Thermoelastic Displacements and Stresses Due to a Heat Source Moving Over the Surface of a Half Plane

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A solution is given for the surface displacement and stresses due to a line heat source that moves at constant speed over the surface of an elastic half plane. The solution is obtained by integration of previous results for the instantaneous point source. The final results are expressed in terms of Bessel functions for which numerically efficient series and asymptotic expressions are given.

Introduction

If two thermally conducting bodies are in sliding contact, the frictional heat generated at the interfaces causes thermoelastic distortion which profoundly influences the extent of the contact area and the distribution of contact stress [1]. This phenomenon – now known as thermoelastic instability – has been observed experimentally in a wide range of practical sliding systems [2, 3].

Several theoretical solutions have been obtained to problems of this kind, e.g., [4–6], but most make the assumption that one of the bodies is either rigid or a nonconductor or both, thus ensuring that the temperature field is stationary in the deformable solid.

In the more general problem, the contact area and hence the temperature field will move with respect to both solids. One way to treat problems of this kind is to use the Green's function for a moving heat source on the surface of the body. The two-dimensional Green's function for a heat source moving over the surface of an elastic half plane was first investigated by Ling and Mow [7], who used Fourier transformation to obtain a simplified solution in which conduction of heat in the direction of motion is neglected. This approximation is justified at large Peclet number – i.e., when

$$\frac{Vx_0}{k} >> 1 \tag{1}$$

where V is the source velocity, x_0 is a characteristic length for the problem and k is the thermal diffusivity of the material.

More recently, Kilaparti and Burton [8] have developed an exact Fourier series solution for a periodic strip heat input. Their series is rather unwieldy, but at large Vx_0/k, it reduces to a form [9] that is simpler than that of Ling and Mow [7].

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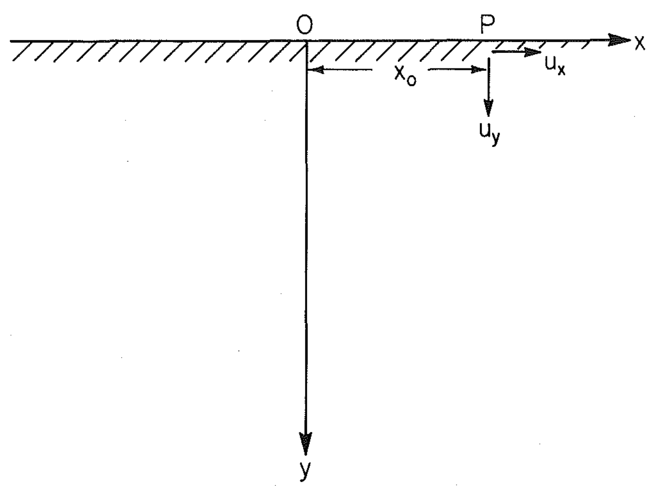

Fig. 1 The heat source is moving to the right at speed *V* and has just reached 0. Stresses and displacements are to be found at *P*.

Tseng and Burton [10] have extended this solution to give the tangential surface stress due to a moving heat source.

In this paper, we shall develop the Green's function for the problem of Kilaparti and Burton, by superposition of previously published results for the instantaneous point heat source [11]. The results obtained are expressed in a simple form and are exact for all values of Vx_0/k in the context of quasi-static, uncoupled thermoelasticity.

Solution

The fundamental problem is illustrated in Fig. 1. A line heat source, q per unit length per unit time, moves from left to right at constant speed V on the traction-free surface of an elastic half plane. We wish to find the components of displacement, u_x, u_y and stress σ_{xx} at the point $P(x_0,0)$ on the surface when the source has reached the origin 0. Plane-strain conditions are assumed.

The moving source can be considered as a sequence of instantaneous sources of strength $q\ \delta t$ at time $(-t)$ and position $(-Vt, 0)$ as t takes all the values in the range $0<t<\infty$.

