

Thermodynamic consistency of the optomechanical master equationM. Tahir Naseem,¹ André Xuereb,² and Özgür E. Müstecaplıoğlu^{1,*}¹*Department of Physics, Koç University, Sarıyer, İstanbul, 34450, Turkey*²*Department of Physics, University of Malta, Msida MSD 2080, Malta*

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We investigate the thermodynamic consistency of the master equation description of heat transport through an optomechanical system attached to two heat baths, one optical and one mechanical. We employ three different master equations to describe this scenario: (i) The standard master equation used in optomechanics, where each bath acts only on the resonator that it is physically connected to; (ii) the so-called dressed-state master equation, where the mechanical bath acts on the global system; and (iii) what we call the global master equation, where both baths are treated nonlocally and affect both the optical and mechanical subsystems. Our main contribution is to demonstrate that, under certain conditions including when the optomechanical coupling strength is weak, the second law of thermodynamics is violated by the first two of these pictures. In order to have a thermodynamically consistent description of an optomechanical system, therefore, one has to employ a global description of the effect of the baths on the system.

DOI: [10.1103/PhysRevA.98.052123](https://doi.org/10.1103/PhysRevA.98.052123)**I. INTRODUCTION**

The field of optomechanics [1,2] investigates composite systems where an optical resonator is coupled to a mechanical oscillator. A significant portion of the studies in the field focus on its promise for testing fundamental quantum laws using macroscopic objects, constructing probes for tiny forces with quantum-limited sensitivity, and generating nonclassical states [3–18]. In optomechanical systems, light interacts with the mechanical motion of an oscillator, typically a micromirror. This effect can be exploited in numerous ways, including for facilitating long-distance quantum communication [19] and quantum information processing [20,21], for performing quantum state transfer between optical pulses and mechanical excitations [22], for the transfer of energy between electromagnetic oscillators with large frequency differences or storing quantum states in mechanical motion [23], and for converting traveling phonons into photons [24]. This radiation pressure interaction also generates quantum correlations between the light and mechanical motion, which are important for force metrology [25].

Attention has recently been devoted to thermodynamic applications of optomechanical systems [26–34], including proposals for optomechanical quantum heat engines [35] and heat transport through optomechanical arrays [36]. Despite this research, and somewhat surprisingly, it appears that a thermodynamically consistent open system description of the optomechanical interaction that is valid at arbitrary coupling strength is still lacking. We remark, however, that significant progress has been made very recently in deriving master equations for open quantum systems that are consistent with the laws of thermodynamics starting from microscopic collision models [37].

In part because of the limitations of current experimental setups, the typical description of an optomechanical system is restricted in validity to the weak coupling regime. The dynamical behavior in this scenario is typically studied using what we will refer to as the standard master equation (SME), where the heat baths connected to the system are assumed to influence only the system that they are attached to [38]. As the coupling strength grows and the system enters the strong-coupling regime, using the so-called dressed-state master equation (DSME) has been suggested [39]. In effect, this description includes the influence of the mechanical heat bath on the optical resonator but neglects the effect of the optical heat bath on the mechanical oscillator; one can say that the optical reservoir is local whereas the mechanical one is global [40]. Recent proposals [41,42] have suggested ways in which this regime may be rendered accessible, highlighting the need for understanding which description of the dynamics is to be used. Indeed, consistency of these different master equations with the laws of thermodynamics is not guaranteed; it is known, for example, that coupled simple harmonic resonators in certain parameter regimes require a fully global treatment of the reservoirs to ensure thermodynamic consistency [40,43–45].

Our main objective in this paper is to systematically examine the heat transport through an optomechanical system from the point of view the first and second laws of thermodynamics. We highlight the failure of both aforementioned master equations in both weak and strong optomechanical coupling to enforce consistency with the second law of thermodynamics. We find nonvanishing heat flow in the system when the coupled baths are at same temperature, independent of the optomechanical coupling strength; i.e., both these master equations violate the second law of thermodynamics even in the weak optomechanical coupling regime, for certain values of other system parameters. We propose a method based upon a global master equation (GME) to ensure consistency at

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arbitrary optomechanical coupling strength. In particular, we provide evidence showing that consistency with the second law of thermodynamics requires phonon sideband modes to be included in the master equation when the temperature of the mechanical bath is greater than that of the optical bath.

This paper is organized as follows. In Sec. II, we outline our basic model and present the three master equations that are the subject of our study. In Sec. III, we compare the three approaches in terms of their consistency with the second law of thermodynamics. We then sum up briefly and give our conclusions in Sec. IV.

II. THE MODEL

Our model consists of a Fabry-Pérot cavity, one of whose end mirrors is allowed to move; this model is shown schematically in Fig. 1 and is representative of a large class of physically equivalent systems containing a localized electromagnetic field mode interacting with a mechanical oscillator. The optical (mechanical) resonator in our model has frequency ω_c (ω_m) and is attached to a thermal bath at temperature T_c (T_m). Both these baths are independent and can possess any non-negative finite temperature. Since the description of the baths depends on the specific approach followed, as detailed below, we defer this discussion to the forthcoming subsections.

The Hamiltonian governing the evolution of the isolated system consisting of the optical and mechanical modes is (we use units in our Hamiltonians such that $\hbar = 1$ for convenience)

$$\hat{H} = \omega_c \hat{a}^\dagger \hat{a} + \omega_m \hat{b}^\dagger \hat{b} - g \hat{a}^\dagger \hat{a} (\hat{b} + \hat{b}^\dagger). \quad (1)$$

The first two terms in \hat{H} are energies of the optical and mechanical modes, respectively, whereas third term denotes the optomechanical interaction with single-photon coupling strength g . We denote the annihilation (creation) operator of the optical mode by \hat{a} (\hat{a}^\dagger) and of the mechanical mode by \hat{b} (\hat{b}^\dagger).

