

# Thermodynamic and cosmological parameters of early stages of the Universe

Amr Abd Al-Rahman Youssef,<sup>1,\*</sup> Gaber Faisal,<sup>2,†</sup> and Hakan Akyildirim<sup>2,‡</sup>

<sup>1</sup> *Mathematics Department, Faculty of Science, Al-Azhar University, Cairo, Egypt.*

<sup>2</sup> *Department of Physics, Faculty of Arts and Sciences,  
Süleyman Demirel University, Isparta, Turkey 32260.*

## Abstract

The early Universe was characterized by the presence of heavy particles that decoupled at different temperatures leading to different phases of the Universe. This had a consequences on the time evolution of the thermodynamic and the cosmological parameters characterizing each phase of the early Universe. In this study, we derive the analytic expressions of the equations governing the time evolution of these parameters in the early eras of the Universe namely, the radiation era, the quark-gluon plasma era, the hadron era and the mixed era. The parameters under concern include the energy density, the entropy density, the temperature, the pressure in addition to Hubble parameter and the scale factor. Having these expressions allows us to give estimations of the times corresponding to the beginning and ending of each era of the Universe as will be presented in this work.

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\* amraay2003@gmail.com

† gaberfaisal@sdu.edu.tr

‡ hakanakyildirim@sdu.edu.tr

## I. INTRODUCTION

It is widely accepted that the Universe is homogeneous and isotropic [1]. As a consequence, the space-time can be parametrized by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric. Upon inserting the metric into the Einstein equations one obtains the Friedmann equations [1]. These equations can be used to get the following equation [2–4]:

$$\frac{\dot{a}}{a} = -\frac{d\varepsilon}{3(\varepsilon + p)} = \sqrt{\frac{8\pi G \varepsilon}{3}} dt \quad (1)$$

where  $a(t)$  is the scale factor. The above differential equation enables us to find the temporal variation of the energy density  $\varepsilon$  once the pressure  $p$  as a function of  $\varepsilon$  is known. It should be noted that the above equation can be rewritten in terms of temperature  $T$  with the help of equations of state and hence one gets solution expressing the temporal evolution of the temperature. This can be the case also for other thermodynamic parameters such as pressure density and entropy density  $s$ .

The scale factor as a cosmological parameter can be calculated with the help of the energy density and pressure upon performing the integration in Eq.(1). On the other hand, knowing the the energy density, the Hubble parameter  $H(t)$  can be estimated using the following equation

$$H(t) = \sqrt{\frac{8\pi G}{3}\varepsilon(t)} \quad (2)$$

Based on the above discussion, the time evolution of the thermodynamic and cosmological parameters depends on the knowledge of the equations of state of the Universe which in turn depend on the phase of the Universe. This can be understood as in each phase of the Universe, the nature of the matter spreading in the Universe and the distribution of the energy in the Universe are different.

It is widely believed that our Universe underwent different cosmological phases started right after the big bang and their time evolution leaded to our current Universe. The early phase of the Universe was dominated by radiation. During this phase the Universe endured several phase transitions due to the cooling process to a temperature  $T \sim m_c$  where  $m_c$  is the charm quark mass. At the end of radiation dominated (RD) era, the Universe experienced another phase transition namely, quark-gluon plasma (QGP) phase. This phase was followed by another phase the so called hadron phase when the temperature

of the Universe was smaller than the critical temperature  $T_C$  which represents the QCD phase transition temperature. In the mixed phase the Universe experienced the co-existence of QGP and hadron phases for a certain time interval. This mixed phase happens when the temperature of the Universe cools down and get close to  $T_C$ . During this phase, the temperature of the Universe was fixed at  $T_C$ .

QGP can be created through colliding ultra-relativistic heavy ions in Colliders as in AGS, SPS, RHIC, LHC. On the theoretical hand side, there are two basic approaches for studying the properties of QCD. The first approach is based on lattice Gauge Theories in which a field theory is formulated and solved on a discrete lattice of space-time points with the help of so powerful computers. As a prediction of the lattice QCD, the nuclear matter experiences a phase transition at a temperature  $T_C$  in the range  $150 - 170 MeV$  and energy density,  $1 GeV/fm^3$ . The second approach adopts phenomenological models to avoid the requirement of the intense numerical calculations needed for lattice QCD. Examples of such models include the bag models with the MIT bag model is the widely used one. With the recent experimental results from heavy ion collisions and the advances in lattice QCD calculations our knowledge of the equation of state of the QGP has been improved.

In the literature, previous studies, related to studying early phases of the Universe, have been carried out where the main interest was directed to the QGP phase [3–14].

In this study we aim to derive the analytic solutions of the Eq.(1). The equation can be casted into energy density or temperature or pressure differential equation and thus can be solved to give the time evolution of the corresponding thermodynamic parameter in the early eras of the Universe as we will show in details in the following. Moreover, we give more attention to show details of estimating the times at which different phase transitions occur and give analytic expressions for estimating these times. We will also derive the expressions governing the time variation of some cosmological parameters in all of the aforementioned phases of the Universe.

## II. THE TIME EVOLUTION OF THE THERMODYNAMIC PARAMETERS IN THE EARLY UNIVERSE

In this section we investigate the thermodynamics and cosmological parameters in early eras of the Universe. Our aim is to derive analytic expressions for these parameters in

each era. These expressions can be used to study the time evolution of the aforementioned parameters which will be presented in section III.

The early Universe is thought to be characterized by very high temperatures. Consequently, massive particles were pair produced, and contributed to the thermal bath. Moreover, particle masses can be neglected providing that  $m \ll T$  where  $m$  and  $T$  denote the mass of the particle and temperature of the Universe respectively. In the Standard Model (SM) the heaviest particle is the top quark with mass  $m_t \simeq 170 \text{ GeV}$ . Searches at the large Hadron Collider (LHC) for particles with heavier masses than the top quark mass exclude particles with masses close to TeV predicted in many theories beyond the SM. Thus, our knowledge about phase transitions occurred in the temperature interval  $T > m_t$  remains model dependent and is uncertain in the same time. Consequently, we adopt the SM as the framework in which the evaluation of the degrees of freedom of particles and bosons, required in this study, are carried out. In the SM, a chemical potential is often associated with baryon number. Due to the fact that the ratio of the net baryon density to the photon density is so tiny, one can neglect that chemical potential when estimating thermodynamic quantities such as the energy density  $\varepsilon$ , the pressure density  $p$  and the entropy density  $s$ . In the following, we will present the equations of state that relate these parameters with temperature and derive the equations governing their time evolution in the early eras of the Universe.