The surface displacements at $(x, 0)$ due to an instantaneous source Q at $(0, 0)$ are given in [11] as

$$u_x = \frac{\alpha Q(1+\nu)}{\pi\rho c(kt)^{1/2}} \frac{(1-e^{-X^2})}{X} \tag{2}$$

$$u_y = -\frac{\alpha Q(1+\nu)}{\pi\rho c(kt)^{1/2}} F_1(R) \tag{3}$$

where

$$F_1(R) = \frac{2}{\pi^{1/2}} \frac{e^{-R^2}}{R} \int_0^R e^{S^2}\, dS \tag{4}$$

$$X = x/(4kt)^{1/2} \tag{5}$$

$$R = |X| \tag{6}$$

and α, ν, ρ, c are the coefficient of thermal expansion, Poisson's ratio, density, and specific heat, respectively, for the material.

If follows that the displacement components for the moving source are

$$u_x = \frac{\alpha(1+\nu)q}{\pi\rho ck^{1/2}} \int_0^\infty \frac{(1-e^{-X^2})}{X} \frac{dt}{t^{1/2}} \tag{7}$$

$$u_y = -\frac{\alpha(1+\nu)q}{\pi\rho ck^{1/2}} \int_0^\infty F_1(R) \frac{dt}{t^{1/2}} \tag{8}$$

where

$$X = (x_0+Vt)/(4kt)^{1/2} \tag{9}$$

We shall evaluate the integrals (7) and (8) by substituting for t in terms of X (or R). It is convenient to consider the cases $x_0 > 0$, $x_0 < 0$ separately.

***(i)* $x_0 > 0$.** The results for $x_0 > 0$ correspond to points P which have not yet been passed by the source. We note from equations (9) and (6) that $R = X$ and that $X \to \infty$ at both ends of the range of integration ($t \to \infty$, $t \to 0$). There is therefore a time t_0 at which X is a minimum, which occurs when

$$\frac{dX}{dt} = \frac{Vt-x_0}{2t(4kt)^{1/2}} = 0 \tag{10}$$

i.e.,

$$t = t_0 = x_0/V. \tag{11}$$

Substituting in equation (9), we find the minimum value of X is

$$X_0 = (Vx_0/k)^{1/2} \tag{12}$$

Solving equation (9) for t, we find

$0 < t < t_0$

$$t = t_1 = k\{2X^2 - X_0^2 - 2\sqrt{X^2(X^2-X_0^2)}\}/V^2 \tag{13}$$

$t > t_0$

$$t = t_2 = k\{2X^2 - X_0^2 + 2\sqrt{X^2(X^2-X_0^2)}\}/V^2 \tag{14}$$

and hence we can rewrite equation (7) in the form

$$u_x = \frac{\alpha(1+\nu)q}{\pi\rho ck^{1/2}} \left[\int_0^{t_0} \frac{(1-e^{-X^2})}{X} \frac{dt_1}{t_1^{1/2}} + \int_{t_0}^\infty \frac{(1-e^{-X^2})}{X} \frac{dt_2}{t_2^{1/2}}\right] \tag{15}$$

$$= \frac{\alpha(1+\nu)q}{\pi\rho ck^{1/2}} \int_{X=X_0}^{X=\infty} \frac{(1-e^{-X^2})}{X} \left\{\frac{dt_2}{t_2^{1/2}} - \frac{dt_1}{t_1^{1/2}}\right\} \tag{16}$$

Finally, differentiating (13) and (14) and substituting for t_1, t_2 into (16) we find

$$\left\{\frac{dt_2}{t_2^{1/2}} - \frac{dt_1}{t_1^{1/2}}\right\} = \frac{4k^{1/2}dX}{V\sqrt{1-X_0^2/X^2}} \tag{17}$$

and hence

$$u_x = \frac{4\alpha(1+\nu)q}{\pi\rho c\ V} \int_{X_0}^\infty \frac{(1-e^{-X^2})dX}{\sqrt{X^2-X_0^2}} \tag{18}$$

Similarly, from equations (8) and (17)

$$u_y = -\frac{4\alpha(1+\nu)q}{\pi c\rho V} \int_{X_0}^\infty \frac{XF_1(X)dX}{\sqrt{X^2-X_0^2}} \tag{19}$$