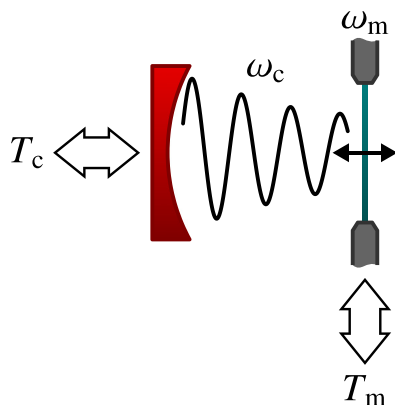


FIG. 1. Schematic illustration of an optomechanical system, consisting of a mechanical resonator with frequency ω_m coupled to an optical resonator with frequency ω_c . In this illustration, an optical cavity has a mobile end mirror (shown on the right) whose motion is assumed to be harmonic. The two resonators are coupled to heat baths, at temperatures T_m and T_c , respectively. We assume that these two baths are independent and can possess any finite non-negative temperature.

A. Three different master equations

We will now proceed to add to \hat{H} the interaction of the two isolated modes with two independent thermal baths. In this subsection, we shall consider three different master equations which may be used to describe the dynamics of the reduced density matrix of the system after the two baths have been traced out.

1. The standard master equation (SME)

Let us first consider the case where each of the two degrees of freedom is coupled to an independent bath mode; these baths can therefore be considered local, in the sense that they interact with the localized field operators \hat{a} and \hat{b} . The Hamiltonian describing the full system is

$$\begin{aligned} \hat{H}_{\text{tot}} = \hat{H} + \sum_{\lambda} [\omega_{c,\lambda} \hat{c}_{c,\lambda}^\dagger \hat{c}_{c,\lambda} + g_{c,\lambda} (\hat{a}^\dagger \hat{c}_{c,\lambda} + \hat{a} \hat{c}_{c,\lambda}^\dagger) \\ + \omega_{m,\lambda} \hat{c}_{m,\lambda}^\dagger \hat{c}_{m,\lambda} + g_{m,\lambda} (\hat{b}^\dagger \hat{c}_{m,\lambda} + \hat{b} \hat{c}_{m,\lambda}^\dagger)]. \quad (2) \end{aligned}$$

The sum in \hat{H}_{tot} runs over the infinite number of bath modes, indexed by λ (which may be regarded as a continuous or discrete index) for both the optical and mechanical baths. The first term in the sum represents the free Hamiltonians of the optical bath modes, $\omega_{c,\lambda}$ being the frequency of the bath mode indexed by λ and \hat{c}_c (\hat{c}_c^\dagger) its annihilation (creation) operator. The second term represents the interaction between these bath modes and the optical resonator, where the interaction with the bath mode indexed by λ is governed by a strength $g_{c,\lambda}$. The last two terms in the sum are analogous to these first two, but describe the mechanical bath modes and their interaction with the mechanical resonator. The optical and mechanical baths are assumed to be at thermal equilibrium at temperatures T_c and T_m , respectively.

The standard way of deriving the master equation starts off by making the Born-Markov approximation, the details of which and whose regime of validity can be found in Refs. [38,46]. In a second step, the weak coupling approximation is made, which finally results in a local master equation without the need to make any secular approximation, i.e.,

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \mathcal{L}_c^{(s)} \hat{\rho} + \mathcal{L}_m^{(s)} \hat{\rho}, \quad (3)$$

where

$$\mathcal{L}_c^{(s)} \hat{\rho} = G_c(\omega_c) D[\hat{a}] \hat{\rho} + G_c(-\omega_c) D[\hat{a}^\dagger] \hat{\rho} \quad \text{and} \quad (4)$$

$$\mathcal{L}_m^{(s)} \hat{\rho} = G_m(\omega_m) D[\hat{b}] \hat{\rho} + G_m(-\omega_m) D[\hat{b}^\dagger] \hat{\rho} \quad (5)$$

are the Liouville superoperators of the optical and mechanical baths, respectively. In these equations,

$$D[\hat{\delta}] \hat{\rho} := \frac{1}{2} (2\hat{\delta} \hat{\rho} \hat{\delta}^\dagger - \hat{\delta}^\dagger \hat{\delta} \hat{\rho} - \hat{\rho} \hat{\delta}^\dagger \hat{\delta}) \quad (6)$$

is the Lindblad dissipator. The spectral density functions $G_x(\omega)$ ($x = c, m$) of the thermal baths are given by

$$G_x(\omega) = \gamma(\omega) [1 + \bar{n}_x(\omega)] \quad \text{and} \quad (7)$$

$$G_x(-\omega) = \gamma(\omega) \bar{n}_x(\omega), \quad (8)$$

where

$$\bar{n}_x(\omega) = \frac{1}{e^{\hbar\omega/(k_B T_x)} - 1} \quad (9)$$

are the Bose-Einstein distributions of the excitations in the baths, with k_B being the Boltzmann constant. The coefficients

$$\gamma_x(\omega) = 2\pi\hbar \frac{f_x(\omega)g_x(\omega)^2}{\omega} \quad (10)$$

are determined by the density of modes of the baths, $f_x(\omega)$, and the interaction strength between the baths and their corresponding resonators, $g_x(\omega)$. In the following, we assume strictly Ohmic baths with a flat density of modes, in which case $\gamma_x(\omega)$ becomes independent of ω and can be denoted by $\gamma_x(\omega) = \kappa_x$.

2. The dressed-state master equation (DSME)

In contrast with the standard master equation, where each bath couples to a local degree of freedom, this is a semiglobal approach. To derive the DSME, the reduced system Hamiltonian is diagonalized by means of a polaron transformation, and the system-bath interaction is described in this new basis. Next, one makes the usual Born-Markov approximation. An assumption is then made whereby the bath attached to the mechanical resonator couples to both degrees of freedom, but where the other bath couples only to the optical resonator. This approximation is valid for sufficiently flat spectral density and $\omega_c \gg \omega_m$, in which case phonon sidebands can be ignored. In the DSME, one further assumes that $\bar{n}_m \gg 1$. A detailed derivation of DSME is presented in Ref. [39] and yields, finally,

$$\frac{d\hat{\rho}}{dt} = -i[\hat{H}, \hat{\rho}] + \mathcal{L}_c^{(d)}\hat{\rho} + \mathcal{L}_m^{(d)}\hat{\rho} + \mathcal{L}_{m,d}^{(d)}\hat{\rho}, \quad (11)$$