### A. Radiation era

The early epoch of the Universe was characterized by temperatures satisfying the relation  $T > m_c$ . In this epoch, the Universe was dominated by radiation and endured several phase transitions as a consequence of the cooling process to a temperature  $T \sim m_c$  where  $m_c$  is the charm quark mass. The equations of state in this era can be approximated as

$$\varepsilon_{RD} = N_{RD} \frac{\pi^2}{30} T^4, \quad p_{RD} = N_{RD} \frac{\pi^2}{90} T^4, \quad s_{RD} = \frac{(\varepsilon_{RD} + p_{RD})}{T} \quad \text{if } m_c < T, \quad (3)$$

here  $N_{RD}$  stands for the effective number of degrees of freedom at temperature  $T$  and can be determined from the relation

$$N_{RD} = \sum_B g_B + \frac{7}{8} \sum_F g_F \quad (4)$$

where  $g_B$  ( $g_F$ ) denotes number of degrees of freedom for a boson  $B$  (a fermion  $F$ ) and the sum runs over all boson and fermion states with masses satisfying  $m \ll T$ . Clearly,  $N_{RD}$  is model dependent. The factor of  $7/8$  in the expression of  $N_{RD}$  accounts for the difference between the Bose and Fermi integrals.

At high temperatures much bigger than the top quark mass  $m_t$ , all the the SM particles were present. Thus, we have 28 bosonic degrees of freedom and 90 fermionic degrees of freedom. The number 28 is the sum of degrees of freedom of the photons  $\gamma$ , the charged gauge bosons  $W^\pm$ , the neutral gauge boson  $Z$ , the gluons  $g$  and the Higgs boson  $H$  where  $g_\gamma = 2, g_{W^-} = g_{W^+} = g_Z = 3, g_g = 16$  and  $g_H = 1$ . On the other hand, the number 90 is the sum of degrees of freedom of all fermions in the SM. After substituting in Eq.(4) we find that  $N_{RD} = 28 + \frac{7}{8} \times 90 = 427/4 = 106.75$

We study the energy density in the time interval starting from  $t_0 = 0$  that corresponds to  $T \sim \infty$  till time  $t_8$  corresponding to  $T_8 = m_c \sim 1 \text{ GeV}$ . First in the mentioned time interval we have  $p = c_s^2 \varepsilon_{RD}(t)$ . Using this in Eq.(1) and setting  $c_s^2 = 1/3$  we get the following differential equation

$$\frac{d\varepsilon_{RD}}{\varepsilon_{RD}\sqrt{\varepsilon_{RD}}} = -\sqrt{\frac{4 \times 32\pi G}{3}} dt \quad (5)$$

after integration we get

$$\varepsilon_{RD}(t) = \frac{4\varepsilon_{RD}(t_i)}{\left[2 + 2\sqrt{\frac{32\pi G}{3}}\varepsilon_{RD}(t_i)(t - t_i)\right]^2} \quad (6)$$

where  $t_i$  represents the initial time. The result agrees with the the corresponding one given in Eq.(9) in Ref. [9]. For  $i \rightarrow 0$  we find that  $\frac{1}{\sqrt{\varepsilon_{RD}(t_0)}} \rightarrow 0$ , as  $T(t_0) \rightarrow \infty$ , and hence we get

$$\varepsilon_{RD}(t) = \frac{3}{32\pi G t^2} \quad (7)$$

The preceding equation gives the expression of the energy density in the time interval starting from  $t = 0$  to  $t$  and for the intervals starting from  $t_i \neq 0$  to  $t_{i+1}$ , the energy density can

be evaluated using Eq.(6). In the temperature range starting at  $T \sim \infty$  which corresponds to the beginning of the Universe and ending at  $T = m_c \sim 1 \text{ GeV}$  several phase transitions occur due to the decoupling of heavy particles and massive bosons. In particular, due to the decoupling of top quark, Higgs boson, massive  $Z$  and  $W^\pm$  bosons,  $b$  quark and  $\tau$  lepton. This in turns will affect the value of  $N_{RD}$ . In order to estimate the time at which these phase transitions occur at temperature  $T_i$ , we can use the result of the integration of Eq.(5) and solve for  $t$ . Thus, we find that

$$t_{i+1} = t_i - \sqrt{\frac{3}{32\pi G}} \left( \frac{1}{\sqrt{\varepsilon_{RD}(t_i)}} - \frac{1}{\sqrt{\varepsilon_{RD}(t_{i+1})}} \right) \quad (8)$$

It should be noted that in the time interval starting from  $t_0 = 0$  till  $t$  Eq.(8) reduces to

$$t = \sqrt{\frac{45}{16\pi^3 G N_{RD}}} \frac{1}{T^2} \quad (9)$$

in agreement with Refs.[10, 15] after setting  $\hbar = c = k_B = 1$ . This relation can be used to estimate the time at which all non standard particles, heavy particles predicted in some classes of new physics beyond standard model, decoupled. Since ongoing search at colliders has not observed such particles up to  $TeV$  energy scale, we can start our estimation of times in the radiation era at  $T = 1 \text{ TeV}$  where only standard model particles exist. In TableI, we present the numerical estimation of the times at which standard model particles decoupled and the corresponding energy densities.

The time dependence of the temperature in the  $RD$  phase of the Universe can be obtained by substituting the definitions of  $\varepsilon_{RD}$  and  $p_{RD}$  given in Eq.(3) into Eq.(1) and hence, we obtain below simple differential equation:

$$T^{-3} dT = -\sqrt{\frac{4\pi^3 G N_{RD}}{45}} dt \quad (10)$$

we find that the solution of the above differential equation can be expressed as

$$T_{RD}(t) = \frac{T_{RD}(t_i)}{\left[ 1 + \sqrt{\frac{16\pi^3 G N_{RD} T_{RD}^4(t_i)}{45}} (t - t_i) \right]^{1/2}} \quad (11)$$

where  $T_{RD}(t_i)$  represents the initial temperature at the start time  $t = t_i$  of the time interval. At  $i = 0$  we have  $T_{RD}(t_0 = 0) = \infty$  and thus we can write

$$T_{RD}(t) = \left[ \sqrt{\frac{16\pi^3 G N_{RD}}{45}} t \right]^{-1/2} \quad (12)$$

i	$T_{i+1}(GeV)$	$4N_{RD}$	$t_{i+1}(s)$	$\varepsilon_{RD}(GeV/fm^3)$
0	1000	427	$2.32 \times 10^{-13}$	$3.5 \times 10^{13}$
1	$m_t$	385	$8.22 \times 10^{-12}$	$3.7 \times 10^{12}$
2	$m_H$	381	$1.57 \times 10^{-11}$	$1.0 \times 10^{12}$
3	$m_{Z^0}$	369	$2.96 \times 10^{-11}$	$2.7 \times 10^{11}$
4	$m_W^\pm$	345	$3.97 \times 10^{-11}$	$1.5 \times 10^{11}$
5	$m_b$	303	$1.56 \times 10^{-8}$	$1.0 \times 10^6$
6	$m_\tau$	289	$9.00 \times 10^{-8}$	$3.0 \times 10^4$
7	$m_c$	247	$1.85 \times 10^{-7}$	$7.3 \times 10^3$

TABLE I. Time in seconds corresponding to decoupling of heavy particles and massive weak gauge bosons.

this gives the evolution of temperature with time in the first interval that ends at  $T = 1 TeV$ . For other intervals corresponding to the times listed in TableI, we can use the relation given in Eq.(11) to estimate the evolution of temperature with time.