***(ii)* $x_0 < 0$.** When the source has passed P and $x_0 < 0$, there is no minimum value of X as defined by equation (10) and X varies monotonically from $-\infty$ to $+\infty$ through the range of integration. Solving equation (9) for t, we find

$0 < t < t_0, \quad X < 0$

$$t = t_1 = k\{2X^2 + X_0^2 - 2\sqrt{X^2(X^2+X_0^2)}\}/V^2 \tag{20}$$

$t > t_0, \quad X < 0$

$$t = t_2 = k\{2X^2 + X_0^2 + 2\sqrt{X^2(X^2+X_0^2)}\}/V^2 \tag{21}$$

where we now redefine

$$t_0 = -x_0/V \tag{22}$$

$$X_0 = (-Vx_0/k)^{1/2} \tag{23}$$

Notice that X_0 is *not* now the minimum value of X – in fact $X \to 0$ as $t \to t_0$, as can be seen on substituting (22) into (9). The time $t = t_0$ corresponds in this case to the instant when the source is passing the point P.

Equation (15) now takes the form

$$u_x = \frac{\alpha(1+\nu)q}{\pi\rho ck^{1/2}} \int_{X=-\infty}^{X=0} \left[\frac{(1-e^{-X^2})}{X} \frac{dt_1}{t_1^{1/2}} + \int_{X=0}^{X=+\infty} \frac{(1-e^{-X^2})}{X} \frac{dt_2}{t_2^{1/2}}\right] \tag{24}$$

and since $(1-e^{-X^2})/X$ is odd in X, the two integrals can be combined to give

$$u_x = \frac{\alpha(1+\nu)q}{\pi\rho\ ck^{1/2}} \int_{X=0}^{X=\infty} \frac{(1-e^{-X^2})}{X} \left\{\frac{dt_2}{t_2^{1/2}} + \frac{dt_1}{t_1^{1/2}}\right\} \tag{25}$$

After substitution from equations (20) and (21), this reduces to

$$u_x = \frac{4\alpha(1+\nu)q}{\pi\rho c\ V} \int_0^\infty \frac{(1-e^{-X^2})dX}{\sqrt{X^2+X_0^2}} \tag{26}$$

We follow a similar procedure to determine u_y, but, since $F_1(X)$ an even function, we find

$$u_y = -\frac{\alpha(1+\nu)q}{\pi\rho ck^{1/2}} \int_{X=0}^{X=\infty} F_1(X) \left\{\frac{dt_2}{t_2^{1/2}} - \frac{dt_1}{t_1^{1/2}}\right\} \tag{27}$$

$$= -\frac{4\alpha(1+\nu)q}{\pi\rho c\ V} \int_0^\infty F_1(X)\, dX \tag{28}$$

after substitution.

Notice that equation (28) is independent of X_0. In other words, the normal displacement u_y remains constant once the source has passed P.

***(iii)* Stress at the Surface.** We anticipate that the maximum thermal stress due to the moving source will occur at the surface and hence we calculate the component σ_{xx} (the other two components σ_{xy}, σ_{yy} being zero because of the zero traction condition). Expressions for σ_{xx} are not given in [11],