where

$$\mathcal{L}_c^{(d)}\hat{\rho} = G_c(\omega_c)D[\hat{a}]\hat{\rho} + G_c(-\omega_c)D[\hat{a}^\dagger]\hat{\rho}, \quad (12)$$

$$\begin{aligned} \mathcal{L}_m^{(d)}\hat{\rho} = & G_m(\omega_m)D[\hat{b} - \alpha\hat{n}_c]\hat{\rho} \\ & + G_m(-\omega_m)D[\hat{b}^\dagger - \alpha\hat{n}_c]\hat{\rho}, \quad \text{and} \end{aligned} \quad (13)$$

$$\mathcal{L}_{m,d}^{(d)}\hat{\rho} = 4(\kappa_m T_m/\omega_m)\alpha^2 D[\hat{n}_c]\hat{\rho}. \quad (14)$$

The first two of these equations describe the dissipation of the optical and mechanical modes, respectively, and the last equation represents dephasing of the optical mode. Furthermore, $\hat{n}_c = \hat{a}^\dagger\hat{a}$ and $\alpha = g/\omega_m$. We note that this master equation reduces identically to the SME in the limit $g \rightarrow 0$.

3. The global master equation (GME)

The derivation in this case is similar to previous case, except that here both baths are treated on an equal (global) footing, and that phonon sidebands are not ignored. In the interaction picture, the coupling between the baths and oscillators is characterized by the interaction Hamiltonians

$$\hat{H}_{c,B} = \hat{a}^\dagger(t)\hat{c}_c(t) + \hat{a}(t)\hat{c}_c^\dagger(t) \quad \text{and} \quad (15a)$$

$$\hat{H}_{m,B} = \hat{b}^\dagger(t)\hat{c}_m(t) + \hat{b}(t)\hat{c}_m^\dagger(t). \quad (15b)$$

Here we have defined $\hat{\delta}(t) = e^{i\hat{H}t}\hat{\delta}e^{-i\hat{H}t}$ and \hat{c}_c (\hat{c}_m) are non-normalized optical (mechanical) bath operators, with $\hat{c}_c = \sum_\lambda g_{c,\lambda}e^{-i\omega_{c,\lambda}t}\hat{c}_{c,\lambda}$ and $\hat{c}_m = \sum_\lambda g_{m,\lambda}e^{-i\omega_{m,\lambda}t}\hat{c}_{m,\lambda}$.

The Hamiltonian of the reduced system, \hat{H} , can be diagonalized using the transformation

$$\hat{S} = e^{-\alpha\hat{a}^\dagger\hat{a}(\hat{b}^\dagger - \hat{b})}, \quad (16)$$

following which the Hamiltonian takes the form

$$\tilde{H} = \omega_c\tilde{a}^\dagger\tilde{a} + \omega_m\tilde{b}^\dagger\tilde{b} - \frac{g^2}{\omega_m}(\tilde{a}^\dagger\tilde{a})^2. \quad (17)$$

The transformed operators then read

$$\tilde{a} = \hat{a}e^{-\alpha(\hat{b}^\dagger - \hat{b})} \quad \text{and} \quad (18)$$

$$\tilde{b} = \hat{b} - \alpha\hat{a}^\dagger\hat{a}. \quad (19)$$

The system operators in Eqs. (15) evaluate to

$$\hat{a}(t) = \tilde{a}e^{-i\omega_c t} \sum_{n=0}^{\infty} \alpha^n (\tilde{b}e^{-i\omega_m t} - \tilde{b}^\dagger e^{i\omega_m t})^n \quad \text{and} \quad (20a)$$

$$\hat{b}(t) = \tilde{b}e^{-i\omega_m t} + \alpha\tilde{a}^\dagger\tilde{a}. \quad (20b)$$

From Eqs. (20), the master equation can be derived by making standard Born-Markov and secular approximations. We emphasize that the system and bath Hamiltonians in Eqs. (15) are the same as those used for the DSME [39] or for the SME. The dissipators associated with each model, however, should be derived carefully. The SME leads to dissipators which are local energy transfer channels between the baths and the corresponding subsystems. The DSME allows for another energy transfer channel from the mechanical bath to the optical subsystem via a nonlocal dissipator, but it still makes use of a local dissipator for the optical subsystem. The differences between these two dissipators emerge after the secular approximation is taken. In what follows, we present a master equation whose dissipators are nonlocal for both the mechanical and optical baths. We shall express them in terms of transformed (dressed) operators, but we emphasize that the dissipators remain nonlocal in terms of the original (bare) operators as well. After the secular approximation is made, one cannot get local dissipators by transformation back to the bare operators.

For simplicity, we consider four sidebands, resulting in the master equation

$$\frac{d\tilde{\rho}}{dt} = \mathcal{L}_c^{(g)}\tilde{\rho} + \mathcal{L}_m^{(g)}\tilde{\rho} + \mathcal{L}_{m,d}^{(g)}\tilde{\rho}, \quad (21)$$

where the dissipative and dephasing terms are given by

$$\begin{aligned} \mathcal{L}_c^{(g)}\tilde{\rho} = & G_c(\omega_c) \left\{ D[\tilde{a}] + \frac{\alpha^4}{4} (D[\tilde{a}\tilde{b}\tilde{b}^\dagger] + D[\tilde{a}\tilde{b}^\dagger\tilde{b}]) \right\} \tilde{\rho} \\ & + G_c(-\omega_c) \left\{ D[\tilde{a}^\dagger] + \frac{\alpha^4}{4} (D[\tilde{a}^\dagger\tilde{b}\tilde{b}^\dagger] + D[\tilde{a}^\dagger\tilde{b}^\dagger\tilde{b}]) \right\} \tilde{\rho} \\ & + \sum_{n=1,2} \frac{\alpha^{2n}}{n^2} \{ G_c(\omega_c + n\omega_m) D[\tilde{a}\tilde{b}^n] \tilde{\rho} \\ & + G_c(-\omega_c - n\omega_m) D[\tilde{a}^\dagger\tilde{b}^n] \tilde{\rho} \\ & + G_c(\omega_c - n\omega_m) D[\tilde{a}\tilde{b}^n] \tilde{\rho} \\ & + G_c(-\omega_c + n\omega_m) D[\tilde{a}^\dagger\tilde{b}^n] \tilde{\rho} \}, \end{aligned} \quad (22)$$

$$\mathcal{L}_m^{(g)} \tilde{\rho} = G_m(\omega_m) D[\tilde{b}] \tilde{\rho} + G_m(-\omega_m) D[\tilde{b}^\dagger] \tilde{\rho}, \quad \text{and} \quad (23)$$

$$\mathcal{L}_{m,d}^{(g)} \tilde{\rho} = G_m(0) \alpha^2 D[\hat{n}_c] \tilde{\rho}. \quad (24)$$