The derivation of an analytic formula for the time variation of the pressure in the radiation era is straightforward following same steps as we did for the case of the energy density. The only difference here is to replace  $\varepsilon_{RD} = 3 p_{RD}$  in Eq.(1) and after performing the integration and setting  $\frac{1}{\sqrt{p_{RD}(t_0=\infty)}} \rightarrow 0$  we get

$$p_{RD}(t) = \frac{1}{32\pi G t^2} \quad (13)$$

For any time interval starting at  $t = t_i$  we find that the pressure is given as

$$p_{RD}(t) = \frac{4p_{RD}(t_i)}{3[2 + 2\sqrt{\frac{32\pi G}{3}\varepsilon_{RD}(t_i)(t - t_i)}]^2} \quad (14)$$

We turn now to derive the expression of the scale factor in the RD era. Using the equations of state given in Eq.(3) and Eq.(1) allows us to write

$$\frac{\dot{a}(t)}{a(t)} = -\frac{\dot{\varepsilon}(t)}{3[\varepsilon(t) + \frac{1}{3}\varepsilon(t)]} = -\frac{1}{4}\frac{\dot{\varepsilon}(t)}{\varepsilon(t)} \quad (15)$$

where we have used  $p(t) = \frac{1}{3}\varepsilon(t)$ . The previous equation can be expressed as

$$\frac{d}{dt} \ln [a(t)] = -\frac{1}{4} \frac{d}{dt} \ln [\varepsilon(t)] \quad (16)$$

Solution of such an equation yields

$$\frac{a(t)}{a(t_i)} = \left[ \frac{\varepsilon(t_i)}{\varepsilon(t)} \right]^{\frac{1}{4}} = \left[ 1 + \sqrt{\frac{32\pi G}{3} \varepsilon_{RD}(t_i) (t - t_i)} \right]^{\frac{1}{2}} \quad (17)$$

where, as before,  $t_i$  stands for the value of the time at the beginning of the time interval in the RD era. It should be noted that Eq.(17) can be expressed in terms of the temperatures or in terms of the times as

$$\frac{a(t)}{a(t_i)} = \left[ \frac{\varepsilon(t_i)}{\varepsilon(t)} \right]^{\frac{1}{4}} = \frac{T(t_i)}{T(t)} = \left( \frac{t}{t_i} \right)^{\frac{1}{2}} \quad (18)$$

The result obtained in the previous equation agrees with the result obtained in Refs.[3, 8].

## B. Quark Gluon Plasma era

The QGP phase of the Universe existed when the temperature of the Universe was in the range  $T_C < T < m_c$ , where  $T_C$  is the critical temperature. In that phase, the Universe was in a state filled with quark-gluon plasma contains lighter quarks in addition to the photons, lighter charged leptons, neutrinos and antineutrinos in thermal equilibrium. It should be noted that the relativistic heavy ion collision experiments at both RHIC and LHC may access to the temperature range  $T_C < T < m_c$  and hence they can shed light on the properties and nature of the plasma formed in this range.

As it is known, quarks as colored particles are confined to each others in bound hadronic states. One of the most successful phenomenological models for quark confinement is the so called MIT bag model [16]. While in the MIT bag model the contributions that arise from the particles in the electroweak sector were not taken into account, here in this work we follow Refs.[3, 9] and include their effects on the effective number of degrees of freedom. The densities corresponding to this epoch of the early universe can be approximated in a bag model  $M_i$  as

$$\varepsilon_{QGP} = N_{QGP} \frac{\pi^2}{30} T^4 + \mathcal{B}, \quad p_{QGP} = \frac{1}{3} N_{QGP} \frac{\pi^2}{30} T^4 - \mathcal{B}, \quad s_{QGP} = 4 N_{QGP} \frac{\pi^2}{90} T^3, \quad (19)$$

here  $\mathcal{B}$  is a bag constant parameter. It represents the exerted external pressure on the bag surface. This pressure balances the internal pressure in the absence of QGP and hence



ensures the stability of the bag. In Eq.(19),  $N_{QGP}$  stands for the effective number of degrees of freedom and can be determined from a relation similar to the one given in Eq.(4) where the summation in this case is carried out for all the bosons and fermions present in the QGP.

The exact analytic solution of the energy density  $\varepsilon_{QGP}$ , temperature and pressure densities can be obtained directly from solving Eq.(1). In the appendix, we show the steps we follow to derive the desired solutions for the three quantities. We find that, the analytic expressions can be expressed as

$$\varepsilon_{QGP}(t) = \chi(t) + \zeta(t) + \sqrt{\left(\chi(t) + \zeta(t)\right)^2 - \zeta^2(t)} \quad (20)$$

The functions  $\zeta(t)$  and  $\chi(t)$  are given in terms of  $\eta(t)$  defined as

$$\eta(t) = \exp\left(4\sqrt{\frac{8\pi\mathcal{B}G}{3}}t + \xi\right) \quad (21)$$

with

$$\xi = \ln\left(\frac{\sqrt{\varepsilon_{RD}(t_8)} + \sqrt{\mathcal{B}}}{\sqrt{\varepsilon_{RD}(t_8)} - \sqrt{\mathcal{B}}}\right) - 4\sqrt{\frac{8\pi\mathcal{B}G}{3}}t_8 \quad (22)$$

where  $\varepsilon_{RD}(t_8)$  is the value of the energy density at the time  $t_8$ . The explicit expressions of  $\zeta(t)$  and  $\chi(t)$  are given in Eq.(49) in the appendix. It should be remarked that, up to our knowledge, our analytic solution of the energy density  $\varepsilon_{QGP}$  given in Eq.(20) was not pointed out in the literature before. Previous studies reexpressed Eq.(1) in terms of temperature and solved it analytically as in Ref.[5] or numerically as in Ref.[8] to obtain the temperature and consequently used the equations of state to evaluate the energy density.

The analytic solution of the temperature for the bag models of QGP discussed above can be written as

$$T_{QGP}(t) = \sqrt{\frac{2\mathcal{B}\kappa(t)}{\left(\frac{\pi^2}{30}N_{QGP} - \mathcal{B}\kappa^2(t)\right)}} \quad (23)$$

where the function  $\kappa(t)$  is given as

$$\kappa(t) = b \exp\left[-\frac{4}{3}\sqrt{6\pi\mathcal{B}G}(t - t_8)\right] \quad (24)$$

with

$$b = T_8^2 \left(\frac{\mathcal{B}}{\frac{\pi^2}{30}N_{QGP}} + \sqrt{\frac{\mathcal{B}}{\frac{\pi^2}{30}N_{QGP}}T_8^4 + \frac{\mathcal{B}^2}{900N_{QGP}^2}}\right)^{-1} \quad (25)$$

here  $T_8 = T_{RD}(t_8)$ .

In Ref.[5], Eq.(1) was written in terms of the critical temperature and solved analytically to obtain the time evolution of the temperature. Here, our result in Eq.(23) has no dependency on the critical temperature but instead depends on the temperature at the beginning of the *QGP* era which is the same one at the end of the radiation era.

