability analysis, thus the simplification of the model by introducing various assumptions leads to the qualitative rather than to the quantitative description of the system.

A. Lumped Parameter Model

The simplest model can be obtained by assuming that the pressure drops in the channel are concentrated at the inlet and outlet orifices. In this case, the characteristic equation takes the form

$$Q^* = 2k_i' + k_e' \left\{ 2\Lambda_3^* + \frac{(1+2V_{gj}^*)}{(1+V_{gj}^*)} \frac{1}{S^*-1} \left[\Lambda_3^* - C_r^* e^{-S^* \tau_3^*} \right] \right\} = 0 \quad \text{(IX4.1)}$$

Substituting Eq. (VIII1.23) and (VIII1.27) and setting S* = jω* for the harmonic oscillation, we obtain two equations corresponding to Eq. (IX3.3) and (IX3.4). Thus

$$Z_{Re}^* = \frac{1}{\omega^*} \left\{ -\left[2K + (2-M)\right]\omega^* + (2-M)\sin\omega^*\tau_{12}^* + \left[2 - MC_r^*\right]\omega^*\cos\omega^*\tau_i^* \right\} = 0 \quad \text{(IX4.2)}$$

and

$$Z_{Im}^* = -\frac{1}{\omega^*} \left\{ -2K\omega^{*2} + (2-M) - (2-M)\cos\omega^*\tau_{12}^* + \left[2 - MC_r^*\right]\omega^*\sin\omega^*\tau_i^* \right\} = 0 \quad \text{(IX4.3)}$$

where K and M are given by

$$K = \frac{k_i' + k_e'}{k_e'} \qquad M = \frac{1 + 2V_{gj}^*}{1 + V_{gj}^*} \qquad \text{(IX4.4)}^*$$

It is well known that the first encirclement of the origin in the Mikhailov diagram for the time delay system is largely governed by the lowest order terms of the polynomials, particularly of the exponential polynomials. This can be explained by the fact that the first crossover of the real axis in the Mikhailov diagram happens with relatively small ω^* due to the large changes in the exponential terms. Therefore, in view of Eq. (IX4.2) and (IX4.3), we may conclude that the time delay in the subcooled liquid region, τ_{12}^*, has more significance than the one in the heated region, τ_{13}^*, except for the case of very small subcooling. Furthermore, the time delays τ_{12}^* and τ_{13}^* are related by Eq. (VIII1.28). Thus, for large τ_{12}^*, we may approximate τ_{13}^* as

$$\tau_{13}^* = \tau_{12}^* + \ln C_r^* \approx \tau_{12}^* \qquad \text{(IX4.5)}$$

By substituting Eq. (IX4.5) into Eq. (IX4.2) and (IX4.3), we can eliminate the trigonometric functions between the two equations. Thus we obtain

$$4K^2(1+\omega^{*2}) + (2-M)^2 = (2 - MC_r^*)^2 \qquad \text{(IX4.6)}$$

---

[*] The recommended expressions for k_i' and k_e' are

$$k_i' = k_i + \frac{f_s}{2D^*}\lambda^* \quad , \quad k_e' = k_e + \frac{f_s C_m}{2D^*}(1-\lambda^*)$$

Equation (IX4.5) expresses the frequency in terms of the system parameters K, M, and C_r*. However, if ω* << 1, Eq. (IX4.6) gives the stability boundary curve. Thus

$$C_r^* = \frac{2}{M}\sqrt{4K^2+(2-M)^2} \qquad \text{(IX4.7)}$$

Recalling Eq. (VIII1.8), (VIII1.10), (VIII2.8), and (VIII2.9), the above equation can be expressed in terms of N_{pch} and N_{sub}, hence the stability criterion becomes

$$\frac{N_{pch}-N_{sub}}{(1+V_{gj}^*)} \leq \frac{2}{M}\sqrt{4K^2+(2-M)^2} - 1 \qquad \text{(IX4.8)}$$

From this relation it can be seen that k_i, the inlet restriction, has a stabilizing effect; on the other hand, k_e, the exit restriction, has a destabilizing effect.

B. Asymptotic Solution for ω* << 1 (High Subcooling Number)

In many practical cases, the effects of the upstream and downstream un-heated regions can be neglected except those of throttlings, as has been explained in Chapter IV. Thus, in the following, we shall consider the dynamics of the heated region with only inlet and exit orifices from the un-heated regions.

In this case the characteristic equation (VIII1.18) can be rewritten as

$$Q^*(S^*) = \frac{1}{S^*(S^*-1)^2(S^*-2)}\Big\{ B_1 S^{*5} + B_2 S^{*4} + B_3 S^{*3} + B_4 S^{*2} + B_5 S^* + B_6 \qquad \text{(IX4.9)}$$

$$+ e^{-S^*\tau_{12}^*}\left(B_7 S^{*4} + B_8 S^{*3} + B_9 S^{*2} + B_{10} S^* + B_{11}\right) +$$

$$+ e^{-S^*\tau_{13}^*}\left(B_{12} S^{*3} + B_{13} S^{*2} + B_{14} S^* + B_{15}\right)\Big\} = 0$$

where the coefficients B_1 to B_{15} are listed in Appendix E. We substitute $S^* = j\omega^*$ into Eq. (IX4.9) and separate the real and imaginary parts; then we get

$$Q^*(j\omega^*) = \frac{1}{[(\omega^{*3} - 5\omega^*)^2 + (4\omega^{*2} - 2)^2}\left\{ \frac{Re}{\omega^*} + j\,\frac{Im}{\omega^*}\right\} = 0 \qquad \text{(IX4.10)}$$

Consequently, we obtain parametric equations for the neutral stability surfaces

$$\frac{Re}{\omega^*} = 0 \qquad \text{(IX4.11)}$$

$$\frac{Im}{\omega^*} = 0 \qquad \text{(IX4.12)}$$

Here Re and Im are complicated trigonometric polynomial functions in ω^*; however, by using the asymptotic condition $\omega^* \ll 1$, they can be simplified to a great extent. As it has been explained in A of this section, the exponential terms have more significance than the polynomial parts for the first crossover of the real axis in the Mikhailov diagram. Thus, by considering the asymptotic case of $\omega^* \ll 1$, the higher order terms of ω^*

can be neglected. Furthermore, using the assumption* of Eq. (IX4.5), i.e., $\tau_{12}^* \approx \tau_{13}^*$, Eq. (IX4.11) and (IX4.12) can be reduced to the following forms.

$$\frac{Re}{\omega^*} \approx -2B_5 - 2(B_{10}+B_{14})\cos\omega^*\tau_{12}^* - \tag{IX4.13}$$

$$- 5B_6(1-\cos\omega^*\tau_{12}^*) - \frac{2B_6}{\omega^*}\sin\omega^*\tau_{12}^* = 0$$

and

$$\frac{I_m}{\omega^*} \approx 2(B_{10}+B_{14})\sin\omega^*\tau_{12}^* - 5B_6\sin\omega^*\tau_{12}^* + \tag{IX4.14}$$

$$+ \frac{2B_6}{\omega^*}(1-\cos\omega^*\tau_{12}^*) = 0$$

where we used the identity

$$B_6 + B_{11} + B_{15} = 0 \tag{IX4.15}$$

which ensured us that

$$\lim_{\omega^*\to 0}\frac{I_m}{\omega^*} = 0 \tag{IX4.16}$$

With Eq. (IX4.15), the excursive stability condition given by the relation (VI6.9) yields

$$B_5 + (B_{10} + B_{14}) + B_6\,\tau_{12}^* < 0 \tag{IX4.17}$$

---

*It can be shown that the order of $\omega^*\tau_{12}^*$, $\omega^*\tau_{13}^*$ should be 2π. Thus $\omega^* \ll 1$ corresponds to $\tau_{12}^* \gg 1$. Under this condition

$$\tau_{13}^* = \tau_{12}^* \ln C_r^* \approx \tau_{12}^*.$$

By rearranging the trigonometric functions in Eq. (IX4.13) and (IX4.14) we obtain

$$-2\left[B_5-(B_{10}+B_{14})+5B_6\right]- \tag{IX4.18}$$

$$-2\cos\left[\frac{\omega^*\tau_{12}^*}{2}\right]\left\{\left[2(B_{10}+B_{14})-5B_6\right]\cos\frac{\omega^*\tau_{12}^*}{2}+\frac{2B_6}{\omega^*}\sin\frac{\tau_{12}^*\omega^*}{2}\right\}=0$$

and

$$2\sin\frac{\omega^*\tau_{12}^*}{2}\left\{\left[2(B_{10}+B_{14})-5B_6\right]\cos\frac{\omega^*\tau_{12}^*}{2}+\frac{2B_6}{\omega^*}\sin\frac{\tau_{12}^*\omega^*}{2}\right\}=0 \tag{IX4.19}$$

The solutions to Eq. (IX4.19) give the crossover frequencies. Thus

$$\omega\tau_{12}^* = 0, \quad 2\pi, \quad 4\pi$$

and

$$\left[2(B_{10}+B_{14})-5B_6\right]\cos\frac{\omega^*\tau_{12}^*}{2}+\frac{2B_6}{\omega^*}\sin\frac{\omega^*\tau_{12}^*}{2}=0 \tag{IX4.20}$$

However, it can be seen from the above roots that the first crossover of the imaginary axis occurs with $\omega^*\tau_{12}^*$ between 0 and 2π. Hence, the smallest positive root for Eq. (IX4.20) gives the first crossover frequency. By substituting Eq. (IX4.20) into Eq. (IX4.18), we obtain

$$\left.\frac{Re}{\omega^*}\right)_c = -2\left[B_5-(B_{10}+B_{12})+5B_6\right]=0 \tag{IX4.21}$$

which gives the first neutral stability surface.

The formal application of the D-Partition method to the full characteristic equation shows that the first neutral stability surface always corresponds to the dynamic stability boundary (see Fig. 15-22). Consequently, in view of Eq. (IX4.17) and (IX4.21), it can be concluded that the necessary condition for the system to be stable is given by

$$B_5 - (B_{10} + B_{14}) + 5\,B_6 < 0 \qquad \text{(IX4.22)}$$

In a parametric form it reduces to

$$C_r^* - 1 < \frac{2\left[k_i + \frac{f_s}{2D^*}\lambda^* + \frac{f_s C_m (1-\lambda^*)}{2D^*} + k_e\right] + \frac{V_{gj}^*}{1+V_{gj}^*}}{1 + \frac{1}{2}\left(\frac{1+2V_{gj}^*}{1+V_{gj}^*}\right)\left(\frac{f_s C_m}{2D^*} + 2k_e\right) - \frac{\upsilon_{fi}^*}{N_{Fr}}\frac{1}{C_r^*} - \rho_g^*\frac{V_{gj}^{*2}}{1+V_{gj}^*}} \qquad \text{(IX4.23)}$$

By defining the dynamic stability number S_d as

$$S_d \equiv \frac{2\left[k_i + \frac{f_s}{2D^*}\lambda^* + \frac{f_s C_m (1-\lambda^*)}{2D^*} + k_e\right] + \frac{V_{gj}^*}{1+V_{gj}^*}}{(C_r^* - 1)\left\{1 + \frac{1}{2}\left(\frac{1+2V_{gj}^*}{1+V_{gj}^*}\right)\left(\frac{f_s C_m}{2D^*} + 2k_e\right) - \frac{\upsilon_{fi}^*}{N_{Fr}}\frac{1}{C_r^*} - \rho_g^*\frac{V_{gj}^{*2}}{1+V_{gj}^*}\right.} \qquad \text{(IX4.24)}$$

The condition for the system to be dynamically stable is given by

$$S_d > 1 \qquad \text{(IX4.25)}$$

On the other hand, the neutral frequency is given by

$$\tan\frac{\omega^*\tau_{12}^*}{2} = -\left\{\frac{2(B_{10}+B_{14})-5B_6}{B_6\tau_{12}^*}\right\}\frac{\omega^*\tau_{12}^*}{2} \qquad \text{(IX4.26)}$$

Recalling the relations (IX4.17) and (IX4.21), it can be shown that

$$-\left\{\frac{2(B_{10}+B_{14})-5B_6}{B_6\tau_{12}^*}\right\} > 1 \qquad \text{(IX4.27)}^*$$

So that the magnitude of the left hand side gives the excursive stability margin. From Eq. (IX4.26) and the inequality (IX4.27) we have

$$0 < \omega^*\tau_{12}^* < \pi \qquad \text{(IX4.28)}$$

However, for the system with sufficiently large excursive stability margin, it can be said that $\omega^*\tau_{12}^*$ is close to π, thus we have

$$\omega^*\tau_{12}^* \lesssim \pi \qquad \text{(IX4.29)}$$

Now let us consider the homogeneous flow with no gravity effect. Then we have

$$V_{gj}^* = 0 \qquad \text{(IX4.30)}$$

$$N_{Fr} = \infty \qquad \text{(IX4.31)}$$

And, furthermore, if the two-phase friction factor coefficient C_m is not

---

*From Appendix E, the coefficient B_6 is positive for most of the cases, thus $B_6 > 0$.

much larger than the unity, the stability criterion (IX4.23) reduces to

$$X_e \frac{\Delta\rho}{\rho_g} = N_{pch} - N_{sub} < \frac{2\left[k_i + \frac{f_s C_m}{2D^*} + k_e\right]}{1 + \frac{1}{2}\left(\frac{f_s C_m}{2D^*} + 2k_e\right)} \qquad \text{(IX4.32)}$$

It should be noted here that the stability boundary expressed by Eq. (IX4.32) gives a <u>straight line in the stability plane, which is parallel to the constant quality line</u>, if C_m is taken as a constant.

IX-5. Excursive Stability Criterion

The excursive stability criterion for the general case has been obtained in Section VI-6 and given by the inequality (VI6.9). Thus, in the dimensionless form, we have

$$\lim_{S^* \to 0} Q^*(S^*) > 0 \qquad \text{(IX5.1)}$$

for the <u>excursive stability</u> of the system.

By substituting Eq. (IX4.9) into (IX5.1) and taking the limit $S^* \to 0$, we obtain

$$B_5 + B_{10} + B_{14} - \tau_{12}^* B_{11} - \tau_{13}^* B_{15} < 0 \qquad \text{(IX5.2)}$$

where the coefficients B are given in Appendix E. Thus, the sufficient condition for the excursive stability of the system is given by the inequality (IX5.2). The condition given by Eq. (IX5.2) ensures that the total pressure drop-inlet flow curve has a positive slope. This criterion has been obtained from the perturbation analysis; however, it can also be derived from the steady state relation between the total pressure drop and the inlet flow velocity.

By substituting the coefficients B into Eq. (IX5.2), we have

$$
\begin{aligned}
&-2k_i - 2\frac{f_s}{2D^*}\lambda^* + \frac{1}{N_{Fr}N_{pch}}\left\{\frac{C_r^*-1}{C_r^*}(1-N_{sub}) - \ln C_r^*\right\} + \\
&+ \frac{f_s C_m}{2D^*}\frac{(C_r^*-1)}{N_{pch}}\left\{(N_{sub}-1) - (1+V_{gj}^*)\left(\frac{C_r^*+1}{2}\right)\right\} + \\
&+ \rho_g^*\frac{V_{gj}^{*2}}{1+V_{gj}^*}(2C_r^*-1)\left[(C_r^*-1)+N_{sub}\right] + \frac{k_e}{1+V_{gj}^*}\left[(C_r^*-1)+N_{sub}-2C_r^*(1+V_{gj}^*)\right] + \\
&+ \frac{1}{1+V_{gj}^*}\left\{N_{sub} - (C_r^*-1)(1+2V_{gj}^*)\right\} < 0
\end{aligned}
\qquad \text{(IX5.3)}
$$

If, for simplicity, we neglect the effect of gravity and relative motion between the two phases, the above criterion reduces to

$$
\begin{aligned}
\lim_{S^*\to 0}\frac{\delta \Delta P_{ex}^*}{\delta v^*} = \frac{f_s}{2D^*}&\left\{\frac{3}{2}C_m\tau_{12}^* - 2(C_m-1)\right\}\tau_{12}^* v_{fi}^{*2} - \\
&-2\left\{\left[\frac{f_s C_m}{2D^*} + k_e\right](\tau_{12}^*-1) - k_i + \tau_{12}^*\right\}v_{fi}^* + \\
&+\left\{\frac{1}{2}\left(\frac{f_s C_m}{2D^*}\right) + k_e + 1\right\} > 0
\end{aligned}
\qquad \text{(IX5.4)}
$$

for the excursive stability of the system.

CHAPTER X

NUMERICAL ANALYSIS

In the previous chapter, the application of the dynamic analysis to the stability problem has been explained. Although the solution for the stability boundary can be expressed in a closed form, the exact parametric study requires the use of a computer due to the complexity of the boundary equations, i.e., Eq. (IX3.3) and (IX3.4). Nevertheless, we could obtain qualitative conclusions on the effects of various parameters for the stability of the system by analytical means. Furthermore, the characteristic equation was simplified under particular conditions and the algebraic stability criteria have been derived.

In this chapter, we shall develop a numerical method not only for the examination of the stability at any particular operating condition, but also for the parametric study of the system. Recalling the discussion in Section IX-3 where the basic steps for the parametric study have been explained, the necessary computer programming can be classified into two groups as

1. the computer programming for the neutral stability curves (called PNS), and

2. the computer programming for the test of the stability at any particular point (called TSP).

In the following, the above two computer programs, (PNS) and (TSP), will be discussed in detail.

X-1. Computer Programming for the Neutral Stability Curves (PNS)

Objective of the Programming (PNS)

The objective of the programming (PNS) is to obtain the functional relation between N_{sub} and N_{pch} from Eq. (IX3.3) and (IX3.4) by cancelling the auxiliary parameter ω^*.

Recalling Appendix E and Eq. (IX3.2), the parametric equations (IX3.3) and (IX3.4) for the neutral surface reduce to the following forms:

$$\begin{aligned} Z_{Re}^* = \frac{1}{\omega^*}\Big\{&\left(B_1\omega^{*5} - B_3\omega^{*3} + B_5\omega^*\right) + \left(-B_8\omega^{*3} + B_{10}\omega^*\right)\cos(\omega^*\tau_{13}^*) \\ &- \left(B_7\omega^{*4} - B_9\omega^{*2} + B_{11}\right)\sin(\omega^*\tau_{12}^*) + \\ &+ \left(-B_{12}\omega^{*3} + B_{14}\omega^*\right)\cos(\omega^*\tau_{13}^*) - \\ &- \left(-B_{13}\omega^{*2} + B_{15}\right)\sin(\omega^*\tau_{13}^*)\Big\} = 0 \end{aligned} \tag{X1.1}$$

and

$$\begin{aligned} Z_{Im}^* = -\frac{1}{\omega^*}\Big\{&\left(B_2\omega^{*4} - B_4\omega^{*2} + B_6\right) + \left(B_7\omega^{*4} - B_9\omega^{*2} + B_{11}\right)\cos(\omega^*\tau_{12}^*) \\ &+ \left(-B_8\omega^{*3} + B_{10}\omega^*\right)\sin(\omega^*\tau_{13}^*) + \left(-B_{13}\omega^{*2} + B_{15}\right)\cos(\omega^*\tau_{13}^*) \\ &+ \left(-B_{12}\omega^{*3} + B_{14}\omega^*\right)\sin(\omega^*\tau_{13}^*)\Big\} = 0 \end{aligned} \tag{X1.2}$$

where the coefficient B's are given in Appendix E.

By taking the limit $\omega^* \to 0$ in Eq. (X1.1) and (X1.2) we get

$$Z_{Re}^{*}(0) = B_5 + B_{10} + B_{14} - B_{11}\tau_{12}^{*} - B_{15}\tau_{13}^{*} \tag{X1.3}$$

and

$$Z_{Im}^{*}(0) = 0 \tag{X1.4}$$

Thus, in view of the inequality (IX5.2) and Eq. (X1.3), the initial value for $Z_{Re}{}^{*}$ decides the excursion stability.

The domain of the positive ω^{*} which can satisfy both Eq. (X1.1) and (X1.2) is bounded by Eq. (X1.1), since

$$\lim_{\omega^{*}\to\infty} Z_{Re}^{*} = +\infty \tag{X1.5}$$

It can be shown from Eq. (X1.1) that

$$0 < \omega^{*} \leq Max\left\{ 1 \,,\, \frac{1}{|B_1|}\left[|B_3| + |B_5| + \sum_{i=7}^{15}|B_i|\right]\right\} \tag{X1.6}$$

is sufficient range of ω^{*} where Eq. (X1.1) and (X1.2) are to be examined.

<u>Input Data</u>

Input data can be provided either in a dimensional or a non-dimensional form depending on the system of interest. Following the discussion in IX-3, the <u>non-dimensional input data</u> for the PNS become

$$N_{\rho}\,(=\rho_g^{*})\,,\; N_d\,(=V_{gj}^{*})\,,\; N_G\,(=D^{*})\,,\; N_{Fr}\,,\; N_{Res}\,,\; k_i\,,\; k_e$$

and $(N_{sub})_{max}$ for the boundary of the stability plane.

The constitutive equation for f_s, C_m or η' must be supplied in addition to the parametric equations for λ^{*} and $C_r{}^{*}$ given by

$$\lambda^* = N_{sub}/N_{pch} \qquad (X1.7)$$

$$C_r^* = 1 + \frac{1}{(1+V_{gj}^*)} \; (N_{pch} - N_{sub}) \qquad (X1.8)$$

In a dimensional form, the input data become

Property;

$$P_s \;,\; \rho_{fs} \;,\; \rho_{gs} \;,\; \mu_{fs} \;,\; \sigma \;,\; \Delta i_{fg} \;,\; \Delta i_s$$

Geometry;

$$D \;,\; \ell \;,\; |g|\cos\theta' \;,\; A_c \;,\; \xi \;,\; k_i \;,\; k_e$$

Inlet Velocity;

$$\bar{v}_{fi}$$

Furthermore, the constitutive equation for V_{gj} and the definitions of the dimensionless groups D*, N_{Fr}, N_{Res}, N_{sub}, N_{pch}, N_d, and N_ρ should be given.

Basic Structure of PNS

The stability plane is bounded by inequality (IX2.1) and (IX2.2), thus we have a finite domain either for N_{sub} or N_{pch}. This plane can be divided by the constant subcooling and quality lines, each of which is equally intervaled, so that the stability plane is covered by the meshes. At each intersection of the above lines inside the operational plane, the coefficients A and B can be calculated. Then we find the crossover frequencies ω_c^* from Eq. (X1.2) by changing ω^* stepwise and using the "Interpolation Method." These ω_c^* will be substituted into Z_{Re}^*, the real part of the characteristic function. If the value of $Z_{Re}*(\omega_c*)$ is non zero, then the system is in a stable or unstable region, which will be decided by the programming TSP. However, when $Z_{Re}*$ is zero, the point of interest in the stability plane is on one of the harmonic frequency curves which constitutes the D-Partition boundaries. Thus by changing the point systematically in the stability plane and repeating the above step, we can construct the complete D-Partition boundaries. It should be noted here that the singularity curve given by Eq. (X1.3) must be added to the harmonic frequency curves, since it specifies the excursive stability boundary.

Output of PNS

The desirable output, therefore, is the functional relations between N_{sub} and N_{pch} which correspond to the excursive stability boundary. However, due to the continuity of the system characteristic, it may be sufficient to obtain the values for $Z_{Re}*(\omega_c*)$ at the properly intervaled points in the stability plane.

X-2. The Computer Programming for the Test of the Stability at Any Particular Operational Condition (TSP)

The test of the stability at any particular point can be achieved by the direct application of the Mikhailov criterion in Section IX-1. Input data for (TSP) will be N_{pch} (or q_w'') and N_{sub} (or Δi_{sub}) in addition to the input data for (PNS) given in Section X-1 .

The computer programming is straightforward and it is basically to calculate the functions $Z_{Im}*(\omega*)$ and $Z_{Re}*(\omega*)$ for the range of $\omega*$ given by the inequality (X1.6). By plotting this result in the $Z*(j\omega*)$ plane and applying the Mikhailov criterion, the stability of the system can be examined. The detailed discussion on the graphical method has been given in Section IX-1 and Section IX-2.

X-3. Results of the Parametric Study from Numerical Analysis and General Conclusions on the Effects of Various Variables

The analytical conclusions on the effects of the various parameters have been drawn from the characteristic equation in Section IX-3. Furthermore, in Section IX-4, simple algebraic criteria which can give not only the qualitative parametric dependence of the stability boundaries, but also the quantitative characteristics are given. In this section, we shall study the effects of various operational variables on the boundary of the stability of the system by using the results from the computer runs. The typical stability maps obtained from the utilization of (PNS) and (TSP) are given in Figures 15 to 22.

It has been found that more than one neutral frequency curve exists and ω^* appears quasi-periodically. However, the most important curve is the first neutral stability curve, i.e., boundary between D(3) and D(5), since it is the stability boundary.

A. Effects of N_{sub} (Subcooling)

In view of Fig. 15 to 22, it can be seen that there is a characterized $(N_{sub})_c$ such that increasing N_{sub} is stabilizing when $N_{sub} > (N_{sub})_c$ and destabilizing when $N_{sub} < (N_{sub})_c$.

For the range of $N_{sub} > (N_{sub})_c$, the stability boundary curve is almost a straight line nearly parallel to the constant exit quality line.

