Thermodynamics of the QCD Plasma and the Large-N Limit

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The equilibrium thermodynamic properties of the SU(N) plasma are studied nonperturbatively in the large-N limit, via high-precision lattice simulations at temperatures from $0.8T_c$ to $3.4T_c$ (T_c being the critical deconfinement temperature). Our results for SU(N) Yang-Mills theories with $N = 3$, 4, 5, 6, and 8 colors show a very mild dependence on N, supporting the idea that the QCD plasma could be described by models based on the large-N limit. We compare our data with various theoretical descriptions, including, in particular, the improved holographic QCD model proposed by Kiritsis and collaborators. We also comment on the relevance of an AdS/CFT description in a phenomenologically interesting temperature range where the system, while still strongly coupled, becomes approximately scale invariant. Finally, we extrapolate our results to the $N \to \infty$ limit.

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Introduction.—The high-energy heavy-ion collisions at SPS and RHIC showed evidence for "a new state of matter" [1], which reaches rapid thermalization, and is characterized by very low viscosity values, making it a nearly ideal fluid [2]. On the theoretical side, however, the understanding of such strongly interacting quark-gluon plasma is still an open issue [3], and investigation from the first principles of QCD largely relies on the numerical approach on the lattice [4]. Although several unquenched studies of finite-temperature QCD have appeared in recent years, the thermodynamics of the pure-gauge sector is still relevant from a fundamental perspective, as it captures the essential qualitative features of the deconfinement phenomenon, is characterized by a well-defined theoretical setup, is computationally less demanding and not hindered by the technical difficulties of simulations with dynamical fermions. Finally, it is relevant for the large-N limit, in which the theory undergoes dramatic analytical simplifications [5] and entails connections with string theory, playing a crucial rôle in the conjectured AdS/CFT correspondence [6] and in models of strongly interacting gauge theories based on gravity duals [7]. All such models aiming at a description of $N = 3$ QCD [or Yang-Mills (YM) theory] in terms of predictions based on the large-N limit implicitly rely on the assumption that the features of the $N = 3$ theory are "close enough" to those of its $N = \infty$ counterpart. While *a priori* this assumption is not guaranteed to be true, there is strong numerical evidence that this is indeed the case [8]. The aim of the work reported here was to study nonperturbatively the equilibrium thermodynamic properties at finite temperature in SU(N) YM theories with $N = 3$, 4, 5, 6, and 8 colors, via lattice simulations; similar studies include Refs. [9]. In particular, here we compare the SU(N) lattice results with the improved holographic QCD (IHQCD) model recently proposed by Kiritsis and collaborators [10]; we also discuss the deficit of the entropy density s with respect to its Stefan-Boltzmann (SB) limit s_0 in a temperature regime where the Yang-Mills plasma, while still strongly interacting, approaches a "quasiconformal" regime characterized by approximate scale invariance (and a related AdS/CFT prediction for the large-N limit of the $\mathcal{N} = 4$ supersymmetric YM theory [11]). Next, we also investigate the possibility that the trace anomaly of the deconfined SU(N) plasma Δ receives contributions proportional to T^2, possibly related to a dimension-two condensate [12]. Finally, we present an extrapolation of our results for the pressure p, trace anomaly Δ, energy density ϵ and entropy density s to the $N \to \infty$ limit.

Lattice formulation.—We simulated SU(N) YM theories with $N = 3$, 4, 5, 6, and 8 colors, regularized on a four-dimensional Euclidean hypercubic, isotropic lattice of spacing a, spacelike volume $V = (aN_s)^3$ at temperature $T = 1/(aN_t)$. The system dynamics was given by the Wilson gauge action [13] and the Markov chains were generated combining heat-bath steps for SU(2) subgroups [14] with full-SU(N) overrelaxation updates [15]. The physical scale for the SU(3) gauge group was set using the r_0 values taken from Ref. [16]. For the other SU($N > 3$) groups, we set the scale interpolating high-precision measurements of the string tension σ [17] or (at the largest β values only) using the method discussed in Ref. [18]. The trace anomaly $\Delta = \epsilon - 3p$ was measured from differences of plaquette expectation values at $T = 0$ (from a lattice of size N_s^4) and at finite T:

$$\Delta = T^5 \frac{\partial}{\partial T} \frac{p}{T^4} = \frac{6}{a^4} \frac{\partial \beta}{\partial \ln a} (\langle U_\square \rangle_0 - \langle U_\square \rangle_T). \quad (1)$$

The pressure (relative to its $T = 0$ vacuum value) was determined using the "integral method" [19]:

$$p = \frac{6}{a^4} \int_{\beta_0}^{\beta} d\beta' (\langle U_\square \rangle_T - \langle U_\square \rangle_0), \quad (2)$$

while the energy and entropy densities were obtained from $\epsilon = \Delta + 3p$ and $s = (\Delta + 4p)/T$, respectively.