Turning now to the pressure, Eq.(1) can be expressed in terms of the pressure using equations of state and then can be solved analytically to obtain the explicit dependency of the pressure on the time as

$$p(t) = \frac{-3\varrho_8^2 \mathcal{B} \exp \left[ -8 \sqrt{\frac{8\pi \mathcal{B} G}{3}} (t - t_8) \right] - 10\varrho_8 \mathcal{B} \exp \left[ -4 \sqrt{\frac{8\pi \mathcal{B} G}{3}} (t - t_8) \right] - 3\mathcal{B}}{3 \left( \varrho_8 \exp \left[ -4 \sqrt{\frac{8\pi \mathcal{B} G}{3}} (t - t_8) \right] + 1 \right)^2} \quad (26)$$

where  $\varrho_8 = \frac{\sqrt{\mathcal{B}} - \sqrt{3p_8 + 4\mathcal{B}}}{\sqrt{\mathcal{B}} + \sqrt{3p_8 + 4\mathcal{B}}}$ .

The scale factor in the *QGP* era can be obtained with the help of the equations of state listed in Eq.(19) and Eq.(1) and performing the integration. We find that

$$\frac{a(t)}{a(t_8)} = \left[ \frac{\varepsilon_{QGP}(t_8) - \mathcal{B}}{\varepsilon_{QGP}(t) - \mathcal{B}} \right]^{\frac{1}{4}} = \frac{T_{QGP}(t_8)}{T_{QGP}(t)} = \frac{T_8}{T_{QGP}(t)} \quad (27)$$

Thus, one can estimate the scale factor using either the expression of the energy density or the expression of the temperature in the *QGP* era.

### C. Hadron era

The critical temperature  $T_C$  represents the QCD phase transition temperature. The transition characterizes the confinement-deconfinement transition between quarks and hadrons where three quarks (anti-quarks) are confined together to form baryon(anti-baryon) and quark anti-quark are confined together to form meson. The formed heavy hadrons are not stable and thus quickly decay to the lightest hadrons i.e. the pions. In the hadron phase, the particle content includes  $e^\pm, \mu^\pm, \nu_{e,\mu,\tau}, \bar{\nu}_{e,\mu,\tau}$  together with photons and pions. The densities corresponding to the massless pion gas are given as

$$\varepsilon_H = N_H \frac{\pi^2}{30} T^4, \quad p_H = N_H \frac{\pi^2}{90} T^4 \quad s_H = 4N_H \frac{\pi^2}{90} T^3 \quad m_\pi < T < T_C \quad (28)$$

where  $m_\pi$  represents the pion mass and  $N_H$  is the effective number of degrees of freedom that can be calculated using the relation

$$N_H = N_f^2 - 1 + N_{EW} \quad (29)$$

where  $N_f$  is the number of flavors and  $N_{EW}$  account for the contributions of  $e^\pm, \mu^\pm, \nu_{e,\mu,\tau}, \bar{\nu}_{e,\mu,\tau}$  together with photons.

Phase equilibrium is achieved when  $p_{QGP} = p_H$  at  $T = T_C$  and thus we find, from Eq.(19) and Eq.(28), that

$$\mathcal{B} = \frac{(N_{QGP} - N_H) \pi^2}{90} T_C^4 \quad (30)$$

Time dependence of energy density in this era can be calculated using Eq.(1) and Eq.(28). It is clear from Eq.(28) that  $p_H = \frac{1}{3}\varepsilon_H$ , hence we have the form

$$-\frac{d\varepsilon_H}{\sqrt{\varepsilon_H}(\varepsilon_H + \frac{1}{3}\varepsilon_H)} = \sqrt{24\pi G} dt \quad (31)$$

Solving this equation we find the evolution of energy density as follows

$$\varepsilon_H(t) = \left[ \frac{1}{\sqrt{\varepsilon_{H_0}}} + \sqrt{\frac{32\pi G}{3}}(t - t_{10}) \right]^{-2} \quad (32)$$

Here  $t_{10}$  is the time at which the hadron era started and  $\varepsilon_{H_0}$  is the initial energy density at the beginning of hadronic era i.e. at  $t_{10}$ . On the other hand, we can obtain the time dependence of temperature in hadron era by substituting the definitions of  $\varepsilon_H$  and  $p_H$  given in Eq.(28) into Eq.(1). We obtain differential equation similar to that one given in Eq.(10) with the only change  $RD \rightarrow H$  and its solution yields

$$\begin{aligned} T_H(t) &= \left[ \frac{1}{T_{H_0}^2} + \sqrt{\frac{16\pi^3 G N_H}{45}}(t - t_{10}) \right]^{-1/2} \\ &= \left[ \frac{1}{T_C^2} + \sqrt{\frac{16\pi^3 G N_H}{45}}(t - t_{10}) \right]^{-1/2} \end{aligned} \quad (33)$$

here  $T_{H_0}$  is the initial temperature corresponding to  $t_{10}$  which is the same temperature  $T_C$  at the end of the mixed era. Following Ref.[5], we define the two quantities  $r$  and  $\lambda$  as

$$\begin{aligned} r &= \frac{N_{QGP}}{N_H} \\ \lambda &= \sqrt{\frac{3}{8\pi G \mathcal{B}}} \end{aligned}$$

As stated in Ref.[5], the quantity  $r$  expresses a number obtained in the pressure equilibrium condition, at  $T = T_C$ , for the QGP and hadron phases while  $\lambda$  is the time scale for the QCD phase transition. In terms of  $r$  and  $\lambda$ , we can obtain a simple expression of the temperature as

$$T_H(t) = T_C \left( 1 + \sqrt{\frac{12}{r-1} \frac{t-t_{10}}{\lambda}} \right)^{-\frac{1}{2}} \quad (34)$$

It should be noted that, in obtaining the above result, we used  $\mathcal{B} = N_H(r-1)\frac{\pi^2}{90}T_C^4$ . Using the expression of  $T_H(t)$  given in Eq.(34) we can obtain the following expressions of the energy density and pressure in the hadron era

$$\begin{aligned} \varepsilon_H(t) &= \frac{\pi^2}{30} N_H T_C^4 \left( 1 + \sqrt{\frac{12}{r-1} \frac{t-t_{10}}{\lambda}} \right)^{-2} \\ p_H(t) &= \frac{\pi^2}{90} N_H T_C^4 \left( 1 + \sqrt{\frac{12}{r-1} \frac{t-t_{10}}{\lambda}} \right)^{-2} \end{aligned} \quad (35)$$

We proceed now to find the equation governing the time evolution of the scale factor in the hadron era. The scale factor, then, can be obtained with the help of the equations of state given in Eq.(28) and Eq.(1). Following the same steps done in the RD era, substituting  $p_H = \frac{1}{3}\varepsilon_H$  and performing the integration we obtain

$$\frac{a(t)}{a(t_{10})} = \frac{T_C}{T_H(t)} \quad (36)$$

using Eq.(34), we finally obtain

$$\frac{a(t)}{a(t_{10})} = \left( 1 + \sqrt{\frac{12}{r-1} \frac{t-t_{10}}{\lambda}} \right)^{\frac{1}{2}} \quad (37)$$

The above result agrees with the corresponding one shown in Ref.[5].