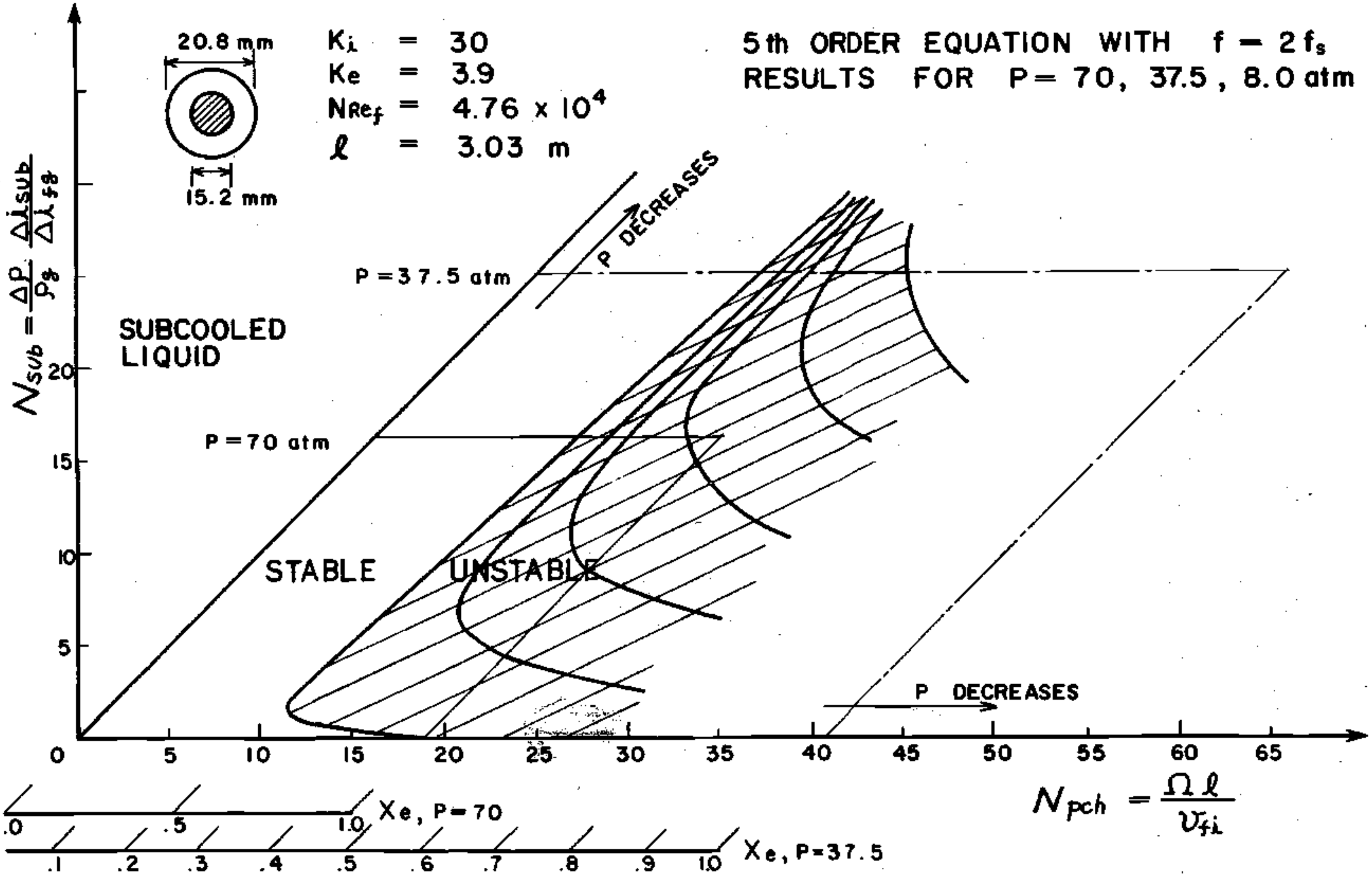


Figure 15. Effect of System Pressure

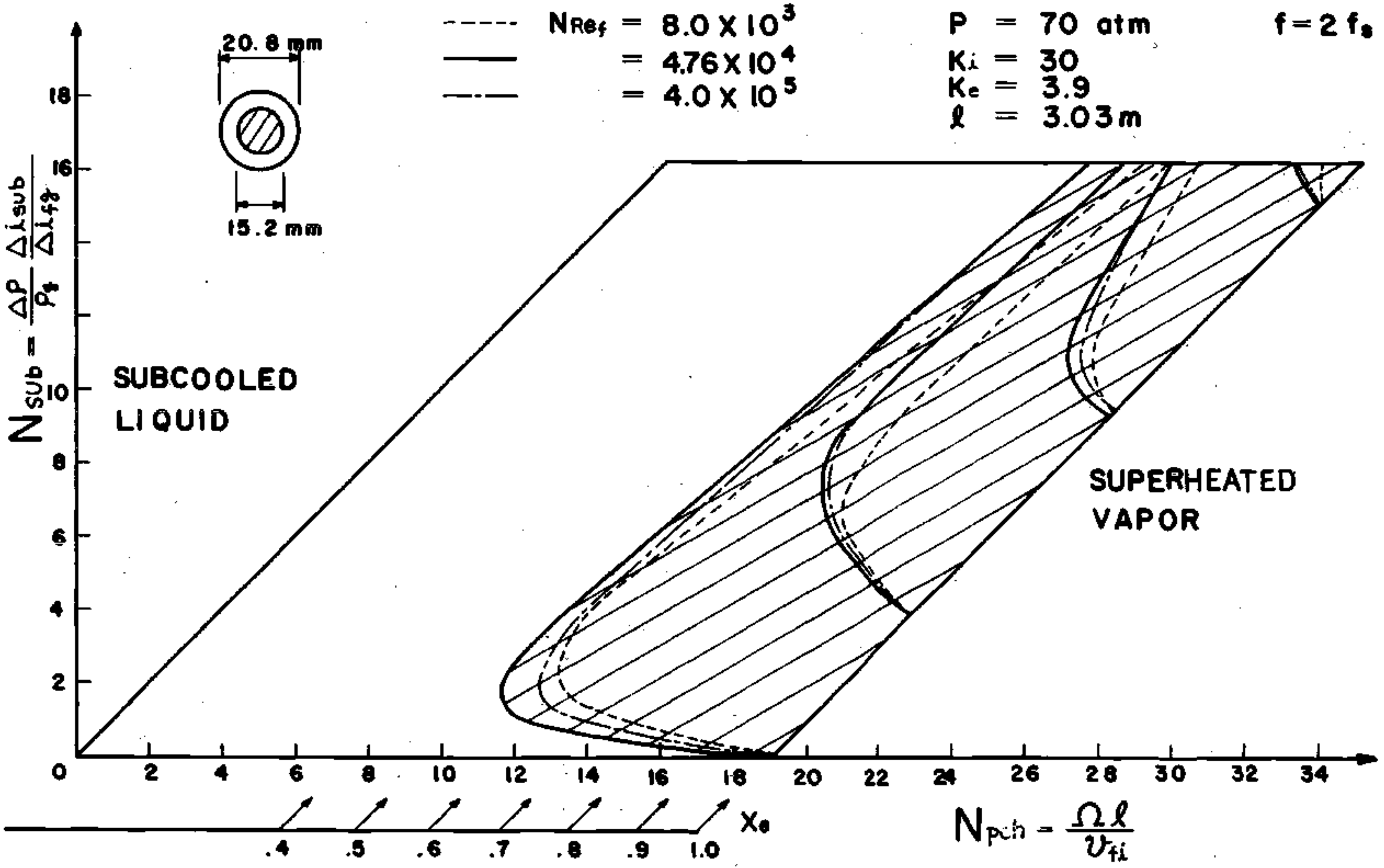


Figure 16. Effect of Reynolds Number

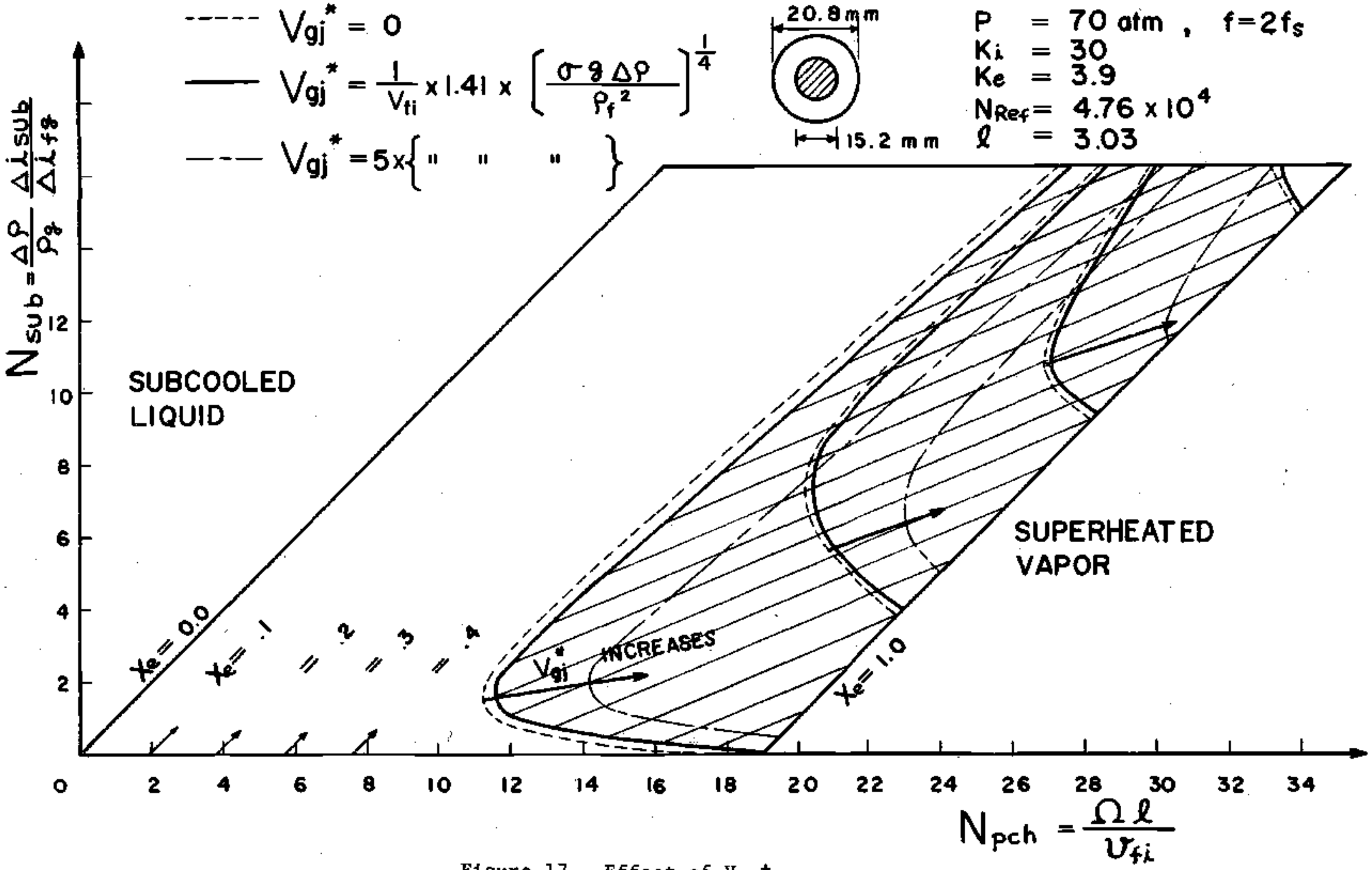


Figure 17. Effect of V_{gj}^*

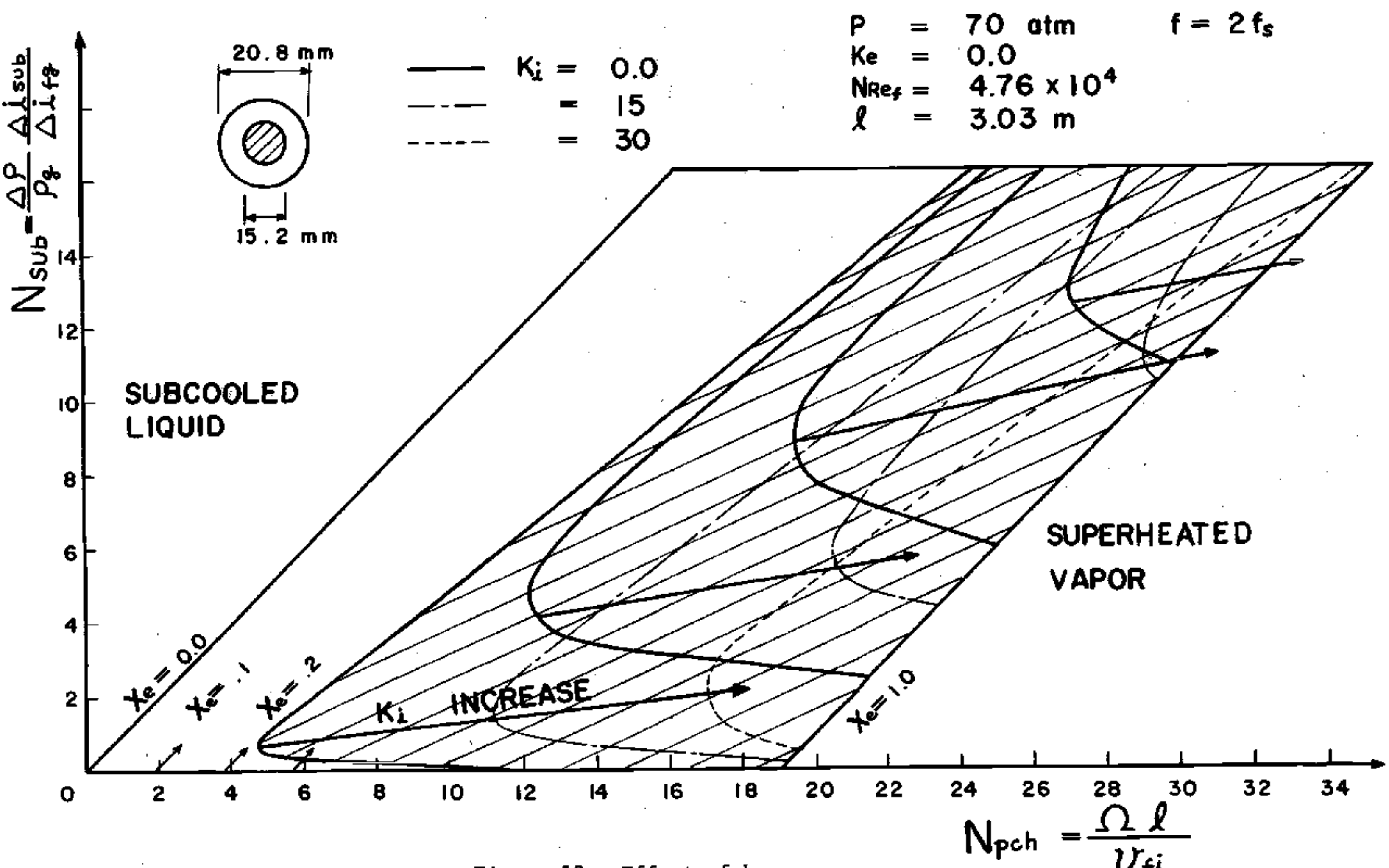


Figure 18. Effect of k_i

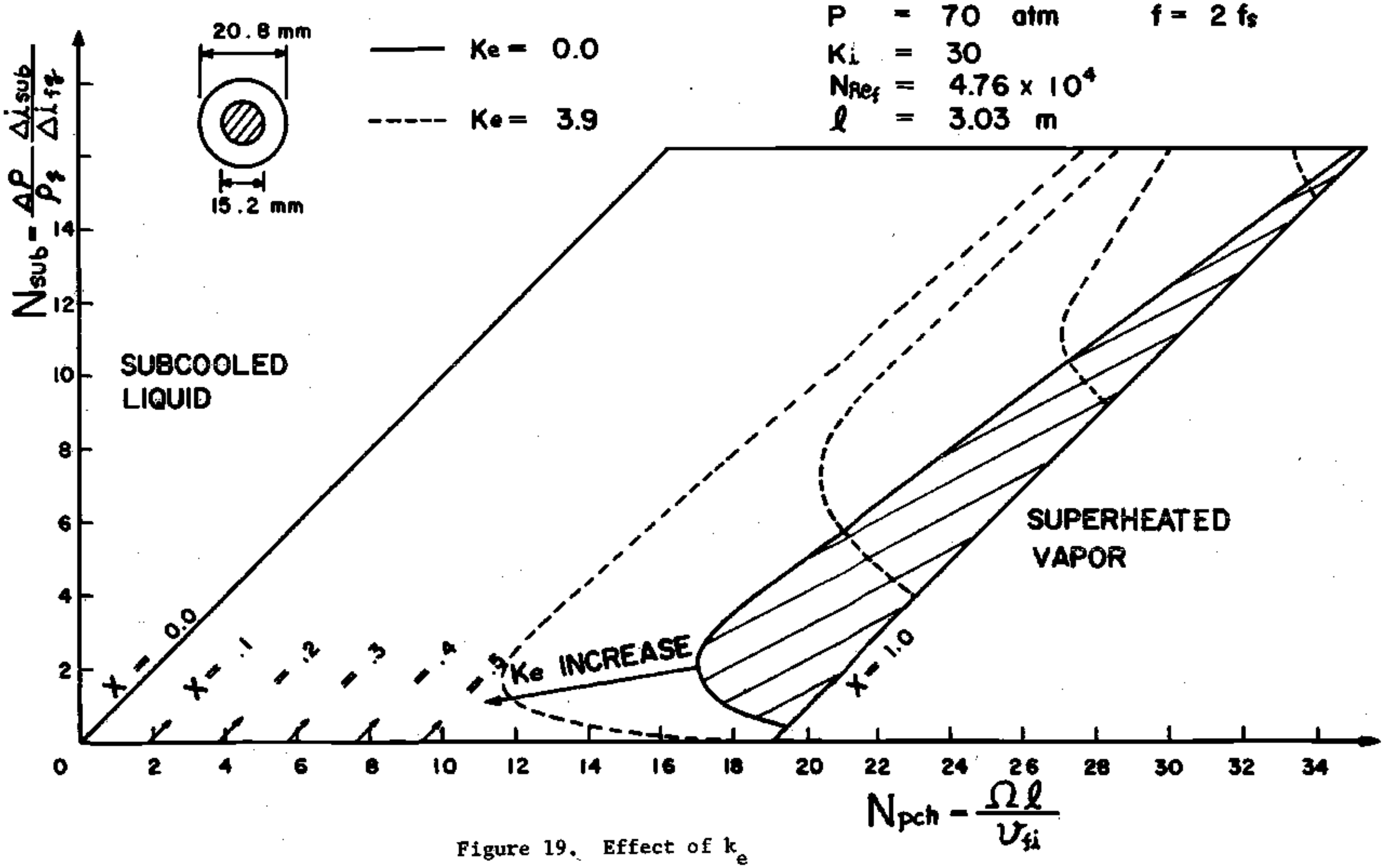


Figure 19. Effect of k_e

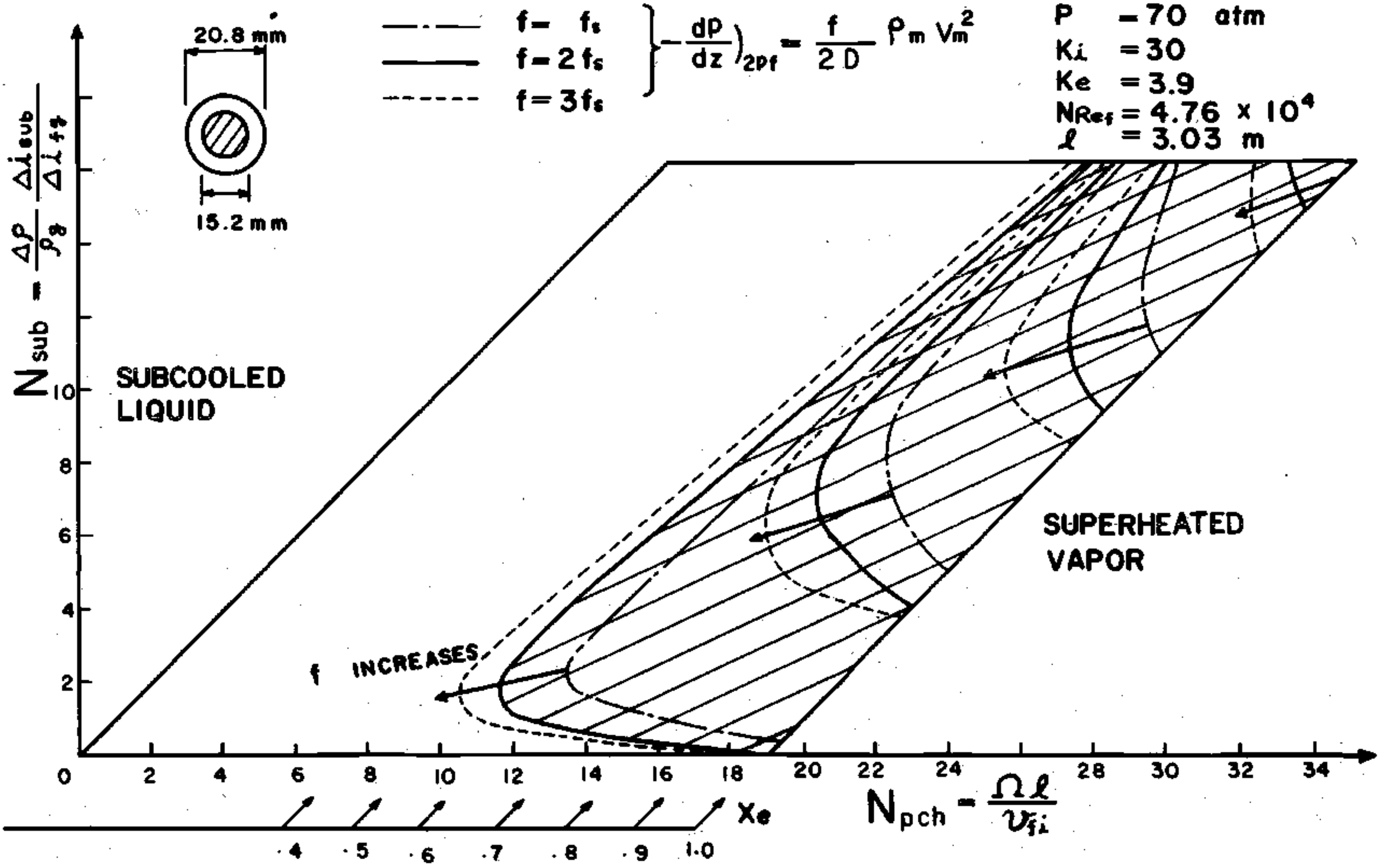


Figure 20. Effect of Friction Factor (magnitude)

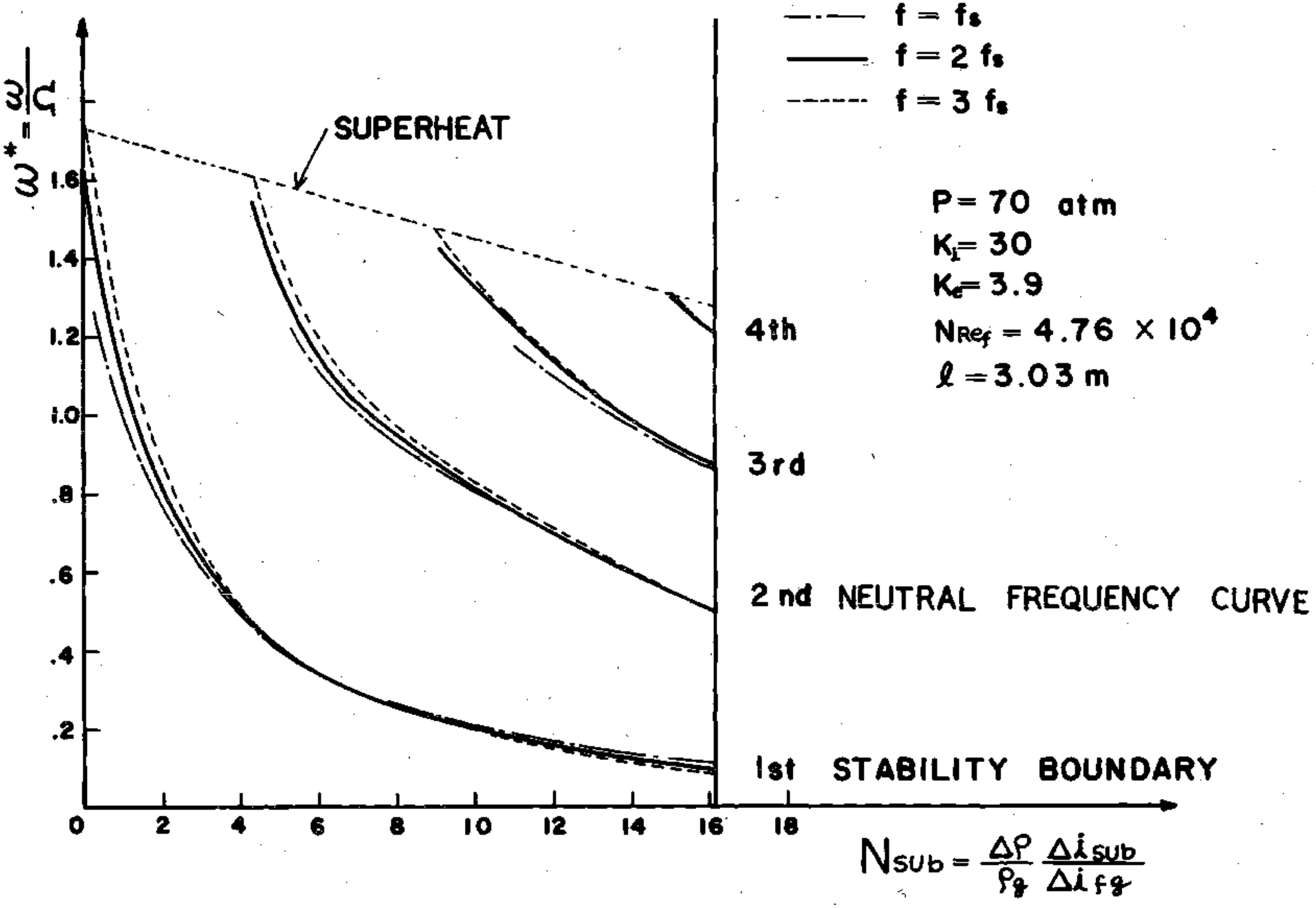


Figure 21. Effect of Friction Factor on ω *

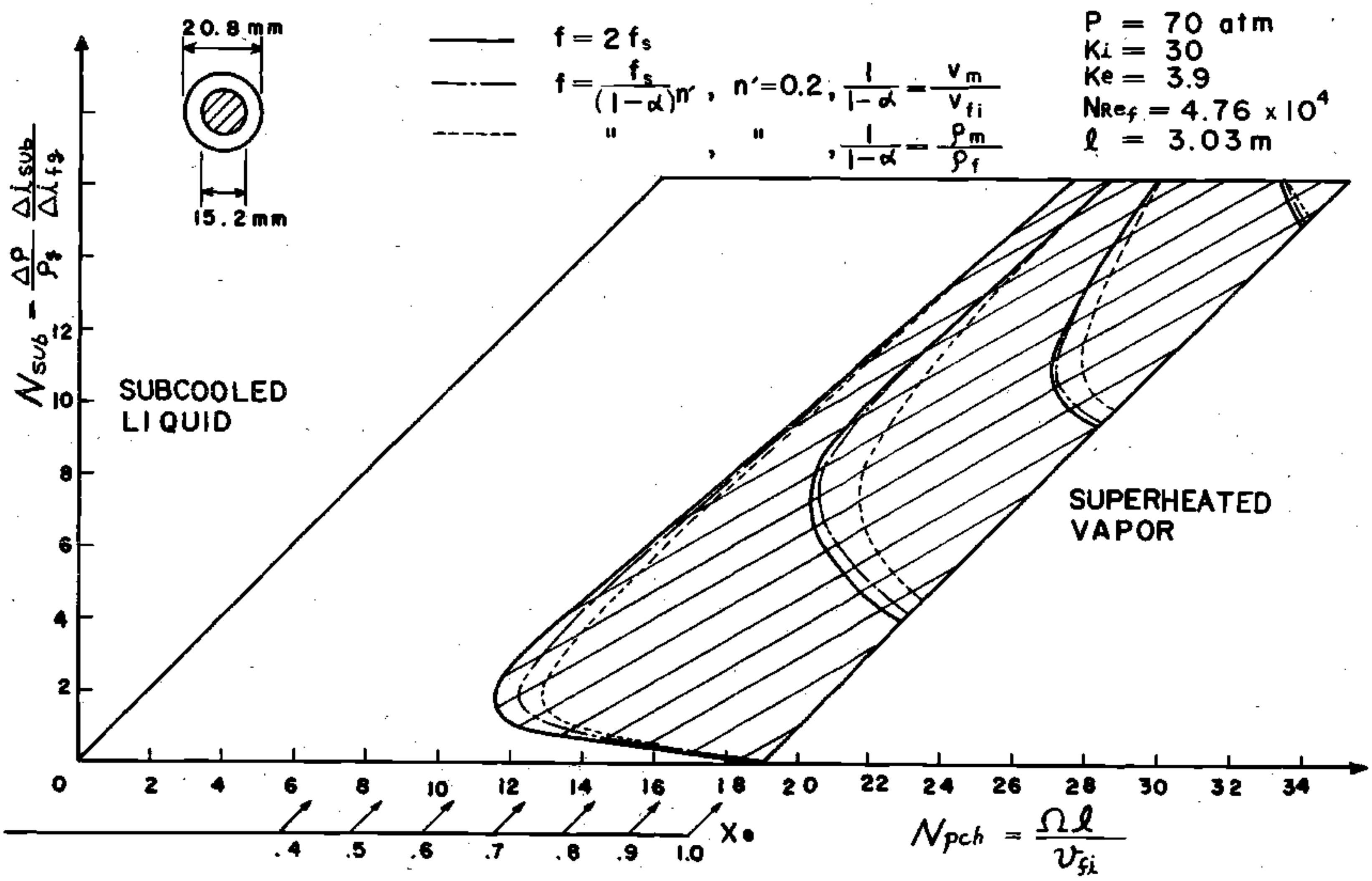


Figure 22. Effect of Friction Factor (Different Models)

This result and the form of the simple criterion (IX4.23) or (IX4.32) suggest that they can be applied for the case of $N_{sub} > (N_{sub})_c$.

The large change of the stability boundary curve at the neighborhood of $(N_{sub})_c$ can be explained by the fact that the frequency ω^* increases as N_{sub} decreases along the curve and thus the higher order polynomial term in the characteristic equation becomes important as ω^* approaches 1. From this argument, it can be said that $(N_{sub})_c$ happens at $\omega^* \approx 1$. By setting $\omega^* = 1$ in Eq. (IX4.26) and eliminating N_{pch} with Eq. (IX4.21), we may obtain $(N_{sub})_c$. On the other hand, from Eq. (IX4.29) we have

$$(N_{sub})_c \lesssim \pi \tag{X3.1}$$

In terms of the subcooling <u>at a given N_{pch}</u>, we conclude that the increase of N_{sub} is stabilizing for $N_{sub} > (N_{sub})_c$ and destabilizing for $N_{sub} < (N_{sub})_c$.

B. Effects of N_{pch} (Heat Flux)

From Fig. 15 to 22 as well as from the simple criterion (IX4.32), it can be seen that <u>increasing N_{pch} is always destabilizing</u>. Thus <u>increasing heat flux at constant inlet velocity shifts the system to the unstable direction</u>.

C. Effects of System Pressure

As has been explained in Section IX-3, the effects of the system pressure are characterized by the density number N_ρ. In the characteristic equation, it appears only in the drift term and in the friction factor coefficients C_m. Thus it has been concluded that the system pressure effects in the stability plane are quite limited. This result is further backed up by the computer run for three different pressure levels, which is shown in Fig. 15, where the stability boundaries obtained on the N_{sub} - N_{pch} plane for different pressure levels cannot be differentiated. From this we conclude that the important part of the system pressure influences on the stability boundaries are taken into account by the subcooling number N_{sub} and phase change number N_{pch}.

D. Effects of N_{Res} (Inlet Velocity).

The analytical conclusion on the effects of inlet velocity is examined by plotting the stability maps for different N_{Res}, i.e., Fig. 16. The results show that the most significant influence of the inlet velocity are accounted for by the phase change number N_{pch}. Thus, increasing the velocity is stabilizing. In terms of the critical heat flux, i.e., maximum heat flux for the system to be stable, it can be said that it is almost inversely proportional to the inlet velocity.

E. Effects of Relative Velocity

By changing the drift number N_d ($= V_{gj}{}^*$), the effects of the relative velocity have been examined. The result is shown in Fig. 17 and it agrees with the analytical conclusion that the relative velocity is a stabilizing effect.

F. Effects of the Inlet Restriction k_i

Figure 18 shows the drastic influence of k_i on the stability boundary. As it has been concluded analytically, increasing k_i is a strong stabilizing factor.

G. Effects of the Exit Restriction k_e

Figure 19 shows the important influence of k_e on the stability of the system. The effect of increasing k_e is shown to be strongly destabilizing as has been expected.

H. Effects of Static Friction Factor

The effect of the static magnitude of the friction factor is examined by changing the value of the friction factor coefficient C_m. As can be seen from Fig. 20, the increase in two-phase frictional pressure drop is destabilizing. The quantitative effect of C_m on the stability boundary largely depends on the values of k_i and k_e. Thus, it can be said that the friction factor becomes more important as the values for k_i and k_e decrease.

I. Effects of Dynamic Friction Factor

The dynamic effects of the two-phase frictional pressure drop have been examined for three different models, i.e., Eq. (VII3.9), (VII3.30), and (VII3.31). The results are shown in Figs. 20, 21, and 22. By comparing the above three stability maps, we can see some differences between them; however, by taking the proper values for C_m and η' , the differences are limited. Thus we may conclude that, for a system with sufficiently large k_i and/or k_e, the difference in friction factor model is not important for operations at high system pressure. However, it also should be noted that, by introducing the complicated dynamic friction factor into the characteristic equation, a numerical instability may be generated due to the complication in the transfer function for the frictional pressure drop.

X-4. Comparison with Experimental Data

The analyses developed in previous chapters have been compared to the experimental data of Levy (5), Solberg (9), and Carver (7). The geometry of the three independent experiments includes both circular and annular tubes at different diameters. The working fluid in all three cases was water. The experimental data reported by these investigations were obtained at relatively high pressures, consequently, they are particularly well suited to test the present analytical results because:

1. The present analysis is limited to the mixture in thermal equilibrium and, furthermore, the pressure drop effects on the fluid properties are neglected.

2. The low frequency oscillation was chosen as the mechanism of instabilities.

Comparisons of the theory to the experimental data are shown in Figs. 23 to 28. The quantitative agreements of the stability boundary as well as of the frequency of the oscillation are excellent. In 90 percent of the cases the agreement in heat flux is within 10 percent. However, it can be seen that, at very small subcooling regions, the experimental data exhibit more unstable systems than the ones predicted by the theory. Two reasons for this discrepancy can be considered. First, at the extremely low subcooling, the stability boundary curve itself has a large change in N_{pch} compared to N_{sub}, which is also confirmed by various experiments (37). Therefore, both the experimental and the numerical points of the starting of the oscillation are difficult to obtain accurately. Second, at the lower subcooling, the effect of subcooled boiling may have a significant influence on the stability of the system,

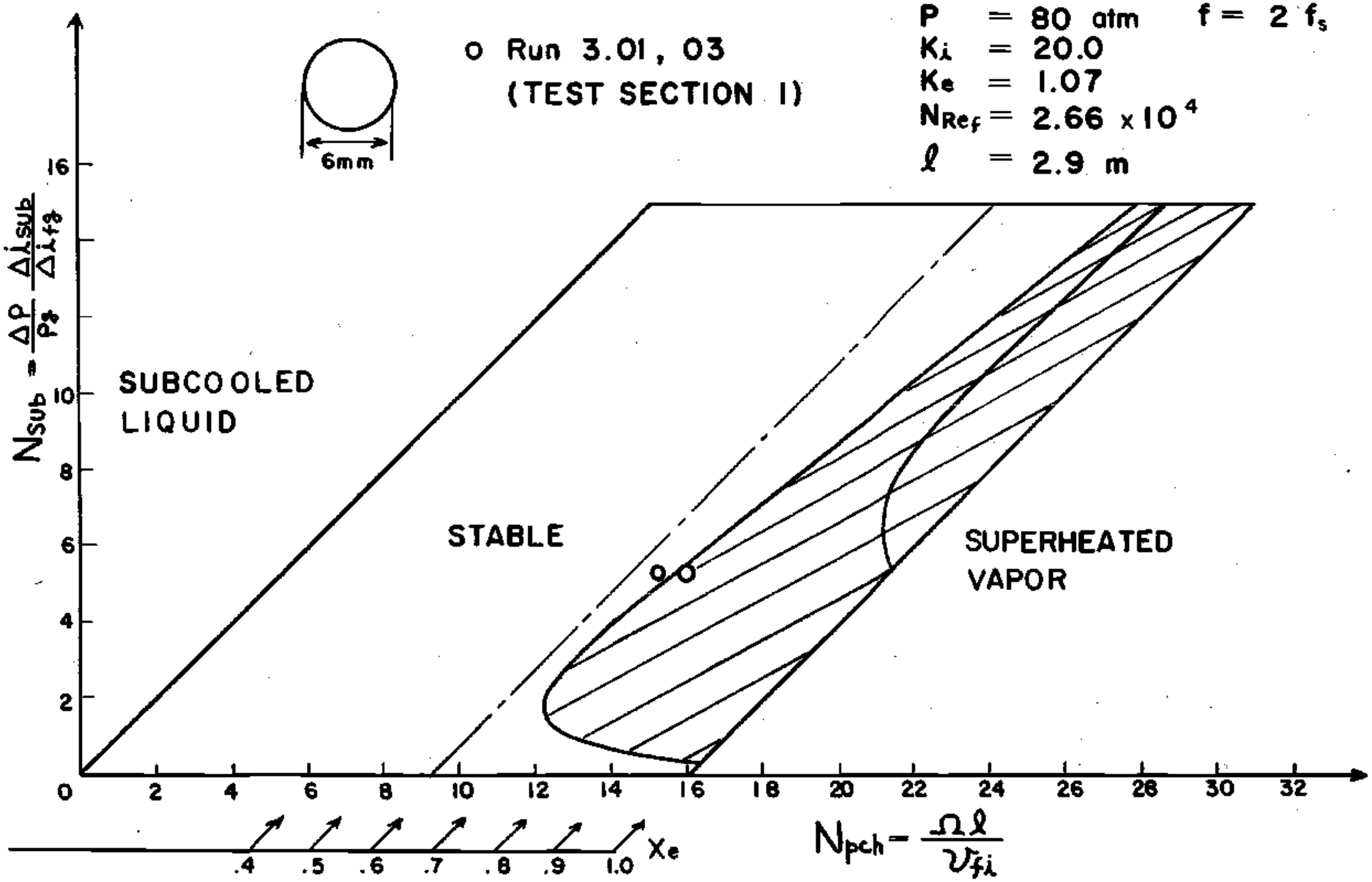


Figure 23. Comparison with Solberg's Experiments; here the Solid Lines Indicate the Computer Solution whereas the Dotted Line Indicates the Simple Criterion Given by (IX.4.32).