Results.—Our results for Δ/T^4, p/T^4, ϵ/T^4, and s/T^3, normalized to the respective SB limits [20] are displayed in Fig. 1, and reveal a very weak dependence on the group rank. It is thus natural to compare these data with the analogous curves obtained in the IHQCD model [10], which is a holographic AdS/QCD model based on an Einstein-dilaton gravity theory in five dimensions. This model involves a dilaton potential which reproduces asymptotic freedom with a logarithmically running coupling in the ultraviolet (UV) and linear confinement with a discrete mass gap in the infrared (IR) limit of the dual SU(N) gauge theory, capturing most of its nonperturbative features (both at zero and finite temperature) at a quantitative level. Strictly speaking, the IHQCD model is expected to hold in the large-N limit only, while at finite N one expects corrections; in particular, the calculations in the gravity model neglect string interactions, that are expected to become important above a cutoff scale—which, for the parameters used in Refs. [10], for SU(3) would be approximately equal to 2.5 GeV. Nevertheless, Fig. 1 shows that the agreement between our SU(N) results and the IHQCD model is very good for all the groups. Note that in Refs. [10], by looking at the comparison with the $N = 3$ lattice results for Δ/T^4 from Ref. [21], it was pointed out that the slight discrepancy between the improved holographic QCD model and the lattice results in the region of the peak [which for the SU(3) lattice data is located at $T \simeq 1.1T_c$, and is slightly lower than the IHQCD curve] was likely to be a finite-volume lattice artifact. The first panel in Fig. 1 indeed confirms this: our results for the SU(3) gauge group are consistent with Ref. [21], while the Δ/T^4 maximum for the $N > 3$ gauge groups is larger and located closer to T_c. This is related to the fact that the deconfining phase transition, which is a weakly first-order one for SU(3), becomes stronger when N is increased [22]—and correspondingly the value of the correlation length at the critical point gets shorter. While the IHQCD model accounts for the running of the coupling, comparing SU(N) results with predictions directly derived from conformal models, such as $\mathcal{N} = 4$ SYM theory, is less straightforward: the left panel of Fig. 2 shows that the regime where the QCD plasma is strongly coupled is far from conformality, and the SU(N) plasma approaches a regime characterized by approximate scale invariance only at temperatures about $3T_c$. Note that, around that temperature, the plasma is still far from the Stefan-Boltzmann limit (top right corner of the diagram), still strongly interacting, and, interestingly, the entropy density deficit with respect to the free limit is very close to the AdS/CFT prediction for the large-N limit of $\mathcal{N} = 4$ supersymmetric YM in the strongly interacting regime [11]:

$$\frac{s}{s_0} = \frac{3}{4} + \frac{45}{32}\zeta(3)(2\lambda)^{-3/2} + \dots \qquad (3)$$

if one identifies the value of the renormalized 't Hooft coupling in the $\overline{\text{MS}}$ scheme with the λ parameter of the supersymmetric model (right panel of Fig. 2) [24]. Another issue we investigated is the possibility that, in the temperature range between T_c and $3T_c$, the trace anomaly may receive nonperturbative contributions proportional to T^2 [12]:

$$\frac{\Delta}{T^4} \simeq \frac{A}{T^2} + B. \qquad (4)$$

From our results, this indeed seems to be a general feature of all the gauge groups studied in this work, as shown in the left-hand side panel of Fig. 3 [25]. Finally, the right-hand side panel of Fig. 3 shows an extrapolation of our results to the $N \to \infty$ limit. This is based on the following parametrization for the trace anomaly [26]:

$$\frac{\Delta}{zT^4} = \left(1 - \frac{1}{\{1 + \exp[\frac{(T/T_c)-f_1}{f_2}]\}^2}\right)\left(\frac{f_3 T_c^2}{T^2} + \frac{f_4 T_c^4}{T^4}\right), \qquad (5)$$

with $z = (N^2 - 1)\pi^2/45$. The results for Δ from the various gauge groups are fitted to Eq. (5), and the resulting parameters are extrapolated to the large-N limit, as functions of N^{-2} [27]. The extrapolated parameters are displayed in Table I, where the first error is statistical and the second is an estimate of the overall systematic uncertainties involved in our calculation (which include discretization effects, finite-volume effects, systematic uncertainties in setting the scale, and in the extrapolation to the large-N limit). Note that $f_2 \to 0$ for $N \to \infty$, which is consistent with the strongly first-order nature of the deconfining