#### D. Mixed era

In the transition from the QGP phase to the hadron phase, the Universe experiences the co-existence of the both phases for a certain time interval. This mixed phase happens when the temperature of the Universe cools down and get close to  $T_C$ . During this time interval, the temperature of the system is fixed at  $T_C$ . This can be understood as the cooling of the

Universe due to its expansion is balanced by the release of the latent heat. In the mixed phase, the energy density can be parameterized as [5]

$$\varepsilon(t) = \varepsilon_H(T_C)f(t) + \varepsilon_{QGP}(T_C)(1 - f(t)) \quad (38)$$

where  $f(t)$  takes the values 0(1) at the start (end) of the co-existence. Regarding the pressure in the mixed era, we find that it can be parameterized in a similar way to the energy density and thus can be written as

$$p(t) = p_H(T_C)f(t) + p_{QGP}(T_C)(1 - f(t)) \quad (39)$$

Using Eqs.(1, 38, 39), see the appendix for detailed derivation, one obtains the following differential equation

$$\frac{df}{dt} = \frac{3}{\lambda} \left( \frac{r}{r-1} - f \right) \sqrt{4(1-f) + \frac{3}{r-1}} \quad (40)$$

The preceding equation has two analytic solutions that can be expressed as

$$f_{\pm}(t) = 1 - \frac{1}{4(r-1)} \left[ \tan^2 \left( \frac{3(t-t_9)}{2\lambda\sqrt{r-1}} \pm \tan^{-1} \sqrt{4r-1} \right) - 3 \right] \quad (41)$$

where  $t_9$  stands for the beginning time of the mixed era. As we will show in the following only  $f_-(t)$  is the acceptable solution and thus we take  $f(t) = f_-(t)$ . Having the expressions of the energy density and pressure in the mixed era, we can now use Eq.(1) to obtain an analytic expression of the scale factor in the mixed era. We find that

$$\frac{\dot{a}(t)}{a(t)} = \frac{\dot{f}}{3\left(\frac{r}{r-1} - f\right)} = \lambda^{-1} \sqrt{4(1-f) + \frac{3}{r-1}} \quad (42)$$

in agreement with Ref.[5]. Details about the derivation of the above equation can be found in the appendix. The equation has an analytic solution that can be obtained upon performing the integration and can be expressed as [5]

$$\frac{a(t)}{a(t_9)} = (4r)^{\frac{1}{3}} \left[ \sin \left( \frac{3(t-t_9)}{2\lambda\sqrt{r-1}} + \sin^{-1} \frac{1}{\sqrt{4r}} \right) \right]^{\frac{2}{3}} \quad (43)$$

### III. NUMERICAL RESULTS AND ANALYSIS

We start our analysis by estimating the approximate times corresponding to the ending of *QGP*, the mixed and the Hadron phases of the early Universe i.e  $t_9$ ,  $t_{10}$  and  $t_{11}$  respectively.

The value of  $t_9$  can be determined from setting  $T = T_C$  in Eq.(23) and solve for  $t_9$ . This leads to

$$\kappa^2(t_9) + \frac{2}{T_C^2}\kappa(t_9) - \frac{N_{QGP}\pi^2}{30\mathcal{B}} = 0 \quad (44)$$

The above equation has two solutions  $\kappa(t_9) = \kappa_1$  and  $\kappa(t_9) = \kappa_2$  where  $\kappa_{1,2} = -\frac{1}{T_C^2} \pm \sqrt{\frac{N_{QGP}\pi^2}{30\mathcal{B}} + \frac{1}{T_C^4}}$ . After solving for  $t$  we get the two solutions

$$t_9 = t_8 - \frac{3}{4\sqrt{6\pi\mathcal{B}G}} \ln\left(\frac{\kappa_{1,2}}{b}\right) \quad (45)$$

After setting  $T_C = 170MeV$ , we find that  $\kappa_2$  is negative. This leads to complex time and so this solution is not accepted. Thus we are left with the other solution  $\kappa_1 \simeq 46.7 GeV^{-2}$  that yields the time corresponds to the end of QGP phase or the beginning of the mixed era as  $t_9 \simeq 11.3 \mu s$ .

The time corresponding to the end of the mixed phase,  $t_{10}$ , can be estimated from solving the equation  $f_{\pm}(t_{10}) = 1$ . Using the expressions of  $f_{\pm}(t)$  given in Eq.(41) and upon setting  $f_{-}(t_{10}) = 1$  we get

$$t_{10} - t_9 = \frac{2\lambda\sqrt{r-1}}{3} \left[ \tan^{-1} \sqrt{(4r-1)} - \tan^{-1} \sqrt{3} \right] \simeq 10.7 \mu s \quad (46)$$

Using the value  $t_9 \simeq 11.3 \mu s$  estimated before we obtain  $t_{10} \simeq 22.0 \mu s$ . It should be noted that setting  $f_{+}(t_{10}) = 1$  one leads to a value of  $t_{10}$  smaller than  $t_9$  which is not acceptable and thus the function  $f(t) = f_{-}(t)$  gives the correct behavior in the mixed era in agreement with the choice of Ref.[5].

The Hadron phase of the universe ends at a time  $t_{11}$  which can be calculated by setting  $T_H(t) = m_{\pi}$ . The reason is attributed to the remark that, for temperatures smaller than this value most of the hadrons undergo either decays or annihilations to final states containing lighter leptons or massless gauge bosons. Moreover, at these temperatures, only small amount of protons and neutrons remain which can be deduced from the ratio  $\frac{n_B}{n_{\gamma}} = 6 \times 10^{-10}$  where  $n_B$  is the net baryon number density and  $n_{\gamma}$  is photon number density. For  $m_{\pi} = 140 GeV$ , we find that the solution of the equation  $T_H(t) = m_{\pi}$  results in  $t_{11} \simeq 31.5 \mu s$ . Having determined all times corresponding to the beginning and the ending of the radiation eras, given in Table I, QGP era, mixed and hadron eras, we are ready now to show our results for the time variation of the thermodynamic and cosmological parameters in all these eras.