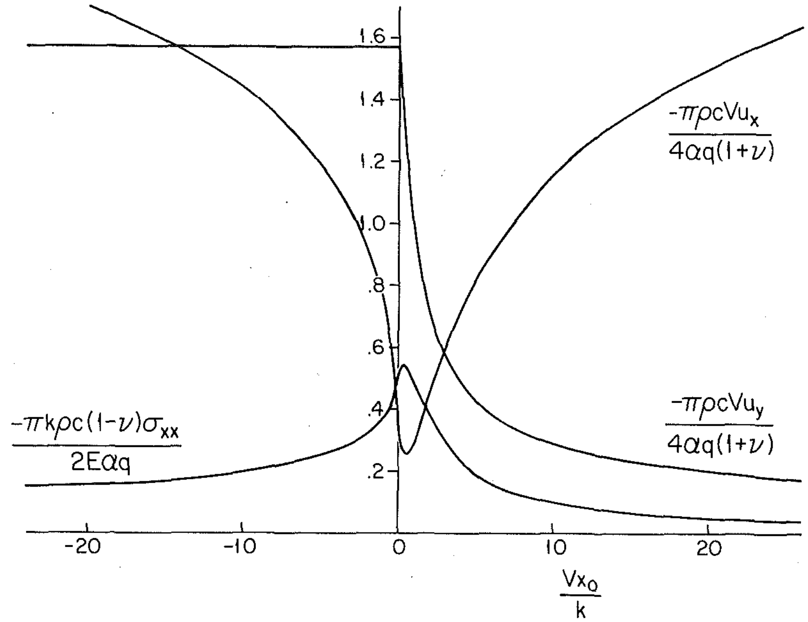

Fig. 2 Displacement u_x, u_y and tangential stress σ_{xx} as functions of Vx_0/k. The source is at 0 and is moving to the right at speed V.

but they are easily obtained in the same manner as the displacement components by superposition of an axisymmetric thermal stress field and an isothermal correction field to leave surface free of traction. The contribution from the axisymmetric field is given by equations (3) and (4) of [11].

If a half plane is subjected to a purely normal traction σ_{yy}, it can be shown that the stress σ_{xx} induced at the surface is everywhere equal to σ_{yy} (except for a possible additive constant). Hence, the corrective traction that cancels the component $\sigma_{\theta\theta}$ (equation (4) of [11]) leaves behind a stress

$$\sigma_{(xx=0)} = \sigma_{rr} - \sigma_{\theta\theta} = -\frac{\alpha QE}{2\pi\rho c(1-\nu)kt}\left[\frac{(1-e^{-R^2})}{R^2} - e^{-R^2}\right] \tag{29}$$

Following the same procedure as in the preceding section, we find that the moving source produces a surface stress

$$\sigma_{xx} = -\frac{\alpha Eq}{2\pi\rho c(1-\nu)k}\int_0^\infty\left[\frac{1-e^{-R^2}}{R^2} - e^{-R^2}\right]\frac{dt}{t} \tag{30}$$

The integrand is even in R and hence the substitution process is similar to that used for u_y in section (*ii*). For $X_0 > 0$,

$$\sigma_{xx} = -\frac{\alpha Eq}{2\pi\rho c(1-\nu)k}\int_{X=X_0}^{X=\infty}\left[\frac{1-e^{-X^2}}{X^2} - e^{-X^2}\right]\left(\frac{dt_2}{t_2} - \frac{dt_1}{t_1}\right) \tag{31}$$

$$= \frac{2\alpha Eq}{\pi\rho c(1-\nu)k}\int_{X_0}^\infty\left[\frac{1-e^{-X^2}}{X^2} - e^{-X^2}\right]\frac{dX}{\sqrt{X^2-X_0^2}} \tag{32}$$

while for $x_0 < 0$,

$$\sigma_{xx} = -\frac{\alpha Eq}{2\pi\rho c(1-\nu)k}\int_{X=0}^{X=\infty}\left[\frac{1-e^{-X^2}}{X^2} - e^{-X^2}\right]\left(\frac{dt_2}{t_2} - \frac{dt_1}{t_1}\right) \tag{33}$$

$$= -\frac{2\alpha Eq}{\pi\rho c(1-\nu)k}\int_0^\infty\left[\frac{1-e^{-X^2}}{X^2} - e^{-X^2}\right]\frac{dX}{\sqrt{X^2+X_0^2}} \tag{34}$$

In each case, definitions of t_1, t_2, and X_0 are as in the appropriate part of section (11).

Results

The integrals in equations (18), (19), (26), (28), (30), and (34) are evaluated in Appendix I, giving the results

$$u_x = -\frac{2\alpha q(1+\nu)}{\pi\rho c\ V}\left\{\log\left|\frac{Vx_0}{k}\right| + e^{-Vx_0/2k}K_0\left(\frac{Vx_0}{2k}\right)\right\} + C \tag{35}$$

$$u_y = -\frac{2\alpha q(1+\nu)}{\rho c\ V}e^{-Vx_0/2k}I_0\left(\frac{Vx_0}{2k}\right); x_0 > 0 \tag{36}$$

$$= -\frac{2\alpha q(1+\nu)}{\rho c\ V};\quad x_0 < 0 \tag{37}$$

$$\sigma_{xx} = -\frac{2\alpha qE}{\pi\rho c(1-\nu)k}\left\{\frac{k}{Vx_0} - \frac{1}{2}e^{-Vx_0/2k}K_1\left(\frac{Vx_0}{2k}\right)\right\}. \tag{38}$$

Equations (35) and (38) are valid for all values of x_0 provided we interpret $K_0(-x) = K_0(x)$, $K_1(-x) = -K_1(x)$.