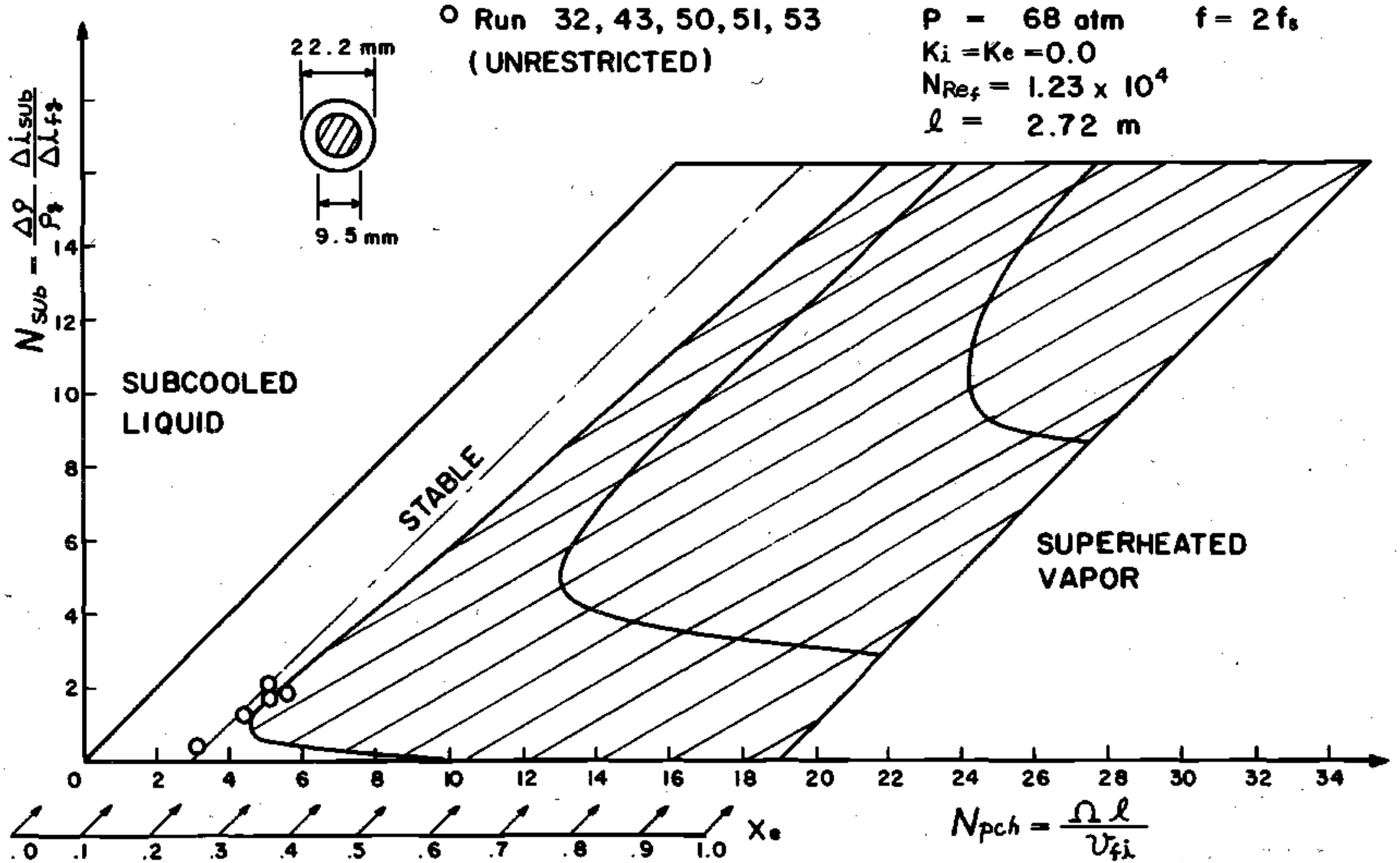


Figure 24. Comparison with Levy's Experiments; here the Solid Lines Indicate the Computer Solution whereas the Dotted Line Indicates the Simple Criterion Given by (IX.4.32).

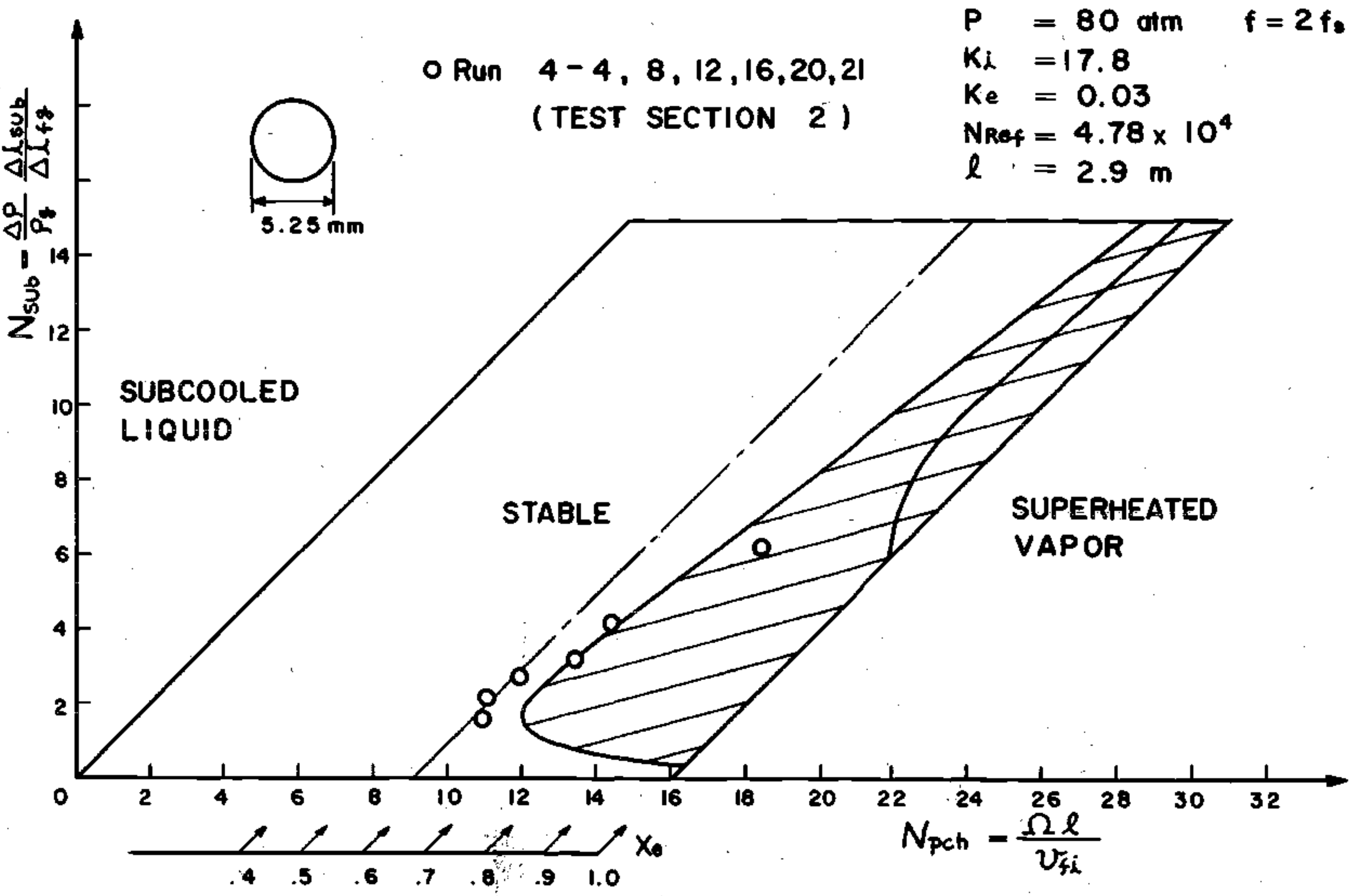


Figure 25. Comparison with Solberg's Experiments; here the Solid Lines Indicates the Computer Solution whereas the Dotted Line Indicates the Simple Criterion Given by (IX.4.32).

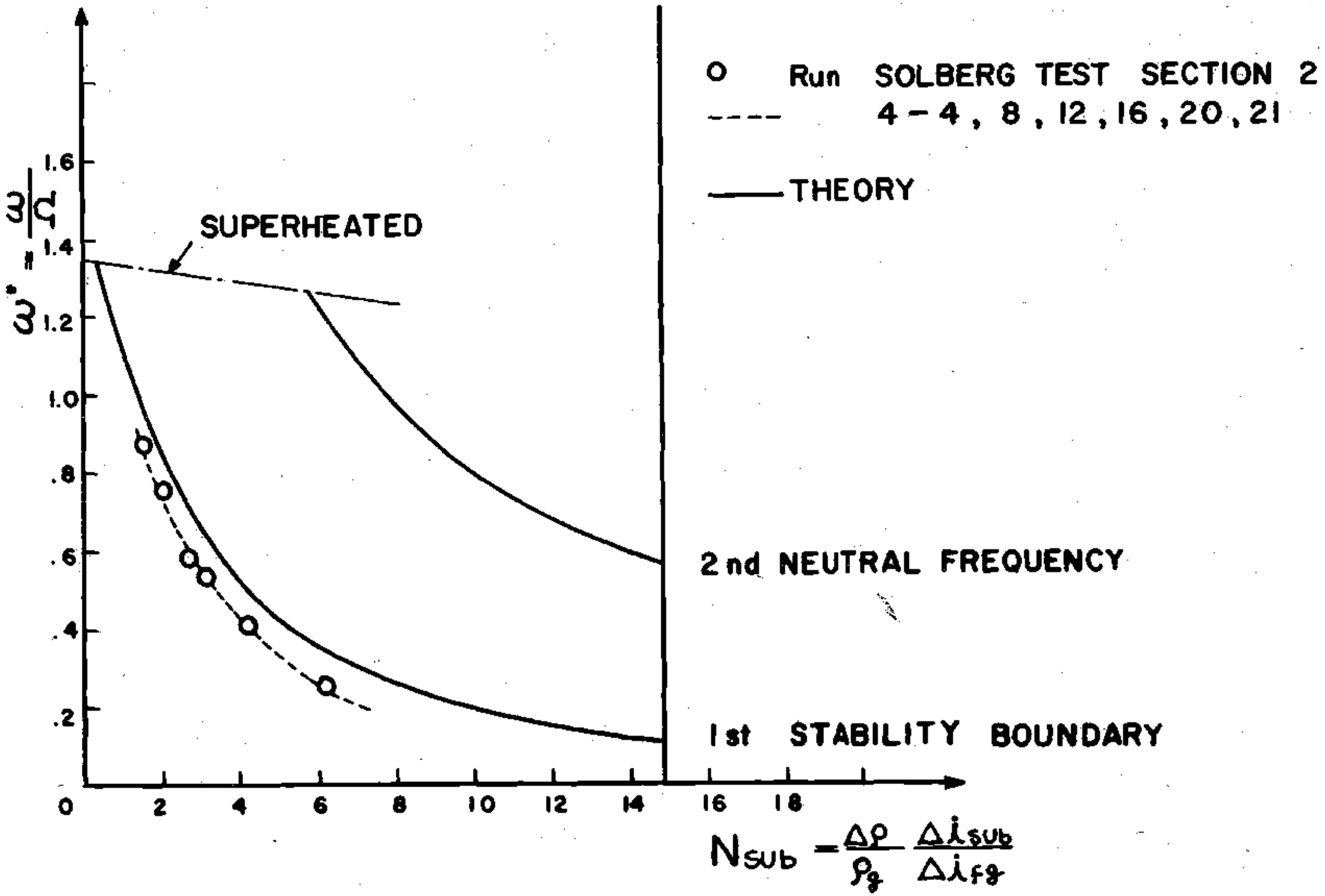


Figure 26. Comparison of Predicted with Measured Frequency

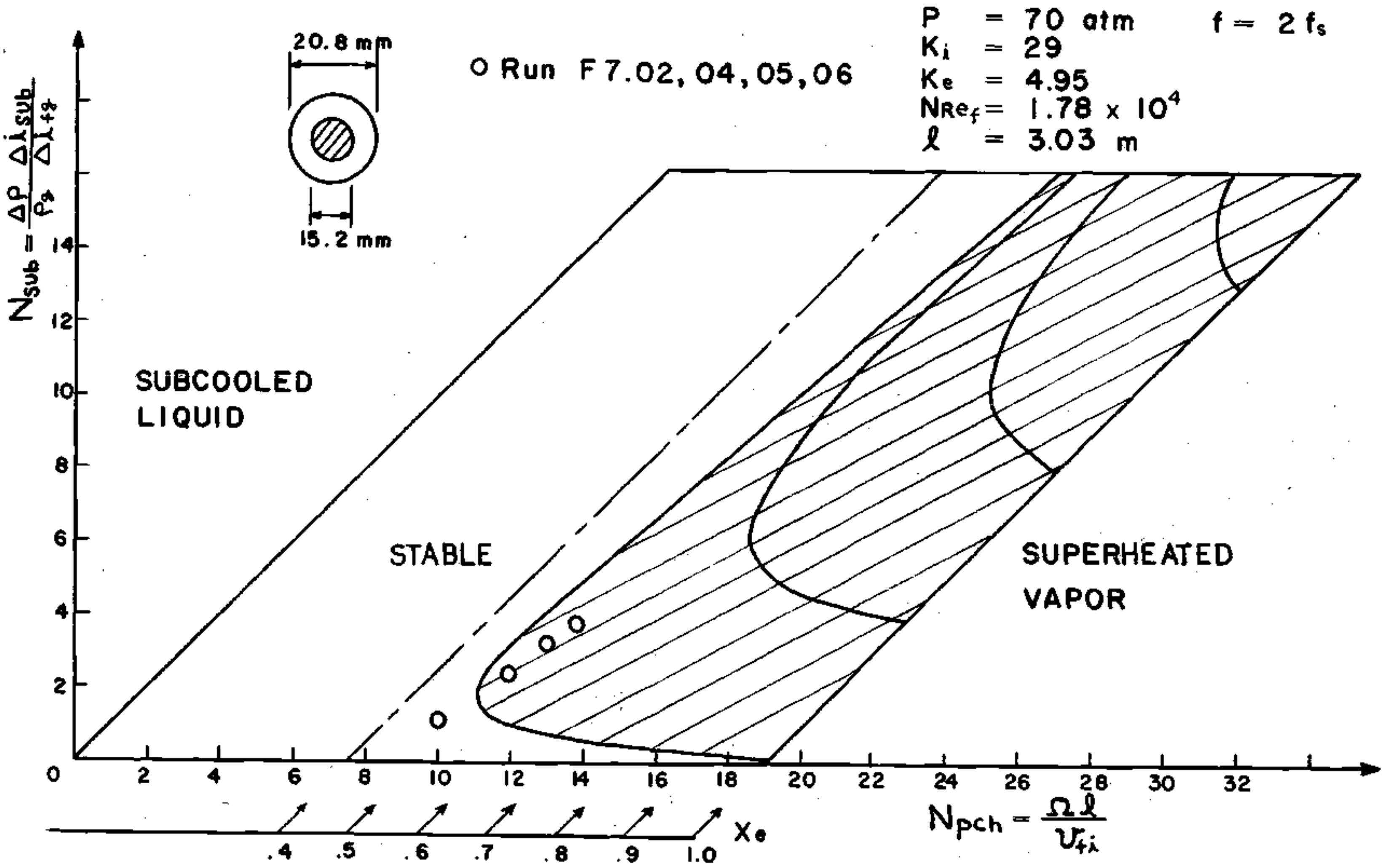


Figure 27. Comparison with FLARE Experiments; here the Solid Lines Indicate the Computer Solution whereas the Dotted Line Indicates the Simple Criterion Given by (IX.4.32).

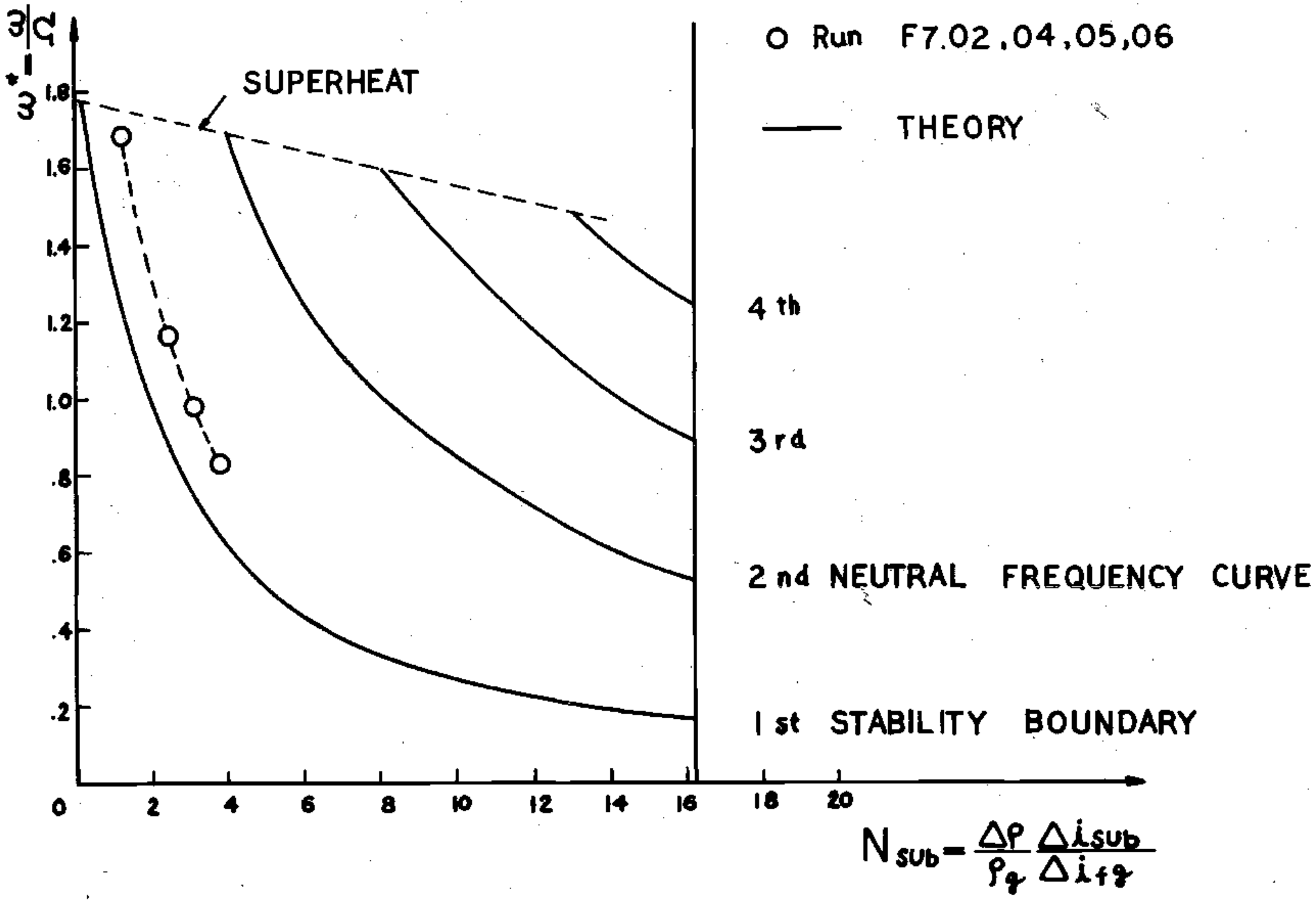


Figure 28. Comparison of Predicted with Measured Frequency

since the amount of heat necessary to remove the subcooling is very small and the portion of the subcooled boiling may dominate the entire liquid heated region of the present analysis. If this is true, the phase change proceeds further than the case of thermal equilibrium, thus the value for N_{pch} increases.

Although the agreement of the theoretically obtained stability boundary to the experimental threshold of oscillations is satisfactory, the analytical description of the unstable state operation itself is limited by the linearization of the mathematical model. However, the present analysis does predict the number of undamped roots in various unstable regions. Such roots appear usually as a couple, thus the number of roots in the unstable regions increases as an even series with an increase in the degree of the instability. For the mathematical model, those roots correspond to the infinite amplification of the oscillation, but for the real physical system, the nonlinearity ultimately becomes important and prevents the unlimited excursion from the equilibrium state. In the unstable regions, it is expected that the undamped roots interact with each other and produce the combined mode of oscillations with finite amplitude. Furthermore, it is also expected that the components of the nonlinear wave have similar frequencies as those of the undamped roots from the linear theory. Thus, in the regions of the higher degree of instabilities, where the system may have the component waves with similar frequencies, <u>the resonance type instability</u> (53) may occur. In such a case, a destructive result is almost certain and it should be avoided even in the transitional operation.

The simplest stability criterion given by inequality (IX4.32) is compared to the computer solution as well as to the experimental results, i.e., Figs. 23 to 28. The solid curve indicates the computer solution, whereas the dotted line indicates the simple criterion.

In view of the considerable simplicity of the criterion, the agreements are rather satisfactory for the range of N_{sub}

$$N_{sub} \gneqq \pi \tag{X4.1}$$

We note here that the significance of the relation (X4.1) is already discussed in Section X-3.A. In the range of $N_{sub} < \pi$, the criterion tends to predict a more unstable system than the exact numerical solution. Thus the criterion (IX4.32) can be used for the first hand design purpose in the entire range for N_{sub}.

The more complicated criterion given by the inequality (IX4.23) is expected to predict more precise parametric dependences of the system, since the former criterion has been obtained by further simplifications on this criterion. By neglecting the stabilizing effect at the extremely lower subcooling, which has been predicted by the exact numerical solution and by recalling the discrepancy between the experimental data and the computer solution in this region, the criterion (IX4.23) is a very good rule in design to avoid the unstable system.

We note here that some details on the method of comparison with experimental data are given in Appendix F.

CHAPTER XI

DISCUSSION

In this analysis, we first derived the field equations for the mixture by statistical and area averaging. Based on this one-dimensional model, the general similarity groups governing the system as well as the relative importance of each term in the field equations have been discussed. Using the above mathematical model, the thermally induced flow instability has been investigated for the case of the mixture in thermal equilibrium. Furthermore, the time lag effect and the density wave propagation have been taken as the main causes of the instability and thus the other instability mechanisms such as the one due to the compressible pressure wave propagation have been eliminated.

Under the above consideration, the density becomes a function only of the enthalpy and not of both the enthalpy and pressure. Therefore, the decoupling of the momentum equation from the energy and continuity equations was possible. Hence the kinematics of the system has been solved first and then the dynamic response of the system has been obtained from the momentum equation.

The perturbation method has been used and the system has been linearized in order to solve the set of partial differential equations analytically. In the present analysis, the relative velocity between the phases, the nonuniform heat flux, the static and dynamic effects of different friction factor models, and the effects of both upstream and downstream un-heated regions have been included.

Part 2 of this thesis has been devoted to the application of the theory divided in Part 1 to the stability analysis of the system. The characteristic equation which describes the onset of the instability has been obtained from the dynamic response of the pressure to the inlet flow perturbation. For simplicity, the uniformly heated system has been considered without a loss of generality, then the dimensionless groups which govern the stability of the system have been obtained from the characteristic equation.

The D-Partition method and the Mikhailov stability criterion have been used for mapping of the stability boundary. The usefulness of the Stability Plane, i.e., $N_{sub} - N_{pch}$ plane, for analyzing two-phase flow stability has been demonstrated.

An extensive parametric study of the stability limits has been carried out both analytically and numerically.

Based on the complete analytical solution, the simple criteria have been obtained under particular conditions; however, the applicability of the criteria for the design purpose is not strictly limited to these operational conditions. This has been demonstrated in comparison with the exact numerical solution. Thus, we have presented not only the exact solution using a high speed computer, IBM 360, but also simple algebraic criteria which require only slide rule calculations. It is important to note here that the input data for the above solutions do not require any additional information than those required for steady state calculations.

Finally, both the exact numerical solution and the simple stability criteria have been compared to the existing experimental data. The theoretically predicted results show a satisfactory agreement at various operational conditions, which confirms the accuracy of the analysis not only on the qualitative aspects, but also on quantitative bases.

By knowing the effect of unperturbed time lag τ_{12} in the uniformly heated system on the stability boundary, now we can return to the discussion on the nonuniformly heated system, i.e., Section VI-8. It has been found there that both the steady state value for the residence time $\bar{\tau}_{12}$ and its fluctuation should be considered in the stability analysis. Furthermore, it has been shown that increasing the heat flux at the boiling boundary in comparison with the non-boiling region heat flux reduces the fluctuation of $\delta\lambda$; thus, it has a stabilizing effect. However, if the above two heat fluxes have the same order, then the weighting function in the integral of Eq. (VI8.3) has a coupling effect with the exponential term, and the influence of the nonuniform heat flux on $\delta\lambda$ cannot be seen directly.

The case of the chopped sine heat flux distribution with a sufficiently high base heat flux, Fig. 8, falls into the above category. Hence, it is necessary to examine the exact characteristic equation in order to establish the dynamic effect of the fluctuating boiling boundary $\delta\lambda$. Apart from this difficulty, it may be said that, for this particular system, the changes in the steady state values of the residence time $\bar{\tau}_{12}$ and the reaction frequency are more important than the effect of $\delta\lambda$, due to the existence of the high base heat flux. By taking into account only the former effects, the characteristic equation basically reduces to a similar form (see Section VII-1) with the following differences between the uniformly and nonuniformly heated systems.

If $\lambda^* < \frac{1}{2}$; $\bar{\tau}_{12(n)} > \bar{\tau}_{12(u)}$, $\Omega_{\infty(n)} > \Omega_{0(u)}$

and $\lambda^* > \frac{1}{2}$; $\bar{\tau}_{12(n)} < \bar{\tau}_{12(u)}$, $\Omega_{\infty(n)} < \Omega_{0(u)}$

where the subscripts (n) and (u) stand for the nonuniformly and the uniformly heated systems. As can be seen from Figs. 15 to 28, the values of N_{sub} at the intersection of the line $\lambda^* = 1/2$ and the dynamic stability boundary are usually higher than $(N_{sub})_c$. If this is the case, the stability boundary for the nonuniformly heated system in the stability plane exhibits a more unstable system than the uniformly heated system <u>as λ^* increases over the value of 1/2</u>.

In the range of $\lambda^* < 1/2$ and $N_{sub} > (N_{sub})_c$, the nonuniform heat flux has a weak stabilizing effect. On the other hand, in the range of $N_{sub} < (N_{sub})_c$, it has an increasingly destabilizing effect as N_{sub} decreases. This result is also in agreement with the experimental observation by Biancone et al. (10).

CHAPTER XII

SUMMARY AND CONCLUSIONS

XII-1. Field Equations

The mixture field equations have been derived by statistical averaging similar to that of the turbulent flow. The results turned out to be quite similar to the field equations for the heterogeneous chemically reacting single phase system, for example, the mixture balance equations of momentum and energy have diffusion terms due to the relative motion of the fluid. However, in contrast to the field equations for chemically reacting single phase mixtures, the field equations for dispersed two-phase flow systems derived in this analysis include also terms which account for the effects of turbulent fluxes and for the existence of interfaces between the two phases. We note that this is the first derivation where these interface terms appear explicitly.

In order to simplify the mathematical model for the dynamic analysis of the channel, the above time averaged field equations were integrated in the radial direction assuming an axially symmetric flow. The resulting one-dimensional model is analogous to the standard single phase flow model in terms of the mixture properties defined in our analysis. The similarity groups governing the two-phase mixture have been obtained from the above model. Basically, eight dimensionless groups scale the two-phase mixture undergoing the phase change, excluding the ones from the covariant terms which are usually neglected even in the single phase flow. The significance and the order of magnitude of each group have been discussed in Appendix C and Section III-2.

Briefly, four similarity groups, i.e., the phase change, drift, density, and surface numbers, characterize the two-phase mixture. The first and second groups take account of the rate of phase change and the drift due to the relative motion of phases, respectively. The density number (or ratio) scales the pressure level of the system and thus the fluid properties. The last group, the surface number, stands for the capillary (body) force due to the surface tension at the interfaces. The equality of phase change number N_{pch} in two different systems ensures that the phase change has progressed equally in both as has been noted by Zuber (33).

XII-2. Formulation of the Problem

The system of interest was subdivided into four regions, i.e., the upstream un-heated, heated liquid, heated mixture, and downstream unheated regions, following the differences in the thermodynamic processes. For each region the system was defined by the appropriate equations of state and the field equations describing the conservation of mass, energy, and momentum.

By introducing two equations of continuity in the mixture region, the effect of the relative motion of the phases has been taken into account. In the mixture region the thermal equilibrium was assumed between two phases and the analysis was concentrated on the mechanism of the kinematic wave propagation (low frequency oscillations).

The dynamic problem was analyzed by perturbing the inlet flow and linearizing the differential equations. First, the kinematic problem was solved by decoupling the momentum equation from the rest of the equations, which was a consequence of the assumption that $\rho_m = \rho_m(i_m)$. Then the dynamic response was obtained by using the solution for the kinematics and integrating the momentum equation along the channel.

XII-3. Characteristic Equation

The singularity of the dynamic response of the system gives the characteristic equation which describes the onset of the instabilities. The problem was solved first for the most general case with arbitrary heat flux distribution and two-phase friction factor model. It can be seen from the general characteristic equation that, depending on the heat flux distribution, the form of the characteristic function may not reduce to the elementary function such as the exponential polynomials. However, it has been suggested that this difficulty may be overcome by approximating the kinematic wave velocity with a linear function in z, since the change of C_k is proportional to the integral of heat flux in z direction. This point is discussed in more detail in Section VI-8. Nevertheless, the effect of chopped sine heat flux distribution was concluded to be destabilizing for a higher subcooling number and for a very small subcooling number. However, it should be noted here that, in the range of $\lambda^* < 1/2$ and $N_{sub} > (N_{sub})_c$, there also exists a stabilizing effect.*

By taking a uniform heat flux, the effects of various friction

---

*This is a substantially important conclusion for nuclear reactor application.

factor models have been examined both from the static and dynamic points of view. It can be said that the perturbation of a complicated two-phase friction factor model such as the one given by Martinelli and Nelson (43) may lead to a numerical instability, due to the fact that such models will generate large numbers of terms in the exponential polynomial.