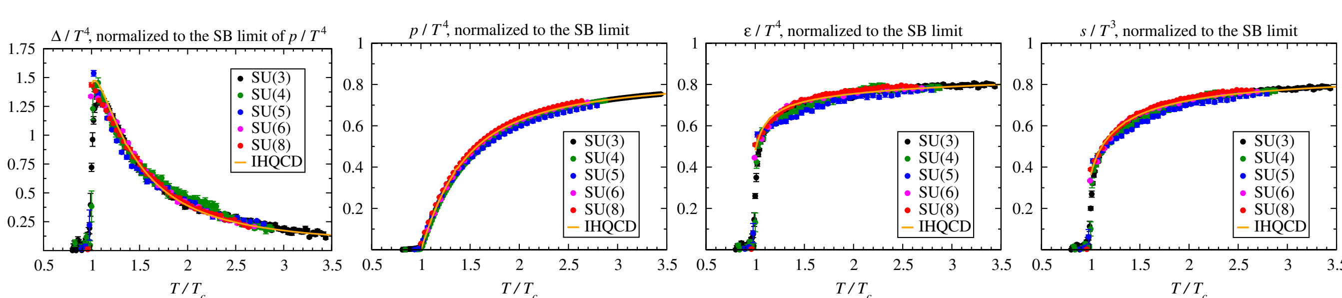

FIG. 1 (color online). Results for the trace anomaly, pressure, energy density, and entropy density evaluated in the various SU(N) gauge groups. All quantities are normalized to their Stefan-Boltzmann (SB) limits (except for Δ/T^4, which is normalized to the SB limit of p/T^4). The results are also compared with the curves obtained using the IHQCD model [10] (yellow solid lines).

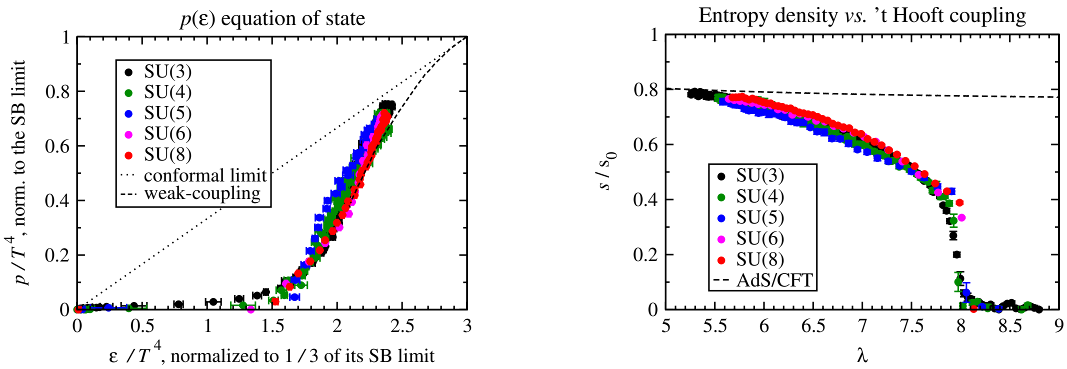

FIG. 2 (color online). Left panel: the equation of state of the SU(N) plasma, displayed as $p(\epsilon)$, reveals strong deviations from the dotted straight line corresponding to conformally invariant models, which is approached only at temperatures around $3T_c$; the dashed line is the weak-coupling expansion for SU(3) [23]. Right panel: the entropy density (normalized to its value in the free limit) as a function of the running 't Hooft coupling in the $\overline{\text{MS}}$ scheme; the dashed line is the corresponding prediction (at the first perturbative order around the strong-coupling limit) in $\mathcal{N} = 4$ SYM [11], obtained identifying the λ parameter with the renormalized YM coupling.

transition in the large-N limit and with the expectation that in this limit the equilibrium thermodynamic quantities considered here vanish for $T < T_c$ (note the N^{-2} normalization factor). From the parameters thus extrapolated, we obtain the curves for the large-N limit of $\Delta/(N^2T^4)$, of $p/(N^2T^4)$ (by numerical integration of the latter over $\ln T$), and of $\epsilon/(N^2T^4)$ and $s/(N^2T^3)$ (through linear combinations of the former two quantities): they are shown in the right panel of Fig. 3, where we also show the large-N limit of the latent heat L_h calculated in Ref. [28], namely: $L_h^{1/4}N^{-1/2}T_c^{-1} = 0.766(40)$. Our estimate for the same quantity is fully consistent: $L_h^{1/4}N^{-1/2}T_c^{-1} = 0.759(19)$.