The expression for u_x is unbounded as $x_0 \to \pm\infty$ and hence the tangential displacement cannot be referred to the point at infinity. For this reason, it might be more convenient in some applications to use the derivative

$$\frac{\partial u_x}{\partial x} = -\frac{2\alpha q(1+\nu)}{\pi\rho ck}\left\{\frac{k}{Vx_0} - \frac{1}{2}e^{-Vx_0/2k}\left[K_1\left(\frac{Vx_0}{2k}\right) + K_0\left(\frac{Vx_0}{2k}\right)\right]\right\} \tag{39}$$

We also record here the surface temperature due to the moving source which is

$$T = \frac{q}{\pi k\rho c}e^{-Vx_0/2k}K_0\left(\frac{Vx_0}{2k}\right) \tag{40}$$

(see Carslaw and Jaeger [12] Section 10.7).

The results for u_x, u_y, and σ_{xx} are plotted in Fig. 2. Notice that the maximum (compressive) value of σ_{xx} and the minimum u_x occur just ahead of the heat source, while the normal displacement u_y is constant behind the source.

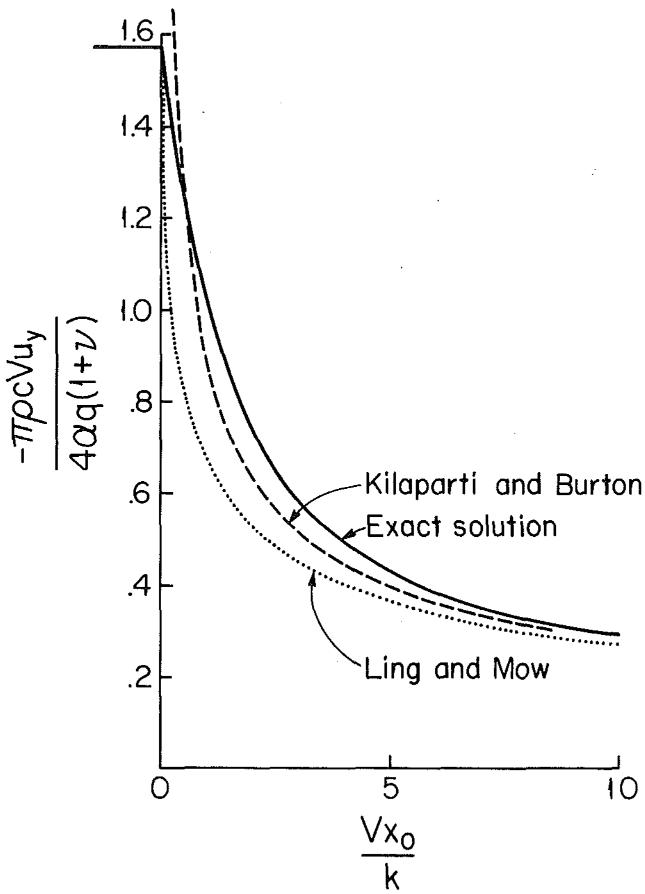

Fig. 3 Comparison of the solution for u_y – equation (36), with the approximate solution of Ling and Mow [7] and the one-term asymptotic expression of Kilaparti and Burton [9]

Numerical Evaluation and Convergence. The principal use of these results is as a Green's function for thermoelastic contact with moving heat sources. Such problems tend to involve extensive numerical computations and it is therefore important to find representations that can be easily and efficiently evaluated numerically.