In view of the characteristic equation, the important factors of the instabilities are the time delay in the liquid subcooled region, thus the space lag, and the time delay in the mixture region. These two exponential terms introduce a phase shift between the system response and the inlet perturbation. The characteristic equation can predict not only the onset of the flow oscillation, but also the excursion instability. Hence it can be used to obtain stability maps and stability criteria by using the standard mathematical theory on the stability.

XII-4. Stability Map

The characteristic equation was non-dimensionalized and the governing set of similarity groups for the stability of the system was obtained. Excluding the geometrical similarity groups from the structural configuration of the system, the governing parameters are N_ρ, N_{sub}, N_{pch}, N_d, N_{Res}, N_{Fr}, k_i, and k_e. The first four groups characterize the two-phase flow system with a subcooled region and are given by the following definitions.

$$N_\rho = \rho_g/\rho_f$$

$$N_{sub} = \frac{\Delta i_{12}}{\Delta i_{fg}} \frac{\Delta \rho}{\rho_g}$$

$$N_{pch} = \frac{\Omega \ell}{\overline{v}_{fi}} = \frac{q_w'' \xi \ell}{A_c \overline{v}_{fi} \Delta i_{fg}} \frac{\Delta \rho}{\rho_g \rho_f}$$

and

$$N_d = \frac{V_{gj}}{\overline{v}_{fi}}$$

The remaining dimensionless groups N_{Res}, N_{Fr}, k_i, and k_e have their standard significance and need no further comment. It was found that the most appropriate domain of stability maps is the N_{sub} and N_{pch} plane, i.e., the stability plane.

The characteristic equation has been solved for the stability boundary with N_{sub} and N_{pch} as parameters. First, all other parameters have been fixed, then the effect of each parameter has been examined by changgin the value for one at once. It was necessary to use a high speed computer in order to solve the characteristic equation in its exact form. Nevertheless, the computing time for the stability test at a particular operational condition is almost negligible compared with the finite difference method previously used in the stability analysis based on the numerical integration of a set of partial differential field equations. Thus, in our analysis, the extensive parametric study could be carried out with relatively short computing time. In solving the characteristic equation for the stability boundary, the D-Partition method and the Mikhailov criterion have been utilized. First, the auxiliary parameter S^* was set equal to $j\omega^*$, then the real and imaginary part of the characteristic function was equated to zero. Then the above two equations have been solved for the neutral frequency curves. It was found that more than one harmonic frequency curve exists; furthermore, ω^* appears quasi-periodically.

Each subdivision is checked by the Mikhailov criterion, which shows that the first neutral frequency curve corresponds to the dynamic stability boundary and other harmonic frequency curves indicate the increase in the degree of instabilities. These higher order instability regions are also observed in the experiment (37, 62). In addition to the dynamic stability boundaries, there are two singularities corresponding to $\omega^* \to 0$ and $\omega^* \to \infty$. Only the first condition is important, since it gives the boundary of an excursive instability. The summary on the effect of various parameters on the stability of the system has been given in Section X-3. Briefly, the conclusions are as follows.

There is a characterized N_{sub} such that increasing N_{sub} is stabilizing when $N_{sub} > (N_{sub})_c$ and destabilizing when $N_{sub} < (N_{sub})_c$. For the range of $N_{sub} > (N_{sub})_c$, the stability boundary curve is almost a straight line nearly parallel to the constant quality line. Furthermore, the value for $(N_{sub})_c$ is given by

$$(N_{sub})_c \lesssim \pi$$

Increasing N_{pch} is always destabilizing whereas the increasing of system pressure stabilizes the system. Furthermore, the important part of the system pressure influences on the stability boundaries in the $N_{sub} - N_{pch}$ plane is taken into account by the coordinate itself, i.e., by N_{sub} and N_{pch}.

Similarily increasing the inlet velocity stabilizes the system. However, the most significant influence of the inlet velocity is accounted for by the phase change number N_{pch}.

Increasing the drift number N_d is stabilizing. Thus the effect of the relative velocity is to stabilize the flow. This result can be explained by the fact that an increase of the relative velocity decreases the void fraction and thereby increases the mixture density which has a stabilizing effect.

Increasing the inlet flow restriction k_i is a strong stabilizing effect, whereas increasing the exit flow restriction k_e drastically destabilizes the system.

Both the static and dynamic effects of the frictional pressure drop were investigated. The results show that increasing the two-phase pressure drop destabilizes the flow. Similarly, the dynamic effects are very important, since by using a different two-phase friction factor model, the order of the exponential polynomial may increase. Furthermore, a complicated parametric dependence of a frictional pressure drop model can generate large numbers of terms in the characteristic equation. It can be said that both effects are directly related to the complication in the numerical calculation and in extreme cases can lead to numerical instabilities.

XII-5. Simple Stability Criteria

The exact application of the characteristic equation to the stability analysis requires the use of a computer, though once the computer program has been established, the stability test can be carried out in a matter of seconds. In Section IX-4, we have developed simple algebraic criteria which can be applied for the first hand design purposes. The simplest criterion (XI4.8) was derived by the lumped parameter method.

Thus, its applications are limited to the case with strong inlet and outlet pressure drop restrictions. More sophisticated criteria were obtained by simplifying the original characteristic equation. In our analysis, we considered the asymptotic case of $\omega^* \ll 1$, which corresponded to the high subcooling number. By neglecting the higher order of ω^*, we were able to obtain a criterion (IX4.23) or (IX4.25). The theoretical limitation on this model is given by the inequality (X4.1). However, the comparison of this criterion with an exact numerical solution and the experimental data suggests the extrapolation into further low subcooling regions. This point has been discussed in more detail in Section X-4. Hence we may conclude that for the design purpose the inequality (IX4.23) is an excellent criterion which can be used to avoid the unstable operation. This criterion gives not only the qualitative parametric dependencies of the stability boundary, but also quantitative aspects such as the stability margin of the system.

A further simplified model (IX4.32) of the above criterion was obtained for a homogeneous flow with no gravity effect. From the inequality (IX4.32), it can be seen that the stability boundary is given by the straight line parallel to the constant exit quality line.

XII-6. Comparison with Experimental Data

The exact numerical solution as well as the simplified criterion have been compared to the experimental data of Levy (5), Solberg (9), and Carver (7). The geometry of these independent experiments includes both circular and annular tubes at different diameters. The working fluid in all three cases is water, and the experimental data are taken at relatively high reduced pressures. The comparisons of the theory to the experimental data are shown in Figs. 23 to 28. The quantitative agreements of the stability boundary as well as of the frequency of the oscillation are satisfactory. However, it can be seen that, at very small subcooling regions, the experimental data exhibit more unstable systems than the ones predicted by theory. Two reasons for this discrepancy have been considered. First, at the extremely low subcooling, the stability boundary curve itself has a larger change in N_{pch} compared to a change in N_{sub}. Thus, both experimental and numerical points of the starting of the oscillation are difficult to obtain accurately. Second, the effect of the subcooled boiling, which gives increased phase change, shifts the coordinate itself. This corresponds to the higher N_{pch}, the phase change number. The effects of the change in τ_{12} are compensated for by the changes in Ω, the reaction frequency. Thus, the effect on N_{sub} is considered to be small.

APPENDIX A

TIME AVERAGE

A-1. Time Domain and Definition of Function

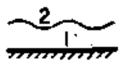
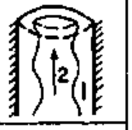
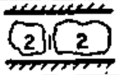
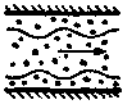
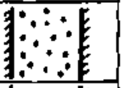
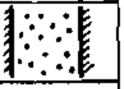
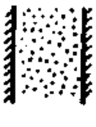
In the following, we shall apply statistical averaging to the two-phase flow field assuming that the occupant of any particular point is alternating randomly between the liquid and the vapor and that the time averaged functions are sufficiently smooth such that they are continuously differentiable.

First we recall that the singular characteristic of two-phase or of two immiscible mixture is the presence of one or several interfaces, between the phases or components (57). Furthermore, whereas single phase flows can be classified according to the geometry of the flow in laminar, transitional and turbulent flow, the flow of two phases or of a mixture of immiscible liquids can be classified according to the geometry of the interface into three classes, i.e., in separated flows, transitional or mixed flows, and dispersed flows. These classes of structured flows are shown in Table 1.

In our analysis of thermally induced flow instabilities, the above three classes of two-phase flows are usually coexisting under turbulent flow conditions, thus it is appropriate to use a statistical average rather than formulating the problem based on the separated flow model.

The discontinuity of properties due to the existence of interfaces is illustrated in Figures A1, A2, and A3 by taking a mass concentration

Table 1. Classification of Structured Flow

	Phase		Single Phase	Two Phase		Geometry
	Component		Two or Multicom.	Single Component	Two or Multicom.	
Class	Type	Regime	Configuration	Configuration	Configuration	
Separated Flows	Plane Flow	Film flow	Liquid A & B A=1, B=2	a) Liquid film in vapor b) Vapor film in liquid (boiling)	a) Liquid film in gas b) Gas film in liquid	
		Stratified flow (layered)	Liquid A,B,C etc. A=1, B=2, C=3	---	Liquid A,C & gas B, etc.	
	Quasi-axisymmetric Flow	Annular flow	Core: liquid A Annulus: liquid B A=2, B=1	a) Core: liquid annulus:vapor b) Core: vapor annulus:liquid	a) Core: liquid annulus: gas b) Core: gas annulus:liquid	
		Jet flow	Liquid A jet in liquid B A=2, B=1	a) Liquid jet in vapor b) Vapor jet in liquid	a) Liquid jet in gas b) Gas jet in liquid	
Mixed or Transitional Flows	Quasi-axisymmetric Flow	Slug flow	Slug: liquid A Continum:liquid B A=2, B=1	Slug: vapor Continum: liquid	Slug: gas Continum: liquid	
		Bubbly annular flow		Core: vapor Annulus: liquid with vapor bubbles	Core: gas Annulus: liquid with gas bubbles	
		Droplet annular flow	Core: liquid A Annulus:liquid B Droplets:liquid B	Core: vapor with liquid droplets Annulus: liquid	Core: gas with liquid droplets Annulus: liquid	
		Bubbly droplet annular flow		Core: vapor with liquid droplets Annulus: liquid with vapor bubbles	Core: gas with liquid droplets Annulus: liquid with vapor bubbles	
	Plane Flow	Film flow with entrainment	Liquid A film in liquid B with droplets A, etc.	Liquid film in vapor with droplets, etc.	Liquid film in gas with droplets, etc.	
Dispersed Flows	Spherical Spheroidal Granular, Etc. Dispersion	Bubbly flow		Bubbles: vapor Continum: liquid	Bubbles: gas Continum: liquid	
		Droplet flow	Droplets:liquid A Continum:liquid B	Droplets: liquid Continum: vapor	Droplets: liquid Continum: gas	
		Particulate flow		a) Particles:solid continum: liquid b) Particles:solid continum: vapor (ice formation)	a) Particles: solid continum: liquid b) Particles: solid Continum: gas	

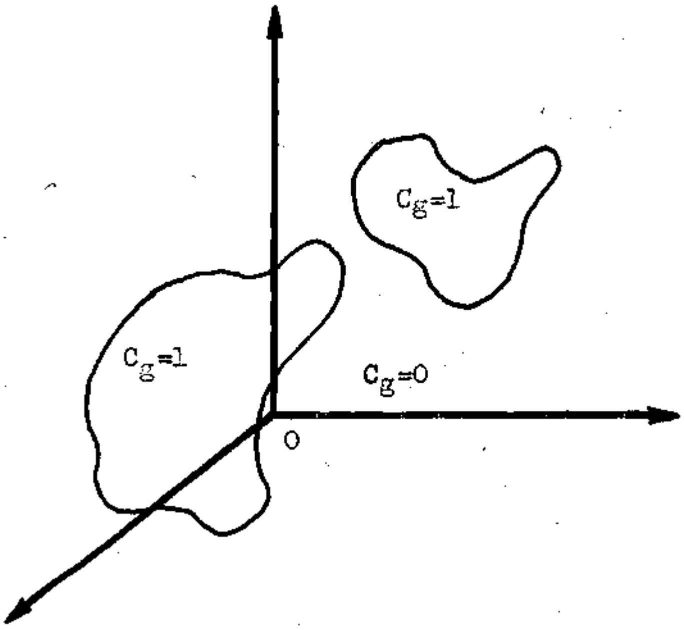


Figure A1. Vapor Concentration in Space at $t = t_o$

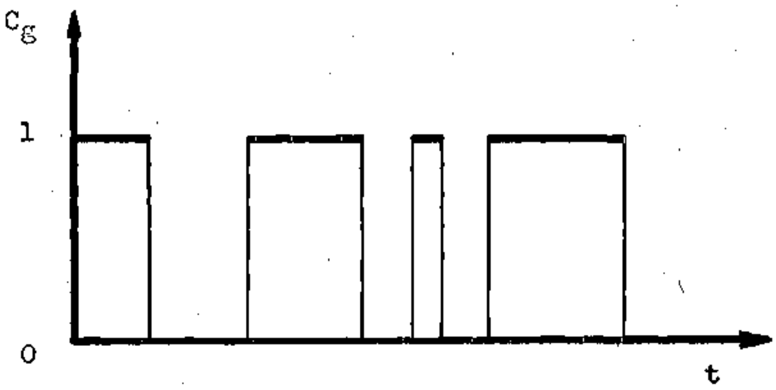


Figure A2. Vapor Concentration in Time at $X = X_o$

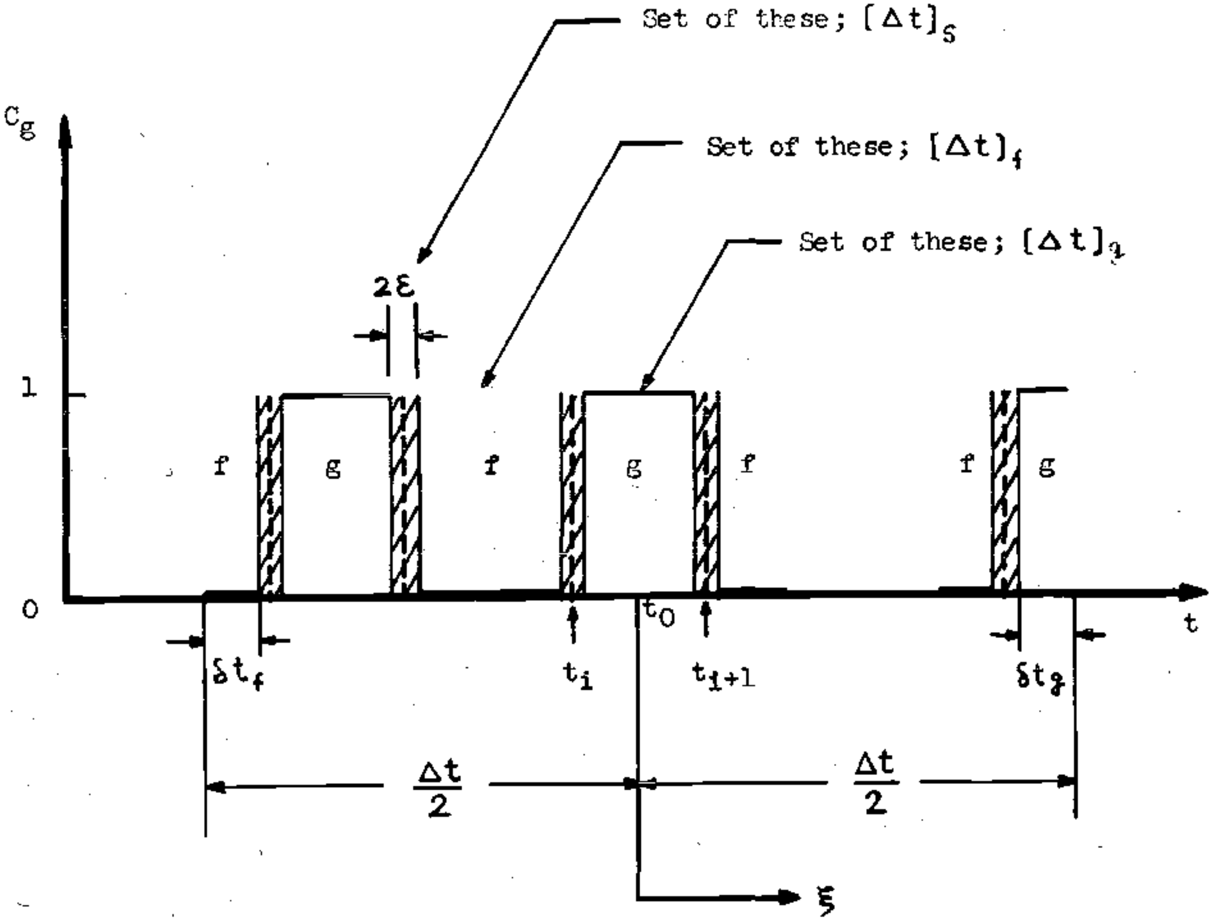


Figure A3. Various Time Intervals

of vapor phase c_g. Figure A1 shows the instantaneous discontinuities of c_g in space, whereas Figures A2 and A3 exhibit the discontinuities in time at some fixed point. From the purpose of statistical averaging, the view from the time coordinate gives a more accurate picture of the problem. It can easily be seen that four distinct processes may occur at that point, which we can classify as follows.

i)	$c_g = 0$ for all t ;	always liquid at Xo
ii)	$c_g = 1$ for all t ;	always vapor at Xo
iii)	c_g changes stepwise between 0 and 1 ;	liquid and vapor alternate at Xo
iv)	c_g is undefined ;	interface stays at Xo for some finite time

It is easy to understand that, following a change of c_g, all properties may change drastically, because the fluid which occupies the point will be different. Since for the case of (i) and (ii) the statistical averaging at that point is trivial, we shall eliminate such cases. Furthermore, as case (iv) is a rather singular configuration of case (iii) unless the interface is stationary, it will be considered later. Hence we shall examine the case (iii) where the vapor concentration alternates stepwise between 0 and 1.

Our purpose here is to time average the fluid properties and field equations in order to treat them as a mixture. First we take a fixed time interval Δt of statistical averaging and assume that it be large enough to smooth out the local variations of properties yet small compared to the macroscopic time constant of the unsteadyness of the bulk fluid. This assumption is identical to that made in analyzing turbulent single phase flow. After choosing any particular reference point and time (X_o, t_o), we have definite times, t_1, t_2, ..., t_i, ... referring to the interfaces which pass the point X_o from time ($t_o - \Delta t/2$) to ($t_o + \Delta t/2$).

By choosing an arbitrarily small time ε we can define the time intervals associated with interfaces, where the characteristics of the interface dominate. The symbol $[\Delta t]_s$ gives the set of such time intervals, i.e.

$$[\Delta t]_s \quad ; \quad t \ni \quad t_i - \varepsilon \le t \le t_i + \varepsilon \qquad i = 1, \cdots, n \tag{A1.1}$$

The remaining part of the Δt interval is given by $[\Delta t]_T$, which can be separated into a vapor and liquid phase time intervals. Thus

$$[\Delta t]_T = [\Delta t]_g + [\Delta t]_f \tag{A1.2}$$

and

$$[\Delta t] \quad ; \quad t \ni \quad t_o - \frac{\Delta t}{2} \le t \le t_o + \frac{\Delta t}{2} \tag{A1.3}$$

Then

$$[\Delta t] = [\Delta t]_s + [\Delta t]_T = [\Delta t]_s + [\Delta t]_g + [\Delta t]_f \tag{A1.4}$$

Now we consider a function F associated with fluids which has the following properties

I) $F = F(X, t)$ $\qquad \frac{\partial F}{\partial t}, \ \frac{\partial^2 F}{\partial x_j^2}, \ \frac{\partial F}{\partial x_j}$, etc.

exist except at the interface

II) for fixed $X = X_o$

$$F(X_o, t) = F_g(X_o, t) \quad , \; F_f(X_o, t) = 0 \quad \text{if} \quad t \in [\Delta t]_g \qquad \text{(A1.5)}$$

$$F(X_o, t) = F_f(X_o, t) \quad , \; F_g(X_o, t) = 0 \quad \text{if} \quad t \in [\Delta t]_f \qquad \text{(A1.6)}$$

The fractions of time occupied by vapor and liquid are defined by taking the limit $\varepsilon \to 0$, as

$$\Delta t_g \equiv \lim_{\varepsilon \to 0}\left[\sum^{n} \left\{ (t_{i+1} - \varepsilon) - (t_i + \varepsilon)\right\} + \delta t_g\right] \quad , \; i = 2m \qquad \text{(A1.7)}$$

$$\Delta t_f \equiv \lim_{\varepsilon \to 0}\left[\sum^{n} \left\{ (t_{i+1} - \varepsilon) - (t_i + \varepsilon)\right\} + \delta t_f\right] \quad , \; i = 2m+1$$

and therefore

$$\Delta t = \Delta t_g + \Delta t_f \quad \text{(If the interfaces are not stationary)} \qquad \text{(A1.8)}$$

Instantaneous Time Scale

For simplicity, let us introduce a new time coordinate with respect to $t = t_o$, i.e.,

$$\xi = t - t_o \qquad \text{(A1.9)}$$

Then our time domain of interest becomes

$$-\frac{\Delta t}{2} \le \xi \le \frac{\Delta t}{2} \qquad \text{(A1.10)}$$

The correspondences of sets $[\Delta t]$, $[\Delta t]_T$, $[\Delta t]_g$, $[\Delta t]_f$, and $[\Delta t]_s$ in the ξ domain are defined by $[\Delta\xi]$, $[\Delta\xi]_T$, $[\Delta\xi]_g$, $[\Delta\xi]_f$, and $[\Delta\xi]_s$.

A-2. Time Averaged and Weighted Mean Values of Functions

In this section we shall define the time averaged and time weighted mean values of functions introduced in the previous section. First let us define the time averaged function F at t_o as follows.

$$\bar{F}(X,t_o) \equiv \lim_{\varepsilon\to 0} \frac{1}{\Delta t}\int_{[\Delta t]_T} F(X,t)\,dt \tag{A2.1}$$

or

$$= \lim_{\varepsilon\to 0} \frac{1}{\Delta t}\int_{[\Delta\xi]_T} F(X,(\xi+t_o))\,d\xi \tag{A2.2}$$

If we assign the integer 1 for vapor and 0 for liquid, the probability of the vapor occupying any given point can be calculated by a statistical average. Thus we have

$$F(X,t) = M(X,t) \tag{A2.3}$$

and

$M(X,t) = 1$ for vapor phase

$M(X,t) = 0$ for liquid phase

We note that, as a consequence of the previous assumption, the interfaces are not stationary and do not occupy a location X for finite time intervals. Then by using definition (A2.1), we can find the time averaged probability of vapor phase α_g as

$$\alpha_g = \bar{M} = \lim_{\varepsilon \to 0} \frac{1}{\Delta t} \int_{[\Delta t]_T} M(x,t)\,dt \qquad \text{(A2.4)}$$

$$= \frac{\Delta t_g}{\Delta t}$$

Similarly for the liquid phase we have

$$\alpha_f = \overline{1-M} = \frac{\Delta t_f}{\Delta t} \qquad \text{(A2.5)}$$

From (A2.4) and (A2.5) we have

$$\alpha_g + \alpha_f = 1 \qquad \text{(A2.6)}$$

which are the results based on the assumption that $\alpha_s = 0$. Hence, we shall introduce the statistical concentration associated with the vapor phase, namely

$$\alpha_g = \alpha = \frac{\Delta t_g}{\Delta t} \qquad \text{(A2.7)}$$

$$\alpha_f = 1 - \alpha = \frac{\Delta t_f}{\Delta t}$$

Now let us proceed and consider the average of the function F. From Eq. (A2.2) we have

$$\bar{\bar{F}} = \lim_{\varepsilon \to 0} \left\{ \frac{1}{\Delta t} \int_{[\Delta t]_T} F_g(x,t)\,dt + \frac{1}{\Delta t} \int_{[\Delta t]_T} F_f(x,t)\,dt \right\}$$

$$= \bar{\bar{F}}_g + \bar{\bar{F}}_f$$

Thus we get

$$\bar{F} = \bar{F}_g + \bar{F}_f \tag{A2.8}$$

The functions F_g and F_f are directly related to instantaneous, local physical or flow variables of each phase; however, $\bar{F}_g$ and $\bar{F}_f$ are averaged over the total time interval Δt; thus, they can be considered as superficially averaged values. From this point of view we introduce more reasonable mean values of the functions which depend on the characteristic of the variables. If a variable G is an intensive property such as the pressure or the temperature, the weighted mean value is defined as

$$\bar{\bar{G}}_k = \frac{1}{\Delta t_k} \lim_{\varepsilon \to 0} \int_{[\Delta t]_k} G_k \, dt = \frac{\bar{G}_k}{\alpha_k} \qquad (k = g, f) \tag{A2.9}$$

Hence

$$\bar{G} = \bar{G}_g + \bar{G}_f = \alpha \bar{\bar{G}}_g + (1-\alpha) \bar{\bar{G}}_f \tag{A2.10}$$

which shows that the intensive variables are weighted by the probability α_k. On the other hand, the volume, energy, enthalpy, etc. are considered to have an extensive characteristic. If the functions F_g and F_f are taken as the variables per unit volume, then they can be expressed also in terms of the variables per unit mass ψ_g and ψ_f as

$$F_k = \rho_k \psi_k \qquad (k = g, f) \tag{A2.11}$$

Here ψ_g and ψ_f are directly related to the instant local physical and flow properties of each phase. The appropriate mean values in this case should be weighted by the averaged densities, thus

$$\bar{\bar{\psi}}_k = \frac{\overline{\rho_k \psi_k}}{\bar{\rho}_k} \qquad (k = g, f) \qquad \text{(A2.12)}$$

In particular for the densities we have

$$\bar{\bar{\rho}}_k = \frac{1}{\Delta t_k} \lim_{\varepsilon \to 0} \int_{[\Delta t]_k} \rho_k \, dt = \frac{\bar{\rho}_k}{\alpha_k} \qquad (k = g, f) \qquad \text{(A2.13)}$$

Thus we get for the mixture density

$$\bar{\rho} = \bar{\rho}_g + \bar{\rho}_f = \alpha \bar{\bar{\rho}}_g + (1-\alpha) \bar{\bar{\rho}}_f \qquad \text{(A2.14)}$$

Using Eq. (A2.11) and (A2.12) we have

$$\alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k = \overline{\rho_k \psi_k} \qquad (k = g, f) \qquad \text{(A2.15)}$$

where $\bar{\bar{\rho}}_k$, $\bar{\bar{\psi}}_k$ are the weighted mean values of ρ_k and ψ_k. The weighted mean values of the mixture are defined by

$$\bar{\bar{\psi}} = \frac{\overline{\rho \psi}}{\bar{\rho}} = \frac{\alpha \bar{\bar{\rho}}_g \bar{\bar{\psi}}_g + (1-\alpha) \bar{\bar{\rho}}_f \bar{\bar{\psi}}_f}{\alpha \bar{\bar{\rho}}_g + (1-\alpha) \bar{\bar{\rho}}_f} \qquad \text{(A2.16)}$$

A-3. Time Average of the Derivatives

In this section the relation between the average of the derivative and the derivative of the average will be obtained.

By the time derivative of the average, we mean

$$\frac{\partial \bar{F}}{\partial t} = \frac{\partial}{\partial t_o}\left\{\frac{1}{\Delta t}\lim_{\varepsilon\to 0}\int_{[\Delta t]_T} F(X_o,t)\,dt\right\} = \frac{1}{\Delta t}\frac{\partial}{\partial t_o}\lim_{\varepsilon\to 0}\int_{[\Delta t]_T} F(X_o,t_o)\,dt \tag{A3.1}$$

where

$$\bar{F} = \bar{F}\,(X_o,\ t_o) \tag{A3.2}$$

In view of Eq. (A3.1), the domain of the integration is discontinuous, hence we subdivide it and apply the Leibniz Rule to Eq. (A3.1). Thus we have

$$\frac{\partial \bar{F}(X_o,t_o)}{\partial t_o} = \frac{\partial}{\partial t_o}\left\{\frac{1}{\Delta t}\lim_{\varepsilon\to 0}\int_{[\Delta \xi]_T} F(X_o,\ \xi + t_o)\,d\xi\right\} \tag{A3.3}$$

$$= \frac{1}{\Delta t}\lim_{\varepsilon\to 0}\int_{[\Delta \xi]_T}\frac{\partial F(X_o,\xi+t_o)}{\partial t_o}\,d\xi\ +$$

$$+\lim_{\varepsilon\to 0}\sum_i \frac{1}{\Delta t}\left\{-\frac{\partial(\xi_i+\varepsilon)}{\partial t_o}F(X_o,\xi_i+\varepsilon+t_o)+\frac{\partial(\xi_i-\varepsilon)}{\partial t_o}F(X_o,\xi_i-\varepsilon+t_o)\right\}$$

Then from the definitions

$$\overline{\frac{\partial F}{\partial t}} = \frac{\partial \bar{F}(X_o,t_o)}{\partial t_o} - \sum_i \frac{1}{\Delta t}\left\{F(X_o,t_i^+) - F(X_o,t_i^-)\right\} \tag{A3.4}$$

On the other hand, the average of the space derivatives such as $\frac{\partial F_{i\ell}}{\partial X_j}$, $\frac{\partial F}{\partial X_j}$ at $X = X_o$ can be written generally as

$$\overline{\frac{\partial F(X_o, t_o)}{\partial X_j}} = \lim_{\varepsilon \to 0} \frac{1}{\Delta t} \int_{[\Delta t]_T} \frac{\partial F(X, t)}{\partial X_j}\bigg|_{X_o} dt \qquad \text{(A3.5)}$$

where X and t are independent, except at the interface where we have a relation

$$t_i = t_i \, (X_o, t_o) \qquad \text{(A3.6)}$$

Therefore, by applying the Leibniz Rule to Eq. (A3.5), we get

$$\overline{\frac{\partial F(X_o, t_o)}{\partial x_j}} = \lim_{\varepsilon \to 0} \frac{\partial}{\partial X_{oj}} \left\{ \frac{1}{\Delta t} \int_{[\Delta t]_T} F(X_o, t)\, dt \right. + \qquad \text{(A3.7)}$$

$$+ \lim_{\varepsilon \to 0} \sum_i \frac{1}{\Delta t} \left\{ - \frac{\partial (t_i - \varepsilon)}{\partial X_{oj}}\bigg|_{t_o} F(X_o, t_i - \varepsilon) + \frac{\partial (t_i + \varepsilon)}{\partial X_{oj}}\bigg|_{t_o} F(X_o, t_i + \varepsilon) \right\}$$

The last term of the right hand side of Eq. (A3.7) is not clear and should be examined in more detail. In view of Fig. A4, we have

$$\lim_{\varepsilon \to 0} \quad t_i + \varepsilon \to t_i^{+}$$

$$\lim_{\varepsilon \to 0} \quad t_i - \varepsilon \to t_i^{-}$$

and correspondingly

$$\lim_{\varepsilon \to 0} \; F(X_o, t_i \pm \varepsilon) = F(X_o, t_i^{\pm})$$

The unit normal vector of the interface is defined such that

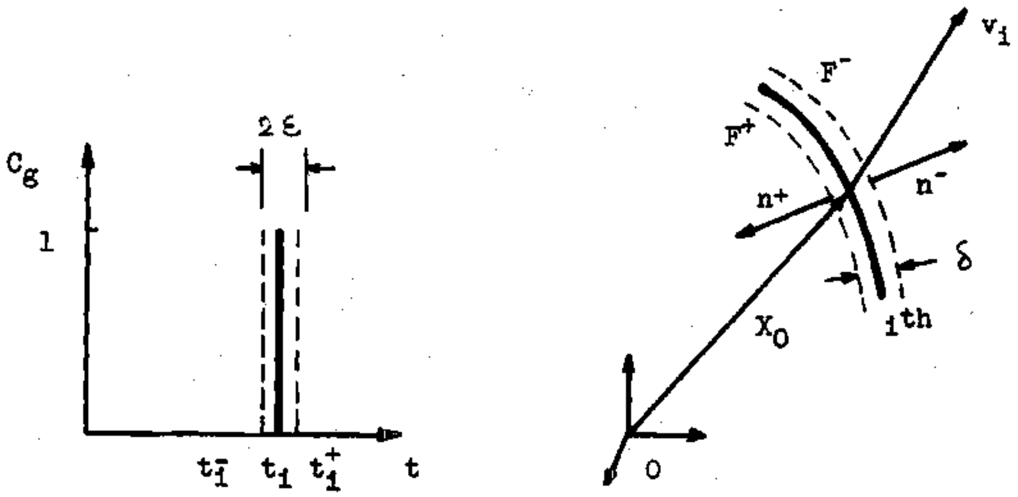


Figure A4. Neighborhood of Interface

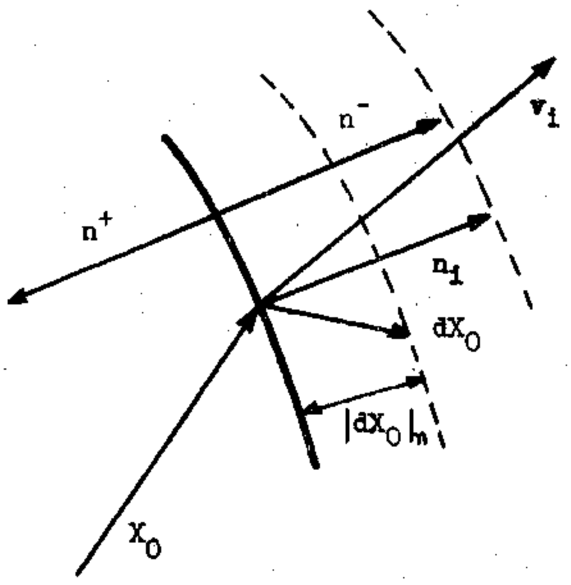


Figure A5. Displacement of Interface

$$n_i \cdot v_i \geq 0 \tag{A3.8}$$

Then

$$n^+ \cdot v_i \geq 0 \tag{A3.9}$$

$$n^- \cdot v_i \leq 0$$

where n^+ and n^- corresponding to the limit outward normal vector of the fluids at each side of the i^{th} interface.