Upon transformation of the dressed operators back to the bare ones, we see that heat flows through both direct and cross-coupled channels between the baths and the subsystems. Inclusion of more sidebands increases the effectiveness of the cross-coupled channels, which allows the system to perform the heat transfer in the direction dictated by the second law.

In principle, all the phonon sidebands should be considered but, as we will show, for consistency with the second law of thermodynamics it is sufficient to consider first few sidebands even for rather strong single-photon coupling. In this work, we will consider up to eight phonon sidebands and refer to Eq. (21) as the global master equation with two and four sidebands as GME2 and GME4, respectively. The dissipators of for six and eight sidebands, giving rise to GME6 and GME8, respectively, are too cumbersome to report here. We note that, as required, in the limit $g \rightarrow 0$, Eq. (21) reduces to SME (3).

B. Entropy production rate and heat current

According to the first law of thermodynamics, the energy of an isolated system is conserved and can be split into heat and work [47]. For a quantum system, the dynamical version of second law of thermodynamics states that the entropy production rate of an isolated system remains non-negative [48]:

$$\xi := \frac{dS}{dt} - \sum_x \frac{\hbar \mathcal{J}_x}{k_B T_x} \geq 0. \quad (25)$$

In this equation, S is the von Neumann entropy, given by $S(\hat{\rho}) = -\text{Tr}(\hat{\rho} \log \hat{\rho})$; in second term \mathcal{J}_x represents the heat flux from the bath, which is given as [48]

$$\mathcal{J}_x = \text{Tr}\{(\mathcal{L}_x^{(\gamma)} \hat{\rho}) \hat{H}\}, \quad (26)$$

where $\mathcal{L}_x^{(\gamma)}$ represents the dissipative terms for the SME ($\gamma = s$), the DSME ($\gamma = d$), or the GME ($\gamma = g$), as the case requires. Noting that $\text{Tr}\{(\mathcal{L}_{m,d}^{(\gamma)} \hat{\rho}) \hat{H}\} = 0$ ($x = d, g$), at steady state we therefore have $\mathcal{J}_c + \mathcal{J}_m = 0$ for all three models, which corresponds to the energy balance dictated by the first law of thermodynamics [48].

In Appendix B, we solve the SME and DSME models, thus yielding the steady-state entropy production rate,

$$\xi^{ss} = \frac{\hbar g \kappa_c}{k_B} \left(\frac{2g\omega_m + \alpha\kappa_m\kappa}{\omega_m^2 + \kappa^2} \right) \bar{n}_c (\bar{n}_c + 1) \left(\frac{1}{T_m} - \frac{1}{T_c} \right), \quad (27)$$

where $\kappa = \kappa_c + \frac{\kappa_m}{2}$ and $\bar{n}_c \equiv \bar{n}_c(\omega_c)$. All the factors in this expression but the last are non-negative, such that the sign of ξ^{ss} is dictated exclusively by the relative magnitude of the non-negative temperatures T_c and T_m . We are not aware of any concise expression for the steady-state entropy production rate predicted by the GME.

III. RESULTS

In this section, we shall make use of the three different master equations developed in the previous section to present a comparative analysis of their consistency with the

first laws of thermodynamics for selected parameters. In our simulations, we use the PYTHON quantum toolbox QuTiP [49] to solve the master equations, and we take parameters relevant to circuit QED optomechanical simulators [50]: $\omega_c = 2\pi \times 10$ GHz, as well as $\omega_m = 2\pi \times 600$ MHz, $\kappa_c = 2\pi \times 200$ MHz, and $\kappa_m = 2\pi \times 50$ MHz. From this point on, all our frequencies will be rescaled by ω_c and thus rendered dimensionless.

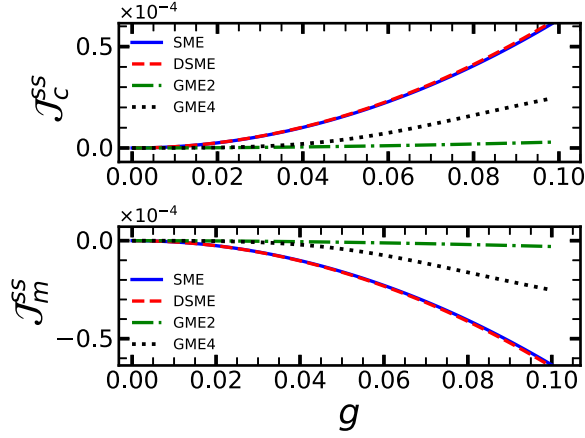
Recall that we are considering the situation when the optical and mechanical resonators are coupled to two distinct thermal baths at temperatures T_c and T_m , respectively. The two baths are independent and can possess any finite non-negative temperature. We shall analyze our models in three different cases: $T_c > T_m$, $T_c = T_m$, and $T_c < T_m$.