The Bessel functions I_0, K_0, K_1, occurring in equations (35)–(40) can be expanded in convergent series when Vx_0/k is small and in asymptotic expansions when it is large. The appropriate expressions are given in Appendix II. Notice that the singular terms in equations (35), (38) are (39) cancel corresponding terms in the Bessel function expansions, leaving a bounded result at $Vx_0/k = 0$.

In all cases, the ranges of the two forms overlap to give good accuracy throughout the range. The series for u_y (equations (36), (*A*22)) is particularly convergent. For example, four-digit accuracy throughout the range can be obtained by using just the first six terms of (*A*22) for $0 < Vx_0/k < 8$ and six terms of the asymptotic expansion (*A*26) for $Vx/k > 8$. By contrast, Kilaparti and Burton [9] used 1000 terms to evaluate their Fourier expansion solution for u_y and Tseng and Burton [10] needed 5000 terms for convergence of their expansion of σ_{xx}.

Comparison With Previous Approximate Solutions. In Fig. 3, Ling and Mow's approximate solution [7] for u_y is compared with the exact solution (equation (36)). Ling and Mow's analysis is restricted to large Peclet number, which for the moving point source is equivalent to $|Vx_0/k| >> 1$. As we would expect, an asymptotic expansion of their result has the same leading term as equations (36), and (*A*26) except for a factor of 2 error in [7], noted already by Kilaparti and Burton [9]. This leading term was also extracted by Kilaparti and Burton [9, equation (25)] and is plotted in Fig. 3. It actually turns out to be a better approximation to the exact solution than Ling and Mow's expression except for $Vx_0/k < 0.4$.

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References

1 Barber, J. R., "Thermoelastic Instabilities in the Sliding of Conforming Solids," *Proc. Roy. Soc.*, Series *A*, Vol. 312, 1969, pp. 381–394.
2 Burton, R. A., ed., "Thermal Deformation in Frictionally Heated Systems," Proceedings of a workshop held at Annapolis Md., June 1979; *Wear*, Vol. 59 1980.
3 Dow, T. A., ed., "Thermomechanical Effects in Wear, "Proceedings of workshop held at Columbus, Ohio, June 1981, *Wear*, Vol. 79, 1982.
4 Burton, R. A., Kilaparti, S. R., and Nerlikar, V., "A Limiting Stationary Configuration With Partially Contacting Surfaces," *Wear*, Vol. 24, 1973, pp. 199–206.
5 Barber, J. R., "Some Thermoelastic Contact Problems Involving Frictional Heating," *Q. J. Mech. Appl. Math*, Vol. 29, 1976, pp. 1–13.
6 Burton, R. A., and Nerlikar, V., "Large Disturbance Solutions for Initially Flat, Frictionally Heated Thermoelastically Deformed Surfaces," ASME *Journal of Lubrication Technology*, Vol. 97, 1975, pp. 539–545.
7 Ling, F. F., and Mow, V. C., "Surface Displacement of a Convective Elastic Half-Space Under an Arbitrarily Distributed Fast-Moving Heat Source," ASME *Journal of Basic Engineering*, Vol. 87, 1965, pp. 729–734.
8 Kilaparti, S. R., and Burton, R. A., "A Moving Hot-Spot Configuration for a Seal-Like Geometry, With Frictional Heating, Expansion and Wear," ASLE *Trans.*, Vol. 20, 1977, pp. 64–70.
9 Kilaparti, S. R., and Burton, R. A., "The Thermoelastic Patch Contact Problem for Large Peclet Number," ASME *Journal of Lubrication Technology*, Vol. 100, 1978, pp. 65–69.
10 Tseng, M. L., and Burton, R. A., "Thermal Stress in a Two-Dimensional (Plane Stress) Half-Space for a Moving Heat Input," *Wear*, Vol. 79, 1982, pp. 1–9.
11 Barber, J. R., and Martin-Moran, C. J., "Green's Functions for Transient Thermoelastic Contact Problems for the Half-Plane," *Wear*, Vol. 79, 1982, pp. 11–19.
12 Carslaw, H. S., and Jaeger, J. C., *Conduction of Heat in Solids*, 2nd Ed., Clarendon Press, Oxford, 1959.
13 Gradshteyn, I. S., and Ryzhik, I. M., (GR), *Tables of Integrals Series and Products*, Academic Press, New York, 1980.