Since $n^+ = n_i$ we have

$$F(X_o,\, t_i^+) = \frac{n^+ \cdot v_i}{n_i \cdot v_i} F^+(X_o,\, t_i) \tag{A3.10*}$$

and

$$F(X_o,\, t_i^-) = -\frac{n^- \cdot v_i}{n_i \cdot v_i} F^-(X_o,\, t_i)$$

Here we assumed that $n_i \cdot v_i \neq 0$ and the particular case when this assumption does not hold will be discussed later. In view of Eq. (A3.7), our interest here is to find the gradient of t_i; however, we have

$$t_i = \xi_i + t_o$$

hence

$$\left.\frac{\partial(t_i \pm \varepsilon)}{\partial X_{oj}}\right|_{t_o} = \left.\frac{\partial(\xi_i \pm \varepsilon)}{\partial X_{oj}}\right|_{t_o} \tag{A3.11}$$

---

*See end of the section for the discussion of the singularity.

This is a change of interface passing time t_i at X_o by shifting the position by dX_o with the fixed time scale, i.e., Fig. A5.

By taking small displacement dX_o we have

$$|dX_o|_n = n_i \cdot dX_o \tag{A3.12}$$

which is a normal distance.

The velocity of the interface in normal direction is given by

$$v_{in} = n_i \cdot v_i \tag{A3.13}$$

Hence we get

$$d\xi_i = \frac{n_i \cdot dX_o}{v_{in}} \tag{A3.14}$$

The gradient of ξ_i becomes in the tensor notation

$$\left.\frac{\partial \xi_i}{\partial X_{oj}}\right| = grad\,[\xi_i] = \frac{n_{i\ell}\, dX_{o\ell}}{dX_{oj}\, n_{im}\, v_{im}}\, e_j \tag{A3.15}$$

However

$$\frac{dX_{o\ell}}{dX_{oj}} = \delta_{\ell j}$$

Thus we get

$$\frac{\partial \xi_i}{\partial X_{oj}} = \frac{n_i}{v_{in}} \tag{A3.16}$$

From Eq. (A3.7), (A3.11), and (A3.16) we have

$$\overline{\frac{\partial F}{\partial X_j}} = \frac{\partial \overline{F}(X_o, t_o)}{\partial X_{o_j}} + \sum_i \frac{1}{\Delta t} \frac{1}{V_{in}} \left\{ n^+ F^+(X_o, t_i) + n^- F^-(X_o, t_i) \right\} \quad \text{(A3.17)}$$

also using Eq. (A3.4) and (A3.10) we obtain

$$\overline{\frac{\partial F}{\partial t}} = \frac{\partial \overline{F}(X_o, t_o)}{\partial t_o} - \sum_i \frac{1}{\Delta t} \frac{1}{V_{in}} \left\{ F^+ n^+ \cdot V_i - F^- n^- \cdot V_i \right\} \quad \text{(A3.18)}$$

Whereas the interface thickness δ corresponding to 2ϵ is given by

$$\delta = 2 \epsilon V_{in} \quad \text{(A3.19)}^*$$

In this section we assumed that $\delta \to 0$ corresponds to $\epsilon \to 0$, where δ is taken to be an arbitrarily small number. This is true under two conditions given below. Since

$$\frac{\partial \epsilon}{\partial t_o} = \epsilon \frac{\partial \ln\left(\frac{1}{V_{in}}\right)}{\partial t_o} \quad \text{(A3.20)}$$

and

$$\frac{\partial \epsilon}{\partial X_{o_j}} = \epsilon \frac{\partial \ln\left(\frac{1}{V_{in}}\right)}{\partial X_{oj}} , \quad \text{(A3.21)}$$

the acceleration and convective acceleration of the interface should be finite in addition to the condition that $V_{in} \neq 0$.

---

*See end of the section.

On the other hand, if the normal velocity of the interface is zero, i.e., $n_i \cdot v_i = 0$, then the interface may stay at X_o for some finite time. Thus

$$\lim_{\varepsilon \to 0} [\Delta t]_s \neq 0$$

In this case we have a statistical concentration of the interface α_s which is given by

$$\alpha_s = \frac{\Delta t_s}{\Delta t}$$

and

$$\Delta t = \Delta t_g + \Delta t_f + \Delta t_s$$

Hence we have

$$\alpha_g + \alpha_f + \alpha_s = 1$$

A-4. Time Averaged Overall Balance

Our time domain of interest can be subdivided into $[\Delta t]_s$ and $[\Delta t]_T$ as shown in Appendix A-1. During $[\Delta t]_T$, the standard balance equations hold, whereas in $[\Delta t]_s$ the interfacial balance equations (jump conditions) hold. Let us first proceed with the analysis in $[\Delta t]_T$.

For time $t \in [\Delta t]_T = [\Delta t]_g + [\Delta t]_f$, we consider the balance of matter ψ per units mass in the following form.

$$V_B = \frac{\partial \rho \psi}{\partial t} + \nabla \cdot \rho \psi v + \nabla \cdot J - \phi_b = 0 \qquad \text{(A4.1)}^*$$

---

[*] For example, ψ stands for the mass, momentum, and energy in which case ψ is 1, v and e, respectively.

Here J and ϕ_b represent the generalized tensor efflux and generation of ψ, respectively. When Eq. (A4.1) is applied for each phase, the subscripts which differentiate two fluids should appear with variables.

For time $t \epsilon [\Delta t]_s$, we use the standard jump conditions with the thin layer assumption for the surface where surface properties are assumed to be constant across the layer.

Then the balance of matter ψ becomes

$$S_B = \left\{ \frac{\partial \rho \psi}{\partial t} + \left(t^j_\alpha v \rho \psi \right)_{,\beta} - \phi_b \right\}_i - \tag{A4.2}$$

$$- \frac{1}{\delta} \left\{ n^+ \cdot \left[\rho^+ (v^+ - v_i) \psi^+ + J^+ \right] + n^- \cdot \left[\rho^- (v^- - v_i) \psi^- + J^- \right] \right.$$

$$\left. + \left(t^j_\alpha \; K \right)_{,\beta} \right\} = 0 \qquad (\alpha, \beta = 1,2 \;\&\; j = 1,2,3)$$

The averaged balance per volume can be obtained by integrating the proper balance equations in time domain. Let us express the balance equation in general by

$$B = 0 \tag{A4.3}$$

where

$$B = V_B = 0 \qquad \text{for} \quad t \epsilon [\Delta t]_T \tag{A4.4}$$

$$B = S_B = 0 \qquad \text{for} \quad t \epsilon [\Delta t]_s \tag{A4.5}$$

By taking a time average of B we have

$$\frac{1}{\Delta t} \int_{[\Delta t]} B \, dt = 0 \tag{A4.5}$$

Substituting Eq. (A4.4) into Eq. (A4.5) we obtain

$$\frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_T} V_B \, dt + \frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_S} S_B \, dt = 0 \tag{A4.6}$$

The first term can be expressed in terms of the averaged values defined in A-3. Hence, from Eq. (A4.1), (A3.17), and (A3.18), we have

$$\begin{aligned} \frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_T} V_B \, dt = {} & \frac{\partial \overline{\rho \psi}}{\partial t} + \nabla \cdot \overline{\rho \psi v} + \nabla \cdot \overline{J} - \overline{\phi}_b + \\ & + \sum_i \frac{1}{\Delta t} \frac{1}{v_{in}} \Big\{ n^+ \cdot [\rho^+ (v^+ - v_i) \psi^+ + J^+] + \\ & + n^- \cdot [\rho^- (v^- - v_i) \psi^- + J^-] \Big\}_i = 0 \end{aligned} \tag{A4.7}$$

Using Eq. (A4.2) and (A3.19), the second term of Eq. (A4.6) becomes

$$\begin{aligned} \frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_S} S_B \, dt = {} & \sum_i \lim_{\varepsilon \to 0} \frac{2\varepsilon}{\Delta t} \Big\{ \frac{\partial \rho \psi}{\partial t} + (t_\alpha^\beta \rho \psi v)_{,\beta} - \phi_b \Big\}_i - \\ & - \sum_i \frac{1}{\Delta t} \frac{1}{v_{in}} \Big\{ n^+ \cdot [\rho^+ (v^+ - v_i) \psi^+ + J^+] + \\ & + n^- \cdot [\rho^- (v^- - v_i) \psi^- + J^-] + (t_\alpha^\beta K)_{,\beta} \Big\}_i = 0 \end{aligned} \tag{A4.8}$$

Substituting Eq. (A4.7 and (A4.8) into (A4.4) we obtain a time averaged general field equation for the matter ψ,

$$\frac{\partial \overline{\rho\psi}}{\partial t} + \nabla\cdot\overline{\rho\psi v} + \nabla\cdot\overline{J} - \overline{\Phi}_b + \sum_i \lim_{\varepsilon\to 0}\frac{\varepsilon}{\Delta t}\left\{\frac{\partial\rho\psi}{\partial t} + (t^j_\alpha \rho\psi v)_{,\beta} - \phi_b\right\}_i \qquad \text{(A4.9)}$$

$$- \frac{1}{\Delta t}\sum_i\left\{\frac{1}{v_{in}}\ (t^j_\alpha\ K)_{,\beta}\right\}_i = 0$$

On the other hand, using the definitions of (A1.5) and (A1.6) we can get

$$\frac{\partial\overline{\rho_k\psi_k}}{\partial t} + \nabla\cdot\overline{\rho_k\psi_k v_k} + \nabla\cdot\overline{J_k} - \overline{\Phi}_{bk} + \qquad \text{(A4.10)}$$

$$+ \sum_i \frac{1}{\Delta t}\left\{\frac{1}{v_{in}}\ n_k\cdot[\rho_k(v_k - v_i)\psi_k + J_k]\right\}_i = 0$$

We define

$$\overline{\phi}_{sk} \equiv - \sum_i \frac{1}{\Delta t}\left\{\frac{1}{v_{in}}\ n_k\cdot[\rho_k(v_k - v_i)\psi_k + J_k]\right\}_i \qquad \text{(A4.11)}$$

$$\overline{\phi}_s \equiv \sum_i \frac{1}{\Delta t}\left\{\frac{1}{v_{in}}\ (t^j_\alpha\ K)_{,\beta}\right\}_i \qquad \text{(A4.12)}$$

Then the interface condition becomes

$$\sum_i \lim_{\varepsilon\to 0}\frac{2\varepsilon}{\Delta t}\left\{\frac{\partial\rho\psi}{\partial t} + (t^j_\alpha\rho\psi v_i)_{,\beta} - \phi_b\right\}_i + \qquad \text{(A4.13)}$$

$$\sum_{k=g,f}\overline{\Phi}_{sk} - \overline{\phi}_s = 0$$

For each phase k = g and f we have

$$\frac{\partial \overline{\rho_k \psi_k}}{\partial t} + \nabla \cdot \overline{\rho_k \psi_k v_k} + \nabla \cdot \overline{J}_k - \overline{\phi}_{bk} = \overline{\phi}_{sk} \qquad \text{(A4.14)}$$

And the mixture field equation is an addition of the above three equations; thus

$$\frac{\partial \overline{\rho \psi}}{\partial t} + \nabla \cdot \overline{\rho \psi v} + \nabla \cdot \overline{J} - \overline{\phi}_b + \sum_i \lim_{\varepsilon \to 0} \frac{2\varepsilon}{\Delta t} \left\{ \frac{\partial \rho \psi}{\partial t} + \right. \qquad \text{(A4.15)}$$

$$\left. + \left(t_\alpha^j \rho \psi v_i \right)_{,\beta} - \phi_b \right\}_i - \overline{\phi}_s = 0$$

If we can neglect the surface capacity of the matter ψ, we get

$$\lim_{\varepsilon \to 0} \frac{2\varepsilon}{\Delta t} \left\{ \frac{\partial \rho \psi}{\partial t} + \left(t_\alpha^j \rho \psi v_i \right)_{,\beta} - \phi_b \right\}_i = 0 \qquad \text{(A4.16)}$$

Then we have only the source term $\overline{\phi}_s, \overline{\phi}_{sk}$ at the interface, thus

$$\sum_{k=g,f} \overline{\phi}_{sk} - \overline{\phi}_s = 0 \qquad \text{(A4.17)}$$

$$\frac{\partial \overline{\rho_k \psi_k}}{\partial t} + \nabla \cdot \overline{\rho_k \psi_k v_k} + \nabla \cdot \overline{J}_k - \overline{\phi}_{bk} = \overline{\phi}_{sk} \quad (k = g, f) \qquad \text{(A4.18)}$$

$$\frac{\partial \overline{\rho \psi}}{\partial t} + \nabla \cdot \overline{\rho \psi v} + \nabla \cdot \overline{J} - \overline{\phi}_b = \overline{\phi}_s \qquad \text{(A4.19)}$$

It should be noted that Eq. (A4.18) and (A4.19) are quite analogous to the heterogeneous chemically reacting system.

A-5. Interfacial Condition

By neglecting the surface capacity term, i.e., Eq. (A5.14), the interfacial condition takes a conventional form of "Korchin's jump condition." Taking only the effect of the surface tension in the line flux term K , the interfacial condition can be written as follows.

$$n_g \cdot [\rho_g (v_g - v_i)\psi_g + J_g] + n_f \cdot [\rho_f (v_f - v_i)\psi_f + J_f] + \phi_s = 0 \quad \text{(A5.1)}$$

This has been done by Slattery (54), Delhaye (55), and a more general case was treated by Scriven (56). The results for mass, momentum, and total energy are as follows.

Table 2. Jump Condition

	ψ_g	J_g	ψ_f	J_f	ϕ_s
Mass	1	0	1	0	0
Momentum	v_g	$-\tau_g$	v_f	$-\tau_f$	$(t^i_\alpha a^{\alpha\beta}\sigma)_{,\beta}$
Total Energy	$\frac{v_g^2}{2} + u_g$	$-\tau_g \cdot v_g + q_g$	$\frac{v_f^2}{2} + u_f$	$-\tau_f \cdot v_f + q_f$	$(t^i_\alpha a^{\alpha\beta}\sigma v_i)_{,\beta}$

The interface conditions for mechanical and internal energy are given in Table 3.

Table 3. Interface Condition

	ψ_g	J_g	ψ_f	J_f	ϕ_s
Mechanical Energy	$\frac{v_g^2}{2}$	$-\tau_g \cdot v_g$	$\frac{v_f^2}{2}$	$-\tau_f \cdot v_f$	$(t^i_\alpha \sigma a^{\alpha\beta})_{,\beta} \cdot \left(\frac{v_g + v_f}{2}\right)$ $+ n_g \cdot \tau_g \cdot \left(\frac{v_g - v_f}{2}\right)$ $+ n_f \cdot \tau_f \cdot \left(\frac{v_f - v_g}{2}\right)$
Internal Energy	u_g	$-q_g$	u_f	q_f	$(t^i_\alpha \sigma a^{\alpha\beta} v_i)_{,\beta}$ $-(t^i_\alpha \sigma a^{\alpha\beta})_{,\beta} \left(\frac{v_g + v_f}{2}\right)$ $- n_g \cdot \tau_g \cdot \left(\frac{v_g - v_f}{2}\right)$ $- n_f \cdot \tau_f \cdot \left(\frac{v_f - v_g}{2}\right)$

A-6. Weighted Mean Values, Turbulent Flux, and Diffusion Flux

I) Following A-2, the density of each phase is defined by

$$\overline{\overline{\rho}}_k = \frac{\overline{\rho_k}}{\alpha_k} \qquad (k = g,\ f) \qquad \text{(A6.1)}$$

II) The density of the mixture

$$\rho_m = \overline{\rho_g} + \overline{\rho_f} = \alpha \overline{\overline{\rho}}_g + (1-\alpha) \overline{\overline{\rho}}_f \qquad \text{(A6.2)}$$

III) The velocity of each phase

$$v_k = \frac{\overline{\rho_k v_k}}{\overline{\rho_k}} = \frac{\overline{\rho_k v_k}}{\alpha_k \overline{\overline{\rho}}_k} \qquad (k = g,\ f) \qquad \text{(A6.3)}$$

IV) The velocity of the mixture

$$U_m = \frac{\overline{\rho U}}{\overline{\rho}} = \frac{\alpha\, \overline{\overline{\rho_g}}\, \overline{\overline{U_g}} + (1-\alpha)\, \overline{\overline{\rho_f}}\, \overline{\overline{U_f}}}{\rho_m} \tag{A6.4}$$

V) The enthalpy of each phase

$$i_k = \frac{\overline{\rho_k i_k}}{\overline{\rho_k}} = \frac{\overline{\rho_k\, i_k}}{\alpha_k \overline{\overline{\rho_k}}} \qquad (k = g,\ f) \tag{A6.5}$$

VI) The enthalpy of the mixture

$$i_m = \frac{\overline{\rho i}}{\overline{\rho}} = \frac{\alpha\, \overline{\overline{\rho_g}}\, \overline{\overline{i_g}} + (1-\alpha)\, \overline{\overline{\rho_f}}\, \overline{\overline{i_f}}}{\rho_m} \tag{A6.6}$$

VII) The convective term

$$\overline{\rho \Psi U} = \overline{\rho_g \Psi_g U_g} + \overline{\rho_f \Psi_f U_f} \tag{A6.7}$$

Let us define instantaneous fluctuation as

$$\rho_k = \overline{\rho_k} + \rho_k' \quad , \Psi_k = \overline{\overline{\Psi_k}} + \Psi_k' \ , \ U_k = \overline{\overline{U_k}} + U_k' \qquad (k = g,\ f) \tag{A6.8}$$

Then

$$\rho_k \Psi_k U_k = (\overline{\rho_k} + \rho_k')(\overline{\overline{\Psi_k}} + \Psi_k')(\overline{\overline{U_k}} + U_k') \tag{A6.9}$$

But

$$\overline{\rho_k'} = 0 \quad , \quad \overline{v_k'} = -\frac{\overline{\rho_k' v_k'}}{\overline{\rho_k}} \quad , \quad \overline{\psi_k'} = -\frac{\overline{\rho_k' \psi_k'}}{\overline{\rho_k}} \tag{A6.10}$$

Hence

$$\overline{\rho_k \psi_k v_k} = \overline{\rho_k}\,\overline{\overline{\psi_k}}\,\overline{\overline{v_k}} + \overline{\rho_k \psi_k' v_k'} \tag{A6.11}$$

Therefore, from Eq. (A6.7) and (A6.11)

$$\overline{\rho \psi v} = \left\{ \alpha\, \overline{\overline{\rho_g}}\,\overline{\overline{\psi_g}}\,\overline{\overline{v_g}} + (1-\alpha)\,\overline{\overline{\rho_f}}\,\overline{\overline{\psi_f}}\,\overline{\overline{v_f}} \right\} + \sum_{k=g,f} \overline{\rho_k \psi_k' v_k'} \tag{A6.12}$$

Let us define diffusion velocity V_{km} by

$$V_{km} = \overline{\overline{v_k}} - v_m \tag{A6.13}$$

Then

$$\sum_{k=g,f} \alpha_k\, \overline{\overline{\rho_k}}\,\overline{\overline{\psi_k}}\,\overline{\overline{v_k}} = \rho_m \psi_m v_m + \sum_k \alpha_k\, \overline{\overline{\rho_k}}\,\overline{\overline{\psi_k}}\, V_{km} \tag{A6.14}$$

By defining diffusion flux J^D and turbulent flux J^T as

$$J^D \equiv \sum_k \alpha_k\, \overline{\overline{\rho_k}}\,\overline{\overline{\psi_k}}\, V_{km} \qquad (k = g,\ f) \tag{A6.15}$$

$$J^T \equiv \sum_k \overline{\rho_k \psi_k' v_k'} \qquad (k = g,\ f) \tag{A6.16}$$

and using Eq. (A6.12), (A6.14) we obtain

$$\rho \psi v = \rho_m \psi_m v_m + J^D + J^T \tag{A6.17}$$

The useful relation between V_{fm} and V_{gm} can be obtained directly from the definition, thus

$$\alpha \bar{\bar{\rho}}_g V_{gm} + (1-\alpha)\bar{\bar{\rho}}_f V_{fm} = 0 \tag{A6.18}$$

VIII) The pressure P_m

$$P_m = \alpha \bar{\bar{P}}_g + (1-\alpha)\bar{\bar{P}}_f \tag{A6.19}$$

and

$$\bar{\bar{P}}_k = \frac{\overline{P_k}}{\alpha_k} \qquad (k = g,\ f)$$

IX) The field force g_m

$$g_m = \frac{\sum_k \alpha_k \bar{\bar{\rho}}_k g_k}{\rho_m} \qquad (k = g,\ f) \tag{A6.20}$$

And, if all the field forces are identically constant, then

$$g_m = g \tag{A6.21}$$

X) The turbulent flux J^T

The turbulent flux is expressed more conveniently through the definition

$$\overline{\overline{\rho_k \psi_k' v_k'}} = \frac{1}{\Delta t_k} \lim_{\varepsilon \to 0} \int_{[\Delta t]_k} \rho_k \psi_k' v_k' \, dt = \frac{\overline{\rho_k \psi_k' v_k'}}{\alpha_k} \tag{A6.22}$$

Then

$$J^T = \sum_k \alpha_k \overline{\overline{\rho_k \psi_k' v_k'}} \qquad (k = g,\ f) \tag{A6.23}$$

A-7. Balance of Mass, Momentum, and Energy

I) Mass Balance

Using Eq. (A4.14) to (A4.17) with

$$\psi = 1 \quad , \quad J = 0 \quad , \quad \phi_b = 0 \quad , \quad \overline{\phi_s} = 0 \tag{A7.1}$$

and defining

$$\Gamma_g \equiv \overline{\phi_{sg}} \qquad , \quad \Gamma_f \equiv \overline{\phi_{sf}} \tag{A7.2}$$

the results for the mass balance can be expressed by mean values of A-6. Hence,

$$\frac{\partial \rho_m}{\partial t} + \nabla\cdot \rho_m v_m = 0 \tag{A7.3}$$

$$\frac{\partial \alpha \overline{\overline{\rho_g}}}{\partial t} + \nabla\cdot(\alpha \overline{\overline{\rho_g}}\, \overline{\overline{v_g}}) = \Gamma_g \tag{A7.4}$$

$$\frac{\partial (1-\alpha) \overline{\overline{\rho_f}}}{\partial t} + \nabla\cdot\left[(1-\alpha) \overline{\overline{\rho_f}}\, \overline{\overline{v_f}}\right] = \Gamma_f \tag{A7.5}$$

$$\Gamma_g + \Gamma_f = 0 \tag{A7.6}$$

and

$$\Gamma_g = -\sum_i \frac{1}{\Delta t}\left\{\frac{1}{v_{in}}\; n_g\cdot\left[\rho_g\,(v_g - v_i)\right]\right\}_i \tag{A7.7}$$

Equation (A7.6) states the conservation of mass at the interface without a source or a sink and Eq. (A7.4) and (A7.5) are continuity equations of each phase.

II) Momentum Balance (Mixture)

By taking

$$\psi = v \quad , \quad J = -\mathcal{T} = PI - \tau^{\mu} , \quad \phi_b = \rho g \ , \ \phi_s = (t_\alpha^j \sigma a^{\alpha\beta})_{,\beta} \qquad \text{(A7.8)}$$

and substituting these expressions into Eq. (A4.17), we obtain the mixture momentum equation.

$$\frac{\partial \rho_m v_m}{\partial t} + \nabla \cdot \rho_m v_m v_m = -\nabla P_m + \nabla \cdot \overline{\tau^{\mu}} + \rho_m g_m + \qquad \text{(A7.9)}$$

$$+ \ \nabla \cdot \left(- \sum_k \alpha_k \overline{\overline{\rho}}_k \overline{\overline{v}}_k V_{km} \right)$$

$$+ \ \nabla \cdot \left(- \sum_k \alpha_k \overline{\overline{\rho_k v_k' v_k'}} \right)$$

$$+ \sum_i \frac{1}{\Delta t} \left\{ \frac{1}{v_{in}} \ (t_\alpha^j \sigma a^{\alpha\beta})_{,\beta} \right\}_i \qquad (k = g, f)$$

Defining:

$$g_m = \frac{\sum \alpha_k \overline{\overline{\rho}}_k \overline{\overline{g}}_k}{\rho_m} \qquad \text{(= g if the body force is constant)} \qquad \text{(A7.10)}$$

$$\tau^D = -\sum_k \alpha_k \overline{\overline{\rho}}_k \overline{\overline{v}}_k V_{km} = -\sum \alpha_k \overline{\rho}_k V_{km} V_{km}$$

$$\tau^T = -\sum_k \overline{\rho_k \psi_k' v_k'} = -\sum \alpha_k \overline{\overline{\rho_k \psi_k' v_k'}}$$

$$f_\sigma = \sum_i \frac{1}{\Delta t} \left\{ \frac{1}{v_{in}} (t_\alpha^j \sigma a^{\alpha\beta})_{,\beta} \right\}_i \qquad (k = g, f)$$

Our time averaged mixture momentum equation becomes,

$$\frac{\partial \rho_m v_m}{\partial t} + \nabla \cdot \rho_m v_m v_m = - \nabla P_m + \nabla \cdot (\overline{\tau^{\mu}} + \tau^{T}) \tag{A7.11}$$

$$+ \rho_m g_m + \nabla \cdot \tau^{D} + f_{\sigma}$$

Equation (A7.12) is analogous to the single fluid equation except the last two terms which are the diffusion stress and the surface force. The momentum equation for the heterogeneous chemically reacting system proposed by various authors (see, for example, von Karman, Truesdell among others) has the same form as Eq. (A7.11) except that the turbulent shear stress and the surface forces do not appear there. The momentum equation for each phase also can be written as

$$\frac{\partial \alpha_k \overline{\overline{\rho_k}}\, \overline{\overline{v_k}}}{\partial t} + \nabla \cdot \alpha_k \overline{\overline{\rho_k}}\, \overline{\overline{v_k}}\, \overline{\overline{v_k}} = - \nabla \alpha_k \overline{\overline{P_k}} + \nabla \cdot (\overline{\tau_k^{\mu}} + \tau_k^{T}) + \tag{A7.12}$$

$$+ \alpha_k \overline{\overline{\rho_k}}\, \overline{\overline{g_k}} - \sum_i \frac{1}{\Delta t} \left\{ \frac{1}{v_{in}} n_k \cdot [\rho_k (v_k - v_i) v_k - \mathcal{T}_k] \right\}_i$$

III) Internal Energy and Enthalpy Equation

By taking

$$\psi = u \quad , \; J = q \, , \qquad \phi_b = - p \nabla \cdot v + \tau^{\mu} : \nabla v \tag{A7.13}$$

and using Eq. (A4.17) with the interfacial condition for internal energy (Table 3), we obtain

$$\frac{\partial \overline{\rho u}}{\partial t} + \nabla \cdot \overline{\rho u v} + \nabla \cdot \bar{q} - \frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_T} \{ -P \nabla \cdot v + \tau^{\mu} : \nabla v \} dt \qquad \text{(A7.14)}$$

$$= - \sum_{i} \frac{1}{\Delta t} \left\{ \frac{1}{v_{in}} \right\} \left\{ n_g \cdot \tau_g \cdot \left(\frac{v_g - v_f}{2} \right) + n_f \cdot \tau_f \cdot \left(\frac{v_f - v_g}{2} \right) - \right.$$

$$\left. - \left(t^i_{\alpha} \sigma a^{\alpha\beta} v_i \right)_{,\beta} + \left(\frac{v_g + v_f}{2} \right) \cdot \left(t^i_{\alpha} \sigma a^{\alpha\beta} \right)_{,\beta} \right\}_i$$

For enthalpy $i = u + \frac{P}{\rho}$, we use the following approach. Averaging the total energy equation in terms of the enthalpy and the mechanical energy equation, then subtracting the second from the first, we can obtain the averaged enthalpy equation. Thus,

$$\frac{\partial \overline{\rho i}}{\partial t} + \nabla \cdot \overline{\rho i v} + \nabla \cdot \bar{q} - \frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_T} \left\{ \frac{DP}{Dt} + \tau^{\mu} : \nabla v \right\} \qquad \text{(A7.15)}$$

$$= - \sum_{i} \frac{1}{\Delta t} \left\{ \frac{1}{v_{in}} \right\} \left\{ n_g \cdot \tau_g \cdot \left(\frac{v_g - v_f}{2} \right) + n_f \cdot \tau_f \cdot \left(\frac{v_f - v_g}{2} \right) - \right.$$

$$\left. - \left(t^i_{\alpha} \sigma a^{\alpha\beta} v_i \right)_{,\beta} + \frac{v_g + v_f}{2} \cdot \left(t^i_{\alpha} \sigma a^{\alpha\beta} \right)_{,\beta} \right\}_i$$

We define the energy dissipation terms by

$$\bar{\Phi}_{de} \equiv \frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_T} \left(\tau^{\mu} : \nabla v \right) dt - \qquad \text{(A7.16)}$$

$$- \frac{1}{\Delta t} \sum_{i} \left(\frac{1}{v_{in}} \right) \left\{ n_g \cdot \left(\frac{\tau^{\mu}_g + \tau^{\mu}_f}{2} \right) \cdot \left(v_g - v_f \right) \right\}_i$$

the compressibility effect on the enthalpy by

$$\overline{\Phi}_{ce} \equiv \frac{1}{\Delta t} \lim_{\varepsilon \to 0} \int_{[\Delta t]_T} \frac{DP}{Dt}\, dt + \tag{A7.17}$$

$$- \frac{1}{\Delta t} \sum_i \left(\frac{1}{v_{in}}\right) \left\{ - n_g \cdot \left(\frac{P_g + P_f}{2}\right)(v_g - v_f) + \right.$$

$$\left. + \left(\frac{v_g + v_f}{2}\right) \cdot \left(t_\alpha^j \sigma a^{\alpha\beta}\right)_{,\beta} - \left(t_\alpha^j \sigma a^{\alpha\beta} v_i\right)_{,\beta} \right\}_i$$

the diffusion heat flux by

$$q^D \equiv \sum_k \alpha_k \overline{\overline{i}}_k V_{km} \qquad (k = g,\ f) \tag{A7.18}$$

and the turbulent heat flux by

$$q^T \equiv \sum_k \overline{\rho_k i_k' v_k'} = \sum_k \alpha_k \overline{\overline{\rho_k i_k' v_k'}} \qquad (k = g,\ f) \tag{A7.19}$$

Using the definitions (A7.16) to (A7.19) and (A6.18) into (A7.15), we get

$$\frac{\partial \rho_m i_m}{\partial t} + \nabla \cdot \rho_m i_m v_m = - \left(\nabla \cdot \overline{q} + \nabla \cdot q^T\right) + \overline{\Phi}_{ce} + \overline{\Phi}_{de} - \nabla \cdot q^D \tag{A7.20}$$

The enthalpy equation (A7.20) is analogous to the single fluid equation except the last term (energy diffusion term) and the expression for the compressibility effect Φ_{ce} and the dissipation term Φ_{de} are modified to account for the effect of the interface.