A. Mechanics colder than optics, $T_c > T_m$

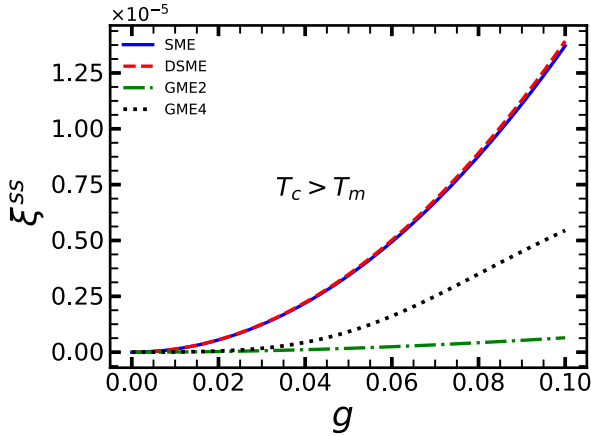
Since the optical and mechanical oscillators are connected to two distinct heat baths, two heat currents, \mathcal{J}_c and \mathcal{J}_m , are present in the system. The heat current equations for the SME and DSME are given in Appendix B, while for the GME we calculate heat currents numerically. Figure 2(a) shows the steady-state heat currents \mathcal{J}_c^{ss} and \mathcal{J}_m^{ss} as a function of the coupling strength g . When the two subsystems are uncoupled, both heat currents are zero as expected. The local (SME, DSME) and global (GME) approaches coincide in the limit $g \rightarrow 0$. This result is in contrast with that in Ref. [45], in which a comparison between local and global master equations is performed for two interacting harmonic oscillators with coupling $(\hat{a}^\dagger \hat{b} + \hat{a} \hat{b}^\dagger)$. In their case, when the two harmonic oscillators are uncoupled and have the same frequency, the SME gives correct (zero) heat currents but the GME yields unphysical (nonzero) heat currents. The divergence of SME and GME in Ref. [45] for the limit $g \rightarrow 0$ is due to failure of secular approximation in this regime. In contrast, in our case the GME reduces identically to the SME in this limit, and we recover consistency with the second law of thermodynamics. However, if $\omega_c = \omega_m$ and the optomechanical coupling strength $g < \kappa_x$ ($x = c, m$) is not very small, then the secular approximation is not well justified and the GME will fail to yield consistent results [45,46]. The GME is also known to fail, due to a failure of the secular approximation, in the case of a one-dimensional spin- $\frac{1}{2}$ Heisenberg chain coupled locally with two independent thermal baths at different temperatures [51]. In this case, it has been shown that heat current vanishes in the presence of a finite-temperature gradient. Since we are considering an optomechanical system in which the parameter regime where secular approximation fails is not accessible, we are justified in using the GME for our dynamical description.

The heat currents \mathcal{J}_c^{ss} and \mathcal{J}_m^{ss} increase as the coupling strength grows. The heat current flows from the hot bath to the cold one, i.e., \mathcal{J}_c^{ss} is positive and \mathcal{J}_m^{ss} is negative. Moreover, both currents are equal at steady state, satisfying the energy conservation requirement. The inclusion of up to four phonon sidebands does not effect the qualitative behavior of the heat currents in this case, but it results in a change of magnitude as compared to the SME and the DSME.

Figure 2(b) shows the entropy production rate, which remains non-negative for all the dynamical equations considered, even in strong coupling regime. Therefore, in the



(a) Steady state heat currents.



(b) Rates of entropy production.

FIG. 2. (a) The steady-state heat currents $\mathcal{J}_c^{\text{ss}}$ and $\mathcal{J}_m^{\text{ss}}$ and (b) the rates of entropy production ξ^{ss} , as a function of the normalized coupling strength g for $T_c > T_m$. The results from the standard master equation (SME) are illustrated by the solid blue curves, those for the dressed-state master equation (DSME) by the red dashed curves, and those for the global master equation (GME) by the green dash-dotted curves (two sidebands) and the black dotted curves (four sidebands). Parameters: $\kappa_c = 0.02$, $\kappa_m = 0.005$, $\omega_c = 1$, $\omega_m = 0.06$, $T_c = 106$ mK, and $T_m = 101$ mK.

case of $T_c > T_m$ all three dynamical equations are thermodynamically consistent for both weak and strong coupling regimes, although they do predict different dynamical behaviors, especially when $g \gtrsim \omega_m$.

It may appear from Fig. 2 that with the addition of more sidebands, the global approach might converge to the local description of the system. While the magnitudes of the heat currents get closer, qualitative differences between the two approaches become more pronounced, as can be seen in Fig. 3, where we consider up to eight sidebands. Such qualitative differences grow dramatically when we consider the cases of equal temperatures for both baths or when the optics is colder than the mechanics.

B. Equal temperatures, $T_c = T_m$

When both baths are at the same temperature, the second law of thermodynamics dictates that the heat currents \mathcal{J}_c and \mathcal{J}_m must stay zero all the time. However, we find that the

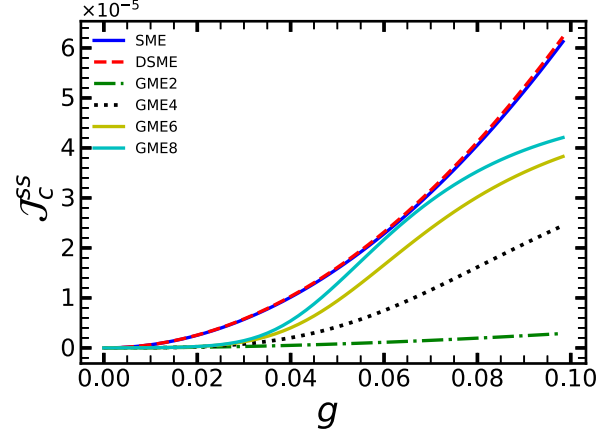


FIG. 3. The steady-state heat current $\mathcal{J}_c^{\text{ss}}$ as a function of the normalized coupling strength g for $T_c > T_m$. From the lowest curve upward, the results shown are those for the global master equation (GME n) with two ($n = 2$), four ($n = 4$), six ($n = 6$), eight ($n = 8$) sidebands, the standard master equation (SME), and for the dressed-state master equation (DSME). Parameters: $\kappa_c = 0.02$, $\kappa_m = 0.005$, $\omega_c = 1$, $\omega_m = 0.06$, $T_c = 106$ mK, and $T_m = 101$ mK.