APPENDIX I

Evaluation of Integrals

In this appendix, we evaluate the integrals in equations (18), (19), (26), (28), (30), and (34) in terms of Bessel functions.

(*i*) The integral

$$I_1 = \int_{X_0}^{\infty} \frac{(1-e^{-X^2})dX}{\sqrt{X^2-X_0^2}} \quad (A1)$$

from equation (18) is unbounded becuase of the behavior of the integrand at infinity. However, we can write $I_1 = J_1 + J_2$ where

$$J_2 = -\int_{X_0}^{\infty} \frac{e^{-X^2}\,dX}{\sqrt{X^2-X_0^2}} \quad (A2)$$

and is bounded, and

$$\frac{dJ_1}{dX_0} = \underset{c\to\infty}{Lt}\ \frac{d}{dX_0}\int_{X_0}^{c} \frac{dX}{\sqrt{X^2-X_0^2}} = -\frac{1}{X_0}. \quad (A3)$$

Substituting $X = X_0\, ch(y)$ in (*A*2), we find

$$J_2 = e^{-X_0^2/2}\int_0^{\infty} e^{-\frac{1}{2}R_0^2\, ch(2y)}\,dy = -1/2\, e^{-\frac{X_0^2}{2}} K_0\left(\frac{X_0^2}{2}\right) \quad (A4)$$

by G R, Section 3.547 [13].

Hence, we can write

$$I_1 = -\log X_0 - \frac{1}{2} e^{-\frac{X_0^2}{2}} K_0(X_0^2/2) + \text{constant} \qquad (A5)$$

(*ii*) From equation (19),

$$I_2 = \int_{X_0}^{\infty} \frac{XF_1(X)dX}{\sqrt{X^2 - X_0^2}} = \frac{2}{\pi^{1/2}} \int_{X_0}^{\infty} \int_0^X \frac{e^{S^2 - X^2} dSdX}{\sqrt{X^2 - X_0^2}} \qquad (A6)$$

$$= \frac{2}{\pi^{1/2}} \int_0^{\pi/2} \int_0^{X_0/\cos\theta} \frac{e^{(S^2 - X_0^2/\cos^2\theta)} dS\, d\theta}{\cos\theta} \qquad (A7)$$

where $\cos\theta = X_0/X$.

Treating S, θ as polar coordinates and reducing to rectangular coordinates through $X = S \cos\theta$, $Y = S \sin\theta$, we obtain

$$I_2 = \frac{2}{\pi^{1/2}} e^{-X_0^2} \int_0^{X_0} \frac{e^{X^2}}{X} \int_0^{\infty} e^{-(X_0^2/X^2 - 1)Y^2} dY\, dX \qquad (A8)$$

$$= \int_0^{X_0} \frac{e^{(X^2 - X_0^2)} dX}{\sqrt{X_0^2 - X^2}} = \frac{1}{2} \int_0^{\pi} e^{-X_0^2/2(1-\cos\phi)} d\phi \qquad (A9)$$

$$= \pi/2 e^{-X_0^2/2} I_0(X_0^2/2) \qquad (A10)$$

by G R, Section 8.431 [13].

(*iii*) From equation (26),

$$I_3 = \int_0^{\infty} \frac{(1 - e^{-X^2})dX}{\sqrt{X^2 + X_0}}$$

$$= -\log X_0 - \frac{1}{2} e^{X_0^2} K_0(X_0^2/2) + \text{constant} \qquad (A11)$$

by a similar procedure to (*i*) in the foregoing, using the substitution $X = X_0\, sh(y)$ and G R, Section 3.547.4 [13].

(*iv*) From equation (28)

$$I_4 = \int_0^{\infty} F_1(X)dX = \pi/2 \qquad (A12)$$

(see [11], equation (14)).