APPENDIX B

AREA AVERAGE

B-1. Definition of the Area Average and the Weighted Mean Values of the Functions

After time averaging, the two phases are distributed in the field and behave like two continua being able to occupy the same point at the same moment, under the assumption that the interfaces are not fixed in the space at any instance. Now our purpose here is to obtain a one-dimensional model* which can be used to analyze the problem where the characteristic length in the main flow direction is much larger than the one in the direction perpendicular to the main flow direction.

We define the average of the function G by

$$\langle G \rangle \equiv \frac{1}{A}\int_A G\, dA \tag{B1.1}$$

Therefore the average mixture density $\langle \rho_m \rangle$ is given by

$$\langle \rho_m \rangle = \frac{1}{A}\int_A \rho_m\, dA = \frac{1}{A}\int_A \left[\alpha\, \bar{\bar{\rho}}_g + (1-\alpha)\bar{\bar{\rho}}_f\right] dA \tag{B1.2}$$

Also the average of the statistical concentration $\langle \alpha \rangle$ is given by

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*There are two different approaches, 1) diffusion model based on the mixture field equations, and 2) two fluid model based on field equations for each phase. Here we follow the first approach.

$$\langle \alpha \rangle = \frac{1}{A}\int_A \alpha \, dA \qquad \text{(B1.3)}$$

and we call $\langle \alpha \rangle$ the void fraction.

The weighted mean density of each phase is given by

$$\langle\!\langle \rho_k \rangle\!\rangle = \frac{\langle \alpha_k \bar{\bar{\rho}}_k \rangle}{\langle \alpha_k \rangle} \qquad (k = g,\ f) \qquad \text{(B1.4)}$$

Substitute Eq. (B1.4) into (B1.2) to obtain

$$\langle \rho_m \rangle = \langle \alpha \rangle \langle\!\langle \rho_g \rangle\!\rangle + \langle 1-\alpha \rangle \langle\!\langle \rho_f \rangle\!\rangle \qquad \text{(B1.5)}$$

And, of course, from the definition we have

$$\langle 1-\alpha \rangle = 1 - \langle \alpha \rangle \qquad \text{(B1.6)}$$

If G is an intensive variable, we have

$$\langle G_m \rangle = \langle \alpha \bar{\bar{G}}_g + (1-\alpha) \bar{\bar{G}}_f \rangle \qquad \text{(B1.7)}$$

By defining

$$\langle\!\langle G_k \rangle\!\rangle \equiv \frac{\langle \alpha G_k \rangle}{\langle \alpha_k \rangle} \qquad (k = g,\ f) \qquad \text{(B1.8)}$$

We get

$$\langle G_m \rangle = \langle \alpha \rangle \langle\!\langle G_g \rangle\!\rangle + \langle 1-\alpha \rangle \langle\!\langle G_f \rangle\!\rangle \qquad \text{(B1.9)}$$

If G is associated with extensive variables, we have

$$G_m = \rho_m \psi_m \tag{B1.10}$$

By defining

$$\langle\langle \psi_k \rangle\rangle \equiv \frac{\langle \alpha_k \overline{\overline{\rho_k}}\, \overline{\overline{\psi_k}} \rangle}{\langle \alpha_k \rho_k \rangle} \qquad (k = g,\ f) \tag{B1.11}$$

and

$$\langle\langle \psi_m \rangle\rangle \equiv \frac{\langle \rho_m \psi_m \rangle}{\langle \rho_m \rangle} \tag{B1.12}$$

we obtain the following expressions

$$\langle \rho_m \psi_m \rangle = \langle \rho_m \rangle \langle\langle \psi_m \rangle\rangle \tag{B1.13}$$

or

$$\langle \rho_m \psi_m \rangle = \langle \alpha \rangle \langle\langle \rho_g \rangle\rangle \langle\langle \psi_g \rangle\rangle + \langle 1-\alpha \rangle \langle\langle \rho_f \rangle\rangle \langle\langle \psi_f \rangle\rangle \tag{B1.14}$$

Hence the weighted mean velocity of each phase is given by

$$\langle\langle v_k \rangle\rangle = \frac{\langle \alpha_k \overline{\overline{\rho_k}}\, \overline{\overline{v_k}} \rangle}{\langle \alpha_k \overline{\overline{\rho_k}} \rangle} \qquad (k = g,\ f) \tag{B1.15}$$

and the weighted mean mixture velocity by

$$\langle\langle v_m \rangle\rangle = \frac{\langle \rho_m v_m \rangle}{\langle \rho_m \rangle} = \frac{\langle \alpha \rangle \langle\langle \rho_g \rangle\rangle \langle\langle v_g \rangle\rangle + \langle 1-\alpha \rangle \langle\langle \rho_f \rangle\rangle \langle\langle v_f \rangle\rangle}{\langle \rho_m \rangle} \tag{B1.16}$$

Similarly, the enthalpy of each phase is given by

$$\langle\langle i_k \rangle\rangle = \frac{\langle \alpha_k \bar{\bar{\rho}}_k \bar{\bar{i}}_k \rangle}{\langle \alpha_k \bar{\bar{\rho}}_k \rangle} \qquad (k = g,\ f) \qquad \text{(B1.17)}$$

and the mixture enthalpy by

$$\langle i_m \rangle = \frac{\langle \rho_m i_m \rangle}{\langle \rho_m \rangle} = \frac{\langle \alpha \rangle \langle\langle \rho_g \rangle\rangle \langle\langle i_g \rangle\rangle + \langle 1-\alpha \rangle \langle\langle \rho_f \rangle\rangle \langle\langle i_f \rangle\rangle}{\langle \rho_m \rangle} \qquad \text{(B1.18)}$$

Furthermore, the vapor phase weighted mixture properties are defined by

$$(G_m)_\alpha \equiv \frac{\langle \alpha \bar{\bar{\rho}}_g G_m \rangle}{\langle \alpha \bar{\bar{\rho}}_g \rangle} \qquad \text{(B1.19)}$$

and

$$(\Psi_m)_\alpha \equiv \frac{\langle \alpha \bar{\bar{\rho}}_g \Psi_m \rangle}{\langle \alpha \bar{\bar{\rho}}_g \rangle} \qquad \text{(B1.20)}$$

B-2. Convective and Diffusion Terms

I) Using the local definition of the diffusion velocities V_{gm} and V_{fm}, the real convective term can be written as

$$\langle \rho_m \Psi_m v_m \rangle + \langle J^D \rangle = \langle \alpha \bar{\bar{\rho}}_g \bar{\bar{\Psi}}_g \bar{\bar{v}}_g + (1-\alpha) \bar{\bar{\rho}}_f \bar{\bar{\Psi}}_f \bar{\bar{v}}_f \rangle \qquad \text{(B2.1)}$$

which can be expressed in terms of the diffusion velocity as

$$\langle \rho_m \psi_m v_m \rangle + \langle J^D \rangle = \langle \rho_m \psi_m v_m \rangle + \sum_k \langle \alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k V_{km} \rangle \qquad (B2.2)$$

We define the covariant of the area average as

$$\mathrm{COV}\left[\rho_m \psi_m v_m\right] \equiv \langle \rho_m \psi_m v_m \rangle - \langle \rho_m \rangle \ll \psi_m \gg \ll v_m \gg \qquad (B2.3)$$

or

$$= \langle \rho_m \psi_m \left(v_m - \ll v_m \gg \right) \rangle \qquad (B2.4)$$

or

$$= \langle \rho_m \left(\psi_m - \ll \psi_m \gg \right) v_m \rangle \qquad (B2.5)$$

Also we define

$$\mathrm{COV}\left[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k V_{km}\right] \equiv \langle \alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k V_{km} \rangle - \langle \alpha_k \rangle \ll \rho_k \gg \ll \psi_k \gg \ll V_{km} \gg \qquad (B2.6)$$

or

$$= \langle \alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k \left(V_{km} - \ll V_{km} \gg \right) \rangle \qquad (B2.7)$$

or

$$= \langle \alpha_k \bar{\bar{\rho}}_k \left(\bar{\bar{\psi}}_k - \ll \psi_k \gg \right) V_{km} \rangle \qquad (B2.8)$$

$(k = g,\ f)$

Here we used the definition

$$\ll V_{km} \gg = \frac{\langle \alpha_k \bar{\bar{\rho}}_k V_{km} \rangle}{\langle \alpha_k \bar{\bar{\rho}}_k \rangle} \qquad (k = g,\ f) \qquad (B2.9)$$

And from Eq. (A6.14), (B1.15), (B1.16), and (B2.9), we obtain

$$\langle \alpha \rangle \ll \rho_g \gg \ll V_{gm} \gg = - \langle 1 - \alpha \rangle \ll \rho_f \gg \ll V_{fm} \gg \qquad (B2.10)$$

By substituting Eq. (B2.3) and (B2.6) into Eq. (B2.2), we get

$$\langle \rho_m \psi_m v_m \rangle + \langle J^D \rangle = \langle \rho_m \rangle \langle\langle \psi_m \rangle\rangle \langle\langle v_m \rangle\rangle + \sum_k \langle \alpha_k \rangle \langle\langle \rho_k \rangle\rangle \langle\langle \psi_k \rangle\rangle \langle\langle V_{km} \rangle\rangle \tag{B2.11}$$

$$+ \operatorname{cov}\left[\rho_m \psi_m v_m \right] + \sum_k \operatorname{cov}\left[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k V_{km} \right]$$

II) Using the diffusion velocity with respect to $\langle\langle v_m \rangle\rangle$

Let us define the diffusion velocity with respect to $\langle\langle v_m \rangle\rangle$ as

$$\tilde{V}_{km} \equiv \bar{\bar{v}}_k - \langle\langle v_m \rangle\rangle \qquad (k = g,\ f) \tag{B2.12}$$

and

$$\langle\langle \tilde{V}_{km} \rangle\rangle \equiv \langle\langle v_k \rangle\rangle - \langle\langle v_m \rangle\rangle = \frac{\langle \alpha_k \bar{\bar{\rho}}_k \tilde{V}_{km} \rangle}{\langle \alpha_k \bar{\bar{\rho}}_k \rangle} \qquad (k = g,\ f) \tag{B2.13}$$

Then the convective and diffusion terms become

$$\langle \rho_m \psi_m v_m \rangle + \langle J^D \rangle = \langle \rho_m \rangle \langle\langle \psi_m \rangle\rangle \langle\langle v_m \rangle\rangle + \sum_k \langle \alpha_k \rangle \langle\langle \rho_k \rangle\rangle \langle\langle \psi_k \rangle\rangle \langle\langle \tilde{V}_{km} \rangle\rangle \tag{B2.14}$$

$$+ \sum_k \operatorname{cov}\left[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k \tilde{V}_{km} \right]$$

and here

$$\operatorname{cov}\left[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k \tilde{V}_{km} \right] = \langle \alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k \left(\tilde{V}_{km} - \langle\langle \tilde{V}_{km} \rangle\rangle \right) \tag{B2.15}$$

$$= \langle \alpha_k \bar{\bar{\rho}}_k \bar{\bar{\psi}}_k \left(\bar{\bar{v}}_k - \langle\langle v_k \rangle\rangle \right) \tag{B2.16}$$

$$= \langle \alpha_k \bar{\bar{\rho}}_k \left(\bar{\bar{\psi}}_k - \langle\langle \psi_k \rangle\rangle \right) \tilde{V}_{km} \rangle \qquad (k = g,\ f) \tag{B2.17}$$

And we have an identity from the definitions (B2.12) and (B2.13)

$$\langle\alpha\rangle \langle\langle \rho_g \rangle\rangle \langle\langle \tilde{V}_{gm} \rangle\rangle = -\langle 1-\alpha\rangle \langle\langle \rho_f \rangle\rangle \langle\langle \tilde{V}_{fm} \rangle\rangle \tag{B2.18}$$

The relation between $\langle\langle \tilde{V}_{gm} \rangle\rangle$ and $\langle\langle V_{gm} \rangle\rangle$ can be found from Eq. (A6.14), (B2.19), (B2.12), and (B2.13). Thus,

$$\langle\langle V_{gm} \rangle\rangle = \langle\langle v_g \rangle\rangle - (v_m)_\alpha \tag{B2.19}$$

$$\langle\langle \tilde{V}_{gm} \rangle\rangle = \langle\langle v_g \rangle\rangle - \langle\langle v_m \rangle\rangle \tag{B2.20}$$

By defining the cofactor C_{om} as

$$C_{om} \langle\langle v_m \rangle\rangle \equiv (v_m)_\alpha = \frac{\langle \alpha \overline{\overline{\rho}}_g v_m \rangle}{\langle \alpha \overline{\overline{\rho}}_g \rangle} \tag{B2.21}$$

we finally get

$$\langle\langle \tilde{V}_{gm} \rangle\rangle - \langle\langle V_{gm} \rangle\rangle = (C_{om} - 1) \langle\langle v_m \rangle\rangle \tag{B2.22}$$

B-3. Volumetric Flux and Drift Velocity

We define the mean volumetric fluxes as

$$\langle\langle j_k \rangle\rangle = \frac{\langle \alpha_k \overline{\overline{\rho}}_k \overline{\overline{v}}_k \rangle}{\langle\langle \rho_k \rangle\rangle} = \langle \alpha_k \rangle \langle\langle v_k \rangle\rangle \qquad (k = g, f) \tag{B3.1}$$

If ρ_g and ρ_f are constant in the cross sectional area, $\langle\langle j_g \rangle\rangle$ and

$\ll \dot{j}_f \gg$ are the same as the volumetric flow rate $\frac{Q_g}{A}$ and $\frac{Q_f}{A}$, respectively. The total volumetric flux $\ll \dot{j} \gg$ is defined by

$$\ll \dot{j} \gg = \ll \dot{j}_g \gg + \ll \dot{j}_f \gg \tag{B3.2}$$

The local drift velocity with respect to $\ll \dot{j} \gg$ is given by

$$\widetilde{V}_{kj} = \overline{\overline{v}}_k - \ll \dot{j} \gg \tag{B3.3}$$

and the mean value of $\widetilde{V}_{kj}$ is defined by

$$\ll \widetilde{V}_{kj} \gg \equiv \frac{\langle \alpha_k \overline{\overline{\rho}}_k \widetilde{V}_{kj} \rangle}{\langle \alpha_k \overline{\overline{\rho}}_k \rangle} = \ll v_k \gg - \ll \dot{j} \gg \qquad (k = g,\ f) \tag{B3.4}$$

Using the definition from (B3.1) to (B3.4), we can obtain a very useful relation between the mixture mean velocity $\ll v_m \gg$ and the total volumetric flux $\ll \dot{j} \gg$.

$$\langle \rho_m \rangle \ll v_m \gg = \langle \rho_m \rangle \ll \dot{j} \gg + \sum_k \langle \alpha_k \rangle \ll \rho_k \gg \ll \widetilde{V}_{kj} \gg \tag{B3.5}$$

Then

$$\ll v_m \gg = \ll \dot{j} \gg + \frac{1}{\langle \rho_m \rangle} \sum_k \langle \alpha_k \rangle \ll \rho_k \gg \ll \widetilde{V}_{kj} \gg \tag{B3.6}$$

We define the mean relative velocity $\ll v_r \gg$ by

$$\langle\!\langle v_r \rangle\!\rangle = \langle\!\langle v_g \rangle\!\rangle - \langle\!\langle v_f \rangle\!\rangle \tag{B3.7}$$

Then from Eq. (B3.3) and (B3.7) we get

$$\langle\!\langle \widetilde{V}_{gj} \rangle\!\rangle = \langle 1-\alpha \rangle \langle\!\langle v_r \rangle\!\rangle \tag{B3.8}$$

and

$$\langle\!\langle \widetilde{V}_{fj} \rangle\!\rangle = -\langle \alpha \rangle \langle\!\langle v_r \rangle\!\rangle \tag{B3.9}$$

Therefore, by combining Eq. (B3.8) and (B3.9), we get

$$\langle\!\langle \widetilde{V}_{fj} \rangle\!\rangle = -\frac{\langle \alpha \rangle}{\langle 1-\alpha \rangle} \langle\!\langle \widetilde{V}_{gj} \rangle\!\rangle \tag{B3.10}$$

The relation between the diffusion velocity $\langle\!\langle V_{km} \rangle\!\rangle$ and the drift velocity $\langle\!\langle \widetilde{V}_{kj} \rangle\!\rangle$ can be found through Eq. (B2.13), (B3.4), (B3.6), and (B3.10). Hence,

$$\langle\!\langle \widetilde{V}_{gj} \rangle\!\rangle = \langle\!\langle \widetilde{V}_{gm} \rangle\!\rangle \frac{\langle \rho_m \rangle}{\langle\!\langle \rho_f \rangle\!\rangle} \tag{B3.11}$$

$$\langle\!\langle \widetilde{V}_{fj} \rangle\!\rangle = \langle\!\langle \widetilde{V}_{fm} \rangle\!\rangle \frac{\langle \rho_m \rangle}{\langle\!\langle \rho_g \rangle\!\rangle} \tag{B3.12}$$

If we define the local drift velocity V_{kj} with respect to the local volumetric flux j, the results will be different. First we define j by

$$j = \alpha \overline{\overline{v}}_g + (1-\alpha)\overline{\overline{v}}_f \tag{B3.13}$$

and

$$V_{gj} \equiv \overline{\overline{v}}_g - j \tag{B3.14}$$

Then

$$\langle\langle V_{gj}\rangle\rangle \equiv \frac{\langle \alpha \overline{\overline{\rho}}_g V_{gj}\rangle}{\langle \alpha \overline{\overline{\rho}}_g\rangle} = \langle\langle v_g\rangle\rangle - (j)_\alpha \tag{B3.15}$$

here

$$(j)_\alpha = \frac{\langle \alpha \overline{\overline{\rho}}_g j\rangle}{\langle \alpha \overline{\overline{\rho}}_g\rangle} \tag{B3.16}$$

By defining cofactor C_{oj} by

$$C_{oj}\langle\langle j\rangle\rangle \equiv (j)_\alpha \tag{B3.17}$$

we obtain

$$\langle\langle \widetilde{V}_{gj}\rangle\rangle - \langle\langle V_{gj}\rangle\rangle = (C_{oj}-1)\langle\langle j\rangle\rangle \tag{B3.18}$$

B-4. Area Average of Various Derivatives

In the following, let us assume that the flow and physical properties obtained by time averaging are axially symmetric and there are no rotational motions of the fluid along the axis. Thus, by taking the polar coordinate (r, θ, z), we have

$$\frac{\partial}{\partial \theta} = 0 \quad , \; \psi_\theta = 0 \tag{B4.1}$$

For simplicity, we also restrict our case for the constant area flow without any mass transfer from the walls. Although the effect of the change in flow area and the wall mass transfer can be easily included in the analysis, these terms usually do not appear in the application for the thermally induced flow oscillations.

Under the above conditions, we can relate the area average of the various derivatives to the derivatives of the area average through utilization of the Leibniz rule.

By taking a concentric annulus with outer radius r_o and inner radius r_i, the cross sectional area for flow A becomes

$$A = \pi\,(r_o^2 - r_i^2) \tag{B4.2}$$

The inner and outer perimeters η_i and η_o are given by

$$\eta_i = 2\pi\, r_i \tag{B4.3}$$

$$\eta_o = 2\pi\, r_o$$

(for the pipe flow $r_i = 0$).

Area Average of the Time Derivatives

Since the boundaries of the area average are constant, we have

$$\frac{1}{A}\int_A \frac{\partial \rho_m \psi_m}{\partial t}\, dA = \frac{\partial}{\partial t}\, \frac{1}{A}\int_A \rho_m \psi_m\, dA \tag{B4.4}$$

From the definition of the area average, i.e., Eq. (B1.1), we can express the area average of the time derivatives as

$$\frac{1}{A}\int_A \frac{\partial \rho_m \Psi_m}{\partial t} = \frac{\partial}{\partial t}\langle \rho_m \Psi_m \rangle \qquad \text{(B4.5)}$$

Area Average of the Scalar Operator in the z Direction

$$\frac{1}{A}\int_A [\nabla S]_z \, dA = \frac{1}{A}\int_{r_i}^{r_o} \frac{\partial S}{\partial z} 2\pi r \, dr = \frac{\partial}{\partial z}\langle S \rangle \qquad \text{(B4.6)}$$

Area Average of the Vector Operator

The tensors appearing in the time averaged field equations are all symmetric if we take both phases as originary Newtonian viscous fluids. Then we can write the area average of the vector operator generally as

$$\frac{1}{A}\int_A (\nabla\cdot\tau)_z \, dA = \frac{1}{A}\int_{r_i}^{r_o}\left[\frac{1}{r}\frac{\partial}{\partial r}(r\tau_{rz}) + \frac{\partial \tau_{zz}}{\partial z}\right] 2\pi r \, dr \qquad \text{(B4.7)}$$

By taking the first part of the intergrand, we have

$$\frac{1}{A}\int_{r_i}^{r_o} 2\pi \frac{\partial}{\partial r}(r\tau_{rz})\, dr = \frac{1}{A}\left\{\eta_o \tau_{rz}(r_o) - \eta_i \tau_{rz}(r_i)\right\} \qquad \text{(B4.8)}$$

And the second part of the integral is given by

$$\frac{1}{A}\int_{r_i}^{r_o}\frac{\partial \tau_{zz}}{\partial z}\,2\pi r\,dr = \frac{\partial}{\partial z}\frac{1}{A}\int \tau_{zz}\,2\pi r\,dr = \frac{\partial}{\partial z}\langle \tau_{zz}\rangle \tag{B4.9}$$

Thus, from Eq. (B4.7) and (B4.9) we have

$$\frac{1}{A}\int (\nabla\cdot\tau)_z\,dA = \frac{\partial}{\partial z}\langle \tau_{zz}\rangle + \frac{1}{A}\left\{\eta_o\tau_{rz}(r_o) - \eta_i\tau_{rz}(r_i)\right\} \tag{B4.10}$$

Particularly for the vector W, we have

$$\frac{1}{A}\int \nabla\cdot W\,dA = \frac{\partial}{\partial z}\langle W_z\rangle + \frac{1}{A}\left\{\eta_o W_r(r_o) - \eta_i W_r(r_i)\right\} \tag{B4.11}$$

B-5. Area Average of the Continuity Equation

The time smoothed or time averaged continuity equations are given in A-7, section (I). Now our purpose here is to area average these equations in order to one-dimensionalize the problem. By taking the area average of Eq. (A7.3), we obtain

$$\frac{1}{A}\int\frac{\partial\rho_m}{\partial t}\,dA + \frac{1}{A}\int\nabla\cdot(\rho_m v_m)\,dA = 0 \tag{B5.1}$$

From Eq. (B4.4) and (B4.11), the above equation can be expressed as

$$\frac{\partial}{\partial t}\langle\rho_m\rangle + \frac{\partial}{\partial z}\langle\rho_m v_m\rangle + \frac{1}{A}\left\{\eta_o\rho_m v_{mr}\Big|_{r_o} - \eta_i\rho_m v_{mr}\Big|_{r_i}\right\} = 0 \tag{B5.2}$$

The last two terms in Eq. (B5.2) express the mass transfer at the walls and, under our assumption, these are zero. Thus we obtain a mixture continuity equation

$$\frac{\partial}{\partial t}\langle \rho_m \rangle + \frac{\partial}{\partial z}\langle \rho_m v_{mz} \rangle = 0 \qquad \text{(B5.3)}$$

Using the definition of the weighted mean velocity $\langle\langle v_m \rangle\rangle$, i.e., Eq. (B1.16), we can get the final form for the averaged mixture continuity equation

$$\frac{\partial}{\partial t}\langle \rho_m \rangle + \frac{\partial}{\partial z}\langle \rho_m \rangle \langle\langle v_{mz} \rangle\rangle = 0 \qquad \text{(B5.4)}$$

Similarly, the continuity equations for each phase can be obtained from Eq. (A7.4) and (A7.5), thus

$$\frac{\partial}{\partial t}\langle \alpha \rho_g \rangle + \frac{\partial}{\partial z}\langle \alpha \rho_g v_g \rangle = \langle \Gamma_g \rangle$$

By substituting the definitions of weighted mean values for the density and velocity, we get

$$\frac{\partial}{\partial t}\langle \alpha \rangle \langle\langle \rho_g \rangle\rangle + \frac{\partial}{\partial z}\langle \alpha \rangle \langle\langle \rho_g \rangle\rangle \langle\langle v_g \rangle\rangle = \langle \Gamma_g \rangle \qquad \text{(B5.5)}$$

For the liquid phase we have

$$\frac{\partial}{\partial t}\langle 1-\alpha \rangle \langle\langle \rho_f \rangle\rangle + \frac{\partial}{\partial z}\langle 1-\alpha \rangle \langle\langle \rho_f \rangle\rangle \langle\langle v_f \rangle\rangle = \langle \Gamma_f \rangle \qquad \text{(B5.6)}$$

And from Eq. (A7.6) and (A7.7) with Eq. (B1.1), we have the following constitutive relations

$$\langle \Gamma_g \rangle = - \langle \Gamma_f \rangle \qquad \text{(B5.7)}$$

and

$$\langle \Gamma_g \rangle = -\frac{1}{A} \int_A \sum_i \frac{1}{\Delta t} \left(\frac{1}{v_{in}} \right) \left\{ n_g \cdot \left[\rho_g \left(v_g - v_i \right) \right] \right\}_i dA \qquad \text{(B5.8)}$$

Equation (B5.7) is a requirement for the conservation of mass and the addition of Eq. (B5.5) and (B5.6) leads to Eq. (B5.4), i.e., the mixture continuity equation. Thus, those four equations are not independent. Equation (B5.8) is a constitutive equation for $\langle \Gamma_g \rangle$, the vapor generation term.

B-6. Area Averaged Momentum Equation in the z Direction

The time averaged momentum equation (A7.11) in the z direction is going to be area averaged under the conditions of uniform gravitational force and no slip at the walls. Using Eq. (B4.5) and (B1.16), the area average of the rate change of momentum becomes,

$$\frac{1}{A} \int_A \frac{\partial}{\partial t} \left(\rho_m v_{mz} \right) dA = \frac{\partial}{\partial t} \langle \rho_m \rangle \langle\langle v_{mz} \rangle\rangle \qquad \text{(B6.1)}$$

All of the flux terms are combined and area averaged by utilization of Eq. (B4.10), which yields

$$\frac{1}{A}\int_A \left[\nabla\cdot\left(\rho_m v_m v_m - \tau^D - \tau^T - \bar{\tau}^\mu + P_m I\right)\right]_z dA \qquad \text{(B6.2)}$$

$$= \frac{\partial}{\partial z}\left\{ < \rho_m\, v_{mz} v_{mz} - \tau_{zz}^D - \tau_{zz}^T - \bar{\tau}^\mu_{zz} + P_m > \right\} +$$

$$+ \frac{1}{A}\left\{ \eta_o \left[-\tau_{rz}^T - \bar{\tau}_{rz}^\mu \right]_{r_o} - \eta_i \left[-\tau_{rz}^T - \bar{\tau}_{rz}^\mu \right]_{r_i} \right\}$$

By using the diffusion velocity with respect to $\ll v_m \gg$, i.e., Eq. (B2.12), the first two terms in the right hand side of Eq. (B6.2) can be written in the form of Eq. (B2.13), thus

$$\frac{\partial}{\partial t}\left\{ < \rho_m v_{mz} v_{mz} - \tau^D > \right\} = \frac{\partial}{\partial t}\left\{ < \rho_m > \ll v_{mz} \gg \ll v_{mz} \gg + \right. \qquad \text{(B6.3)}$$

$$\left. + \sum_k \langle \alpha_k \rangle \ll \rho_k \gg \ll v_k \gg \ll \tilde{V}_{km} \gg + \sum_k \mathrm{cov}\left[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{v}}_{kz} V_{kmz}\right] \right\}$$

Thus the area average of the momentum equation in the z direction becomes,

$$\frac{\partial}{\partial t} < \rho_m > \ll v_{mz} \gg + \frac{\partial}{\partial z} < \rho_m > \ll v_{mz} \gg \ll v_{mz} \gg = \qquad \text{(B6.4)}$$

$$= -\frac{\partial}{\partial z} < P_m > + \frac{\partial}{\partial z}\left\{ < \bar{\tau}^\mu_{zz} > + < \tau_{zz}^T > \right\} + < \rho_m > g_z + < f_{\sigma z} > -$$

$$- \frac{\partial}{\partial z} \sum_k < \alpha_k > \ll \rho_k \gg \ll v_{kz} \gg \ll \tilde{V}_{km} \gg - \frac{\partial}{\partial z} \sum_k \mathrm{cov}\left[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{v}}_{kz} V_{kmz}\right] +$$

$$+ \frac{1}{A}\left\{ \eta_o \left[\tau_{rz}^T + \bar{\tau}_{rz}^\mu \right]_{r_o} - \eta_i \left[\tau_{rz}^T + \bar{\tau}_{rz}^\mu \right]_{r_i} \right\}$$

The expression for $\langle f_{\sigma z} \rangle$ is given by

$$\langle f_{\sigma z} \rangle = \frac{1}{A} \int_A \sum_i \frac{1}{\Delta t} \left(\frac{1}{v_{in}} \right) \left\{ \left(t_\alpha^z \, \sigma a^{\alpha\beta} \right)_{,\beta} \right\}_i \, dA \tag{B6.5}$$

We introduce a two-phase Darcy-type friction factor f_m defined by

$$f_m \equiv \frac{1}{\frac{1}{2} \langle \rho_m \rangle \langle\langle v_{mz} \rangle\rangle^2} \, D \left\{ -\frac{1}{A} \left[\eta_o \left(\tau_{rz}^T + \bar{\tau}_{rz}^\mu \right)_{r_o} - \eta_i \left(\tau_{rz}^T + \tau_{rz}^\mu \right)_{r_i} \right] \right\} \tag{B6.6}$$

Here D is a hydraulic diameter and is defined by

$$D \equiv \frac{4A}{\eta} = \frac{4A}{\eta_o + \eta_i} \tag{B6.7}$$

Then the mixture momentum equation becomes

$$\begin{aligned} \frac{\partial}{\partial t} \langle \rho_m \rangle \langle\langle v_{mz} \rangle\rangle + \frac{\partial}{\partial z} \langle \rho_m \rangle \langle\langle v_{mz} \rangle\rangle \langle\langle v_{mz} \rangle\rangle = -\frac{\partial}{\partial z} \langle P_m \rangle - \\ - \frac{f_m}{2D} \langle \rho_m \rangle \langle\langle v_{mz} \rangle\rangle^2 + \langle \rho_m \rangle g_z + \langle f_{\sigma z} \rangle - \\ - \frac{\partial}{\partial z} \left\{ \sum_k \langle \alpha_k \rangle \langle\langle \rho_k \rangle\rangle \langle\langle v_{kz} \rangle\rangle \langle\langle \tilde{V}_{kmz} \rangle\rangle \right\} + \\ + \frac{\partial}{\partial z} \left\{ \langle \bar{\tau}_{zz}^\mu \rangle + \langle \tau_{zz}^T \rangle - \sum_k \mathrm{cov}\left[\alpha_k \overline{\overline{\rho_k}} \, \overline{\overline{v_{kz}}} \, \tilde{V}_{kmz} \right] \right\} \end{aligned} \tag{B6.8}$$

The three terms in the last bracket of the right hand side may be neglected. Since the first two of the three terms represent the momentum transfer due to the normal stress, they are almost always neglected if the characteristic dimension in the main flow direction is much larger than the one in the radial direction. The third term which is a covariant due to the velocity profile, needs closer examination. From Eq. (B2.16) we have

$$\mathrm{cov}\left[\alpha_k \overline{\overline{\rho}}_k \overline{\overline{v}}_{kz} \tilde{V}_{kmz}\right] = \left\langle \alpha_k \overline{\overline{\rho}}_k \overline{\overline{v}}_{kz}\left(\overline{\overline{v}}_{kz} - \langle\langle v_{kz}\rangle\rangle\right)\right\rangle \tag{B6.9}$$

Thus, for turbulent flow where the velocity profile is fairly flat, it is common to neglect this term in comparison with the average convective term in the left hand side.