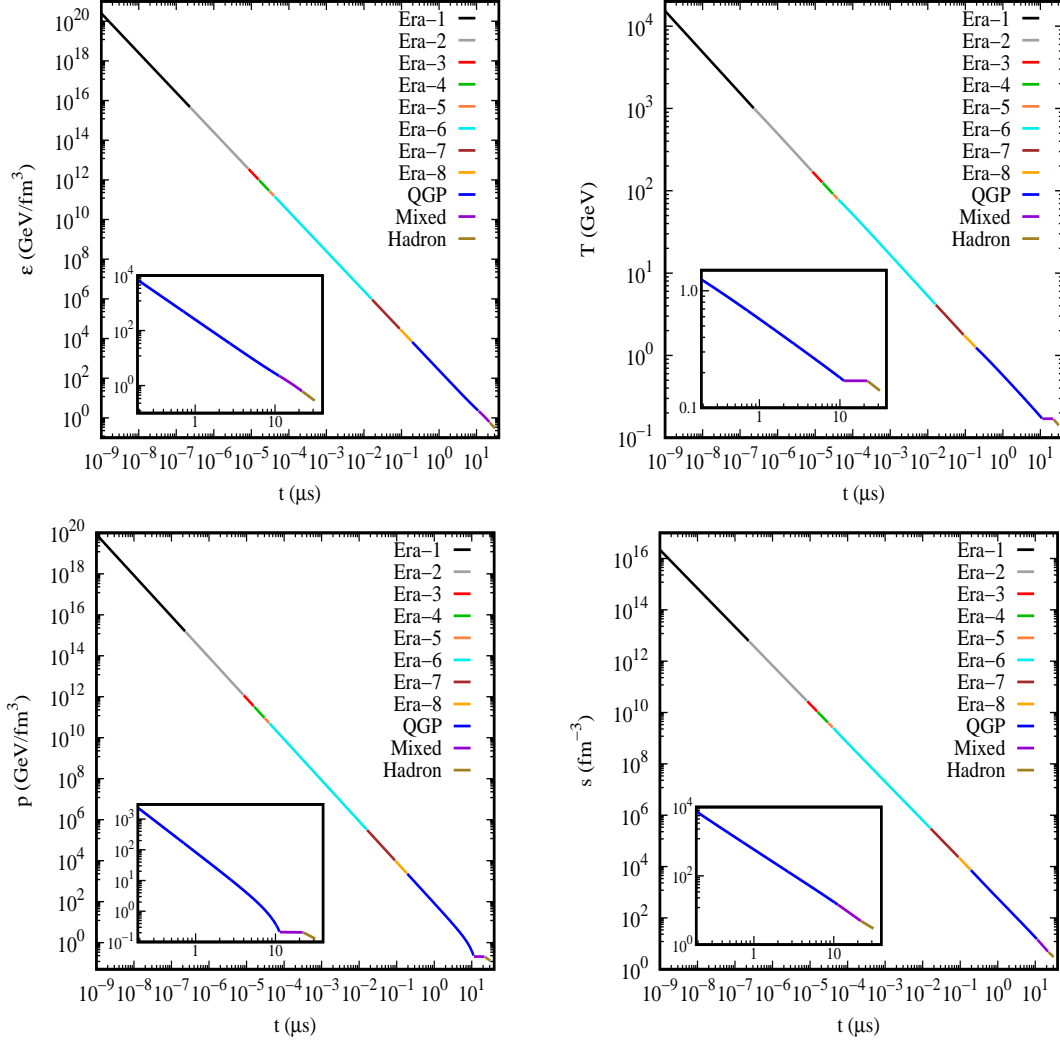


FIG. 1. Time evolution of the energy density, pressure density, entropy density and temperature in the four eras of the Universe.

In Fig(1) we show the evolution of the energy density  $\varepsilon$ , temperature  $T$ , pressure density  $p$  and entropy density  $s$  with time where different colors correspond to the different time intervals in the early eras of the universe. Clearly, in all eras of the early Universe, these thermodynamic parameters decrease with increasing time except at the mixed era where temperature and pressure are constant. The temperature at mixed era is constant and equals to  $T_C$ . This can be explained as stated in Ref.[5] that the release of the latent heat recoups the cooling of the Universe due to the expansion. Regarding the energy density and the pressure in the mixed era, we show in Fig.(2) their time evolution together with the

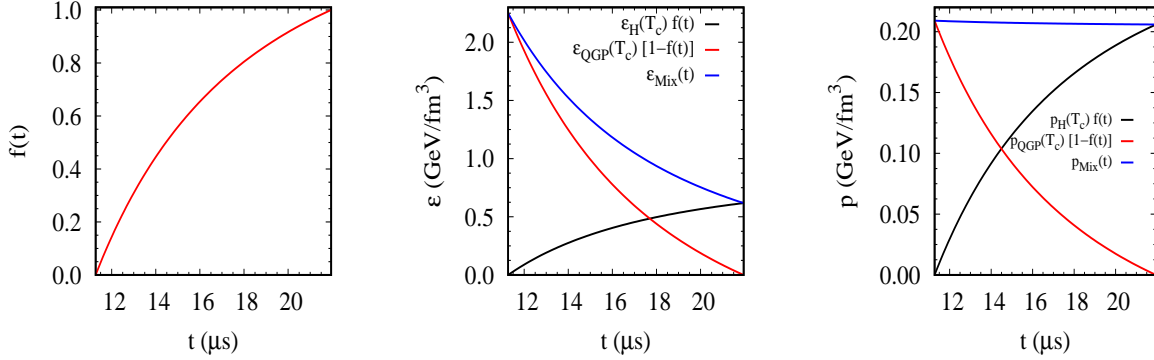


FIG. 2. Time evolution of  $f(t)$ , the energy density and the pressure density the mixed era of the Universe.

the function  $f(t)$ . Clearly, from the figure, the energy density decreases also with time in this era while the pressure is nearly constant with varying the time. In the corresponding plots, the contributions proportional to  $\varepsilon_H(T_C)$  and  $p_H(T_C)$  increase as time runs while those proportional to  $\varepsilon_{QGP}(T_C)$  and  $p_{QGP}(T_C)$  decrease as time runs. This can be attributed to the behavior of  $f(t)$  seen in the left plot in Fig.(2).

In Fig.(3), we show the plots of the time evolution of  $\frac{a(t)}{a(t_i)}$  in the different eras of the early Universe. Here,  $a(t_i)$  stands for the scale factor at the beginning of the concern era of the Universe. Clearly, from Fig.(3), the ratio  $\frac{a(t)}{a(t_i)}$  increase in each era indicating expansion of the Universe. At the end of RD,  $QGP$ , mixed and hadron eras we find that  $\frac{a(t_8)}{a(t_0)} \simeq 1.38 \times 10^4$ ,  $\frac{a(t_9)}{a(t_8)} \simeq 7.85$ ,  $\frac{a(t_{10})}{a(t_9)} \simeq 1.44$  and  $\frac{a(t_{11})}{a(t_{10})} \simeq 1.21$ . In Fig(4), we show the time evolution of the Hubble parameter for the studied eras of the universe where the different colors represent the different eras. Clearly, the Hubble parameter decreases with the increase of the time. Recall that, from Eq.(2), the Hubble parameter is directly proportional to the square root of the energy density. Due to the expansion of the Universe, the volume of the Universe increases. Since the amount of the total energy is constant, it turns that the energy density decreases.