(*v*) From equation (32)

$$I_5 = \int_{X_0}^{\infty} \left[\frac{1 - e^{-X^2}}{X^2} - e^{-X^2} \right] \frac{dX}{\sqrt{X^2 - X_0^2}} = \frac{1}{X_0^2} - J_1 - J_2 \qquad (A13)$$

where

$$J_1 = \int_{X_0}^{\infty} \frac{e^{-X^2} dX}{X^2 \sqrt{X^2 - X_0^2}}; \quad J_2 = \int_{X_0}^{\infty} \frac{e^{-X^2} dX}{\sqrt{X^2 - X_0^2}} \qquad (A14)$$

Integrating by parts,

$$\int \frac{e^{-X^2} dX}{\sqrt{X^2 - X_0^2}} = \frac{\sqrt{X^2 - X_0^2}}{X} e^{-X^2} + \int \sqrt{X^2 + X_0^2} e^{-X^2}\, dX$$

$$+ \int \frac{\sqrt{X^2 + X_0^2} e^{-X^2} dX}{X^2} \qquad (A15)$$

It follows that

$$J_2 = 2 \int_{X_0}^{\infty} \sqrt{X^2 - X_0^2} e^{-X^2}\, dX + \int_{X_0}^{\infty} \frac{\sqrt{X^2 - X_0^2} e^{-X^2} dX}{X^2} \qquad (A16)$$

$$= 2 \int_{X_0}^{\infty} \sqrt{X^2 - X_0^2} e^{-X^2}\, dX + J_2 - X_0^2 J_1 \qquad (A17)$$

and hence

$$J_1 = \frac{2}{X_0^2} \int_{X_0}^{\infty} \sqrt{X^2 - X_0^2} e^{-X^2}\, dX. \qquad (A18)$$

The integrals J_1, J_2 can now be evaluated as in (*i*), in the foregoing using the substitution $X = X_0\, ch(y)$, giving

$$I_5 = \frac{1}{X_0^2} - \frac{1}{2} e^{-\frac{X_0^2}{2}} K_1(X_0^2/2) \qquad (A19)$$

(*vi*) We use the same procedure for the integral

$$I_6 = \int_0^{\infty} \left[\frac{1 - e^{-X^2}}{X^2} - e^{-X^2} \right] \frac{dX}{\sqrt{X^2 + X_0^2}} \qquad (A20)$$

from equation (34), obtaining

$$I_6 = -\frac{1}{X_0^2} + \frac{1}{2} e^{\frac{X_0^2}{2}} K_1(X_0^2/2). \qquad (A21)$$

APPENDIX II

Series and Asymptotic Forms for Bessel Functions

For small arguments (positive or negative) we can use the convergent series

$$I_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2} \qquad (A22)$$

$$K_0(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{(k!)^2} [\psi(k+1) - \log |x/2|] \qquad (A23)$$

$$K_1(x) = \frac{1}{x} + \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!} \left[\frac{1}{2(k+1)} + \log |x/2| - \psi(k+2) \right] \qquad (A24)$$

where

$$\psi(k+1) = -c + \sum_{i=1}^{k} \frac{1}{i} \qquad (A25)$$

and $c = 0.577215\ldots$ is Euler's constant.
(see Gradshteyn and Ryzhik [13], Sections 8.447, 8.486.18, 8.365.4, and 9.73).

Asymptotic expansions for large positive arguments are

$$I_0(x) = \frac{e^x}{(2\pi x)^{1/2}} \left[1 + \frac{1^2}{8x} + \frac{1^2 \cdot 3^2}{2!(8x)^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{3!(8x)^3} + \ldots \right] \qquad (A26)$$

$$K_0(x) = \left(\frac{\pi}{2x} \right)^{1/2} e^{-x} \left[1 - \frac{1^2}{8x} + \frac{1^2 \cdot 3^2}{2!(8x)^2} - \frac{1^2 \cdot 3^2 \cdot 5^2}{3!(8x)^3} + \ldots \right] \qquad (A27)$$

$$K_1(x) = \left(\frac{\pi}{2x} \right)^{1/2} e^{-x} \left[1 + \frac{(4 - 1^2)}{8x} + \frac{(4 - 1^2)(4 - 3^2)}{2!(8x)^2} + \ldots \right] \qquad (A28)$$

(see [12], Appendix III).