SME and the DSME predict nonzero values of $\mathcal{J}_c^{\text{ss}}$ and $\mathcal{J}_m^{\text{ss}}$, as shown in Fig. 4; this represents a violation of the second law of thermodynamics as stated above. This sort of violation of second law when baths attached to system are kept at same temperature has been reported for Fermionic transport models [52,53] and two interacting harmonic oscillators [40]. We find that this unphysical result disappears if the system is described by the GME; when including the phonon sideband terms in the master equation, both heat currents \mathcal{J}_c and \mathcal{J}_m become zero. In Ref. [45], it was reported that when the baths are at same temperature the global approach captures an accurate description of the steady state of the system,

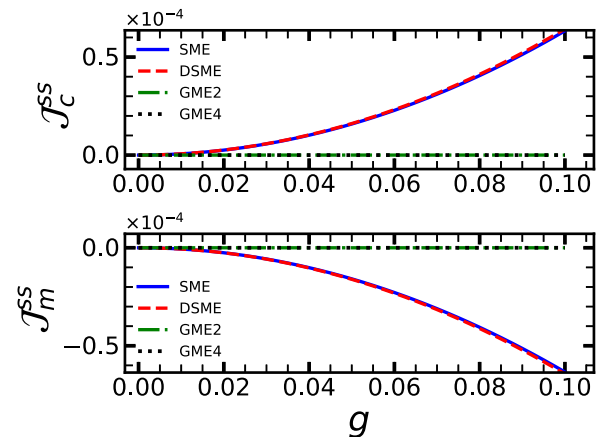


FIG. 4. The steady-state heat currents $\mathcal{J}_c^{\text{ss}}$ and $\mathcal{J}_m^{\text{ss}}$ as a function of normalized coupling strength g for $T_c = T_m$. The results from the standard master equation (SME) are illustrated by the solid blue curves, those for the dressed-state master equation (DSME) by the red dashed curves, and those for the global master equation (GME) by the green dash-dotted curves (two sidebands) and the black dotted curves (four sidebands). Parameters: $T_c = 106$ mK, and $T_m = 106$ mK, with the rest of the parameters as in Fig. 2.

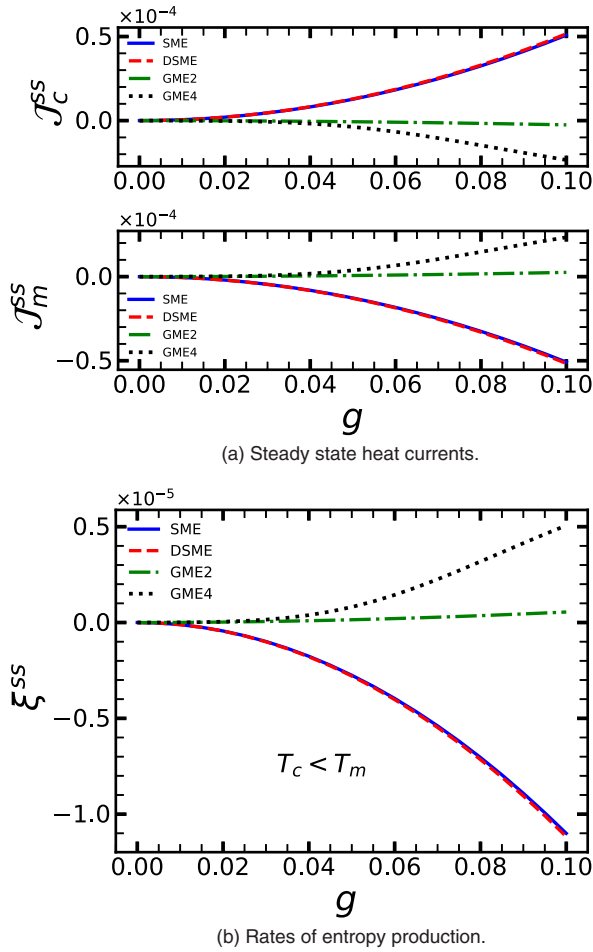


FIG. 5. (a) The steady-state heat currents \mathcal{J}_c^{ss} and \mathcal{J}_m^{ss} , and (b) the rates of entropy production ξ^{ss} as a function of normalized coupling strength g for $T_c < T_m$. The results from the standard master equation (SME) are illustrated by the solid blue curves, those for the dressed-state master equation (DSME) by the red dashed curves, and those for the global master equation (GME) by the green dash-dotted curves (two sidebands) and the black dotted curves (four sidebands). Parameters: $T_c = 101$ mK and $T_m = 106$ mK, with the rest of the parameters as in Fig. 2.

whereas the local approach fails even in weak coupling. We expect, and indeed observe, similar results to hold in our case as well.

C. Optics colder than mechanics, $T_c < T_m$

For the case where $T_c < T_m$, Fig. 5(a) shows the steady-state heat currents \mathcal{J}_c^{ss} and \mathcal{J}_m^{ss} as a function of the normalized coupling strength g . We see that for all values of g , the two heat currents are equal and are therefore consistent with energy conservation. Moreover, for $g = 0$ there is no heat flow into or out of the system, as required, and as g increases the heat currents increase in magnitude. According to the Clausius statement of second law, heat must flow from the hot to the cold bath. Figure 5(a), however, shows that if the dynamics of the system is described by the SME or the DSME, heat flows from the cold to the hot bath, independent of the optomechanical coupling strength. It is

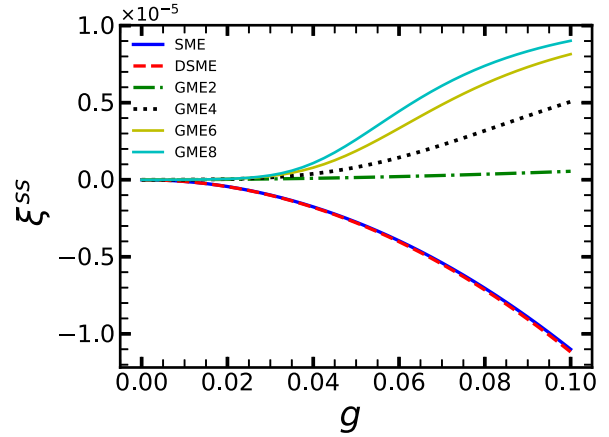


FIG. 6. The rates of entropy production ξ^{ss} as a function of normalized coupling strength g . From the lowest curve upward, the results shown are those for the dressed-state master equation (DSME), the standard master equation (SME), and for the global master equation (GME n) with two ($n = 2$), four ($n = 4$), six ($n = 6$), and eight ($n = 8$) sidebands. All parameters are as in Fig. 5.

only when we include the phonon side modes in the master equation that the direction of the heat current becomes correct and the violation of the second law of thermodynamics disappears.