#### IV. CONCLUSION

In this study we derived the analytic expressions governing the time evolution of the thermodynamic and the cosmological parameters in early eras of the Universe namely, the



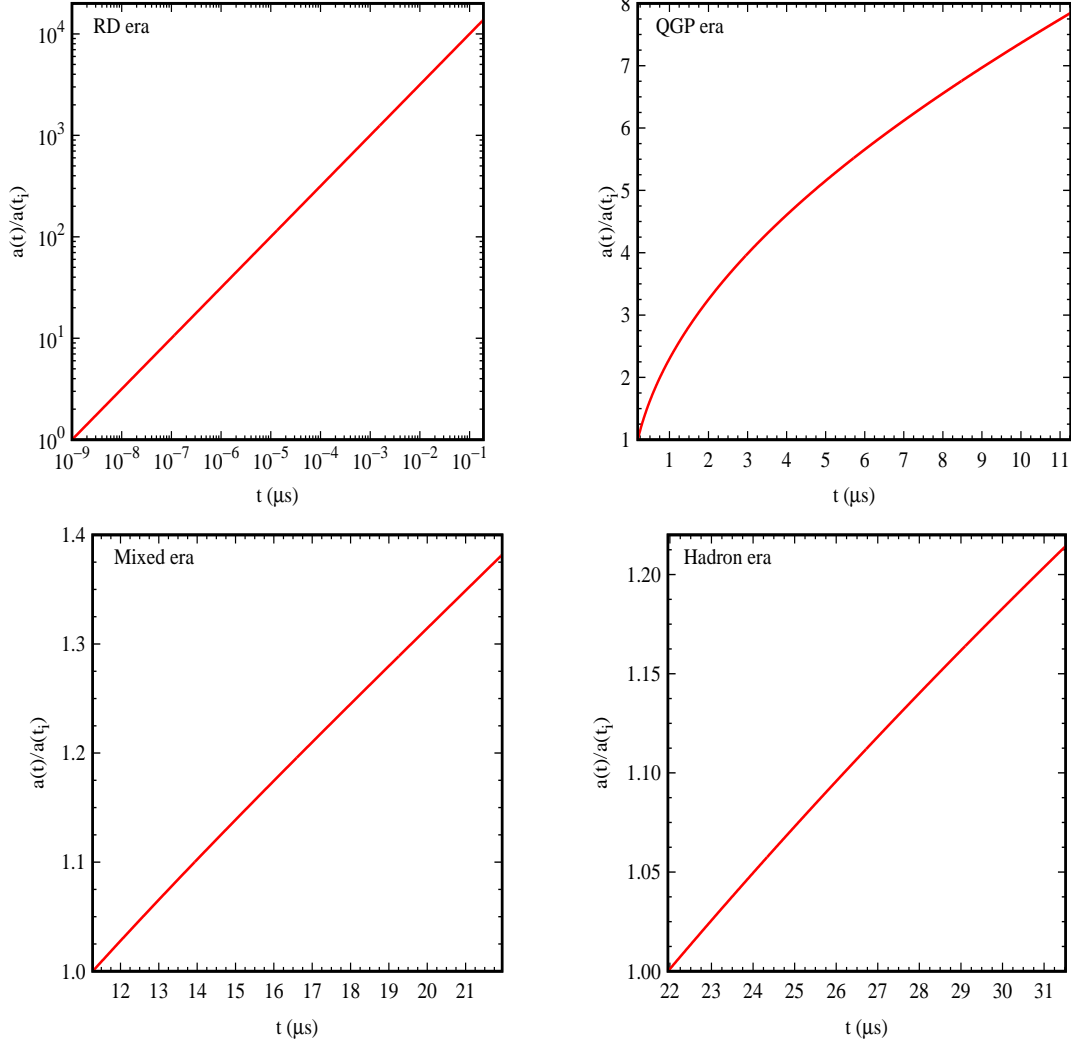


FIG. 3. Time evolution of  $\frac{a(t)}{a(t_i)}$  in the four eras of the Universe where  $i$  stands for the beginning time of the given era.

radiation era, the quark-gluon plasma era and the hadron era. In particular, these parameters include the energy density, the entropy density, the temperature, the pressure in addition to Hubble parameter and the scale factor. The values of the time corresponding to the beginning and ending of these eras were also derived in this work.

Using the aforementioned expressions, we investigated the time variation of the energy density, entropy density, pressure and temperature in all these eras. Moreover, we showed the behaviors of the Hubble parameter and scale factor with the variation of time in the considered eras. In studying the *QGP* era, we adopted simple bag models for the equations

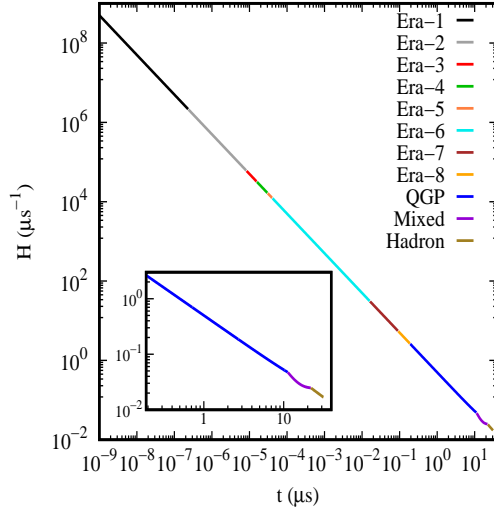


FIG. 4. The time evolution of the Hubble constant in early eras of the Universe.

of state of the thermodynamic parameters based on the MIT bag model. However, adopting other models with complicated equations of state of the thermodynamic parameters can be included in our formalism even for the cases of obtaining numerical solutions rather than analytic solutions for the Friedmann differential equations.

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## APPENDIX

### A. Analytic solutions for the time evolution of the thermodynamic parameters in the QGP era

In this subsection we derive analytic solutions for the time evolution differential equations of the energy density, temperature and pressure.

In order to study the evolution of the energy density with time, in the early Universe, we need to solve the differential equation given in Eq.(1). To do this, firstly, we eliminate the

temperature from Eq.(19) to get:

$$p = \frac{1}{3}(\varepsilon - 4\mathcal{B}) \quad (47)$$

After doing the integration and making some simplifications, the exact analytic solution of the differential equations given in Eq.(1) can be expressed as

$$\varepsilon_{QGP}(t) = \chi(t) + \zeta(t) + \sqrt{\left(\chi(t) + \zeta(t)\right)^2 - \zeta^2(t)} \quad (48)$$

The functions  $\zeta(t)$  and  $\chi(t)$  are defined as

$$\begin{aligned} \zeta(t) &= -(1 + \eta(t))(1 - \eta(t))^{-1}\mathcal{B} \\ \chi(t) &= 2\left(1 - \eta(t)\right)^{-2}\mathcal{B} \end{aligned} \quad (49)$$

where

$$\eta(t) = \exp\left(4\sqrt{\frac{8\pi\mathcal{B}G}{3}}t + \xi\right) \quad (50)$$

the quantity  $\xi$  is defined through

$$\xi = \ln\left(\frac{\sqrt{\varepsilon_{RD}(t_8)} + \sqrt{\mathcal{B}}}{\sqrt{\varepsilon_{RD}(t_8)} - \sqrt{\mathcal{B}}}\right) - 4\sqrt{\frac{8\pi\mathcal{B}G}{3}}t_8 \quad (51)$$

where  $\varepsilon_{RD}(t_8)$  will be equivalent to the initial value of the energy density at the time  $t_8$  of the beginning of the QGP era.