The term $\langle f_{\sigma z}\rangle$ represents the capillary body force in Eq. (B6.8). In view of Eq. (B6.5) and under quasi-isothermal conditions, it is proportional to the surface volume ratio, to the average total curvature of the surface, and to the surface tension.

By neglecting the normal stresses and the covariant term, we obtain

$$\begin{aligned} \frac{\partial}{\partial t}\langle\rho_m\rangle\langle\langle v_{mz}\rangle\rangle + \frac{\partial}{\partial z}\langle\rho_m\rangle\langle\langle v_{mz}\rangle\rangle\langle\langle v_{mz}\rangle\rangle = \qquad \text{(B6.10)} \\ = -\frac{\partial}{\partial z}\langle P_m\rangle - \frac{f_m}{2D}\langle\rho_m\rangle\langle\langle v_{mz}\rangle\rangle^2 + \langle\rho_m\rangle g_z + \\ + \langle f_{\sigma z}\rangle - \frac{\partial}{\partial z}\sum_k \langle\alpha_k\rangle\langle\langle\rho_k\rangle\rangle\langle\langle v_{kz}\rangle\rangle\langle\langle \tilde{V}_{kmz}\rangle\rangle \end{aligned}$$

As we can see in Appendix A-7.2, two momentum equations for each phase with the correct interface condition provide the necessary and sufficient information on the momentum balance. Instead of using two momentum equations for the liquid and vapor phases, we added up these two equations and obtained the mixture momentum equation, which exhibits the similarity to the single-phase flow equation (see, for example, Eq. (B6.10)). Since we started from two momentum balances, we need one additional piece of information, which can be either of the phase momentum equations.

By adding this equation, we can satisfy the mathematical completeness, but on the other hand the complication of the problem is not much less than the starting point. In contrast to adding up the momentum equations for each phase (which gave us the mixture momentum equation), we also can subtract the one from the other which describes the relative motion of each phase. If we carry out this subtraction from Eq. (A7.12) and area average the resulting relative equation of motion, then it will express the dynamic character of the relative motion. Instead of doing so, if we can assume that relative motion can be expressed in a statical way, we may replace this relative equation of motion with a constitutive equation, as it has been shown experimentally that such is the case if we take a drift velocity of the vapor as a parameter. Thus, we write another momentum equation as

$$\langle\langle \widetilde{V}_{gj3} \rangle\rangle = f\left(\langle\alpha\rangle, \langle\rho_m\rangle, \langle\langle v_{m3} \rangle\rangle, \sigma, g_3, \text{etc.}\right) \qquad \text{(B6.11)}$$

Furthermore, information on the mechanical state between two phases is also required; hence, we have

$$\Delta \langle\langle P_{gf} \rangle\rangle \equiv \langle\langle P_g \rangle\rangle - \langle\langle P_f \rangle\rangle = f(\sigma, \langle \Gamma_g \rangle, \text{etc.}) \qquad \text{(B6.12)}^*$$

B-7. Area Averaged Energy (Enthalpy) Equation

By taking the area average of Eq. (A7.20), the time rate change of the enthalpy becomes

$$\frac{1}{A}\int_A \frac{\partial \rho_m i_m}{\partial t} = \frac{\partial}{\partial t}\langle \rho_m i_m \rangle \qquad \text{(B7.1)}$$

By using Eq. (B1.18) we have

$$\frac{1}{A}\int_A \frac{\partial \rho_m i_m}{\partial t} = \frac{\partial}{\partial t}\langle \rho_m \rangle \langle\langle i_m \rangle\rangle \qquad \text{(B7.2)}$$

All of the flux terms are combined and expressed through Eq. (B4.11), thus

$$\frac{1}{A}\int_A \nabla \cdot \left(\rho_m i_m v_m + q^D + \bar{q} + q^T \right) dA = \qquad \text{(B7.3)}$$

$$= \frac{\partial}{\partial z}\left\{ \langle \rho_m i_m v_{mz} \rangle + q_z^D + \bar{q}_z + q_z^T \right\} +$$

$$+ \frac{1}{A}\left\{ \eta_o \left[\bar{q}_r + q_r^T \right]_{r_o} - \eta_i \left[\bar{q}_r + q_r^T \right]_{r_i} \right\}$$

And the first two terms of the right hand side of Eq. (B7.3) can be rewritten through Eq. (B2.13) as

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[*] This condition is necessary for the uniqueness of the solution and corresponds to the thermodynamic condition of the phase change at the interface; for example, in the case of a mechanical equilibrium $\Delta \langle\langle P_{gf} \rangle\rangle = 0$. The exact form of Eq. (B6.12) can be obtained from the area average of Eq. (A4.17).

$$\frac{\partial}{\partial z}\langle \rho_m i_m v_{mz} + q_z^D \rangle = \frac{\partial}{\partial z}\Big\{ \langle \rho_m \rangle \langle\langle i_m \rangle\rangle \langle\langle v_{mz} \rangle\rangle + \tag{B7.4}$$

$$+ \sum_k \langle \alpha_k \rangle \langle\langle \rho_k \rangle\rangle \langle\langle i_k \rangle\rangle \langle\langle \tilde{V}_{kmz} \rangle\rangle + \sum_k \mathrm{cov}\big[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{i}}_k V_{kmz}\big]\Big\}$$

Thus, by using Eq. (B7.2), (B7.3), and (B7.4), the area average of Eq. (A7.20) becomes

$$\frac{\partial}{\partial t}\langle \rho_m \rangle \langle\langle i_m \rangle\rangle + \frac{\partial}{\partial z}\langle \rho_m \rangle \langle\langle i_m \rangle\rangle \langle\langle v_{mz} \rangle\rangle = \tag{B7.5}$$

$$= -\frac{\partial}{\partial z}\left\{ \langle \bar{q}_z \rangle + \langle q_z^T \rangle \right\} - \frac{\partial}{\partial z}\Big\{ \sum_k \langle \alpha_k \rangle \langle\langle \rho_k \rangle\rangle \langle\langle i_k \rangle\rangle \langle\langle \tilde{V}_{kmz} \rangle\rangle \Big\} -$$

$$- \frac{1}{A}\left\{ \eta_o \left[\bar{q}_r + q_r^T \right]_{r_o} - \eta_i \left[\bar{q}_r + q_r^T \right]_{r_i} \right\} -$$

$$- \frac{\partial}{\partial z}\Big\{ \sum_k \mathrm{cov}\big[\alpha_k \bar{\bar{\rho}}_k \bar{\bar{i}}_k \tilde{V}_{kmz}\big] \Big\} + \langle \bar{\Phi}_{ce} \rangle + \langle \bar{\Phi}_{de} \rangle$$

We define the average heat flux from the wall q_w'' as

$$q_w'' = \frac{1}{\xi_h}\left\{ -\eta_o \left[\bar{q}_r + q_r^T \right]_{r_o} + \eta_i \left[\bar{q}_r + q_r^T \right]_{r_i} \right\} \tag{B7.6}$$

where ξ_h is the heated perimeter.

Equation (B7.5) becomes

$$\frac{\partial}{\partial t}\langle \rho_m \rangle \langle\langle i_m \rangle\rangle + \frac{\partial}{\partial z}\langle \rho_m \rangle \langle\langle i_m \rangle\rangle \langle\langle v_{mz} \rangle\rangle = \frac{q_w'' \xi_h}{A} + \tag{B7.7}$$

$$+ \langle \bar{\Phi}_{ce} \rangle + \langle \bar{\Phi}_{de} \rangle - \frac{\partial}{\partial z}\left\{ \langle \bar{q}_z \rangle + \langle q_z^T \rangle \right\} -$$

(Continued)

$$-\frac{\partial}{\partial z}\left\{\sum_{k}\langle\alpha_k\rangle\langle\langle\rho_k\rangle\rangle\langle\langle i_k\rangle\rangle\langle\langle V_{kmz}\rangle\rangle\right\}-\frac{\partial}{\partial z}\left\{\sum_{k}\mathrm{cov}\left[\alpha_k\bar{\bar{\rho}}_k\bar{\bar{i}}_k V_{kmz}\right]\right\}$$

Now, let us consider the right hand side terms in Eq. (B7.7). The first term is the energy input from the wall. The second and third terms give the compressibility effect and the viscous dissipation in the mixture and can be written as

$$\langle\bar{\Phi}_{ce}\rangle=\frac{1}{A}\int_A\frac{1}{\Delta t}\left\{\int_{[\Delta t]}\frac{DP}{Dt}\,dt-\sum_i\left(\frac{1}{v_{in}}\right)\left\{-n_g\frac{P_g+P_f}{2}\cdot(v_g-v_f)+\right.\right. \tag{B7.8}$$

$$\left.\left.+\left(\frac{v_g+v_f}{2}\right)\cdot\left(\sigma t^i_\alpha a^{\alpha\beta}\right)_{,\beta}-\left(t^i_\alpha\sigma a^{\alpha\beta}v_i\right)_{,\beta}\right\}_i dA$$

$$\langle\bar{\Phi}_{de}\rangle=\frac{1}{A}\int_A\frac{1}{\Delta t}\left\{\int_{[\Delta t]}\tau^{\mu}:\nabla v\,dt-\right. \tag{B7.9}$$

$$\left.-\sum_i\left(\frac{1}{v_{in}}\right)\left\{n_g\cdot\frac{\tau_g^{\mu}+\tau_f^{\mu}}{2}\cdot(v_g-v_f)\right\}_i dA$$

The fourth and fifth terms in Eq. (B7.7) give the heat transfer in the normal direction and can be neglected in the mixture region due to the small temperature gradient in the axial direction. The sixth term takes into account the energy transport due to the relative motion of each phase. The last term is a covariant due to the velocity and the enthalpy profiles, and for turbulent flow it is common to neglect it, since both profiles are fairly flat and thus it is much smaller than the average convective term in the left hand side.

By neglecting the axial conduction due to the molecular and turbulent diffusion and the covariant term in Eq. (B7.7), we obtain

$$\frac{\partial}{\partial t} \langle \rho_m \rangle \langle\langle i_m \rangle\rangle + \frac{\partial}{\partial z} \langle \rho_m \rangle \langle\langle i_m \rangle\rangle \langle\langle v_{mz} \rangle\rangle = \frac{q_w'' \xi_h}{A_c} + \tag{B7.10}$$

$$+ \langle \Phi_{ce} \rangle + \langle \Phi_{de} \rangle - \frac{\partial}{\partial z} \left\{ \sum_k \langle \alpha_k \rangle \langle\langle \rho_k \rangle\rangle \langle\langle i_k \rangle\rangle \langle\langle \tilde{V}_{kmz} \rangle\rangle \right\}$$

In section B-6 we discussed the necessity of the constitutive equation for the relative velocity in addition to the mixture momentum equation. From a similar consideration here, we also should supply a constitutive equation for the relative energy, i.e., the difference of the fluid and vapor enthalpy. Under the assumption that we may express this equation statically, we write

$$\Delta \langle\langle i_{fg} \rangle\rangle = \langle\langle i_g \rangle\rangle - \langle\langle i_f \rangle\rangle = f(\alpha, \langle \rho_m \rangle, \langle P_m \rangle, \text{etc.}) \tag{B7.11}$$

APPENDIX C

SIMILARITY GROUPS FOR THE TWO-PHASE ONE-DIMENSIONAL FLOW

C-1. Summary of Field Equations

The time-smoothed and then area averaged field equations are given in Sections B-5 to B-7. They are expressed through averaged and weighted mean values of the original single phase variables. For simplicity, from now on let us omit all symbols of averaging and the subscripts for the vector component z. Thus we can write Eq. (B5.4) to Eq. (B5.7), i.e., the mass balances, in the following form,

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m v_m}{\partial z} = 0 \tag{C1.1}$$

$$\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial \alpha \rho_g v_g}{\partial z} = \Gamma_g \tag{C1.2}$$

$$\frac{\partial (1-\alpha) \rho_f}{\partial t} + \frac{\partial (1-\alpha) \rho_f v_f}{\partial z} = \Gamma_f \tag{C1.3}$$

and

$$\Gamma_g = -\Gamma_f = -\frac{1}{A}\int_A \sum_i \frac{1}{\Delta t}\left(\frac{1}{v_{in}}\right)\left\{n_g \cdot \left[\rho_g (v_g - v_i)\right]\right\}_i dA \tag{C1.4}$$

Here it should be noted that Eq. (C1.1) can be obtained from Eq. (C1.2) to (C1.4).

From the mixture momentum balance, i.e., Eq. (B6.10), and using Eq. (C1.1) we have

$$\rho_m \left\{ \frac{\partial v_m}{\partial t} + v_m \frac{\partial v_m}{\partial z} \right\} = - \frac{\partial P_m}{\partial z} - \frac{f_m}{2D} \rho_m v_m^2 + \qquad \text{(C1.5)}$$

$$+ \left(g \rho_m + f_\sigma \right) - \frac{\partial}{\partial z} \left\{ \frac{\alpha}{1-\alpha} \frac{\rho_g \rho_f}{\rho_m} V_{gj}^2 \right\}$$

Here we used Eq. (B3.8), (B3.11), (B3.12), and (B2.17) in the diffusion stress term of Eq. (B6.10) and transformed it into the drift stress term, i.e., in terms of V_{gj}. The term f_σ is a capillary (body) force and is given by Eq. (B6.5). Thus,

$$f_\sigma = \frac{1}{A} \int_A \sum_i \frac{1}{\Delta t} \left(\frac{1}{v_{in}} \right) \left\{ \left(t_\alpha^z \sigma a^{\alpha\beta} \right)_{,\beta} \right\}_i dA \qquad \text{(B6.5)}$$

The above term has been obtained from a proper application of the momentum jump condition at the interface. On the other hand, f_m stands for the viscous and the turbulent friction force at the wall and is given by Eq. (B6.6). It should be noted here that this is the first derivation where this term, i.e., the capillary body force, appears explicitly in the time averaged mixture momentum equation. From the mixture enthalpy balance, i.e., Eq. (B7.10), with Eq. (C1.1), we obtain

$$\rho_m \left\{ \frac{\partial i_m}{\partial t} + v_m \frac{\partial i_m}{\partial z} \right\} = \frac{q_w'' \xi}{A} + \Phi_{ce} + \Phi_{de} - \frac{\partial}{\partial z} \left\{ \alpha \frac{\rho_g \rho_f}{\rho_m} V_{gj} \Delta i_{fg} \right\} \qquad \text{(C1.6)}$$

Here we used Eq. (B3.11), (B3.12), and (B2.17) in the energy diffusion term in Eq. (B7.10) and transformed it in terms of V_{gj} and Δi_{fg}.

The terms Φ_{ce} and Φ_{de} are the compressibility effect and the dissipation effect on the enthalpy, respectively. The detailed form of these two terms is given by Eq. (B7.8) and (B7.9).

$$\phi_{ce} = \frac{1}{A}\int_A \left\{ \frac{1}{\Delta t}\int_{[\Delta t]} \frac{DP}{Dt}\,dt - \sum_i \left(\frac{1}{v_{in}}\right)\left[-n_g \cdot \left(\frac{P_g + P_f}{2}\right)(v_g - v_f) + \right.\right. \tag{B7.8}$$

$$\left.\left. + \left(\frac{v_g + v_f}{2}\right) \cdot \left(t_\alpha^i \sigma a^{\alpha\beta}\right)_{,\beta} - \left(t_\alpha^i \sigma a^{\alpha\beta} v_i\right)_{,\beta} \right]\right\} dA$$

and

$$\phi_{de} = \frac{1}{A}\int_A \frac{1}{\Delta t}\left\{ \int_{[\Delta t]} \tau^\mu : \nabla v\,dt \; - \right. \tag{B7.9}$$

$$\left. - \sum_i \left(\frac{1}{v_{in}}\right)\left[n_g \cdot \left(\frac{\tau_g + \tau_f}{2}\right) \cdot (v_g - v_f)\right]\right\}_i dA$$

C-2. Scaling Parameters and Dimensionless Groups

In the following analysis, the subscript o denotes the reference parameters chosen to be constant. The axial characteristic length is ℓ. Below we define dimensionless parameters whose order of magnitude is considered to be 1.

$$\rho_m^* = \frac{\rho_m}{\rho_{mo}} \;,\; v_m^* = \frac{v_m}{v_{mo}} \;,\; t^* = \frac{t}{\tau} = t \Big/ \left(\frac{\ell}{v_{mo}}\right) \;,\; D^* = \frac{D}{\ell} \;,\; z^* = \frac{z}{\ell} \tag{C2.1}$$

$$\rho_g^* = \frac{\rho_g}{\rho_{go}} \;,\; \rho_f^* = \frac{\rho_f}{\rho_{fo}} \;,\; V_{gj}^* = \frac{V_{gj}}{V_{gjo}} \;,\; \Gamma_g^* = \frac{\Gamma_g}{\Gamma_{go}} \;,\; P_m^* = \frac{P_m}{\rho_{mo} v_{mo}^2}$$

$$i_m^* = \frac{i_m - i_{fo}}{\Delta i_{fgo}} \;,\; \Delta i_{fg}^* = \frac{\Delta i_{fg}}{\Delta i_{fgo}} \;,\; f_\sigma^* = \frac{f_\sigma}{f_{\sigma o}} \;,\; \phi_{ce}^* = \frac{\phi_{ce}}{\rho_{mo} v_{mo}^2}\left(\frac{\ell}{v_{mo}}\right)$$

$$\phi_{de}^* = \phi_{de} \Big/ \frac{f_m}{4}\left(\frac{\rho_{mo} v_{mo}^2}{2}\right)\left(\frac{v_{mo}}{D}\right) \;,\; q_w^* = \frac{q_w''}{q_{wo}}$$

The choice of the time scale actually comes from the mixture continuity equations. By taking the arbitrary time scale τ, Eq. (C1.1) becomes

$$\frac{1}{\tau}\left(\frac{\ell}{v_{mo}}\right)\frac{\partial \rho_m^*}{\partial t^*} = -\frac{\partial \rho_m^* v_m^*}{\partial z^*}$$

The orders of both dimensionless time and space derivatives should be one, thus

$$\tau = \frac{\ell}{v_{mo}} \tag{C2.2}$$

Hence we take a time scale by a residence time of the particle τ.

I) Dimensionless Groups from Continuity Equation

Subtracting Eq. (C1.3) from (C1.2) and using Eq. (B2.13) and (B3.11) we can write Eq. (C1.3) in terms of ρ_m, v_m, Γ_g, V_{gj}, ρ_g, ρ_f, and α. Hence we obtain

$$\frac{\partial \rho_m}{\partial t} + \frac{\partial \rho_m v_m}{\partial z} - \left\{\frac{\partial \alpha \rho_g}{\partial t} + \frac{\partial \alpha \rho_g v_m}{\partial z}\right\} = -\Gamma_g + \frac{\partial}{\partial z}\left\{\frac{\alpha \rho_g \rho_f}{\rho_m} V_{gj}\right\} \tag{C2.3}$$

Using Eq. (C2.1) in Eq. (C2.3) and dividing the whole equation by $\rho_{mo} v_{mo}/\ell$ we obtain

$$\begin{aligned}&\left\{\frac{\partial \rho_m^*}{\partial t^*} + \frac{\partial \rho_m^* v_m^*}{\partial z^*}\right\} - \frac{\rho_{go}}{\rho_{mo}}\left\{\frac{\partial \alpha \rho_g^*}{\partial t^*} + \frac{\partial \alpha \rho_g^* v_m^*}{\partial z^*}\right\} = - \\ &-\left(\frac{\rho_{go}}{\rho_{mo}}\right)\left(\frac{\Gamma_{go}\,\ell}{\rho_{go}\,v_{mo}}\right)\Gamma_g^* + \left(\frac{\rho_{go}}{\rho_{mo}}\right)\left(\frac{\rho_{fo}}{\rho_{mo}}\,\frac{V_{gjo}}{v_{mo}}\right)\frac{\partial}{\partial z^*}\left\{\alpha\,\frac{\rho_g^*\rho_f^*}{\rho_m^*}\,V_{gj}^*\right\}\end{aligned} \tag{C2.4}$$

Thus we obtain the following dimensionless groups for the kinematic similarity

$$C_{I} = \frac{\Gamma_{go}}{\rho_{go}} \frac{\ell}{v_{mo}} \tag{C2.5}$$

$$C_{II} = \left(\frac{\rho_{fo}}{\rho_{mo}} \frac{V_{gjo}}{v_{mo}} \right)$$

$$C_{III} = \frac{\rho_{go}}{\rho_{mo}}$$

By substituting Eq. (C2.5) into Eq. (C2.4) we obtain

$$\frac{\partial \rho_m^*}{\partial t^*} + \frac{\partial \rho_m^* v_m^*}{\partial z^*} - C_{III} \left\{ \left[\frac{\partial \alpha \rho_g^*}{\partial t^*} + \frac{\partial \alpha \rho_g^* v_m^*}{\partial z^*} \right] - \right. \tag{C2.6}$$

$$\left. - C_I \Gamma_g^* + C_{II} \frac{\partial}{\partial z^*} \left[\frac{\alpha \rho_g^* \rho_f^*}{\rho_m^*} V_{gj}^* \right] \right\} = 0$$

II) Dimensionless Groups from Momentum Equations

We use the dimensionless parameters defined by Eq. (C2.1) in the mixture momentum equation (C1.5) and divide the whole equation by $\frac{\rho_{mo} v_{mo}^2}{\ell}$. We get

$$\rho_m^* \left\{ \frac{\partial v_m^*}{\partial t^*} + v_m^* \frac{\partial v_m^*}{\partial z^*} \right\} = - \frac{\partial P_m^*}{\partial z^*} - \left(\frac{f_m}{2D^*} \right) \rho_m^* v_m^{*2} + \left[\frac{g \ell}{v_{mo}^2} \right] \rho_m^* + \tag{C2.7}$$

$$+ \left[\frac{f_{\sigma o} \ell}{\rho_{mo} v_{mo}^2} \right] f_\sigma^* - \left(\frac{\rho_{go}}{\rho_{fo}} \right) \left[\frac{\rho_{fo}}{\rho_{mo}} \frac{V_{gjo}}{v_{mo}} \right]^2 \frac{\partial}{\partial z^*} \left\{ \frac{\alpha}{1-\alpha} \frac{\rho_f^* \rho_g^*}{\rho_m^*} V_{gj}^{*2} \right\}$$

Thus we obtain the following dimensionless groups for the dynamic similarity

$$M_I = \frac{f_m}{2D^*} \tag{C2.8}$$

(Continued)

$$M_{II} = \frac{g\ell}{v_{mo}^2}$$

$$M_{III} = \frac{f_{\sigma o}\,\ell}{\rho_{mo}\,v_{mo}^2}$$

$$M_{IV} = \frac{\rho_{go}}{\rho_{fo}}$$

Substitution of Eq. (C2.8) and (C2.5) into Eq. (C2.7) gives

$$\rho_m^*\left\{\frac{\partial v_m^*}{\partial t^*} + v_m^*\frac{\partial v_m^*}{\partial z^*}\right\} = -\frac{\partial P_m^*}{\partial z^*} - M_I\,\rho_m^*\,v_m^{*2} + \tag{C2.9}$$

$$+ M_{II}\,\rho_m^* + M_{III}\,f_\sigma^* - M_{IV}\,C_{II}^2\,\frac{\partial}{\partial z^*}\left\{\frac{\alpha}{1-\alpha}\,\frac{\rho_g^*\rho_f^*}{\rho_m^*}\,V_{gj}^{*2}\right\}$$

Scaling of capillary (body) force, $f_{\sigma o}$. From Eq. (B6.6) we have

$$f_{\sigma z} = \frac{1}{A}\int_A \sum_i \frac{1}{\Delta t}\left(\frac{1}{v_{in}}\right)\left\{\left(t^j_\alpha\,\sigma\,a^{\alpha\beta}\right)_{,\beta}\right\}_i dA$$

Thus, under quasi-isothermal conditions

$$f_{\sigma o} \sim \left(\frac{\overline{N}}{\Delta t}\right)\frac{\sigma_o\,\overline{H}}{\overline{v}_{in}}$$

where $\overline{N}$ can be considered to be an average number of surfaces passing the point in Δt. Consequently, $\overline{N}/\Delta t$ respresnts the statistical concentration of the surfaces. On the other hand, $\overline{H}$ is an average total curvature of the interface. Thus, it is related to the dimension of the droplets and/or bubbles.

Then the average number of surface per length, i.e., the surface concentration, becomes

$$\bar{m} \sim \frac{\bar{N}}{\Delta t\, \bar{v}_{in}} \tag{C2.10}$$

Hence we have

$$f_{\sigma o} = \bar{m}\, \sigma_o \bar{H} \tag{C2.11}$$

Using Eq. (C2.8) and (C2.11), M_{III} becomes

$$M_{III} = \frac{\bar{m}\, \sigma_o \bar{H}}{\rho_{mo} v_{mo}^2/\ell} = M_{II}\left(\frac{\bar{m}\, \sigma_o \bar{H}}{g\, \rho_{mo}}\right) = M_{II} M_{III}' \tag{C2.12}$$

where M_{III}' is the ratio of the capillary (body) force to the gravitational body force.

<u>III) Dimensionless Groups from Energy Equation</u>

Using the mixture energy equation (C1.16) together with the definition of the dimensionless parameters, Eq. (C2.1), we get

$$\rho_m^* \left\{ \frac{\partial i_m^*}{\partial t^*} + v_m^* \frac{\partial i_m^*}{\partial z^*} \right\} = \left[\frac{q_{wo}'' \xi_n \ell}{\rho_{mo} v_{mo} \Delta i_{fgo} A} \right] q_w^* + \tag{C2.13}$$

$$+ \left[\frac{v_{mo}^2}{\Delta i_{fgo}} \right] \Phi_{ce}^* + \frac{1}{4} \left[\frac{f_m}{2D^*} \right] \left[\frac{v_{mo}^2}{\Delta i_{fgo}} \right] \Phi_{de}^*$$

$$- \left[\frac{\rho_{go}}{\rho_{mo}} \right] \left[\frac{\rho_{fo}}{\rho_{mo}} \frac{V_{gjo}}{v_{mo}} \right] \frac{\partial}{\partial z^*} \left\{ \frac{\alpha \rho_g^* \rho_f^*}{\rho_m^*} V_{gj}^* \Delta i_{fg}^* \right\}$$

Thus we obtain the following dimensionless groups

$$E_{I} = \frac{q_{wo}'' \xi_h \ell}{A \rho_{mo} v_{mo} \Delta i_{fgo}} \tag{C2.14}$$

$$E_{II} = \frac{v_{mo}^2}{\Delta i_{fgo}}$$

By substituting Eq. (C2.5), (C2.7), and (C2.14) into Eq. (C2.13), we obtain

$$\rho_m^* \left\{ \frac{\partial i_m^*}{\partial t^*} + v_m^* \frac{\partial i_m^*}{\partial z^*} \right\} = E_I q_w^* + E_{II} \left(\Phi_{ce}^* + \frac{1}{4} M_I \Phi_{de}^* \right) \tag{C2.15}$$

$$- C_{II} C_{III} \frac{\partial}{\partial z^*} \left\{ \frac{\alpha \rho_g^* \rho_f^*}{\rho_m^*} V_{gj}^* \Delta i_{fg}^* \right\}$$

C-3. Discussion of the Dimensionless Groups

The physical interpretation of the dimensionless groups obtained in Section C-2 is as follows.

$$N_{pch} = C_I = \left(\frac{\Gamma_{go}}{\rho_{go}} \right) \left(\frac{\ell}{v_{mo}} \right) = \frac{\text{fluid residence time}}{\text{reaction time}} \tag{C3.1}$$

$$= \text{phase change number}$$

$$N_d = C_{II} = \frac{\rho_{fo} V_{gjo}}{\rho_{mo} v_{mo}} = \frac{\text{drift transport of mass}}{\text{convective trans. of mass}} \tag{C3.2}$$

$$= \text{drift number}$$

Since the mixture density is a function of void fraction α and of fluid densities ρ_g and ρ_g, the dimensionless groups associated with the density ratio, i.e., C_{III} and M_{IV}, cannot be chosen independently once the boundary condition for the differential equation is specified. Hence we have

$$C_{III} = \frac{\rho_{go}}{\rho_{mo}} = f\left(\alpha_o ,\ \frac{\rho_{go}}{\rho_{fo}}\right)$$

In view of the definition of M_{IV}, the group C_{III} can be represented by

$$C'_{III} = \alpha_o$$

which is a boundary condition for the continuity equation for the vapor phase. Thus, it is not necessary to include it in the similarity group.* As a conclusion we have only one density ratio given by

$$N_\rho = M_{IV} = \left(\frac{\rho_{go}}{\rho_{fc}}\right) = \frac{\text{density of vapor}}{\text{density of liquid}} \qquad \text{(C3.3)}$$

$$= \text{density ratio}$$

Thus, only N_ρ is sufficient from the two density ratio.