Conclusions.—The high-precision lattice study of finite-temperature SU(N) gauge theories with $N = 3, 4, 5, 6$, and 8 colors presented here, reveals that the main equilibrium thermodynamic observables (per gluon) have a very weak N dependence—except for a tendency towards a more strongly first-order deconfining transition when N is increased, as it is expected on general grounds [22]—and that the SU(3) results are close to the $N \to \infty$ limit. This is important for AdS/CFT or holographic QCD models, which aim at providing an analytical description of the strongly interacting QCD plasma produced in relativistic collisions of heavy ions, using techniques based on the mathematical simplifications occurring in the large-N limit. We compared our results with the IHQCD model [10], finding very good agreement, and with predictions for the entropy density deficit from $\mathcal{N} = 4$ supersymmetric YM using the AdS/CFT correspondence, in particular, in a temperature regime where the SU(N) plasma gets nearly scale invariant, while still strongly interacting. Our data also show that the T^2 behavior observed in the SU(3) trace anomaly [12] appears to be a generic feature of all theories

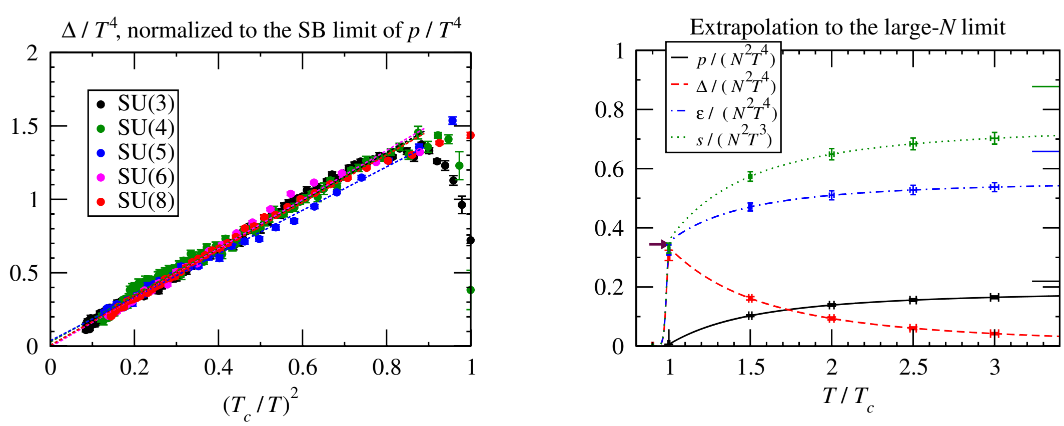

FIG. 3 (color online). Left panel: the Δ/T^4 ratio, plotted against $(T_c/T)^2$, is compatible with the behavior described by Eq. (4) in the range $(T_c/T)^2 \leq 0.9$. Right panel: extrapolation of $p/(N^2T^4)$ (black solid curve), $\Delta/(N^2T^4)$ (red dashed curve), $\epsilon/(N^2T^4)$ (blue dash-dotted curve) and $s/(N^2T^3)$ (green dotted curve) to the $N \to \infty$ limit; the error bars (including statistical and systematic uncertainties) at some reference temperatures are also shown. The horizontal bars on the right-hand side of this plot show the SB limits for the pressure, energy density, and entropy density, from bottom to top; the maroon arrow denotes the large-N limit of the latent heat L_h, as calculated in Ref. [28].

TABLE I. Results of the large-N extrapolation of the parameters appearing in Eq. (5). For each parameter, the first (second) error is the statistical (systematic) one.

Parameter	Extrapolated value
f_1	0.9918(20)(87)
f_2	0.0090(5)(58)
f_3	1.768(36)(71)
f_4	−0.244(46)(29)

investigated here (and of the large-N limit thereof). The extrapolation of our results for $N \to \infty$ strongly suggests the possibility that the hot QCD plasma admits a very simple (yet to be discovered) theoretical description in this limit.

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