We turn now to derive analytic solution of the temperature for the bag model of QGP discussed above. Firstly, the differential equation governs the time evolution of the temperature, after eliminating the pressure and energy density using the equations of the state in Eq.(19) and upon substituting in Eq.(1), can be written as

$$\frac{dT}{T\sqrt{T^4 + \frac{\mathcal{B}}{N_{QGP}\frac{\pi^2}{30}}}} = -\frac{2}{3}\sqrt{N_{QGP}\frac{\pi^3}{5}}G dt, \quad (52)$$

We can integrate Eq.(52) to get

$$\int_{T_8}^T \frac{dT}{T\sqrt{T^4 + \frac{\mathcal{B}}{N_{QGP}\frac{\pi^2}{30}}}} = -\frac{2}{3}\sqrt{N_{QGP}\frac{\pi^3}{5}}G (t - t_8), \quad (53)$$

with  $t_8 \simeq 1.85 \times 10^{-7} s$  and  $T_8 = T_{RD}(t_8)$  are the initial time and initial temperature at the beginning of the QGP phase respectively. We find that the differential equation above has a solution in the form

$$\ln \left( \frac{T^2}{\frac{\mathcal{B}}{\frac{\pi^2}{30} N_{QGP}} + \sqrt{\frac{\mathcal{B}}{\frac{\pi^2}{30} N_{QGP}} T^4 + \frac{\mathcal{B}^2}{\frac{\pi^4}{900} N_{QGP}^2}}} \right) - \ln \left( \frac{T_8^2}{\frac{\mathcal{B}}{\frac{\pi^2}{30} N_{QGP}} + \sqrt{\frac{\mathcal{B}}{\frac{\pi^2}{30} N_{QGP}} T_8^4 + \frac{\mathcal{B}^2}{\frac{\pi^4}{900} N_{QGP}^2}}} \right) = -\frac{4}{3} \sqrt{6\pi \mathcal{B} G} (t - t_8) \quad (54)$$

which can be expressed as

$$T_{QGP}(t) = \sqrt{\frac{2 \mathcal{B} \kappa(t)}{\left(\frac{\pi^2}{30} N_{QGP} - \mathcal{B} \kappa^2(t)\right)}} \quad (55)$$

where the function  $\kappa(t)$  is given as

$$\kappa(t) = b \exp \left[ -\frac{4}{3} \sqrt{6\pi \mathcal{B} G} (t - t_i) \right] \quad (56)$$

with

$$b = T_8^2 \left( \frac{\mathcal{B}}{\frac{\pi^2}{30} N_{QGP}} + \sqrt{\frac{\mathcal{B}}{\frac{\pi^2}{30} N_{QGP}} T_8^4 + \frac{\mathcal{B}^2}{\frac{\pi^4}{900} N_{QGP}^2}} \right)^{-1} \quad (57)$$

The time evolution of the pressure can be derived using Eqs.(47) as we show in the following. Recall that from Eqs.(47) we have

$$p = \frac{1}{3} (\varepsilon - 4\mathcal{B}) \quad (58)$$

so we get

$$\varepsilon = 3p + 4\mathcal{B} \quad (59)$$

after substituting in the differential equation given in Eq.(1) we get

$$\frac{dp}{(p + \mathcal{B}) \sqrt{3(p + \mathcal{B}) + \mathcal{B}}} = -4 \sqrt{\frac{8\pi G}{3}} dt \quad (60)$$

defining  $y = p + \mathcal{B}$  we obtain

$$\frac{dy}{y \sqrt{3y + \mathcal{B}}} = -4 \sqrt{\frac{8\pi G}{3}} dt \quad (61)$$

After performing the integration we find that

$$\left\{ \ln \left[ \frac{\sqrt{\mathcal{B}} - \sqrt{3p + 4\mathcal{B}}}{\sqrt{\mathcal{B}} + \sqrt{3p + 4\mathcal{B}}} \right] \right\}_{p_8}^p = -4 \sqrt{\frac{8\pi \mathcal{B} G}{3}} (t - t_8) \quad (62)$$

which can be written as

$$\frac{\sqrt{\mathcal{B}} - \sqrt{3p + 4\mathcal{B}}}{\sqrt{\mathcal{B}} + \sqrt{3p + 4\mathcal{B}}} = \frac{\sqrt{\mathcal{B}} - \sqrt{3p_8 + 4\mathcal{B}}}{\sqrt{\mathcal{B}} + \sqrt{3p_8 + 4\mathcal{B}}} \exp \left[ -4 \sqrt{\frac{8\pi\mathcal{B}G}{3}} (t - t_8) \right] \quad (63)$$

defining  $\varrho_8 = \frac{\sqrt{\mathcal{B}} - \sqrt{3p_8 + 4\mathcal{B}}}{\sqrt{\mathcal{B}} + \sqrt{3p_8 + 4\mathcal{B}}}$  and  $k = \exp \left[ -4 \sqrt{\frac{8\pi\mathcal{B}G}{3}} (t - t_8) \right]$  and solve for  $p$  we get

$$p = \frac{-3\varrho_8^2 \mathcal{B} k^2 - 10\varrho_8 \mathcal{B} k - 3\mathcal{B}}{3(\varrho_8 k + 1)^2} \quad (64)$$

The explicit dependency of the pressure on the time is clear as, after substituting  $k$ ,  $p$  have finally the form

$$p = \frac{-3\varrho_8^2 \mathcal{B} \exp \left[ -8 \sqrt{\frac{8\pi\mathcal{B}G}{3}} (t - t_8) \right] - 10\varrho_8 \mathcal{B} \exp \left[ -4 \sqrt{\frac{8\pi\mathcal{B}G}{3}} (t - t_8) \right] - 3\mathcal{B}}{3 \left( \varrho_8 \exp \left[ -4 \sqrt{\frac{8\pi\mathcal{B}G}{3}} (t - t_8) \right] + 1 \right)^2} \quad (65)$$

## B. Derivation of some relations in the Mixed era

As discussed in subsection IID, the energy density and the pressure in the mixed phase can be parameterized as

$$\begin{aligned} \varepsilon_{mix}(t) &= \varepsilon_H(T_C) f(t) + \varepsilon_{QGP}(T_C) (1 - f(t)) \\ p_{mix}(t) &= p_H(T_C) f(t) + p_{QGP}(T_C) (1 - f(t)) \end{aligned} \quad (66)$$

In terms of  $\mathcal{B}$  and  $r$  the the energy density and the pressure in the above equations are given as:

$$\begin{aligned} \varepsilon_{mix} &= \mathcal{B} \left( 1 - 4f + \frac{3r}{r-1} \right) \\ p_{mix} &= \mathcal{B} \left( \frac{r}{r-1} - 1 \right) \\ \frac{d\varepsilon_{mix}}{dt} &= -4\mathcal{B} \frac{df}{dt} \end{aligned} \quad (67)$$

additionally, from Eq.(1), we can write

$$-\frac{d\varepsilon_{mix}/dt}{3\sqrt{\varepsilon_{mix}}(\varepsilon_{mix} + p_{mix})} = \frac{1}{\lambda\sqrt{\mathcal{B}}} \quad (68)$$

Substituting the components of Eq.(67) into Eq.(68) we obtain the below differential equation for  $f(t)$

$$\frac{df}{dt} = \frac{3}{\lambda} \left( \frac{r}{r-1} - f \right) \sqrt{4(1-f) + \frac{3}{r-1}} \quad (69)$$

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