The entropy production rate is plotted as a function of the coupling strength g in Fig. 5(b). Since the entropy production must be non-negative, this figure yet again demonstrates that the local (SME) and the semiglobal (DSME) approaches are inconsistent with second law of thermodynamics, even in the weak coupling regime. Inclusion of phonon sidebands, as per the GME, recovers consistency with the second law of thermodynamics in both weak and strong coupling regimes. Further investigation reveals that the source of this inconsistency is the assumption of an unphysical flat bath spectrum and the presence of nonsecular terms, both of which are relaxed when deriving the GME. Indeed, if we reduce the number of sidebands or consider a flat response function for the baths, then even the dynamical description of the system by the GME results in a violation of the second law of thermodynamics.

To demonstrate the robustness of our investigation and the convergence of the GME as the number of sidebands is increased, we plot in Fig. 6 the rates of entropy production for a number of different situations. Convergence of the GME requires an increasing number of sidebands as g increases, but the higher sidebands do not contribute significantly in the regime where $g \ll \omega_m$. In any case, the inclusion of more sidebands does not change the qualitative behavior of the heat currents or the steady-state entropy production.

Similarly to Fig. 3, we can see from Fig. 5(a) that the magnitude of the heat current in the global approach gets closer to that of the local approach when we consider more phonon sidebands. However, their behaviors are qualitatively different when the optics is colder than the mechanics. In particular, since the signs of the heat currents differ, these qualitative differences may in fact increase when more sidebands are taken into account.

IV. CONCLUSIONS

We have studied a generic optomechanical system coupled to two heat baths, one optical and one mechanical. Our focal point was the consistency of its dynamics with second law of thermodynamics, with respect to local and nonlocal approaches to obtaining the master equation describing this dynamics. When the two baths attached to the system are kept at the same temperature, a nonzero heat current persists in steady state in the standard and dressed-state pictures, which violates the second law of thermodynamics. On the other hand, when the system is described by means of a nonlocal dynamical equation where each heat bath acquires a global character, the heat currents vanish. When the mechanical bath is held at a higher temperature, we have also seen that the steady-state heat current flows from the cold to the hot bath and that there is a negative entropy production rate in steady state, under the first two descriptions. These violations are present in both the weak and strong coupling regimes that are typically investigated using these descriptions. In order to obtain a thermodynamically consistent and physically plausible result, we accounted for the phonon sidebands in the master equation while taking into account the frequency dependence of the bath occupation number. This corrects the direction of the heat flow and restores consistency with the second law of thermodynamics.

The fact that dramatic differences between the local and global approaches emerge even in the weak coupling regime means that our results are testable in current cavity optomechanical systems. Further differences can be tested by using systems that can simulate strong optomechanical-like coupling, such as superconducting transmission line resonators [50].

In closing, we note that the implications of our results are rather broad. Regardless of the regime in which one operates, thermodynamic quantities such as the rate of entropy production deduced from the standard or dressed-state master equations differ quantitatively, if not qualitatively, from those deduced from the global master equation. Because of the consistency of the latter with the second law of thermodynamics, we are drawn to the conclusion that when the mechanical bath is hotter than the optical bath, and especially outside the weak-coupling regime, any thermodynamic prediction for optomechanical systems should be based on the global master equation.

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APPENDIX A: DYNAMICS OF THE SYSTEM USING THE SME OR THE DSME

The equations of motions for the relevant dynamical observables of our system are determined from the DSME, Eq. (11). The equations of motion read

$$\begin{aligned}
\frac{d}{dt}\langle\hat{n}_c\rangle &= \kappa_c(\bar{n}_c - \langle\hat{n}_c\rangle), \\
\frac{d}{dt}\langle\hat{p}\rangle &= -\omega_m\langle\hat{q}\rangle - \frac{\kappa_m}{2}\langle\hat{p}\rangle + 2g\langle\hat{n}_c\rangle, \\
\frac{d}{dt}\langle\hat{q}\rangle &= \omega_m\langle\hat{p}\rangle - \frac{\kappa_m}{2}\langle\hat{q}\rangle + \kappa_m\alpha\langle\hat{n}_c\rangle, \\
\frac{d}{dt}\langle(\Delta\hat{n}_c)^2\rangle &= \kappa_c\bar{n}_c + (2\kappa_c\bar{n}_c + \kappa_c)\langle\hat{n}_c\rangle - 2\kappa_c\langle(\Delta\hat{n}_c)^2\rangle, \\
\frac{d}{dt}\langle\hat{n}_c, \hat{p}\rangle &= -\kappa\langle\hat{n}_c, \hat{p}\rangle - \omega_m\langle\hat{n}_c, \hat{q}\rangle + 2g\langle(\Delta\hat{n}_c)^2\rangle, \\
\frac{d}{dt}\langle\hat{n}_c, \hat{q}\rangle &= -\kappa\langle\hat{n}_c, \hat{q}\rangle + \omega_m\langle\hat{n}_c, \hat{p}\rangle + \kappa_m\alpha\langle(\Delta\hat{n}_c)^2\rangle, \\
\frac{d}{dt}\langle\hat{n}_m\rangle &= -\kappa_m(\langle\hat{n}_m\rangle - \bar{n}_m) + g(\langle\hat{n}_c, \hat{p}\rangle + \langle\hat{n}_c\rangle\langle\hat{p}\rangle) \\
&\quad + \frac{\kappa_m}{2}\alpha(\langle\hat{n}_c, \hat{q}\rangle + \langle\hat{n}_c\rangle\langle\hat{q}\rangle), \\
\frac{d}{dt}\langle\hat{n}_c\hat{p}\rangle &= -\kappa\langle\hat{n}_c\hat{p}\rangle - \omega_m\langle\hat{n}_c\hat{q}\rangle + 2g\langle\hat{n}_c^2\rangle + \kappa_c\bar{n}_c\langle\hat{p}\rangle, \\
\frac{d}{dt}\langle\hat{n}_c\hat{q}\rangle &= -\kappa\langle\hat{n}_c\hat{q}\rangle + \alpha\kappa_m\langle\hat{n}_c^2\rangle + \kappa_c\bar{n}_c\langle\hat{q}\rangle \\
&\quad + \omega_m\langle\hat{n}_c\hat{p}\rangle, \quad \text{and} \\
\frac{d}{dt}\langle\hat{n}_c^2\rangle &= \kappa_c\bar{n}_c - 2\kappa_c\langle\hat{n}_c^2\rangle + \kappa_c(4\bar{n}_c + 1)\langle\hat{n}_c\rangle,
\end{aligned}$$