From the momentum equation for the mixture, we obtain

$$N_f = M_I = \frac{f_m}{2D^*} = \frac{\text{wall friction force}}{\text{inertia force}} = \text{friction number} \qquad \text{(C3.4)}$$

and

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*Of course, it is possible to scale the vapor and liquid void fractions by α_o and $1-\alpha_o$ in the balance equations and, thus, obtain dimensionless groups including α_o explicitly. Since order of α_o is unity, this does not affect our analysis, except when we examine the order of the drift stress term. This point is discussed in Section III.2.

$$\frac{1}{N_{Fr}} = M_{II} = \frac{g\ell}{v_{mo}^{2}} = \frac{\text{gravity force}}{\text{inertia force}} = \frac{1}{\text{Froude number}} \qquad \text{(C3.5)}$$

The above two groups are familiar in a standard single-phase flow analysis. Furthermore, the first similarity group is usually related to the Reynolds number which is the ratio of the viscous to the inertia forces. In addition to the above two numbers, we obtain

$$N_s = M_{III} = \frac{f_{\sigma o}\,\ell}{\rho_{mo} v_{mo}^{2}} = \frac{\text{capillary body force}}{\text{inertia force}} = \text{surface number} \qquad \text{(C3.6)}$$

Instead of the number defined by Eq. (C3.6), we may use the group

$$M_{III}' = \frac{\overline{m}\,\sigma_o \overline{H}}{g\,\rho_{mo}} = \frac{\text{capillary body force}}{\text{gravity force}}$$

From the energy equation we have

$$N_{St} = E_I = \frac{q_{wo}''\,\xi_h\,\ell/A}{\rho_{mo} v_{mo}\,\Delta i_{fgo}} = \frac{\text{heat transfer from wall}}{\text{heat trans. by convection}} \qquad \text{(C3.7)}$$

$$= \text{Stanton number}$$

and

$$N_{Ec} = E_{II} = \frac{v_{mo}^{2}}{\Delta i_{fgo}} = \frac{\text{K.E. change by convection}}{\text{enthalpy change by convection}} \qquad \text{(C3.8)}$$

$$= \text{Eckert number}$$

Hence, under the assumption that the constitutive functional relations are the same, the necessary and sufficient conditions for the similarity of the two different systems are N_{pch}, N_d, N_ρ, N_f, N_{Fr}, N_s, N_{St}, N_{Ec}.

The kinematic similarity gives the first three groups, N_{pch}, N_d, and N_ρ. If the number N_{pch} is much larger than N_d, the system is controlled by the change in phase. On the other hand, if $N_d \gg N_{pch}$, then the system is controlled by the change in distributions of each phase, i.e., the diffusion of phases. For forced convection boiling with high pressure, the effect of N_{pch} seems to dominate the diffusion effect N_d, whereas for the low pressure or the natural convection boiling, N_d may be also very important.

From dynamic similarity we obtain N_ρ, N_d, N_f, N_{Fr}, N_s, and α_0. The friction number N_f is the most important among them, because of the large frictional pressure drop in two-phase flow. If the analogous frictional law holds between the single phase and thw two-phase flow, N_f is related to the same power of the Reynolds number. The Froude number N_{Fr} takes account of the effect of the gravity, whereas the surface force number N_s measures the importance of the capillary body forces which are usually neglected in the long pipe flows with a sufficiently large diameter.* The drift momentum transport gives $N_\rho N_d^2$ in the dimensionless momentum equation. If the order of N_d is less than unity and the density ratio N_ρ is very small, then this effect also may be neglected.

From the energy similarity we obtain N_{St}, N_{Ec}, N_d, and N_ρ. For the boiling heat transfer problem, the Stanton number N_{St} will dominate the right hand side of the energy equation. Except for high speed flow

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*In comparison with the equilibrium bubble diameter, for example.

the Eckert number N_{Ec} is small, thus the compressibility and dissipative effect can be neglected compared with wall heat transfer. Energy drift (or diffusion) has the same order as the drift (or diffusion) of mass, thus they should be treated under the same condition.

APPENDIX D

CHARACTERISTIC EQUATION CORRESPONDING TO EQ. (VII1.13)

Under the assumptions introduced in Section VII.1, the various pressure drop responses are given by:

For the upstream un-heated region:

$$\frac{\delta \Delta P_{01}}{\delta v} = \Lambda_i(S) = \rho_f \left\{ \left(\frac{A_c}{A_o}\right) S \ell_o + \frac{f_c \ell_c}{2 D_c} 2 \left(\frac{A_c}{A_o}\right)^2 \bar{v}_{fi} + 2 k_i \bar{v}_{fi} \right\} \qquad \text{(D1)}$$

For the heated liquid region:

$$\frac{\delta \Delta P_{12}}{\delta v} = \Lambda_f(S) = \rho_f \left\{ S \bar{\lambda} + \frac{f_s}{2D} 2 \bar{v}_{fi} \bar{\lambda} \right\} + \qquad \text{(D2)}$$

$$+ \rho_f \left(g + \frac{f_s}{2D} \bar{v}_{fi}^{\,2} \right) \Lambda_2(S)$$

For the heated mixture region:

the acceleration term

$$\frac{\delta \Delta P_{23a}}{\delta v} = \Lambda_g(S) = \rho_f \left\{ \left(1 + \frac{V_{gj}}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{S - \Omega_\infty}\right) C_k(\bar{\lambda}) \ln(C_r^*) \frac{S}{\Omega_\infty} \Lambda_3(S) \qquad \text{(D3)}\right.$$

$$\left. + V_{gj} \frac{S}{S - \Omega_\infty} \left(1 - \Lambda_3 \frac{S}{S - \Omega_\infty}\right)\left(1 - C_r^* e^{-S \bar{\tau}_{12}}\right) \right\}$$

the convective term

$$\frac{\delta \Delta P_{23c}}{\delta v} = \Lambda_9(s) = \rho_f \bar{v}_{fi}^2 \left\{ -\frac{1}{C_k(\bar{\lambda})}(1-\Lambda_3) + \right. \tag{D4}$$

$$+ \frac{1}{\bar{v}_{fi}} \frac{s}{s-\Omega_\infty} \ln (C_r^*) \, \Lambda_3(s) +$$

$$\left. + \left(1 - \Lambda_3 \frac{s}{s-\Omega_\infty}\right)\left(C_r^* e^{-s\bar{\tau}_{12}} - 1\right)\left(\frac{1}{C_k(\bar{\lambda})} + \frac{2}{C_k(\bar{\lambda})} \frac{V_{gj}}{\bar{v}_{fi}} - \frac{1}{\bar{v}_{fi}} \frac{s}{s-\Omega_\infty}\right)\right\}$$

the gravitational term

$$\frac{\delta \Delta P_{23g}}{\delta v} = \Lambda_{10}(s) - g \rho_f \Lambda_2 = \tag{D5}$$

$$= g \rho_f \left\{ \frac{1}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{s-\Omega_\infty} \frac{(\ell-\bar{\lambda})}{C_r^*} \Lambda_3 + \right.$$

$$\left. + \frac{1}{s}\left(1 - \Lambda_3 \frac{s}{s-\Omega_\infty}\right)\left(1 - e^{-s\bar{\tau}_{21}}\right)\right\} - g \rho_f \Lambda_2(s)$$

the frictional term

$$\frac{\delta \Delta P_{23f}}{\delta v} = \Lambda_{11}(s) - \frac{f_s}{2D} \rho_f \bar{v}_{fi}^2 \Lambda_2(s) \tag{D6}$$

$$= \frac{\bar{f}_m}{2D} \rho_f \bar{v}_{fi}^2 \left\{ \frac{\Omega_\infty}{s-\Omega_\infty} \frac{(\ell-\bar{\lambda})}{C_k(\bar{\lambda})} \Lambda_3 + \frac{2}{\bar{v}_{fi}}\left(1 + \frac{V_{gj}}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{s-\Omega_\infty}\right)(\ell-\bar{\lambda}) \Lambda_3 + \right.$$

$$\left. + \left(1 + \frac{2 V_{gj}}{\bar{v}_{fi}}\right) \frac{1}{s - 2\Omega_\infty}\left(1 - \Lambda_3 \frac{s}{s-\Omega_\infty}\right)\left(1 - C_r^{*2} e^{-s\bar{\tau}_{23}}\right)\right\} -$$

$$- \frac{f_s}{2D} \rho_f \bar{v}_{fi}^2 \Lambda_2(s)$$

and

the drift term

$$\frac{\delta \Delta P_{23d}}{\delta \upsilon} = \Lambda_{12}(s) = (1 - 2C_r^*)\rho_f V_{gj}^2 \left\{ \frac{1}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{s - \Omega_\infty} \Lambda_3 + \right. \tag{D7}$$

$$\left. + \frac{C_r^*}{C_k(\bar{\lambda})} \left[1 - \Lambda_3 \frac{s}{s - \Omega_\infty} \right] e^{-s\bar{\tau}_{23}} \right\}$$

<u>For the downstream un-heated region:</u>

the exit orifice term

$$\frac{\delta \Delta P_{34e}}{\delta \upsilon} = \Lambda_{13}(s) = \rho_f \bar{\upsilon}_{fi} k_e \left\{ \left[2 + \frac{1}{C_k(\bar{\lambda})} (\bar{\upsilon}_{fi} + \right. \right. \tag{D8}$$

$$\left. \left. + 2V_{gj}) \frac{\Omega_\infty}{s - \Omega_\infty} \right] \Lambda_3 + \frac{C_r^*}{C_k(\bar{\lambda})} (\bar{\upsilon}_{fi} + 2V_{gj}) e^{-s\bar{\tau}_{23}} \left(1 - \Lambda_3 \frac{s}{s - \Omega_\infty}\right) \right\}$$

the acceleration term

$$\frac{\delta \Delta P_{34a}}{\delta \upsilon} = \Lambda_{14}(s) = \frac{\rho_f}{C_r^*} l_e \left(\frac{A_c}{A_e}\right) s \left\{ \left[1 + \frac{V_{gj}}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{s - \Omega_\infty} \right] \Lambda_3 + \right. \tag{D9}$$

$$\left. + C_r^* \frac{V_{gj}}{C_k(\bar{\lambda})} \left[1 - \Lambda_3 \frac{s}{s - \Omega_\infty} \right] e^{-s\bar{\tau}_{23}} \right\}$$

the gravitational term

$$\frac{\delta \Delta P_{34g}}{\delta \upsilon} = \Lambda_{15}(s) = g_e \rho_f \bar{\upsilon}_{fi} \left(\frac{A_c}{A_e}\right) \left(\frac{1 - e^{-s\bar{\tau}_{34}}}{s}\right) \left\{ \frac{1}{C_r^*} \frac{1}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{s - \Omega_\infty} \Lambda_3 + \right. \tag{D10}$$

$$\left. + \frac{1}{C_k(\bar{\lambda})} \left[1 - \Lambda_3 \frac{s}{s - \Omega_\infty} \right] e^{-s\bar{\tau}_{23}} \right\}$$

the frictional term

$$\frac{\delta \Delta P_{34f}}{\delta \upsilon} = \Lambda_{16}(S) = \frac{\bar{f}_{me}}{2D_e} \rho_f \bar{\upsilon}_{fi} \left(\frac{A_c}{A_e}\right)^2 \left\{ 2\ell_e \left[\left(1 + \frac{V_{gj}}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{S-\Omega_\infty}\right) \Lambda_3 + \right.\right. \tag{D11}$$

$$\left. + C_r^* \frac{V_{gj}}{C_k(\bar{\lambda})} e^{-S\bar{\tau}_{23}} \left(1 - \Lambda_3 \frac{S}{S-\Omega_\infty} \right) \right] +$$

$$\left. + \bar{\upsilon}_{fi}^{\,2} \left[\frac{1}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{S-\Omega_\infty} \Lambda_3 + \frac{C_r^*}{C_k(\bar{\lambda})} \left(1 - \Lambda_3 \frac{S}{S-\Omega_\infty} \right) e^{-S\bar{\tau}_{23}} \right] \left(\frac{1 - e^{-S\bar{\tau}_{34}}}{S} \right) \right\}$$

and

the drift term

$$\frac{\delta \Delta P_{34d}}{\delta \upsilon} = (1 - 2C_r^*) \rho_g V_{gj}^{\,2} \left\{ \frac{1}{C_k(\bar{\lambda})} \frac{\Omega_\infty}{S-\Omega_\infty} \Lambda_3 + \right. \tag{D12}$$

$$\left. + \frac{C_r^*}{C_k(\bar{\lambda})} \left[1 - \Lambda_3 \frac{S}{S-\Omega_\infty} \right] e^{-S\bar{\tau}_{23}} \right\} \left(e^{-S\bar{\tau}_{34}} - 1 \right)$$

Consequently, the characteristic equation can be obtained from the above equations by adding the responses of the pressure drop. Hence

$$Q(S) = \rho_f \left\{ 2k_i \bar{\upsilon}_{fi} + \left[\frac{A_c}{A_o} \ell_o + \bar{\lambda} \right] S + \left[\frac{f_o \ell_o}{2D_o} \left(\frac{A_c}{A_o} \right)^2 + \frac{f_s}{2D} \bar{\lambda} \right] 2\bar{\upsilon}_{fi} \right\} \tag{D13}$$

$$+ \left\{ \Lambda_8(S) + \Lambda_9(S) + \Lambda_{10}(S) + \Lambda_{11}(S) + \Lambda_{12}(S) \right\}$$

$$+ \left\{ \Lambda_{13}(S) + \Lambda_{14}(S) + \Lambda_{15}(S) + \Lambda_{16}(S) + \Lambda_{12}(S) \left(e^{-S\bar{\tau}_{34}} - 1 \right) \right\} = 0$$

APPENDIX E

COEFFICIENTS OF THE CHARACTERISTIC EQUATION CORRESPONDING TO B_i IN EQ. (IX4.9)

E-1. The Case with $f_m = C_m f_s$

The characteristic equation for the uniformly heated system with a constant two-phase friction factor model is given by Eq. (VIII1.18) with Eq. (VIII1.19 to VIII1.28). In many practical cases, the effects of the upstream and downstream un-heated regions can be neglected except those of throttlings, i.e., of the orifice coefficients k_i and k_e, as it has been explained in Chapter IV. Thus we shall consider the dynamics of the heated region with only inlet and exit orifices. Under these conditions, the characteristic equation becomes

$$Q^*(S^*) = \frac{1}{S^*(S^*-1)^2(S^*-2)}\Big\{B_1 S^{*5} + B_2 S^{*4} + B_3 S^{*3} + B_4 S^{*2} + B_5 S^* + B_6 + \tag{E1}$$

$$+ e^{-S^*\tau_{12}^*}\left(B_7 S^{*4} + B_8 S^{*3} + B_9 S^{*2} + B_{10} S^* + B_{11}\right)$$

$$+ e^{-S^*\tau_{13}^*}\left(B_{12} S^{*3} + B_{13} S^{*2} + B_{14} S^* + B_{15}\right)$$

$$= 0$$

We define the following parameters A_i as

$$A_1 = \mathrm{Nsub} = \tau_{/2}^{*} \qquad \text{(E2)}$$

$$A_2 = (1 + V_{gj}^{*})\, \ln C_r^{*}$$

$$A_3 = V_{gj}^{*} \ln C_r^{*}$$

$$A_4 = V_{gj}^{*}$$

$$A_5 = V_{gj}^{*}\, C_r^{*}$$

$$A_6 = 2\left[k_i + \frac{f_s}{2D^{*}}\, \bar{\lambda}^{*} \right]$$

$$A_7 = \frac{1}{1 + V_{gj}^{*}}$$

$$A_8 = \ln C_r^{*} + 2\, \frac{f_s C_m}{2D^{*}} (1 - \bar{\lambda}^{*}) + 2\, k_e$$

$$A_9 = \ln C_r^{*} + \frac{1}{N_{Fr}\, N_{pch}} \left(1 - \frac{1}{C_r^{*}}\right) + \frac{f_s C_m}{2D^{*}} (1 - \bar{\lambda}^{*}) \left(2 - \frac{1}{1 + V_{gj}^{*}}\right) -$$

$$- \rho_g^{*} \frac{(V_{gj}^{*})^2}{1 + V_{gj}^{*}} \left(2 C_r^{*} - 1\right) + k_e \left(2 - \frac{1}{1 + V_{gj}^{*}}\right)$$

$$A_{10} = 1.0$$

$$A_{11} = \frac{V_{gj}^{*}}{1 + V_{gj}^{*}}$$

$$A_{12} = \frac{1}{N_{Fr} \cdot N_{pch}}$$

$$A_{13} = \frac{f_s C_m}{2D^*} \frac{1}{N_{pch}} \left(2V_{gj}^* + 1 \right)$$

$$A_{14} = C_r^*$$

$$A_{15} = C_r^* \left(\frac{V_{gj}^*}{1 + V_{gj}^*} \right)$$

$$A_{16} = \frac{f_s C_m}{2D^*} \frac{1}{N_{pch}} \left(2V_{gj}^* + 1 \right) C_r^{*2}$$

$$A_{17} = \rho_g^* \frac{\left(V_{gj}^* \right)^2}{1 + V_{gj}^*} \left(2C_r^* - 1 \right) C_r^*$$

$$A_{18} = k_e \left(\frac{2V_{gj}^* + 1}{V_{gj}^* + 1} \right) C_r^*$$

Then the coefficients of the characteristic equation (E1) become as follows.

$$B_1 = A_1 + A_2 \tag{E3}$$

$$B_2 = -4A_1 - 5A_2 + A_3 + A_6 + A_8$$

$$B_3 = 5A_1 + 9A_2 - 4A_3 - 4A_6 - A_7 - 5A_8 + A_9$$

$$B_4 = -2A_1 - 7A_2 + 5A_3 + 5A_6 + 4A_7 + 9A_8 - 4A_9$$

$$B_5 = 2A_2 - 2A_3 - 2A_6 - 5A_7 - 7A_8 + 5A_9$$

$$B_6 = 2A_7 + 2A_8 - 2A_9$$

$$B_7 = A_2$$

$$B_8 = -4A_2 + A_3 - A_4 + A_7 + A_8 + A_{11}$$

$$B_9 = 5A_2 - 3A_3 + 3A_4 - 4A_7 - 4A_8 + A_9 - A_{10} - 3A_{11} - A_{12} - A_{13}$$

$$B_{10} = -2A_2 + 2A_3 + 5A_7 + 5A_8 - 3A_9 + 2A_{10} + 2A_{11} + 3A_{12} + A_{13}$$

$$B_{11} = -2A_7 - 2A_8 + 2A_9 - 2A_{12}$$

$$B_{12} = A_5 - A_{15} + A_{17} - A_{18}$$

$$B_{13} = -2A_5 + A_{12} + A_{14} + 3A_{15} + A_{16} - 3A_{17} + 3A_{18}$$

$$B_{14} = -3A_{12} - 2A_{14} - 2A_{15} - A_{16} + 2A_{17} - 2A_{18}$$

$$B_{15} = 2A_{12}$$

E-2. The Case with $f_m = f_s \left(\frac{\rho_f}{\rho_m}\right)^{n'}$ or $f_s \left(\frac{V_m}{\bar{V}_{fi}}\right)^{n'}$

The characteristic equation with the dynamic friction factor can be easily obtained by substituting Eq. (VIII3.37) or (VIII3.40) for the transfer function $\Lambda_{II}(s)$.

By taking the density type friction factor, we have

$$f_m = f_s \left(\frac{\rho_f}{\rho_m}\right)^{n'} , \qquad 0 < n' < 1 \tag{E4}$$

The coefficients A_8, A_9, A_{13}, and A_{16} should be modified in Eq. (E2) as

$$A_8 = \ln C_r^* + \frac{2 f_s}{2D^*} \frac{(1-\bar{\lambda}^*)}{(n'+1)} \frac{(C_r^{*\,n'+1} - 1)}{(C_r^* - 1)} \tag{E5}$$

$$A_9 = \ln C_r^* + \frac{1}{N_{Fr} N_{pch}} \left(1 - \frac{1}{C_r^*}\right) + \frac{f_s}{2D^*} \frac{(1-\bar{\lambda}^*)}{(n'+1)} \left[\frac{C_r^{*\,n'+1} - 1}{C_r^* - 1}\right] \left(2 - \frac{1+n'}{1+V_{gj}^*}\right) \tag{E6}$$

(Continued)

$$- \rho_g^* \frac{V_{gj}^{*2}}{1+V_{gj}^*} (2C_r^* - 1) + k_e \left(2 - \frac{1}{1+V_{gj}^*}\right)$$

$$A_{13} = \frac{f_s}{2D^*} \frac{1}{N_{pch}} \left(2V_{gj}^* + 1 - n'\right) \tag{E7}$$

$$A_{16} = \frac{f_s}{2D^*} \frac{1}{N_{pch}} \left(2V_{gj}^* + 1 - n'\right) \left[C_r^*\right]^{(2+n')} \tag{E8}$$

On the other hand, by taking the <u>velocity type friction factor</u>, we have

$$f_m = f_s \left(\frac{v_m}{\bar{v}_{fi}}\right)^{n'} , \quad 0 < n' < 1 \tag{E9}$$

The coefficients A_8, A_9, A_{13}, and A_{16} become

$$A_8 = \ln C_r^* + \frac{f_s}{2D^*} \frac{(2+n')}{(1+n')} (1 - \bar{\lambda}^*) \left[\frac{C_r^{*(n'+1)} - 1}{C_r^* - 1}\right] + 2k_e \tag{E10}$$

$$A_9 = \ln C_r^* + \frac{1}{N_{Fr} N_{pch}} \left(1 - \frac{1}{C_r^*}\right) + \frac{f_s}{2D^*} \frac{(1-\bar{\lambda}^*)}{(n'+1)} \left[\frac{C_r^{*\,n'+1} - 1}{C_r^* - 1}\right] \left(2 + n' - \frac{1+n'}{1+V_{gj}^*}\right) \tag{E11}$$

$$- \rho_g^* \frac{V_{gj}^{*2}}{1+V_{gj}^*} (2C_r^* - 1) + k_e \left(2 - \frac{1}{1+V_{gj}^*}\right)$$

$$A_{13} = \frac{f_s}{2D^*} \frac{1}{N_{pch}} \left[(2+n')V_{gj}^* + 1\right] \tag{E12}$$

$$A_{16} = \frac{f_s}{2D^*} \frac{1}{N_{pch}} \left[(2+n')V_{gj}^* + 1\right] \left[C_r^*\right]^{(2+n')} \tag{E13}$$

In both cases the functional form of Eq. (E1) and (E3) is preserved.

APPENDIX F

METHOD OF COMPARISON WITH EXPERIMENTAL DATA

The characteristic equation for the uniformly heated system with a constant two-phase friction factor model has been compared to the existing experimental data (5, 9, 7). For simplicity, both the downstream and upstream un-heated region effects have been neglected, except the inlet and exit orificing effects. Hence, in this case, the characteristic equation is given by Eq. (E1). The necessary input data to the computer programs (PNS) and (TSP) are given in Section X-1.

The fluid used in the experiments and the system pressure level fix the domain of operations in the stability plane, i.e., $N_{sub} - N_{pch}$ plane. From this information, all the saturation properties can also be obtained from a property table. The geometric parameters such as D, ℓ, $|g|\cos\theta$, Ac, and ξ are specified in the experimental data. The data are then grouped such that each set of data is taken at the same system pressure, geometry, and inlet velocity.

The experimental data of (5, 9, 7) do not specify the inlet and outlet orifice coefficients explicitly; however, they do give the total pressure drops at the upstream and downstream un-heated regions. On the other hand, the theoretical steady state pressure drops in the above regions are given by Eq. (VI1.4) and (VI4.25). At the specified inlet subcoolings and heat flux, these can be calculated according to the presently obtained steady state solutions. Then, by subtracting the gravity and frictional pressure drops from the experimental total pressure drop, the orifice coefficients k_i' and k_e' could be obtained. The upstream and downstream un-heated regions have been lumped in this characteristic equation as mentioned before; therefore, the frictional pressure drops should also be included in the virtual orifice coefficients, which can be obtained by adding the friction number effects to k_i and to k_e. This gives the corresponding exact value for k_i in the liquid region, since there is no time delay in this region. On the other hand, the corresponding k_e becomes an approximate value due to the time delay in the density response. However, the velocity in region (D) is relatively high and the length of the duct is small; therefore, this approximation is valid.

The values of k_i and k_e obtained by the above method are again compared among the experimental data. Eventually, we could classify all of the experimental data into subgroups which have the same P_s geometry, N_{Res}, k_i, and k_e within each set. Due to the small scattering in experimental values of N_{Res}, k_i, and k_e, we have permitted about 5 percent ranges for each of the above parameters within a set of data. Otherwise, we can plot only one data point in the stability plane.

By knowing P_s, $\overline{v}_{fi}$, D^*, $|g|\cos\theta$, k_i, and k_e, the stability boundaries can be found from the computer programs (PNS) and (TSP). However, we should supply two constitutive relations for V_{gj} and C_m. In the present analysis, we have used the following expressions.

$$f_m = C_m f_s \tag{F1}$$

with

$$C_m = 2 \qquad \text{and} \qquad f_s = \frac{0.184}{N_{Res}^{0.2}} \tag{F2}$$

and

$$V_{gj} = 1.41 \left[\frac{\sigma\,\Delta\rho g}{\rho_f^2}\right]^{1/4} \tag{F3}$$

The above expression for V_{gj} has been obtained for bubbly and churn turbulent bubbly flow regimes in (32). According to their experiments and analysis, this form of a constitutive equation for V_{gj} can be applied to a wide range of α. The value 2 for C_m gives a slightly higher pressure drop than the Martinelli and Nelson correlation; however, the present model agrees quite well with the experimental data of (5, 9, 7). The order of magnitude of $V_{gj}{}^*$ calculated from Eq. (G3) for Figs. 23 to 28 is in the range of 0.1 to 0.4.

APPENDIX G

THE CONSTANT PROPERTY ASSUMPTIONS IN THE MIXTURE REGION

In the present analysis, the fluid properties of each phase have been assumed to be constant throughout the mixture region, though the changes in mixture properties due to the phase change have been taken into account.

In this appendix we shall examine this assumption in some detail. By considering the mixture in thermodynamic equilibrium, we have, for the thermal equation of state

$$\rho_m(C, P) = \frac{1}{C/\rho_g + (1-C)/\rho_f} \tag{G1}$$

and for the caloric equation of state

$$i_m(C, P) = C i_g + (1-C) i_f \tag{G2}$$

where the fluid properties of each phase, ρ_g, ρ_f, i_g, and i_f, change along the saturation line. In the differential form, the above two equations become

$$-\frac{1}{\rho_m}\frac{D\rho_m}{Dt} = -\rho_m \Delta v_{fg}\frac{DC}{Dt} + \frac{1}{\rho_m a_m^2}\frac{DP}{Dt} \tag{G3}$$

and

$$\rho_m \frac{Di_m}{Dt} = \rho_m \Delta i_{fg} \frac{DC}{Dt} + \rho_m \left[C_m - \frac{T \Delta v_{fg}}{\Delta i_{fg}} \right] \frac{DP}{Dt} \qquad \text{(G4)}$$

where the saturated mixture sound velocity a_m and the heat capacity C_m are given by

$$\left.\frac{\partial \rho_m}{\partial P}\right)_c \equiv \frac{1}{a_m^2} \qquad , \qquad \left.\frac{\partial i_m}{\partial T}\right)_c \equiv C_m \qquad \text{(G5)}$$

We note here that a_m and C_m can be related to the sound velocity, the isothermal compressibility, and the specific heat of each phase.

Now let us consider the effects of two different wave propagations on the mixture density and the mixture enthalpy. The kinematic waves of frequency ω_k propagating with velocity C_k are given by

$$\frac{\delta C}{C_o} = \delta e^{j\left(\frac{\omega_k}{C_k} z - \omega_k t\right)} \qquad \text{(G6)}$$

whereas the acoustic waves of frequency ω_a propagating with velocity C_a are given by

$$\frac{\delta P}{\rho_{mo} v_{mo}^2} = \delta e^{j\left(\frac{\omega_a}{C_a} z \mp \omega_a t\right)} \qquad \text{(G7)}$$

By assuming the perturbations due to these waves are small, we only consider the first order effect in δ.

Effect on the Density

By substituting Eq. (G6) and (G7) into Eq. (G3), we obtain the following expressions for the zeroth order of δ

$$-\overline{\frac{1}{\rho_m}\frac{D\rho_m}{Dt}} = -\rho_m \Delta v_{fg}\, v_m \frac{dC}{dz} + \frac{1}{\rho_m a_m^2}\, v_m \frac{dP}{dz} \tag{G8}$$

and for the first order of δ

$$-\delta\left(\frac{1}{\rho_m}\frac{D\rho_m}{Dt}\right) = -\rho_m \Delta v_{fg}\left\{\left[-\delta\cdot\omega_k\left(1-\frac{v_m}{C_k}\right)\right]C_0 + \delta\cdot v_{m0}\frac{dC}{dz}\right\}$$

$$+\frac{1}{\rho_m a_m^2}\left\{\left[\mp\delta\cdot\omega_a\left(\frac{1}{1\pm M}\right)\right]\rho_m v_{m0}^2 + \delta\cdot v_{m0}\frac{dP}{dz}\right\} \tag{G9}$$

Making the order of magnitude analysis on Eq. (G8), the condition that the pressure drop effect can be neglected with respect to the effect of the phase change yields

$$\rho_{m0}\,\Delta v_{fg0}\, C_0 \gg \frac{M^2}{1\pm M} \tag{G9}$$

where M is the mixture Mach number given by $\frac{v_m}{a_m}$. On the other hand, from Eq. (G9), we obtain the condition under which, if

$$\frac{\rho_{m0}\,\Delta v_{fg0}\, C_0(1\pm M)}{M^2} \gg \frac{\omega_a}{\omega_k} \tag{G10}$$

then the acoustic wave effect can be neglected with respect to the effect of the kinematic waves.

Effect on the Enthalpy

Similarly, from the caloric equation of state, Eq. (G4), the condition that the pressure drop effect can be neglected with respect to the phase change yields

$$\frac{1}{N_{Ec}}\left[\frac{C_0 \Delta i_{fg0}}{\rho_{m0} C_{m0} T_0 \Delta v_{fg0}}\right] \gg 1 \tag{G11}$$

On the other hand, the acoustic wave effect on the enthalpy can be neglected, if

$$\frac{1}{N_{Ec}}\left[\frac{C_0 \Delta i_{fg0}}{\rho_{m0} C_{m0} T_0 \Delta v_{fg0}}\right] \gg \frac{\omega_a}{\omega_k} \tag{G12}$$

Low Frequency Analysis with Constant Properties

The preceding analysis shows the conditions when the low frequency analysis with constant property assumption for each phase can be applied. Such conditions can be attained under low Mach numbers. However, if the period of the oscillations considerably exceeds the residence time of the particle, this assumption breaks down.

Furthermore, it is known that thermodynamic second derivatives such as the sound velocity and the specific heat become singular at the critical points. Hence, as system operational conditions approach very near to the critical point, the assumption cannot be used.

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*Abbreviations used herein follow the form given in The Engineering Index Annual, Engineering Index Inc., New York, U.S.A.

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In September 1969, he transferred from New York University to the Georgia Institute of Technology, School of Mechanical Engineering, following his thesis advisor and continued his study toward the degree. In the latter institution, Mamoru worked as a research assistant under the National Science Foundation grant from September 1969.

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