where $\alpha = g/\omega_m$, $\hat{q} = \hat{b} + \hat{b}^\dagger$, $\hat{p} = i(\hat{b} - \hat{b}^\dagger)$, and $\kappa = \kappa_c + \kappa_m/2$. The correlation functions between any two operators \hat{o}_1 and \hat{o}_2 are denoted by $\langle\hat{o}_1, \hat{o}_2\rangle := \langle\hat{o}_1\hat{o}_2\rangle - \langle\hat{o}_1\rangle\langle\hat{o}_2\rangle$. The steady-state solutions of these dynamical equations in the long time limit are as follows:

$$\begin{aligned}
\langle\hat{n}_c\rangle^{\text{ss}} &= \bar{n}_c, \\
\langle\hat{q}\rangle^{\text{ss}} &= \left(\frac{8g\omega_m + 2\alpha\kappa_m^2}{4\omega_m^2 + \kappa_m^2}\right)\langle\hat{n}_c\rangle^{\text{ss}}, \\
\langle\hat{p}\rangle^{\text{ss}} &= \left(\frac{4g\kappa_m - 4\alpha\omega_m\kappa_m}{4\omega_m^2 + \kappa_m^2}\right)\langle\hat{n}_c\rangle^{\text{ss}}, \\
\langle(\Delta\hat{n}_c)^2\rangle^{\text{ss}} &= \bar{n}_c(\bar{n}_c + 1), \\
\langle\hat{n}_c, \hat{p}\rangle^{\text{ss}} &= \frac{2g\kappa_c}{\omega_m^2 + \kappa^2}\langle(\Delta\hat{n}_c)^2\rangle^{\text{ss}}, \\
\langle\hat{n}_c, \hat{q}\rangle^{\text{ss}} &= \left(\frac{2g\omega_m + \alpha\kappa_m\kappa}{\omega_m^2 + \kappa^2}\right)\langle(\Delta\hat{n}_c)^2\rangle^{\text{ss}}, \\
\langle\hat{n}_m\rangle^{\text{ss}} &= \bar{n}_m + \frac{g}{\kappa_m}(\langle\hat{n}_c, \hat{p}\rangle^{\text{ss}} + \langle\hat{n}_c\rangle^{\text{ss}}\langle\hat{p}\rangle^{\text{ss}}) \\
&\quad + \frac{\alpha}{2}(\langle\hat{n}_c, \hat{q}\rangle^{\text{ss}} + \langle\hat{n}_c\rangle^{\text{ss}}\langle\hat{q}\rangle^{\text{ss}}), \\
\langle\hat{n}_c\hat{q}\rangle^{\text{ss}} &= A[\kappa_c\bar{n}_c(\kappa\langle\hat{q}\rangle^{\text{ss}} + \omega_m\langle\hat{p}\rangle^{\text{ss}}) \\
&\quad + (\kappa\alpha\kappa_m + 2g\omega_m)\langle\hat{n}_c^2\rangle^{\text{ss}}],
\end{aligned}$$

$$\begin{aligned} \langle \hat{n}_c \hat{p} \rangle^{ss} &= A [\kappa_c \bar{n}_c (\kappa \langle p \rangle^{ss} - \omega_m \langle q \rangle^{ss}) \\ &\quad + (2\kappa g - \alpha \kappa_m \omega_m) \langle \hat{n}_c^2 \rangle], \quad \text{and} \\ \langle \hat{n}_c^2 \rangle^{ss} &= \bar{n}_c (2\bar{n}_c + 1), \end{aligned}$$

where $A = 1/(\kappa^2 + \omega_m^2)$ and $\bar{n}_c \equiv \bar{n}_c(\omega_c)$. The corresponding equations for the SME are found by setting $\alpha = 0$.

APPENDIX B: HEAT CURRENTS USING THE SME OR THE DSME

The heat currents to the optical and mechanical baths to which the optomechanical system is attached are,

respectively,

$$\begin{aligned} \mathcal{J}_c &= \kappa_c (\omega_c - g \langle \hat{q} \rangle) (\bar{n}_c - \langle \hat{n}_c \rangle) + g \kappa_c \langle \hat{n}_c, \hat{q} \rangle, \quad \text{and} \\ \mathcal{J}_m &= \omega_m \kappa_m (\bar{n}_m - \langle \hat{n}_m \rangle) + g \kappa_m (\langle \hat{n}_c, \hat{q} \rangle + \langle \hat{n}_c \rangle \langle \hat{q} \rangle) \\ &\quad - g \alpha \kappa_m (\langle (\Delta \hat{n}_c)^2 \rangle + \langle \hat{n}_c \rangle^2). \end{aligned}$$

At steady state, the heat currents are therefore given by

$$\begin{aligned} \mathcal{J}_c^{ss} &= g \kappa_c \langle \hat{n}_c, \hat{q} \rangle^{ss} \quad \text{and} \\ \mathcal{J}_m^{ss} &= -g \kappa_c \langle \hat{n}_c, \hat{q} \rangle^{ss}. \end{aligned}$$

The results for the SME are once again found by setting $\alpha = 0$.

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